Sampling numbers and function spaces

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Abstract

We want to recover a continuous function $f : (0,1)^d \to \mathbb{C}$ using only its function values. Let us assume, that f is from the unit ball of some function space (for example a fractional Sobolev space or a Besov space) and the precision of the reconstruction is measured in the norm of another function space of this type. We describe the rate of convergence of the optimal sampling method (linear as well as nonlinear) in this setting.

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1 Introduction

We study the following question. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $B^s_{pq}(\Omega)$ denote the scale of Besov spaces on Ω, see Definition A.1 and Definition A.3 for details. We try to approximate $f \in B^{s_1}_{p_1q_1}(\Omega)$ in the norm of another Besov space, say $B^{s_2}_{p_2q_2}(\Omega)$, by a linear sampling method

$$S_n f = \sum_{j=1}^{n} f(x_j) h_j, \tag{1.1}$$

where $h_j \in B^{s_2}_{p_2q_2}(\Omega)$ and $x_j \in \Omega$. First of all, we have to give a meaning to the pointwise evaluation in (1.1). For this reason, we shall restrict ourselves to the case

$$s_1 > \frac{d}{p_1},$$

which guarantees the continuous embedding $B^{s_1}_{p_1q_1}(\Omega) \hookrightarrow C(\bar{\Omega})$. Second, we always assume that the embedding $B^{s_1}_{p_1q_1}(\Omega) \hookrightarrow B^{s_2}_{p_2q_2}(\Omega)$ is compact, which holds if and only if

$$s_1 - s_2 > d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ .$$

We measure the worst case error of $S_n f$ by

$$\sup\{||f - S_n f|B^{s_2}_{p_2q_2}(\Omega)|| : ||f|B^{s_1}_{p_1q_1}(\Omega)|| \leq 1\}. \tag{1.2}$$

The same worst case error may also be considered for nonlinear sampling methods

$$S_n f = \varphi(f(x_1), \ldots, f(x_n)), \tag{1.3}$$

where $\varphi : \mathbb{C}^n \to B^{s_2}_{p_2q_2}(\Omega)$ is an arbitrary mapping. In this paper, we discuss the decay of (1.2) for linear (1.1) and nonlinear (1.3) sampling methods.

In some cases we restrict ourselves to the case $\Omega = I^d = (0,1)^d$. This allows to describe the optimal sampling operator more explicitly. However, we conjecture, that many of these results can be generalised to general bounded Lipschitz domains.

Let $L_p(\Omega)$ stand for the usual Lebesgue space and $W^k_p(\Omega), k \in \mathbb{N}$, denotes the classical Sobolev space over Ω. Then it is well known that

$$\inf_{S_n} \sup\{||f - S_n f|L_{p_2}(\Omega)|| : ||f|W^k_{p_1}(\Omega)|| \leq 1\} \approx n^{-\frac{k}{d}+(\frac{1}{p_1}-\frac{1}{p_2})_+}, \tag{1.4}$$

where the infimum in (1.4) runs over all linear sampling operators S_n, see (1.1) (cf. [5] or [10]). The result remains true if we switch to the general situation where nonlinear methods S_n are allowed. In [12], this statement has been proved for arbitrary bounded Lipschitz domain, but with the Sobolev spaces replaced by the more general scales of Besov and Triebel-Lizorkin spaces. The target space was always given by $L_{p_2}(\Omega)$. The proof given there uses the simple structure of the Lebesgue space. It is the main aim of this paper to generalise (1.4) and to investigate also other "target" spaces.

Let us present our main results. If $s_2 > 0$, then the quantity

$$\inf_{S_n} \sup\{||f - S_n f|B^{s_2}_{p_2q_2}(\Omega)|| : ||f|B^{s_1}_{p_1q_1}(\Omega)|| \leq 1\} \tag{1.5}$$

behaves like

$$n^{-\frac{s_1-s_2}{d}+(\frac{1}{p_1}-\frac{1}{p_2})_+}$$

in both, the linear as well as the nonlinear setting. We prove this result only for the special case of $\Omega = (0,1)^d$. However in this situation we are able to give an explicit description of in order optimal operator which we are going to introduce now. Namely, if $n \approx 2^{kd}$, where $k \in \mathbb{N}$ is fixed, we use a smooth decomposition of unity $\{\psi_{k,\nu}\}$ such that $\sum_\nu \psi_{k,\nu}(x) = 1$ for $x \in (0,1)^d$ where the support of $\psi_{k,\nu}$ is concentrated around $2^{-k}\nu$. Then we approximate f locally on $\operatorname{supp}\psi_{k,\nu}$ by a polynomial $g_{k,\nu}$ and define

$$S_n f = \sum_\nu g_{k,\nu}\psi_{k,\nu}.$$

To calculate each of the $2^{(k+2)d}$ functions $g_{k,\nu}$ we need to combine $\binom{M+d-1}{d}$ function values of f in a linear way. Altogether, we need $2^{(k+2)d}\binom{M+d-1}{d} \approx 2^{kd} \approx n$ function values of f to obtain $S_n f$. Here, $M > s_1$ is a fixed natural number. The generalisation of this construction to bounded Lipschitz domains remains a subject of further study.

If $s_2 < 0$, we give the following characterisation of (1.5). If $p_1 \geq p_2$ or $p_1 < p_2$ and $\dfrac{d}{p_2} - \dfrac{d}{p_1} > s_2$, then (1.5) decays like

$$n^{-\frac{s_1}{d}}$$

and if $p_1 < p_2$ and $0 > s_2 > \dfrac{d}{p_2} - \dfrac{d}{p_1}$, then (1.5) behaves like

$$n^{-\frac{s_1}{d}+\frac{s_2}{d}+\frac{1}{p_1}-\frac{1}{p_2}}.$$

All these results hold for linear as well as nonlinear methods S_n.

These estimates can be applied in connection with elliptic differential operators, which was the actual motivation for this research, c.f. [6] and [7]. Let us briefly introduce this setting. Let

$$\mathcal{A} : H \to G$$

be a bounded linear operator from a Hilbert space H to another Hilbert space G. We assume that $\mathcal{A}$ is boundedly invertible, hence

$$\mathcal{A}(u) = f$$

has a unique solution for every $f \in G$. A typical application is an operator equation, where $\mathcal{A}$ is an elliptic differential operator, and we assume that

$$\mathcal{A} : H_0^s(\Omega) \to H^{-s}(\Omega),$$

where Ω is a bounded Lipschitz domain, $H_0^s(\Omega)$ is a function space of Sobolev type with fractional order of smoothness $s > 0$ of functions vanishing on the boundary and H^{-s} is a function space of Sobolev type with negative smoothness $-s < 0$. The classical example is the Poisson equation

$$-\Delta u = f \quad \text{in} \quad \Omega \qquad \text{and} \qquad u = 0 \quad \text{on} \quad \partial\Omega.$$

Here, $s = 1$ and

$$\mathcal{A} = -\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$$

is bounded and boundedly invertible. We want to approximate the *solution operator* $u = S(f)$ using only function values of f.
We define the n-th linear sampling number of the identity $id : H^{-1+t}(\Omega) \to H^{-1}(\Omega)$ by

$$g_n^{\rm lin}(id : H^{-1+t}(\Omega) \to H^{-1}(\Omega)) = \inf_{S_n} ||id - S_n|\mathcal{L}(H^{-1+t}(\Omega), H^{-1}(\Omega))||, \qquad (1.6)$$

where t is a positive real number with $-1+t > \frac{d}{2}$, and the n-th linear sampling number of $S : H^{-1+t}(\Omega) \to H^1(\Omega)$ by

$$g_n^{\rm lin}(S : H^{-1+t}(\Omega) \to H^1(\Omega)) = \inf_{S_n} ||S - S_n|\mathcal{L}(H^{-1+t}(\Omega), H^1(\Omega))||. \qquad (1.7)$$

The infimum in (1.6) and (1.7) runs over all linear operators S_n of the form (1.1) and $\mathcal{L}(X, Y)$ stands for the space of bounded linear operators between two Banach spaces X and Y, equipped with the classical operator norm.
It turns out that these quantities are equivalent (up to multiplicative constants which do not depend neither on f nor on n) and are of the asymptotic order

$$g_n^{\rm lin}(S : H^{-1+t}(\Omega) \to H^1(\Omega)) \approx g_n^{\rm lin}(id : H^{-1+t}(\Omega) \to H^{-1}(\Omega)) \approx n^{-\frac{-1+t}{d}}.$$

We refer to [6] and [7] for a detailed discussion of this approach. The estimates of sampling numbers of embedding between two function spaces translates therefor into estimates of sampling numbers of the solution operator S. We observe that the more regular f, the faster is the decay of the linear sampling numbers of the solution operator S. Let us also point out that optimal linear methods (not restricted to use only the function values of f) achieve asymptotically a better rate of convergence, namely $n^{-\frac{t}{d}}$. Hence, the limitation to the sampling operators results in a serious restriction. One has to pay at least $n^{1/d}$ in comparison with optimal linear methods.
Using our estimates of sampling numbers of identities between Besov and Triebel-Lizorkin spaces, this result may be generalised as follows.[1] If $p \geq 2$, $1 \leq q \leq \infty$ and $-1+t > \frac{d}{p}$ then

$$g_n^{\rm lin}(S : B_{pq}^{-1+t}(\Omega) \to H^1(\Omega)) \approx g_n^{\rm lin}(id : B_{pq}^{-1+t}(\Omega) \to H^{-1}(\Omega)) \approx n^{-\frac{-1+t}{d}}.$$

If $p < 2$ with $\dfrac{1}{p} > \dfrac{1}{d} + \dfrac{1}{2}$, $1 \leq q \leq \infty$ and $-1+t > \frac{d}{p}$ then

$$g_n^{\rm lin}(S : B_{pq}^{-1+t}(\Omega) \to H^1(\Omega)) \approx g_n^{\rm lin}(id : B_{pq}^{-1+t}(\Omega) \to H^{-1}(\Omega)) \approx n^{-\frac{t}{d}+\frac{1}{p}-\frac{1}{2}}.$$

Finally, if $p < 2$ with $\dfrac{1}{p} < \dfrac{1}{d} + \dfrac{1}{2}$, $1 \leq q \leq \infty$ and $-1+t > \frac{d}{p}$ then

$$g_n^{\rm lin}(S : B_{pq}^{-1+t}(\Omega) \to H^1(\Omega)) \approx g_n^{\rm lin}(id : B_{pq}^{-1+t}(\Omega) \to H^{-1}(\Omega)) \approx n^{-\frac{-1+t}{d}}.$$

We prove the same results also for the nonlinear sampling numbers $g_n(S)$. Altogether, the regularity information of f may now be described by an essentially broader scale of function spaces.

[1] Although the results are stated only for Besov spaces, they are proved also for Triebel-Lizorkin spaces, which include also fractional Sobolev spaces as a special case.

All the unimportant constants are denoted by the letter c, whose meaning may differ from one occurrence to another. If $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are two sequences of positive real numbers, we write $a_n \lesssim b_n$ if, and only if, there is a positive real number $c > 0$ such that $a_n \le c\, b_n, n \in \mathbb{N}$. Furthermore, $a_n \approx b_n$ means that $a_n \lesssim b_n$ and simultaneously $b_n \lesssim a_n$.

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2 Sampling numbers

The notation and basic facts about function spaces, which we shall need later on, are included in the Appendix.

We now introduce the concept of sampling numbers.

Definition 2.1. Let Ω be a bounded Lipschitz domain. Let $G_1(\Omega)$ be a space of continuous functions on Ω and $G_2(\Omega) \subset D'(\Omega)$ be a space of distributions on Ω. Suppose, that the embedding

$$id : G_1(\Omega) \hookrightarrow G_2(\Omega)$$

is compact.

For $\{x_j\}_{j=1}^n \subset \Omega$ we define the *information map*

$$N_n : G_1(\Omega) \to \mathbb{C}^n, \qquad N_n f = (f(x_1), \dots, f(x_n)), \quad f \in G_1(\Omega).$$

For any (linear or nonlinear) mapping $\varphi_n : \mathbb{C}^n \to G_2(\Omega)$ we consider

$$S_n : G_1(\Omega) \to G_2(\Omega), \qquad S_n = \varphi_n \circ N_n.$$

(i) Then, for all $n \in \mathbb{N}$, the $n-$th *sampling number* $g_n(id)$ is defined by

$$g_n(id) = \inf_{S_n} \sup\{||f - S_n f|G_2(\Omega)|| : ||f|G_1(\Omega)|| \le 1\}, \tag{2.1}$$

where the infimum is taken over all n-tuples $\{x_j\}_{j=1}^n \subset \Omega$ and all (linear or nonlinear) φ_n.

(ii) For all $n \in \mathbb{N}$ the $n-$th *linear sampling number* $g_n^{\rm lin}(id)$ is defined by (2.1), where now only linear mappings φ_n are admitted.

2.1 The case $s_2 > 0$

In this subsection, we discuss the case where $\Omega = I^d = (0,1)^d$ is the unit cube, $G_1(\Omega) = A^{s_1}_{p_1q_1}(\Omega)$ and $G_2(\Omega) = A^{s_2}_{p_2q_2}(\Omega)$ with $s_1 > \dfrac{d}{p_1}$ and $s_1 - d\Big(\dfrac{1}{p_1} - \dfrac{1}{p_2}\Big)_+ > s_2 > 0$. Here, $A^s_{pq}(\Omega)$ stands either for a Besov space $B^s_{pq}(\Omega)$ or a Triebel-Lizorkin space $F^s_{pq}(\Omega)$, see Definition A.3 for details. We start with the most simple and most important case, namely when $p_1 = p_2 = q_1 = q_2$.

Proposition 2.2. *Let $\Omega = I^d = (0,1)^d$. Let $G_1(\Omega) = B^{s_1}_{pp}(\Omega)$ and $G_2(\Omega) = B^{s_2}_{pp}(\Omega)$ with $1 \le p \le \infty$,*

$$s_1 > \frac{d}{p}, \quad \textit{and} \quad s_1 > s_2 > 0.$$

Then

$$g_n^{\rm lin}(id) \lesssim n^{-\frac{s_1 - s_2}{d}}.$$

Proof. First, we introduce necessary notation. Let $a > 0$, $z \in \mathbb{R}^d$ and $U \subset \mathbb{R}^d$. Then

$$aU = \{ax : x \in U\} \text{ and } z + aU = \{z + ax : x \in U\}. \tag{2.2}$$

Furthermore, if $k \in \mathbb{N}_0$ and $\nu \in \mathbb{Z}^d$, we set

$$Q_{k,\nu} = \{x \in \mathbb{R}^d : 2^{-k}\nu_i < x_i < 2^{-k}(\nu_i + 1)\},$$
$$Q^{k,\nu} = \{x \in I^d : 2^{-k}\Big(\nu_i - \frac{1}{2}\Big) < x_i < 2^{-k}\Big(\nu_i + \frac{3}{2}\Big)\}.$$

We point out, that (up to a set of measure zero)

$$I^d = \bigcup \{Q_{k,\nu} : 0 \le \nu_i \le 2^k - 1, i = 1, 2, \dots, d\}.$$

Next, we introduce smooth decomposition of unity, first on $\mathbb{R}^d$ and then its restriction to I^d. Let $\tilde{\psi} \in S(\mathbb{R}^d)$ with

$$\operatorname{supp} \tilde{\psi} \subset \Big(-\frac{1}{2}, \frac{3}{2}\Big)^d \quad \text{and} \quad \sum_{\nu \in \mathbb{Z}^d} \tilde{\psi}(x - \nu) = 1, \quad x \in \mathbb{R}^d.$$

Then we define

$$\psi_{k,\nu}(x) = \begin{cases} \tilde{\psi}(2^k x - \nu), & \text{if } x \in I^d, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Let us denote $A_k = \{-1, 0, \dots, 2^k\}^d$. By (2.3), the following identities are true for every $k \in \mathbb{N}$:

$$\sum_{\nu \in \mathbb{Z}^d} \psi_{k,\nu}(x) = \sum_{\nu \in A_k} \psi_{k,\nu}(x) = \chi_{I_d}(x) = \begin{cases} 1, & \text{if } x \in I^d, \\ 0 & \text{otherwise,} \end{cases}$$
$$\operatorname{supp} \psi_{k,\nu} \subset Q^{k,\nu}, \qquad \nu \in A_k.$$

Now we define linear approximation operators $\tilde{S}_k$. Take $f \in G_1(I^d)$ and consider the decomposition

$$f = \sum_{\nu \in A_k} f \psi_{k,\nu}.$$

To each $Q_{k,\nu}$ we associate $g_{k,\nu} \in \mathcal{P}^M(Q^{k,\nu})$ such that $g_{k,\nu}(2^{-k}\cdot)$ approximates $f(2^{-k}\cdot)$ on $2^k Q^{k,\nu}$ according to Corollary A.6, see the Appendix,

$$||(f - g_{k,\nu})(2^{-k}\cdot)|B^{s_1}_{pp}(2^k Q^{k,\nu})|| \lesssim \Big(\int_0^1 t^{-s_1 p} ||d_t^{M,2^k Q^{k,\nu}}(f(2^{-k}\cdot))(x)|L_p(2^k Q^{k,\nu})||^p \frac{\mathrm{d}t}{t}\Big)^{1/p}. \tag{2.4}$$

The operators $\tilde{S}_k : G_1(I^d) \to G_2(I^d)$ are defined by

$$\tilde{S}_k f = \sum_{\nu \in A_k} g_{k,\nu} \psi_{k,\nu}, \qquad k \in \mathbb{N}. \tag{2.5}$$

Trivially, the right-hand side of (2.5) belongs to $G_1(I^d)$ and hence also to $G_2(I^d)$. The operators $\tilde{S}_k$ use $\binom{M+d-1}{d} \cdot (2^k + 2)^d \approx 2^{kd}$ points. So, it is enough to prove the estimate

$$|| \sum_{\nu \in A_k} (f - g_{k,\nu}) \psi_{k,\nu} | B^{s_2}_{pp}(I^d)|| \lesssim 2^{-k(s_1 - s_2)} ||f|B^{s_1}_{pp}(I^d)||.$$

We use the dilation property (cf. [9, Prop. 2.2.1]) as well as the embedding $B^{s_1}_{pp}(\mathbb{R}^d) \hookrightarrow B^{s_2}_{pp}(\mathbb{R}^d)$ and obtain

$$\begin{aligned}
&\Big|\Big|\sum_{\nu\in A_k}(f-g_{k,\nu})\psi_{k,\nu}|B^{s_2}_{pp}(I^d)\Big|\Big| \\
&\qquad \lesssim 2^{k\left(s_2-\frac{d}{p}\right)}\Big|\Big|\sum_{\nu\in A_k}(f-g_{k,\nu})(2^{-k}\cdot)\psi_{k,\nu}(2^{-k}\cdot)|B^{s_2}_{pp}(2^kI^d)\Big|\Big| \\
&\qquad \lesssim 2^{k\left(s_2-\frac{d}{p}\right)}\Big|\Big|\sum_{\nu\in A_k}(f-g_{k,\nu})(2^{-k}\cdot)\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(2^kI^d)\Big|\Big|.
\end{aligned} \tag{2.6}$$

We claim that

$$\Big|\Big|\sum_{\nu\in A_k}(f-g_{k,\nu})(2^{-k}\cdot)\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(2^kI^d)\Big|\Big| \lesssim \left(\sum_{\nu\in A_k}\big|\big|(f-g_{k,\nu})(2^{-k}\cdot)|B^{s_1}_{pp}(2^kQ^{k,\nu})\big|\big|^p\right)^{1/p}. \tag{2.7}$$

To prove (2.7), we first decompose $\sum_{\nu\in A_k}$ into $\sum_{\alpha=1}^{K}\sum_{\nu\in A_k^\alpha}$ with the number $K\in\mathbb{N}$ (independent of $k\in\mathbb{N}$) so that

$$\operatorname{dist}(\operatorname{supp}\psi_{k,\nu_1}(2^{-k}\cdot),\operatorname{supp}\psi_{k,\nu_2}(2^{-k}\cdot))>1 \tag{2.8}$$

for every $\nu_1,\nu_2\in A_k^\alpha$ and every $\alpha=1,\dots,K$.
To every $\nu\in A_k^\alpha$ we associate $\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}\cdot))$ defined on $\mathbb{R}^d$ such that

$$\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}x))=(f-g_{k,\nu})(2^{-k}x),\qquad x\in 2^kQ^{k,\nu}, \tag{2.9}$$

$$\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}x))=0\quad\text{if}\quad x\in\operatorname{supp}\psi_{k,\nu'}(2^{-k}\cdot) \tag{2.10}$$

if $\nu'\in A_k^\alpha,\nu'\neq\nu$ and

$$||\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}x))|B^{s_1}_{pp}(\mathbb{R}^d)||\le c\,||(f-g_{k,\nu})(2^{-k}x)|B^{s_1}_{pp}(2^kQ^{k,\nu})||. \tag{2.11}$$

The existence of $\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}\cdot))$ satisfying (2.9)-(2.11) follows directly from the Definition A.3, possibly combined with some smooth cut-off function and the pointwise multiplier assertion, cf. [15, Theorem 2.8.2].
Denoting

$$\tilde\psi_{k,\nu}(x)=\tilde\psi(2^kx-\nu),\qquad x\in\mathbb{R}^d,\quad k\in\mathbb{N},\quad \nu\in\mathbb{Z}^d, \tag{2.12}$$

we get

$$\begin{aligned}
&\Big|\Big|\sum_{\nu\in A_k}(f-g_{k,\nu})(2^{-k}\cdot)\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(2^kI^d)\Big|\Big| \\
&\qquad \lesssim \sum_{\alpha=1}^{K}\Big|\Big|\sum_{\nu\in A_k^\alpha}(f-g_{k,\nu})(2^{-k}\cdot)\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(2^kI^d)\Big|\Big| \\
&\qquad \lesssim \sum_{\alpha=1}^{K}\Big|\Big|\sum_{\nu\in A_k^\alpha}\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}\cdot))\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(\mathbb{R}^d)\Big|\Big|.
\end{aligned}$$

By (2.8) and the so called *localisation property*, c.f. [16, Chapter 2.4.7], we may estimate the last expression from above by

$$\sum_{\alpha=1}^{K}\Big(\sum_{\nu\in A_k^\alpha}\Big|\Big|\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}\cdot))\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(\mathbb{R}^d)\Big|\Big|^p\Big)^{1/p}$$

$$\lesssim\Big(\sum_{\alpha=1}^{K}\sum_{\nu\in A_k^\alpha}\Big|\Big|\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}\cdot))\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(\mathbb{R}^d)\Big|\Big|^p\Big)^{1/p}$$

$$=\Big(\sum_{\nu\in A_k}\Big|\Big|\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}\cdot))\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(\mathbb{R}^d)\Big|\Big|^p\Big)^{1/p}.$$

Together with Lemma A.7 and (2.11) this finally leads to

$$\Big|\Big|\sum_{\nu\in A_k}(f-g_{k,\nu})(2^{-k}\cdot)\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(2^kI^d)\Big|\Big|$$

$$\lesssim\Big(\sum_{\nu\in A_k}\Big|\Big|\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}\cdot))|B^{s_1}_{pp}(\mathbb{R}^d)\Big|\Big|^p\cdot\Big|\Big|\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(\mathbb{R}^d)\Big|\Big|^p\Big)^{1/p}$$

$$\lesssim\Big(\sum_{\nu\in A_k}\Big|\Big|\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}\cdot))|B^{s_1}_{pp}(\mathbb{R}^d)\Big|\Big|^p\Big)^{1/p}$$

$$\lesssim\Big(\sum_{\nu\in A_k}\Big|\Big|(f-g_{k,\nu})(2^{-k}\cdot)|B^{s_1}_{pp}(2^kQ^{k,\nu})\Big|\Big|^p\Big)^{1/p},$$

which finishes (2.7).

We insert (2.7) into (2.6) and use (2.4) together with (A.4)

$$\Big|\Big|\sum_{\nu\in A_k}(f-g_{k,\nu})\psi_{k,\nu}|B^{s_2}_{pp}(I^d)\Big|\Big|$$

$$\lesssim 2^{k\left(s_2-\frac{d}{p}\right)}\Big(\sum_{\nu\in A_k}\int_0^1 t^{-s_1p}||(d_t^{M,2^kQ^{k,\nu}}f(2^{-k}\cdot))(x)|L_p(2^kQ^{k,\nu})||^p\frac{\mathrm{d}t}{t}\Big)^{1/p}$$

$$\lesssim 2^{k\left(s_2-\frac{d}{p}\right)}\Big(\sum_{\nu\in A_k}\int_0^1 t^{-s_1p}||(d_{2^{-k}t}^{M,Q^{k,\nu}}f)(2^{-k}x)|L_p(2^kQ^{k,\nu})||^p\frac{\mathrm{d}t}{t}\Big)^{1/p}.$$

The rest is done by direct substitutions and Theorem A.4

$$\Big|\Big|\sum_{\nu\in A_k}(f-g_{k,\nu})\psi_{k,\nu}|B^{s_2}_{pp}(I^d)\Big|\Big|$$

$$\lesssim 2^{k\left(s_2-s_1-\frac{d}{p}\right)}\Big(\sum_{\nu\in A_k}\int_0^{2^{-k}}\xi^{-s_1p}||(d_\xi^{M,Q^{k,\nu}}f)(2^{-k}x)|L_p(2^kQ^{k,\nu})||^p\frac{\mathrm{d}\xi}{\xi}\Big)^{1/p}$$

$$\lesssim 2^{k(s_2-s_1)}\Big(\sum_{\nu\in A_k}\int_0^{2^{-k}}\xi^{-s_1p}||(d_\xi^{M,Q^{k,\nu}}f)(x)|L_p(Q^{k,\nu})||^p\frac{\mathrm{d}\xi}{\xi}\Big)^{1/p}$$

$$\lesssim 2^{-k(s_1-s_2)}\Big(\int_0^{2^{-k}}\xi^{-s_1p}||(d_\xi^{M,I^d}f)(x)|L_p(I^d)||^p\frac{\mathrm{d}\xi}{\xi}\Big)^{1/p}$$

$$\lesssim 2^{-k(s_1-s_2)}||f|B^{s_1}_{pp}(I^d)||.$$

$\square$

Next we consider the case of general integrability and summability parameters.

Proposition 2.3. *Let $\Omega = I^d = (0,1)^d$. Let $G_1(\Omega) = A^{s_1}_{p_1q_1}(\Omega)$ and $G_2(\Omega) = A^{s_2}_{p_2q_2}(\Omega)$ with $1 \le p_1, p_2, q_1, q_2 \le \infty$ ($p_1, p_2 < \infty$ in the F-case),*

$$s_1 > \frac{d}{p_1}, \quad \text{and} \quad s_1 - d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+ > s_2 > 0. \tag{2.13}$$

Then

$$g_n^{\mathrm{lin}}(id) \lesssim n^{-\frac{s_1-s_2}{d} + \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+}. \tag{2.14}$$

Proof. First, we deal with the case $p_1 = p_2 = p$ and $p \ne q_1$ and/or $p \ne q_2$. We use the well-known real interpolation formula, c.f. [13], [1], [15] and [17]

$$B^r_{pq}(\mathbb{R}^d) = \big(B^{r_0}_{pp}(\mathbb{R}^d), B^{r_1}_{pp}(\mathbb{R}^d)\big)_{\theta,q}$$

and its counterpart

$$B^r_{pq}(I^d) = \big(B^{r_0}_{pp}(I^d), B^{r_1}_{pp}(I^d)\big)_{\theta,q}$$

for

$$1 \le p, q \le \infty, \quad 0 < \theta < 1, \quad r_0 < r_1, \quad r = (1-\theta)r_0 + \theta r_1.$$

If, for example, $p \ne q_2$, we find two different real numbers s_2' and s_2'' such that

$$s_1 > s_2', s_2'' > 0, \qquad s_2 = (1-\theta)s_2' + \theta s_2''$$

and apply Proposition 2.2 to embeddings id' and id'' in the following diagram

$$\begin{tikzcd} & B^{s_2'}_{pp}(I^d) \\ B^{s_1}_{pp}(I^d) \arrow[ur, "id'"] \arrow[r, "id"] \arrow[dr, "id''"'] & B^{s_2}_{pq_2}(I^d) \\ & B^{s_2''}_{pp}(I^d) \end{tikzcd}$$

Using the same approximation operator $\tilde{S}_k$, we may interpolate the estimates for $||f - \tilde{S}_k f | B^{s_2'}_{pp}(I^d)||$ and $||f - \tilde{S}_k f | B^{s_2''}_{pp}(I^d)||$ and obtain (2.14).
If also $p \ne q_1$, we proceed in the same way.
If $p_1 \le p_2$ we define s_0 by

$$s_1 > s_0 := s_2 + d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) > s_2 > 0$$

and use the chain of embeddings

$$B^{s_1}_{p_1q_1}(I^d) \hookrightarrow B^{s_0}_{p_1q_2}(I^d) \hookrightarrow B^{s_2}_{p_2q_2}(I^d).$$

The first embedding provides the estimate

$$g_n^{\mathrm{lin}}(id) \lesssim n^{-\frac{s_1-s_0}{d}} = n^{-\frac{s_1-s_2}{d} + \frac{1}{p_1} - \frac{1}{p_2}},$$

the second one is bounded.
If $p_1 \geq p_2$, we use the embedding

$$B^{s_1}_{p_1q_1}(I^d) \hookrightarrow B^{s_2}_{p_1q_2}(I^d) \hookrightarrow B^{s_2}_{p_2q_2}(I^d).$$

The second embedding is bounded, the first one together with Proposition 2.2 gives the result.
This finishes the proof in the B-case. The F-case then follows through trivial embeddings, c.f. [15, 2.3.2]

$$F^{s_1}_{p_1q_1}(I^d) \hookrightarrow B^{s_1}_{p_1,\infty}(I^d) \hookrightarrow B^{s_2}_{p_2,1}(I^d) \hookrightarrow F^{s_2}_{p_2q_2}(I^d).$$

□

Theorem 2.4. *Let $\Omega = I^d = (0,1)^d$. Let $G_1(\Omega) = A^{s_1}_{p_1q_1}(\Omega)$ and $G_2(\Omega) = A^{s_2}_{p_2q_2}(\Omega)$ with $1 \leq p_1, p_2, q_1, q_2 \leq \infty$ ($p_1, p_2 < \infty$ in the F-case) and (2.13) Then*

$$g_n(id) \approx g_n^{\text{lin}}(id) \approx n^{-\frac{s_1-s_2}{d}+\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+}. \tag{2.15}$$

Proof. According to the Proposition 2.3, it is enough to prove that

$$g_n(id) \gtrsim n^{-\frac{s_1-s_2}{d}+\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+}. \tag{2.16}$$

We use the following simple observation, (c.f. [12, Proposition 20]). For $\Gamma = \{x_j\}_{j=1}^n \subset \Omega$ we denote

$$G_1^\Gamma(\Omega) = \{f \in G_1(\Omega) : f(x_j) = 0 \quad \text{for all} \quad j = 1, \dots, n\}.$$

Then

$$g_n(id) \approx \inf_\Gamma \sup\{||f|G_2(\Omega)|| : f \in G_1^\Gamma(\Omega), ||f|G_1(\Omega)|| = 1\} \tag{2.17}$$

$$= \inf_\Gamma ||id : G_1^\Gamma(\Omega) \hookrightarrow G_2(\Omega)||, \tag{2.18}$$

where both the infima extend over all sets $\Gamma = \{x_j\}_{j=1}^n \subset \Omega$.
To prove (2.16), we construct for every $\Gamma = \{x_j\}_{j=1}^{2^{ld}}, l \in \mathbb{N}$, a function $\psi_l \in G_1^\Gamma(\Omega)$ with

$$||\psi_l|G_1(\Omega)|| \lesssim 1 \quad \text{and} \quad ||\psi_l|G_2(\Omega)|| \gtrsim 2^{l\left(s_2-s_1+d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+\right)}, \tag{2.19}$$

where the constants of equivalence do not depend on $l \in \mathbb{N}$.
We rely on the wavelet characterisation of the spaces $A^s_{pq}(\mathbb{R}^n)$, as described in [18, Section 3.1]. Let

$$\psi_F \in C^K(\mathbb{R}) \quad \text{and} \quad \psi_M \in C^K(\mathbb{R}), \quad K \in \mathbb{N},$$

be the Daubechies compactly supported K-wavelets on $\mathbb{R}$ with K large enough. Then we define

$$\Psi(x) = \prod_{i=1}^d \psi_M(x_i), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d$$

and

$$\Psi^j_m(x) = \Psi(2^j x - m), \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n.$$

Then the function

$$\psi_j(x) = \sum_m \lambda_{jm} \Psi^j_m(x), \quad j \in \mathbb{N} \tag{2.20}$$

satisfies

$$||\psi_j|A^s_{pq}(\Omega)|| \approx 2^{j(s-\frac{d}{p})} \left(\sum_m |\lambda_{jm}|^p \right)^{1/p} \tag{2.21}$$

with constants independent on $j \in \mathbb{N}$ and on the sequence $\lambda = \{\lambda_{jm}\}$. The summation in (2.20) and (2.21) runs over those $m \in \mathbb{Z}^n$ for which the support of Ψ^j_m is included in Ω. The proof of (2.21) is based on [18, Theorem 3.5]. First, this theorem tells us that the $A^s_{pq}(\Omega)$-norm of (2.20) may be estimated from above by the right-hand side of (2.21). On the other hand, considering another extension of ψ_j to $\mathbb{R}^d$ and its (unique) wavelet decomposition, we get the opposite inequality.

There is a number $k \in \mathbb{N}$ with the following property. For any $l \in \mathbb{N}$ and any $\Gamma = \{x_j\}_{j=1}^{2^{ld}}$, there are $m_j \in \mathbb{Z}^d$, $j = 1, \ldots, 2^{ld}$ such that

$$\operatorname{supp} \Psi^{k+l}_{m_j} \subset \Omega \quad \text{and} \quad \operatorname{supp} \Psi^{k+l}_{m_j} \cap \Gamma = \emptyset, \quad \text{for} \quad j = 1, \ldots, 2^{ld}.$$

Step 1: $p_1 \le p_2$. In this case, we take in (2.20) $\lambda_{k+l,m_1} = 2^{-j(s-\frac{d}{p})}$ and $\lambda_{k+l,m_n} = 0, n = 2, \ldots, 2^{ld}$ and apply (2.21) twice to verify (2.19).

Step 2: $p_1 > p_2$. In this case, we take $\lambda_{k+l,m_n} = 2^{-js}$, $n = 1, \ldots, 2^{ld}$ in (2.20) and apply again (2.21) twice to prove (2.19). □

2.2 The case $s_2 = 0$

In the case $s_2 = 0$, new phenomena come into play. First we point out that Lemma A.8 for $s = 0$ gives an immediate counterpart of (2.6) and this leads to the following result.

Theorem 2.5. *Let* $\Omega = I^d = (0,1)^d$. *Let*

$$id : G_1(\Omega) \hookrightarrow G_2(\Omega)$$

with

$$G_1(\Omega) = B^s_{p_1q_1}, \quad G_2(\Omega) = B^0_{p_2q_2}$$

and

$$1 \le p_1, q_1, p_2, q_2 \le \infty, \quad s > \frac{d}{p_1}.$$

Then

$$n^{-\frac{s}{d}+(\frac{1}{p_1}-\frac{1}{p_2})_+} \lesssim g_n(id) \lesssim g^{\text{lin}}_n(id) \lesssim n^{-\frac{s}{d}+(\frac{1}{p_1}-\frac{1}{p_2})_+}(1+\log n)^{1/q_2}, \qquad n \in \mathbb{N}. \tag{2.22}$$

If the target space is a Lebesgue space, this can be improved, cf. [12].

Theorem 2.6. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^d$. *Let*

$$id : G_1(\Omega) = A^s_{pq}(\Omega) \hookrightarrow L_r(\Omega) = G_2(\Omega)$$

with

$$1 \le p, q \le \infty, \quad s > \frac{d}{p} \quad \textit{and} \quad 1 \le r \le \infty$$

($p<\infty$ in the F-case). Then

$$g_n(id) \approx g_n^{\rm lin}(id) \approx n^{-\frac{s}{d}+(\frac{1}{p}-\frac{1}{r})_+}, \qquad n\in\mathbb{N}.$$

Remark 2.7. We show in one example, that the logarithmic factor cannot be removed in general. Let $\Omega = I^d = (0,1)^d$ and consider the embedding

$$id: B^s_{1,1}(\Omega) \to B^0_{1,1}(\Omega).$$

Finally, take $\psi\in S(\mathbb{R}^d)$ with $\operatorname{supp}\psi\subset\Omega$ and $\widehat{\psi}(0)\neq 0$. For every $k\in\mathbb{N}$ and every $\Gamma=\{x_j\}_{j=1}^n\subset\Omega, n=2^{kd}$, we set $f_k^\Gamma(x)=\psi(2^{k+1}(x-x^\Gamma))$, where x^Γ is chosen such that $\operatorname{supp} f_k^\Gamma\cap\Gamma=\emptyset$ and $\operatorname{supp} f_k^\Gamma\subset\Omega$. We claim that

$$||f_k^\Gamma|B^s_{1,1}(I^d)|| \le c\,2^{k(s-d)} \tag{2.23}$$

and

$$||f_k^\Gamma|B^0_{1,1}(I^d)|| \ge c\,k\,2^{-kd}. \tag{2.24}$$

Combining (2.23) with (2.24), it follows that

$$g_n(id) \approx g_n^{\rm lin}(id) \approx n^{-\frac{s}{d}}(1+\log n), \quad n\in\mathbb{N}.$$

The proof of (2.23) follows directly from Lemma A.8. To prove (2.24), let $l\in\mathbb{N}$ be the smallest natural number such that

$$\widehat{\psi}(\xi)\neq 0 \quad \text{for} \quad |\xi|\le 2^{-l}$$

and write for $k\ge 2l$

$$\begin{aligned}
||f_k^\Gamma|B^0_{1,1}(I^d)|| &\ge c\,||f_k^\Gamma|B^0_{1,1}(\mathbb{R}^d)|| = c\sum_{j=0}^\infty ||(\varphi_j\widehat{f_k^\Gamma})^\vee|L_1(\mathbb{R}^d)||\\
&\ge c\sum_{j=0}^{k-l-1} ||(\varphi_1(2^{-j}\xi)2^{(-k-1)d}\widehat{\psi}(2^{-k-1}\xi)e^{-i\xi\cdot x^\Gamma})^\vee|L_1(\mathbb{R}^d)||\\
&= c\,2^{(-k-1)d}\sum_{j=0}^{k-l-1} ||(\varphi_1(2^{-j}\xi)\widehat{\psi}(2^{-k-1}\xi))^\vee|L_1(\mathbb{R}^d)|| \qquad (2.25)\\
&= c\sum_{j=0}^{k-l-1} ||(\varphi_1(2^{-j+k+1}\xi)\widehat{\psi}(\xi))^\vee(2^{k+1}x)|L_1(\mathbb{R}^d)||\\
&= 2^{(-k-1)d}\sum_{j=0}^{k-l-1} ||(\varphi_1(2^{-j+k+1}\xi)\widehat{\psi}(\xi))^\vee(x)|L_1(\mathbb{R}^d)||.
\end{aligned}$$

To estimate each of the summands from below, we consider the function

$$(\varphi_1(2^{-j+k+1}\cdot))^\vee = (\varphi_1(2^{-j+k+1}\cdot)\cdot\widehat{\psi}\cdot\frac{1}{\widehat{\psi}}\cdot\varphi_0(2^l\cdot))^\vee$$

and use Young's inequality to estimate its L_1-norm.

$$\begin{aligned}
||\varphi_1^\vee|L_1(\mathbb{R}^d)|| &= ||(\varphi_1(2^{-j+k+1}\cdot))^\vee|L_1(\mathbb{R}^d)|| \qquad (2.26)\\
&\le ||(\varphi_1(2^{-j+k+1}\cdot)\cdot\widehat{\psi})^\vee|L_1(\mathbb{R}^d)||\cdot||(\frac{\varphi_0(2^l\cdot)}{\widehat{\psi}})^\vee|L_1(\mathbb{R}^d)||.
\end{aligned}$$

Now, (2.24) is a combination of (2.25) and (2.26).

2.3 The case $s_2 < 0$

As the last case, we consider the situation $s_2 < 0$.

Theorem 2.8. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let*

$$id : G_1(\Omega) = A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow G_2(\Omega) = A^{s_2}_{p_2q_2}(\Omega)$$

with $1 \leq p_1, p_2, q_1, q_2 \leq \infty$ (with $p_1, p_2 < \infty$ in the F-case) and

$$s_1 > \frac{d}{p_1}, \qquad s_2 < 0.$$

If $p_1 \geq p_2$, then

$$g_n(id) \approx g_n^{\text{lin}}(id) \approx n^{-\frac{s_1}{d}}. \tag{2.27}$$

If $p_1 < p_2$ and $s_2 > \frac{d}{p_2} - \frac{d}{p_1}$, then

$$g_n(id) \approx g_n^{\text{lin}}(id) \approx n^{-\frac{s_1}{d}+\frac{s_2}{d}+\frac{1}{p_1}-\frac{1}{p_2}}. \tag{2.28}$$

If $p_1 < p_2$ and $\frac{d}{p_2} - \frac{d}{p_1} > s_2$, then

$$g_n(id) \approx g_n^{\text{lin}}(id) \approx n^{-\frac{s_1}{d}}. \tag{2.29}$$

Proof. Step 1. In this step, we prove two estimates from below. First, using the method from the proof of Theorem 2.4, we obtain

$$g_n^{\text{lin}}(id) \gtrsim g_n(id) \gtrsim n^{-\frac{s_1-s_2}{d}+\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}$$

exactly as in the case $s_2 > 0$. To prove the second estimate from below, namely

$$g_n^{\text{lin}}(id) \gtrsim g_n(id) \gtrsim n^{-\frac{s_1}{d}}, \tag{2.30}$$

we proceed as follows. We rely on atomic decomposition of $A^{s_1}_{p_1q_1}(\mathbb{R}^d)$ spaces as described in [18, Chapter 1.5]. For every set $\Gamma \subset \Omega$ with $|\Gamma| = 2^{jd}$ we construct a function

$$\psi_j(x) = \sum_{m=1}^{M_j} \lambda_{jm} a_{jm}(x), \quad x \in \mathbb{R}^d,$$

where $M_j \approx 2^{jd}$, $\lambda_{jm} = 2^{-j\frac{d}{p_1}}$ for $m = 1, \ldots, M_j$ and a_{jm} are positive atoms in the sense of [18, Definition 1.15]. As $s_1 > 0$, no moment conditions are needed. We suppose that $\operatorname{supp} a_{jm} \cap \Gamma = \emptyset$ and $\operatorname{supp} a_{jm} \subset \Omega$. Altogether, we get

$$||\psi_j|A^{s_1}_{p_1q_1}(\Omega)|| \leq ||\psi_j|A^{s_1}_{p_1q_1}(\mathbb{R}^d)|| \lesssim 1$$

and

$$||\psi_j|L_1(\Omega)|| = \int_{I^d} \psi_j(x)\mathrm{d}x \approx \sum_{m=1}^{M_j} \lambda_{jm}||a_{jm}(x)|L_1(\mathbb{R}^d)|| \approx 2^{jd} \cdot 2^{-j\frac{d}{p_1}} \cdot 2^{-jd} \cdot 2^{-j(s-\frac{d}{p_1})} = 2^{-js_1}.$$

Finally, we choose a non-negative function $\varrho \in S(\mathbb{R}^d)$ such that the mapping

$$f \to \int_\Omega \varrho(x) f(x) \mathrm{d}x$$

yields a linear bounded functional on $A^{s_2}_{p_2q_2}(\Omega)$, supp $\varrho \subset \Omega$ and $\int \varrho(x)\psi_j(x)\mathrm{d}x \gtrsim \int \psi_j(x)\mathrm{d}x$. This leads to

$$2^{-js_1} \approx ||\psi_j|L_1(\Omega)|| \lesssim \int_\Omega \varrho(x)\psi_j(x)\mathrm{d}x \lesssim ||\psi_j|A^{s_2}_{p_2q_2}(\Omega)||.$$

Hence, (2.30) is proved and it implies all estimates from below included in the theorem.

Step 2.

If $p_1 \geq p_2$ we use the following chain of embeddings

$$A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow L_{p_1}(\Omega) \hookrightarrow A^{s_2}_{p_2q_2}(\Omega) \tag{2.31}$$

and obtain

$$g^{\mathrm{lin}}_n(id) \leq g^{\mathrm{lin}}_n(id' : A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow L_{p_1}(\Omega)) \cdot ||id'' : L_{p_1}(\Omega) \hookrightarrow A^{s_2}_{p_2q_2}(\Omega)|| \lesssim n^{-\frac{s_1}{d}}. \tag{2.32}$$

If $p_1 < p_2$ and $0 > \dfrac{d}{p_2} - \dfrac{d}{p_1} > s_2$, then (2.31) holds true as well and, consequently, also (2.32) remains true.

If $p_1 < p_2$ and $0 > s_2 > \dfrac{d}{p_2} - \dfrac{d}{p_1}$, we define $r > 0$ by $\frac{1}{r} := -\frac{s_2}{d} + \frac{1}{p_2}$. It follows that $p_1 < r < p_2$. Using the embeddings

$$A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow L_r(\Omega) \hookrightarrow A^{s_2}_{p_2p_2}(\Omega) \tag{2.33}$$

we get

$$\begin{aligned} g^{\mathrm{lin}}_n(id) &\leq g^{\mathrm{lin}}_n(id' : A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow L_r(\Omega)) \cdot ||id'' : L_r(\Omega) \hookrightarrow A^{s_2}_{p_2p_2}(\Omega)|| \\ &\lesssim n^{-\frac{s_1}{d}+\frac{1}{p_1}-\frac{1}{r}} = n^{-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{p_2}}. \end{aligned}$$

This proves the upper estimate in (2.28) if $p_2 = q_2$. The general case follows then by interpolation, similar to the proof of Proposition 2.3. $\square$

2.4 Comparison with approximation numbers

In this closing part we wish to compare the sampling numbers of

$$id : B^{s_1}_{p_1q_1}(\Omega) \to B^{s_2}_{p_2q_2}(\Omega) \tag{2.34}$$

for $\Omega = (0,1)^d$ with corresponding approximation numbers. Let us first recall their definition.

Definition 2.9. Let A, B be Banach spaces and let T be a compact linear operator from A to B. Then for all $n \in \mathbb{N}$ the kth approximation number $a_n(T)$ of T is defined by

$$a_n(T) = \inf\{||T - L|| : L \in L(A,B), \mathrm{rank}\ L \leq n\}, \tag{2.35}$$

where rank L is the dimension of the range of L.

Obviously, $a_n(id)$ represents the approximation of id by linear operators with the dimension of the range smaller or equal to n, in general not restricted to involve only function values. Hence

$$a_n(id) \leq g_n^{\mathrm{lin}}(id), \qquad n \in \mathbb{N}.$$

We again assume that

$$s_1 > \frac{d}{p_1}, \qquad s_1 - s_2 > d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+, \tag{2.36}$$

which ensures that (2.34) is compact and its sampling numbers are well defined. The approximation numbers of (2.34) are well known, we refer to [2], [14], [4] and [18] for details. We wish to discuss, when the equivalence $a_n(id) \approx g_n^{\mathrm{lin}}(id)$ holds true. The comparison of our results with the known results for $a_n(id)$ shows, that this is the case if either

1. $s_2 > 0$ and $1 \leq p_2 \leq p_1 \leq \infty$ or
2. $s_2 > 0$ and $1 \leq p_1 \leq p_2 \leq 2$ or $2 \leq p_1 \leq p_2 \leq \infty$ or
3. $0 > s_2 > d\left(\frac{1}{p_2} - \frac{1}{p_1}\right)$ and $1 \leq p_1 \leq p_2 \leq 2$ or $2 \leq p_1 \leq p_2 \leq \infty$.

A Function spaces on domains

A.1 Function spaces on $\mathbb{R}^d$

We use standard notation: $\mathbb{N}$ denotes the collection of all natural numbers, $\mathbb{R}^d$ is the Euclidean d-dimensional space, where $d \in \mathbb{N}$, and $\mathbb{C}$ stands for the complex plane. Let $S(\mathbb{R}^d)$ be the Schwartz space of all complex-valued rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^d$ and let $S'(\mathbb{R}^d)$ be its dual - the space of all tempered distributions.
Furthermore, $L_p(\mathbb{R}^d)$ with $1 \leq p \leq \infty$, are the Lebesgue spaces endowed with the norm

$$||f|L_p(\mathbb{R}^d)|| = \begin{cases} \left(\int_{\mathbb{R}^d} |f(x)|^p \mathrm{d}x\right)^{1/p}, & 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |f(x)|, & p = \infty. \end{cases}$$

For $\psi \in S(\mathbb{R}^d)$ we denote by

$$\widehat{\psi}(\xi) = (F\psi)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i<x,\xi>} \psi(x) \mathrm{d}x, \quad x \in \mathbb{R}^d,$$

its Fourier transform and by $\psi^\vee$ or $F^{-1}\psi$ its inverse Fourier transform.
We give a Fourier-analytic definition of Besov and Triebel-Lizorkin spaces, which relies on the so-called *dyadic resolution of unity*. Let $\varphi \in S(\mathbb{R}^d)$ with

$$\varphi(x) = 1 \quad \text{if} \quad |x| \leq 1 \quad \text{and} \quad \varphi(x) = 0 \quad \text{if} \quad |x| \geq \frac{3}{2}. \tag{A.1}$$

We put $\varphi_0 = \varphi$ and $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$ for $j \in \mathbb{N}$ and $x \in \mathbb{R}^d$. This leads to identity

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1, \qquad x \in \mathbb{R}^d.$$

Definition A.1. (i) Let $s \in \mathbb{R}, 1 \le p, q \le \infty$. Then $B^s_{pq}(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|B^s_{pq}(\mathbb{R}^d)|| = \left(\sum_{j=0}^{\infty} 2^{jsq}||(\varphi_j \widehat{f})^\vee|L_p(\mathbb{R}^d)||^q\right)^{1/q} < \infty \tag{A.2}$$

(with the usual modification for $q = \infty$).
(ii) Let $s \in \mathbb{R}, 1 \le p < \infty, 1 \le q \le \infty$. Then $F^s_{pq}(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|F^s_{pq}(\mathbb{R}^d)|| = \left|\left|\left(\sum_{j=0}^{\infty} 2^{jsq}|(\varphi_j \widehat{f})^\vee(\cdot)|^q\right)^{1/q}|L_p(\mathbb{R}^d)\right|\right| < \infty \tag{A.3}$$

(with the usual modification for $q = \infty$).

Remark A.2. These spaces have a long history. In this context we recommend [13], [15], [16] and [18] as standard references. We point out that the spaces $B^s_{pq}(\mathbb{R}^d)$ and $F^s_{pq}(\mathbb{R}^d)$ are independent of the choice of ψ in the sense of equivalent norms. Special cases of these two scales include Lebesgue spaces, Sobolev spaces, Hölder-Zygmund spaces and many other important function spaces. We omit any detailed discussion.

A.2 Function spaces on domains

Let Ω be a bounded domain. Let $D(\Omega) = C_0^\infty(\Omega)$ be the collection of all complex-valued infinitely-differentiable functions with compact support in Ω and let $D'(\Omega)$ be its dual - the space of all complex-valued distributions on Ω.
Let $g \in S'(\mathbb{R}^d)$. Then we denote by $g|\Omega$ its restriction to Ω:

$$(g|\Omega) \in D'(\Omega), \qquad (g|\Omega)(\psi) = g(\psi) \quad \text{for} \quad \psi \in D(\Omega).$$

Definition A.3. Let Ω be a bounded domain in $\mathbb{R}^d$. Let $s \in \mathbb{R}$, $1 \le p, q \le \infty$ with $p < \infty$ in the F-case. Let A^s_{pq} stand either for B^s_{pq} or F^s_{pq}. Then

$$A^s_{pq}(\Omega) = \{f \in D'(\Omega) : \exists g \in A^s_{pq}(\mathbb{R}^d) : g|\Omega = f\}$$

and

$$||f|A^s_{pq}(\Omega)|| = \inf ||g|A^s_{pq}(\mathbb{R}^d)||,$$

where the infimum is taken over all $g \in A^s_{pq}(\mathbb{R}^d)$ such that $g|\Omega = f$.

We collect some important properties of spaces $A^s_{pq}(\Omega)$ which will be useful later on. For this reason, we have to restrict to bounded Lipschitz domains. We use a standard definition of the notion of Lipschitz domain, the reader may consult for example [18, Chapter 1.11.4].
Let $x \in \mathbb{R}^d, h \in \mathbb{R}^d$ and $M \in \mathbb{N}$. Then

$$(\Delta_h^{M+1} f)(x) = (\Delta_h^1 \Delta_h^M f)(x) \quad \text{with} \quad (\Delta_h^1 f)(x) = f(x+h) - f(x),$$

are the usual differences in $\mathbb{R}^d$. For $x \in \Omega$ we consider the differences with respect to Ω:

$$(\Delta_{h,\Omega}^M f)(x) = \begin{cases} (\Delta_h^M f)(x) & \text{if } x + lh \in \Omega \text{ for } l = 0, \dots, M, \\ 0 & \text{otherwise.} \end{cases}$$

We also need to adapt the classical ball means of differences to bounded domains. Let $M \in \mathbb{N}, t > 0, x \in \Omega$. Then we define

$$V^M(x,t) = \{h \in \mathbb{R}^d : |h| < t, x + \tau h \in \Omega \text{ for } 0 < \tau \leq M\}$$

and

$$d_t^{M,\Omega} f(x) = t^{-d} \int\limits_{V^M(x,t)} |(\Delta_h^M f)(x)| \mathrm{d}h.$$

We shall also use the simple relation (cf. [12, (4.10)])

$$(d_t^{M,\Omega} f(\tau\cdot))(x) = (d_{\tau t}^{M,\tau\Omega} f)(\tau x), \quad x \in \Omega, \quad 0 < \tau, t < \infty. \tag{A.4}$$

The following theorem connects the classical definition of Besov and Triebel-Lizorkin spaces using differences with Definition A.3. We refer to [8] and [18, 1.11.9] for details and references to this topic.

Theorem A.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let $1 \leq p, q \leq \infty$ and*

$$0 < s < M \in \mathbb{N}.$$

Then $B^s_{pq}(\Omega)$ is the collection of all $f \in L_p(\Omega)$ such that

$$||f|L_p(\Omega)|| + \left(\int_0^1 t^{-sq} ||d_t^{M,\Omega} f|L_p(\Omega)||^q \frac{\mathrm{d}t}{t}\right)^{1/q} < \infty \tag{A.5}$$

in the sense of equivalent norms (usual modification if $q = \infty$).

We present a modification of the preceding theorem, which suits better for our needs.
Let $M \in \mathbb{N}$. Let $\mathcal{P}^M(\mathbb{R}^d)$ be the space of all complex-valued polynomials of degree smaller than M and let $\mathcal{P}^M(\Omega)$ be its restriction to Ω. We denote

$$D_M = \dim \mathcal{P}^M(\mathbb{R}^d) = \dim \mathcal{P}^M(\Omega) = \binom{M+d-1}{d}.$$

We say, that $\{x_j\}_{j=1}^{D_M} \subset \mathbb{R}^d$ is a $M-$*regular set* if for every $\{y_j\}_{j=1}^{D_M} \in \mathbb{R}^{D_M}$ there exists (unique) $p \in \mathcal{P}^M(\mathbb{R}^d)$ such that $p(x_j) = y_j, j = 1, \ldots, D_M$. In particular, if $p(x_j) = 0$ for $p \in \mathcal{P}^M(\mathbb{R}^d)$ and all $j = 1, 2, \ldots, D_M$ then $p \equiv 0$. One may observe directly (or consult [11]) that the set

$$\{m \in \mathbb{Z}^d : 0 \leq m_i \leq M \text{ for } i = 1, 2, \ldots, d \text{ and } \sum_{i=1}^d m_i \leq M\}$$

and all its translations, dilations and rotations are $M-$regular.

Theorem A.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $M \in \mathbb{N}$ and let $\{x_j\}_{j=1}^{D_M}$ be a $M-$regular set in Ω.*
Let $1 \leq p, q \leq \infty$ and

$$\frac{d}{p} < s < M \in \mathbb{N}. \tag{A.6}$$

Then $B^s_{pq}(\Omega)$ is the collection of all $f \in L_p(\Omega)$ such that

$$\sum_{j=1}^{D_M} |f(x_j)| + \left(\int_0^1 t^{-sq} ||d_t^{M,\Omega} f|L_p(\Omega)||^q \frac{\mathrm{d}t}{t} \right)^{1/q} < \infty \tag{A.7}$$

in the sense of equivalent norms (usual modification if $q = \infty$).

Proof. According to (A.6), the following embedding is true:

$$B^s_{pq}(\Omega) \hookrightarrow C(\bar{\Omega})$$

and for every $x \in \Omega$

$$|f(x)| \le ||f|C(\bar{\Omega})|| \lesssim ||f|B^s_{pq}(\Omega)||.$$

This shows that the left-hand side of (A.7) is (up to some constant) smaller than the left-hand side of (A.5).

We prove the reverse inequality be contradiction. We denote the left side of (A.7) by $||f|B^s_{pq}(\Omega)||'$. We suppose, that there is no $c > 0$ such that

$$||f|L_p(\Omega)|| \le c\,||f|B^s_{pq}(\Omega)||' \quad \text{for all} \quad f \in B^s_{pq}(\Omega).$$

Then there is a sequence $\{f_n\}_{n=1}^\infty \subset B^s_{pq}(\Omega)$ such that

$$||f_n|L_p(\Omega)|| = 1 \quad \text{and} \quad ||f_n|B^s_{pq}(\Omega)||' < \frac{1}{n}, \quad n \in \mathbb{N}. \tag{A.8}$$

This shows, that $\{f_n\}_{n=1}^\infty$ is bounded in $B^s_{pq}(\Omega)$ and hence precompact in $C(\bar{\Omega})$. We may therefore assume that

$$f_n \to f \text{ in } C(\bar{\Omega}).$$

From (A.8) it follows that

$$\sum_{j=1}^{D_M} |f(x_j)| = 0 \quad \text{and} \quad (d_t^{M,\Omega} f)(x) = 0, \text{ for a. e. } x \in \Omega. \tag{A.9}$$

The second part of (A.9) gives that $f \in \mathcal{P}^M(\Omega)$. Furthermore, the definition of $M-$regular sets and the first part of (A.9) implies that $f = 0$. This contradicts (A.8). □

This characterisation has a direct corollary.

Corollary A.6. *Under the assumptions of Theorem A.5,*

$$\inf_{g \in \mathcal{P}^M(\Omega)} ||f - g|B^s_{pq}(\Omega)|| \approx \left(\int_0^1 t^{-sq} ||d_t^{M,\Omega} f|L_p(\Omega)||^q \frac{\mathrm{d}t}{t} \right)^{1/q}.$$

Proof. Consider some M-regular set $\{x_j\}_{j=1}^{D_M}$ and $g \in \mathcal{P}^M(\Omega)$ such that

$$g(x_j) = f(x_j), \quad j = 1, \ldots, D_M.$$

Let us mention, that the polynomial g is uniquely determined and its definition combines the function values $f(x_1), \ldots, f(x_{D_M})$ in a linear way. The rest of the proof follows directly from Theorem A.5. □

We also recall the fact that the spaces $B^s_{pq}(\mathbb{R}^d)$ are *multiplication algebras* if $s > \frac{d}{p}$, c.f. [15, 2.8.3].

Lemma A.7. *Let $1 \le p, q \le \infty$ and $s > \frac{d}{p}$. Then*

$$||h_1 \cdot h_2 | B^s_{pq}(\mathbb{R}^d)|| \le c\, ||h_1 | B^s_{pq}(\mathbb{R}^d)|| \cdot ||h_2 | B^s_{pq}(\mathbb{R}^d)||,$$

where the constant c does not depend on h_1 and h_2.

Finally, we consider the dilation operator $T_k : f \to f(2^k \cdot),\ k \in \mathbb{N}$, and its behaviour on the scale of Besov spaces. For the proof, we refer to [3, 1.7] and [9, 2.3.1].

Lemma A.8. *Let $s \ge 0$, $1 \le p, q \le \infty$ and $k \in \mathbb{N}$. Then the operator T_k is bounded on $B^s_{p,q}(\mathbb{R}^d)$ and its norm is bounded by $c\, 2^{k(s-\frac{d}{p})}$ if $s > 0$ and by $c\, 2^{-k\frac{d}{p}}(1+k)^{1/q}$ if $s = 0$. The constant c does not depend on $k \in \mathbb{N}$.*

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