Traces of Functions with a Dominating Mixed Derivative in $\mathbb{R}^3$

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November 4, 2005

Abstract

We investigate traces of functions, belonging to a class of functions with dominating mixed smoothness in $\mathbb{R}^3$, with respect to planes in oblique position. In comparison with the classical theory for isotropic spaces a few new phenomenona occur. We shall present two different approaches. One is based on the use of the Fourier transform and restricted to $p = 2$. The other one is applicable in the general case of Besov-Lizorkin-Triebel spaces and based on atomic decompositions.

AMS Classification: 42B35, 46E35

Keywords and phrases: Sobolev spaces of dominating mixed smoothness, Besov and Lizorkin-Triebel classes of dominating mixed smoothness, Fourier analytic characterizations, atomic decompositions, traces on hyperplanes in oblique position.

1 Introduction

Sobolev spaces with dominating mixed smoothness $S_p^{\overline{r}}W(\mathbb{R}^d)$ have been introduced in 1962 by S.M. Nikol'skij, see [Ni1, Ni2], originally in connection with some partial differential equations. Later on there has been some interest in these type of spaces as special cases of vector-valued Sobolev spaces ($S_p^{r,\ldots,r}W(\mathbb{R}^d)$ can be interpreted as an iterated version of the Sobolev spaces $W_p^r(\mathbb{R})$), see Grisvard [Gr], Sparr [Sp] and Schmeißer [Sc]. At the end of the eighties Triebel [Tr1], motivated by problems in connection with eigenvalue distributions of integral operators, investigated the trace problem with respect to the diagonal $x_1 = x_2$ for the Besov spaces $S_{p,1}^{r,r}B(\mathbb{R}^2)$. In recent years there is an increasing interest in function spaces with a dominating mixed derivative in connection with the numerical solution of some special partial differential equations or integral equations, see e.g. Griebel, Oswald, Schiekofer [GOS], Yserentant [Ys1, Ys2], Nitzsche [Ni] or Bungartz and Griebel [BG].

We are interested in the description of the trace classes of $S_p^{r_1,r_2,r_3}W(\mathbb{R}^3)$ (and more general function spaces) with respect to an hyperplane in oblique position. Since at least twenty years the situation is well understood if the trace is taken with respect to hyperplanes parallel to the coordinate axes, cf. e.g. the monographs Amanov [Am], Gelman, Maz'ya [GM] ($p = 2$) and Schmeißer, Triebel [ST]. However, there is an essential difference in case that the hyperplane is in an oblique position. First observations in this direction have been made by Triebel [Tr1] in the two-dimensional case, later continued by Rodriguez [Ro] and complemented by the first author, see [Vy3]. To our own surprise the problem for $d = 3$ turned out to be more complicated. New phenomenona occur. Whereas for $d = 2$ almost all trace classes of Sobolev and Besov-Lizorkin-Triebel spaces are again Besov or Lizorkin-Triebel classes (in some limiting cases of generalized smoothness, see [Vy3]) the situation changes with $d = 3$. Here it turns out that the trace classes can be described as the sum of three different function spaces of dominating mixed smoothness. In proving such a statement we offer two different approaches. The first one is restricted to $p = 2$ and uses elementary properties of the Fourier transform. In this simplified situation we are also able to establish a characterization of the trace class of $S_2^{r_1,r_2,r_3}W(\mathbb{R}^3)$ as a L_2-space with a weight in the Fourier image. For $p \neq 2$ one is confronted with difficult Fourier multiplier assertions. To circumvent this we apply the characterization of these classes by atoms which works also in the more general case of Besov and Lizorkin-Triebel spaces. However, the description of the trace classes found in this way is not very transparent. Here some further progress would be desirable.

To explain a part of the difficulties let us consider an example. We equip the hyperplane $x_1 + x_2 + x_3 = 0$ with an orthogonal basis

$$\mathcal{O} = \{\vec{\sigma}_1, \vec{\sigma}_2\}, \quad \vec{\sigma}_1 = (\sigma_{1,1}, \sigma_{1,2}, \sigma_{1,3}) \in \Gamma, \quad \vec{\sigma}_2 = (\sigma_{2,1}, \sigma_{2,2}, \sigma_{2,3}) \in \Gamma, \quad \vec{\sigma}_1 \perp \vec{\sigma}_2 \,. \tag{1.1}$$

Then we associate to this basis the corresponding "orthogonal" trace operator

$$(\mathrm{tr}_{\mathcal{O}}\, f)(z_1, z_2) = f(z_1\vec{\sigma}_1 + z_2\vec{\sigma}_2), \quad z_1, z_2 \in \mathbb{R}. \tag{1.2}$$

Now we consider the following family of functions

$$f_\lambda(x_1, x_2, x_3) := \psi(x_1)\, \psi(x_2)\, \psi(x_3)\, |x_3|^\lambda\,, \qquad (x_1, x_2, x_3) \in \mathbb{R}^3\,, \quad \lambda \in \mathbb{R}\,,$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth cut-off function supported around the origin. Such a function f_λ belongs to $S_p^{r,r,r}W(\mathbb{R}^3)$ if $\lambda > r - 1/p$. But the regularity of the function

$$g_\lambda(z_1, z_2) = \psi(\sigma_{1,1}z_1 + \sigma_{2,1}z_2)\, \psi(\sigma_{1,2}z_1 + \sigma_{2,2}z_2)\, \psi(\sigma_{1,3}z_1 + \sigma_{2,3}z_2)\, |\sigma_{1,3}z_1 + \sigma_{2,3}z_2|^\lambda$$

depends on $\mathcal{O}$. The function g_λ belongs to $S_p^{r,r}W(\mathbb{R}^2)$, $\lambda > r - 1/p$, if either $\sigma_{1,3} = 0$ or $\sigma_{2,3} = 0$. If $\sigma_{1,3} \cdot \sigma_{2,3} \neq 0$, then g_λ belongs to $S_p^{t,t}W(\mathbb{R}^2)$, $\lambda > 2\,t - 1/p$. As a consequence the description of the traces of $S_p^{r_1,r_2,r_3}W(\mathbb{R}^3)$ to the hyperplane $x_1 + x_2 + x_3 = 0$ must depend on the chosen basis $\mathcal{O}$.

The paper is organized as follows. In Section 2 we start with a general discussion of the notion of the trace and continue with a detailed investigation of the trace problem for the Sobolev spaces of dominating smoothness built on $L_2(\mathbb{R}^3)$. Here we shall apply methods from Fourier analysis. In case $p \neq 2$, treated in Section 3, the situation becomes more complicated and we switch to the powerful but less transparent method of decompositions of functions into small building blocks like atoms. By means of those decompositions we are able to describe the trace classes for the general case of Besov and Lizorkin-Triebel classes. Our main results are contained in Theorems 2.11, 3.10, and 3.14.

2 The Trace of Sobolev Spaces of Dominating Mixed Smoothness $S_2^{\overline{r}}(\mathbb{R}^3)$

2.1 Sobolev Spaces of Dominating Mixed Smoothness

Let $1 < p < \infty$ and $\overline{r} = (r_1, \ldots, r_d) \in \mathbb{N}_0^d$ ($\mathbb{N}_0$ denotes the natural numbers including 0). The Sobolev space of dominating mixed smoothness $\overline{r} = (r_1, \ldots, r_d)$ is the collection of all functions $f \in L_p(\mathbb{R}^d)$ such that

$$D^\alpha f \in L_p(\mathbb{R}^d)\,, \qquad 0 \le \alpha_i \le r_i\,, \qquad i = 1, \ldots, d\,,$$

endowed with the norm

$$\| \, f \, | S_p^{\overline{r}} W(\mathbb{R}^d) \| := \sum_{\alpha \le \overline{r}} \| \, D^\alpha f \, | L_p(\mathbb{R}^d) \| \,. \tag{2.1}$$

Here $\alpha \le \overline{r}$ means $\alpha_i \le r_i$, $i = 1, \ldots, d$.

The mixed derivative $\frac{\partial^{r_1 + \ldots + r_d} f}{\partial x_1^{r_1} \ldots \partial x_d^{r_d}}$ plays a dominant part here and this fact is responsible for the name of these classes. Based on a Fourier multiplier theorem of Lizorkin one can prove a characterization of these classes using the Fourier transfom. Let $\mathcal{S}(\mathbb{R}^d)$ denote the class of all complex-valued rapidly decreasing infinitely differentiable functions defined on $\mathbb{R}^d$. By $\mathcal{S}'(\mathbb{R}^d)$ we mean the collection of all tempered distributions and $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, respectively, both defined on $\mathcal{S}'(\mathbb{R}^d)$. Then $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $S_p^{\overline{r}} W(\mathbb{R}^d)$ if and only if

$$\mathcal{F}^{-1}\Big((1 + |\xi_1|^2)^{r_1/2} \, \ldots \, (1 + |\xi_d|^2)^{r_d/2} \, \mathcal{F} f(\xi)\Big)(\,\cdot\,) \in L_p(\mathbb{R}^d)\,.$$

Furthermore, the norms $\| \, f \, | S_p^{\overline{r}} W(\mathbb{R}^d) \|$ and

$$\| \, f \, | S_p^{\overline{r}} W(\mathbb{R}^d) \|^* := \Big\| \mathcal{F}^{-1}\Big(\prod_{i=1}^d (1 + |\xi_i|^2)^{r_i/2} \, \mathcal{F} f(\xi)\Big)(\,\cdot\,) \Big| L_p(\mathbb{R}^d) \Big\| \tag{2.2}$$

are equivalent, cf. e.g. [ST, 2.3.1]. The Fourier-analytic description can be taken to generalize these Sobolev spaces to fractional and negative order of smoothness, cf. [ST, Chapt. 2]. We will take (2.2) as the definition of $S_p^{\overline{r}} W(\mathbb{R}^d)$ if $r = (r_1, \ldots, r_d)$, $r_i \in \mathbb{R}$, $i = 1, \ldots, d$.

2.2 Some new Function Spaces

As it will become clear below the description of the trace spaces will lead to some new Sobolev-type spaces. For us it will be sufficient to introduce these classes in the two-dimensional setting. For the rest of this section we concentrate on $p = 2$.
Let $\mathcal{M}$ be a 2×2-matrix,

$$\mathcal{M} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad \det \mathcal{M} \neq 0\,, \quad \text{and let} \quad \vec{\eta}_1 = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad \vec{\eta}_2 = \begin{pmatrix} c \\ d \end{pmatrix}. \tag{2.3}$$

Definition 2.1. *Let $\mathcal{M}, \vec{\eta_1}, \vec{\eta_2}$ be as in (2.3). Let $r_1, r_2 \in \mathbb{R}$. Then $S_2^{r_1,r_2}W(\mathcal{M},\mathbb{R}^2)$ denotes the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^2)$ such that $f \circ \mathcal{M} \in S_2^{r_1,r_2}W(\mathbb{R}^2)$. We endow this class with the norm*

$$\| f \,|S_2^{r_1,r_2}W(\mathcal{M},\mathbb{R}^2)\| := \| f \circ \mathcal{M} \,|S_2^{r_1,r_2}W(\mathbb{R}^2)\| \,.$$

The following properties of these classes are immediate.

Lemma 2.2. *Let $\mathcal{M}, \vec{\eta_1}, \vec{\eta_2}$ be as in (2.3). Let $r_1, r_2 \in \mathbb{R}$.*
(i) *The classes $S_2^{r_1,r_2}W(\mathcal{M},\mathbb{R}^2)$ are Banach spaces continuously embedded into $\mathcal{S}'(\mathbb{R}^2)$.*
(ii) *$C_0^\infty(\mathbb{R}^2)$ is a dense subset of $S_2^{r_1,r_2}W(\mathcal{M},\mathbb{R}^2)$.*
(iii) *A function $f \in \mathcal{S}'(\mathbb{R}^2)$ belongs to $S_2^{r_1,r_2}W(\mathcal{M},\mathbb{R}^2)$ if and only if*

$$(1+|a\,\xi_1+b\,\xi_2|^2)^{r_1/2}\,(1+|c\,\xi_1+d\,\xi_2|^2)^{r_2/2}\,|\mathcal{F}f(\xi)| \in L_2(\mathbb{R}^2)\,.$$

Furthermore, the expression

$$\Big\|\,(1+|a\,\xi_1+b\,\xi_2|^2)^{r_1/2}\,(1+|c\,\xi_1+d\,\xi_2|^2)^{r_2/2}\,|\mathcal{F}f(\xi)|\,\Big|L_2(\mathbb{R}^2)\Big\|$$

yields an equivalent norm in $S_2^{r_1,r_2}W(\mathcal{M},\mathbb{R}^2)$.

For $r_1, r_2 \in \mathbb{N}_0$ (here $\mathbb{N}_0$ denotes the natural numbers including 0) there is an other interpretation. As usual, by $\frac{\partial f}{\partial \vec{\eta}}$ we denote the weak directional derivative of f in direction $\vec{\eta}$.

Definition 2.3. *Let $\vec{\eta_1}, \vec{\eta_2}$ be linearly independent vectors in $\mathbb{R}^2$. Let $r_1, r_2 \in \mathbb{N}_0$. Then $S_2^{r_1,r_2}W(\vec{\eta_1},\vec{\eta_2})$ denotes the collection of all functions $f \in L_2(\mathbb{R}^2)$ such that*

$$\frac{\partial^{\alpha_1+\alpha_2} f}{\partial\vec{\eta_1}^{\alpha_1}\,\partial\vec{\eta_2}^{\alpha_2}} \in L_2(\mathbb{R}^2) \qquad \text{for all} \quad \alpha_i \le r_i,\ i=1,2\,.$$

We endow this class with the norm

$$\| f \,|S_2^{r_1,r_2}W(\vec{\eta_1},\vec{\eta_2})\| := \sum_{\alpha_1=0}^{r_1}\sum_{\alpha_2=0}^{r_2}\Big\|\,\frac{\partial^{\alpha_1+\alpha_2} f}{\partial\vec{\eta_1}^{\alpha_1}\,\partial\vec{\eta_2}^{\alpha_2}}\,\Big|L_2(\mathbb{R}^2)\Big\|\,.$$

Remark 2.4. Obviously, these classes $S_2^{r_1,r_2}W(\vec{\eta_1},\vec{\eta_2})$ are Banach spaces. Let e_1, e_2 denote the elements of the canonical basis of $\mathbb{R}^2$. Then we have $S_2^{r_1,r_2}W(e_1,e_2) = S_2^{r_1,r_2}W(\mathbb{R}^2)$. Furthermore, $C_0^\infty(\mathbb{R}^2)$ is a dense set in $S_2^{r_1,r_2}W(\vec{\eta_1},\vec{\eta_2})$ for arbitrary vectors $\vec{\eta_1},\vec{\eta_2}$.

For a smooth function f it follows

$$\frac{\partial}{\partial x_1}(f\circ\mathcal{M})(x) =< \nabla f(\mathcal{M}x), \vec{\eta_1} >= \frac{\partial f}{\partial\vec{\eta_1}}(\mathcal{M}x)\,.$$

By an induction argument we conclude

$$\frac{\partial^{\alpha_1+\alpha_2}}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}}(f\circ\mathcal{M})(x) = \frac{\partial^{\alpha_1+\alpha_2} f}{\partial\vec{\eta_1}^{\alpha_1}\,\partial\vec{\eta_2}^{\alpha_2}}(\mathcal{M}x)\,.$$

Using the density of smooth compactly supported functions this proves the following.

Lemma 2.5. *Let $\mathcal{M}, \vec{\eta_1}$, and $\vec{\eta_2}$ be as in (2.3). A function $f \in L_2(\mathbb{R}^2)$ belongs to $S_2^{r_1,r_2}W(\vec{\eta_1},\vec{\eta_2})$ if and only if the function $f\circ\mathcal{M}$ belongs to $S_2^{r_1,r_2}W(\mathbb{R}^2)$. Furthermore, the norms $\| f \,|S_2^{r_1,r_2}W(\vec{\eta_1},\vec{\eta_2})\|$ and $\| f\circ\mathcal{M} \,|S_2^{r_1,r_2}W(\mathbb{R}^2)\|$ are equivalent.*

2.3 The Trace with Respect to an Arbitrary Orthogonal Basis of the Hyperplane

Let $A_1(\mathbb{R}^3)$ be a class of functions (distributions) defined on $\mathbb{R}^3$ and let $C(\mathbb{R}^3)$ be the collection of all continuous functions on $\mathbb{R}^3$. By $\widetilde{\Gamma}$ we denote a hyperplane in $\mathbb{R}^3$. Then we consider the mapping

$$T:\, f \to f_{|_{\widetilde{\Gamma}}}$$

which is well-defined in case of a continuous function f. The aim of this paper consists in determining a class of functions $A_2(\mathbb{R}^2) \hookrightarrow \mathcal{S}'(\mathbb{R}^2)$ such that T, originally defined on $A_1(\mathbb{R}^3) \cap C(\mathbb{R}^3)$, extends to a linear, continuous and surjective mapping belonging to $\mathcal{L}(A_1(\mathbb{R}^3), A_2(\mathbb{R}^2))$. In case, that there exists a linear and continuous operator $\mathrm{ext} \in \mathcal{L}(A_2(\mathbb{R}^2), A_1(\mathbb{R}^3))$ such that $T \circ \mathrm{ext} = I$ (identity on $A_2(\mathbb{R}^2)$), we shall call T a retraction and ext its corresponding coretraction.
In the monographs Amanov [Am, 9.5], Gelman, Maz'ya [GM, 2.3] and Schmeißer, Triebel [ST, 2.4.2] the traces of function spaces with dominating mixed smoothness on hyperplanes parallel to the coordinate axes were studied. For simplicity let the hyperplane be given by $x_3 = 0$. Then the result is the following.

Proposition 2.6. *Let $r_3 > 1/2$. Then the mapping*

$$T:\, f(x_1, x_2, x_3) \to f(x_1, x_2, 0)$$

extends to a retraction from $S_2^{r_1,r_2}W(\mathbb{R}^3)$ onto $S_2^{r_1,r_2}W(\mathbb{R}^2)$.

Remark 2.7. A few comments are in order. First of all, $\mathcal{S}(\mathbb{R}^3)$ is dense in the class $S_p^{r_1,r_2}W(\mathbb{R}^3)$. So the trace operator is the unique linear extension of the mapping T. Secondly, there is a natural coordinate system on the hyperplane $x_3 = 0$ to measure the smoothness of the trace, namely that one induced by the unit vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Notice that the spaces $S_p^{r_1,r_2}W(\mathbb{R}^2)$ are not invariant under rotations in general.

In this paper we investigate the trace with respect to the hyperplane

$$\Gamma := \Big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \quad x_1 + x_2 + x_3 = 0\Big\}.$$

with Γ as a model case for a hyperplane in an oblique position. However, taking the trace with respect to the hyperplane

$$\Gamma_{\overline{\gamma}} := \{(x_1, x_2, x_3) : \quad \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 = 0\}, \qquad \overline{\gamma} = (\gamma_1, \gamma_2, \gamma_3)\,,$$

where $\gamma_1 \cdot \gamma_2 \cdot \gamma_3 \neq 0$, would give us the same result (up to the norms of considered operators). This statement relies on the fact, that the mapping

$$f(x_1, x_2, x_3) \to f(\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3), \quad \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \neq 0,$$

is a bounded bijective mapping of $S_2^{\overline{r}}W(\mathbb{R}^3)$ onto itself.
The "natural" trace operators

$$(\mathrm{tr}_1\, f)(x_2, x_3) = f(-x_2 - x_3, x_2, x_3), \tag{2.4}$$

$$(\mathrm{tr}_2\, f)(x_1, x_3) = f(x_1, -x_1 - x_3, x_3), \tag{2.5}$$

$$(\mathrm{tr}_3\, f)(x_1, x_2) = f(x_1, x_2, -x_1 - x_2) \tag{2.6}$$

and the trace operator $\mathrm{tr}_{\mathcal{O}}\, f$, see (1.1) and (1.2), are connected through

$$\begin{aligned} (\mathrm{tr}_{\mathcal{O}}\, f)(z_1, z_2) &= f(z_1\vec{\sigma}_1 + z_2\vec{\sigma}_2) = f\left(\sigma_{1,1}z_1 + \sigma_{2,1}z_2, \sigma_{1,2}z_1 + \sigma_{2,2}z_2, \sigma_{1,3}z_1 + \sigma_{2,3}z_2\right) \\ &= (\mathrm{tr}_1\, f)\left(\sigma_{1,2}z_1 + \sigma_{2,2}z_2, \sigma_{1,3}z_1 + \sigma_{2,3}z_2\right) \\ &= (\mathrm{tr}_1\, f)(\mathcal{R}_1\vec{z}), \end{aligned} \tag{2.7}$$

where

$$\mathcal{R}_1 = \begin{pmatrix} \sigma_{1,2} & \sigma_{2,2} \\ \sigma_{1,3} & \sigma_{2,3} \end{pmatrix} \quad \text{and} \quad \vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{2.8}$$

Analoguously one obtains

$$(\mathrm{tr}_{\mathcal{O}}\, f)(z_1, z_2) = (\mathrm{tr}_2\, f)(\mathcal{R}_2\vec{z}) = (\mathrm{tr}_3\, f)(\mathcal{R}_3\vec{z})\,, \tag{2.9}$$

with

$$\mathcal{R}_2 = \begin{pmatrix} \sigma_{1,1} & \sigma_{2,1} \\ \sigma_{1,3} & \sigma_{2,3} \end{pmatrix}, \quad \mathcal{R}_3 = \begin{pmatrix} \sigma_{1,1} & \sigma_{2,1} \\ \sigma_{1,2} & \sigma_{2,2} \end{pmatrix}. \tag{2.10}$$

The linear independence of the vectors $\vec{\sigma}_1, \vec{\sigma}_2$, combined with $\vec{\sigma}_1, \vec{\sigma}_2 \in \Gamma$, ensure that these matrices $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ are regular.
In what follows we shall determine the regularity of $\mathrm{tr}_{\mathcal{O}}\, f$ as well as of $\mathrm{tr}_i\, f$, $i = 1, 2, 3$.
Above we considered all orthogonal bases of Γ. Probably it would be more natural to restrict to orthonormal bases. However, all function spaces under consideration here remain invariant under the change from an orthogonal to the associated orthonormal basis (up to equivalent quasi-norms). The greater generality leads to nothing new but it simplifies the calculations. For that reason we shall work with orthogonal bases.

2.4 The Regularity of $\mathrm{tr}_{\mathcal{O}}\, f$

2.4.1 A Description of the General Case

Let $f \in C_0^\infty(\mathbb{R}^3)$. Now we introduce a very useful decomposition of f. Let $\mathcal{X}_i$ denote the characteristic function of the set

$$M_i := \Big\{(\tau_1, \tau_2, \tau_3) : \quad |\tau_i| = \min(|\tau_1|, |\tau_2|, |\tau_3|)\Big\}, \quad i = 1, 2, 3.$$

Hence

$$|M_i \cap M_j| = 0\,, \qquad i \neq j\,, \qquad \text{and} \qquad \bigcup_{i=1}^{3} M_i = \mathbb{R}^3\,,$$

(here $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^3$). We put

$$f_i(x) := \mathcal{F}^{-1}[\mathcal{X}_i(\xi)\,\mathcal{F}f(\xi)](x)\,,$$

and obtain $f = f_1 + f_2 + f_3$. We continue by defining

$$g_i(x_1, x_2) = (\mathrm{tr}_i\, f_i)(x_1, x_2)\,, \qquad i = 1, 2, 3\,. \tag{2.11}$$

Elementary properties of the Fourier transform yield

$$\begin{aligned}
\mathcal{F}_2 g_1(\xi_1,\xi_2) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathcal{F}_3 f_1(\tau_1,\xi_1+\tau_1,\xi_2+\tau_1)\,d\tau_1\\
\mathcal{F}_2 g_2(\xi_1,\xi_2) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathcal{F}_3 f_2(\xi_1+\tau_2,\tau_2,\xi_2+\tau_2)\,d\tau_2\\
\mathcal{F}_2 g_3(\xi_1,\xi_2) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathcal{F}_3 f_3(\xi_1+\tau_3,\xi_2+\tau_3,\tau_3)\,d\tau_3\,,
\end{aligned}\tag{2.12}$$

where $\mathcal{F}_2 g$ denotes the Fourier transform in $\mathbb{R}^2$ and $\mathcal{F}_3 f$ the Fourier transform in $\mathbb{R}^3$, respectively. Now we are going to check the regularity of the functions g_i. To begin with we investigate $i=1$. Let $r_1 > 1/2$. By using Hölder's inequality we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} & (1+\xi_2^2)^{r_2}(1+\xi_3^2)^{r_3}\Big|\int_{\mathbb{R}} \mathcal{F}_3 f_1(\tau_1,\xi_2+\tau_1,\xi_3+\tau_1)\ d\tau_1\Big|^2\,d\xi_1\,d\xi_2\\
&\le c_1\int_{\mathbb{R}^3}(1+\xi_2^2)^{r_2}(1+\xi_3^2)^{r_3}(1+\tau_1^2)^{r_1}\,|\mathcal{F}_3 f_1(\tau_1,\xi_2+\tau_1,\xi_3+\tau_1)|^2\,d\tau_1\,d\xi_1\,d\xi_2\\
&= c_1\int_{\mathbb{R}^3}[1+(\tau_2-\tau_1)^2]^{r_2}[1+(\tau_3-\tau_1)^2]^{r_3}(1+\tau_1^2)^{r_1}|\mathcal{F}_3 f_1(\tau_1,\tau_2,\tau_3)|^2\,d\vec{\tau}
\end{aligned}$$

with $c_1 = \int_{\mathbb{R}}(1+\tau_1^2)^{-r_1}\,d\tau_1 < \infty$. Finally, we observe that if $|\tau_1|\le\min(|\tau_2|,|\tau_3|)$, then $|\tau_2-\tau_1|\le 2|\tau_2|$, $|\tau_3-\tau_1|\le 2|\tau_3|$ and

$$[1+(\tau_2-\tau_1)^2]^{r_2}\,[1+(\tau_3-\tau_1)^2]^{r_3}\le 4^{r_2+r_3}\,(1+\tau_2^2)^{r_2}\,(1+\tau_3^2)^{r_3}.$$

Because of $\operatorname{supp}\mathcal{F}_3 f_1\subset\{(\tau_1,\tau_2,\tau_3)\in\mathbb{R}^3 : |\tau_1|\le\min(|\tau_2|,|\tau_3|)\}$, we finally conclude

$$\|\ \mathrm{tr}_1\, f_1\,|S_2^{r_2,r_3}W(\mathbb{R}^2)\|\le c_2\,\|\,f_1\,|S_2^{\vec{r}}W(\mathbb{R}^3)\|\le c_2\,\|\,f\,|S_2^{\vec{r}}W(\mathbb{R}^3)\|\,. \tag{2.13}$$

This proves $\mathrm{tr}_1\, f_1\in S_2^{r_2,r_3}W(\mathbb{R}^2)$. Similarly one obtains $\mathrm{tr}_2\, f_2\in S_2^{r_1,r_3}W(\mathbb{R}^2)$ (if $r_2>1/2$) and $\mathrm{tr}_3\, f_3\in S_2^{r_1,r_2}W(\mathbb{R}^2)$ (if $r_3>1/2$), respectively. To summarize our findings we need to recall a further notion. For three quasi-Banach spaces $A_1,A_2,A_3\hookrightarrow\mathcal{S}'(\mathbb{R}^2)$ of tempered distributions we put

$$A_1+A_2+A_3 := \Big\{g\in\mathcal{S}'(\mathbb{R}^2):\quad \exists g_i\in A_i,\ i=1,2,3,\quad \text{s.t. } g=g_1+g_2+g_3\Big\}.$$

We equip this space with a quasi-norm by taking

$$\|\,g\,|A_1+A_2+A_3\| := \inf\Big\{\sum_{i=1}^{3}\|\,g_i|A_i\|\ :\quad g=g_1+g_2+g_3\,,\ g_i\in A_i,\ i=1,2,3\Big\}.$$

Lemma 2.8. *Let $\mathcal{O}$ be an orthogonal basis of Γ and let $\mathcal{R}_i, i=1,2,3$ be matrices associated with $\mathcal{O}$ by* (1.1), (2.8) *and* (2.10).
Suppose $\min(r_1,r_2,r_3)>1/2$. *Then* $\mathrm{tr}_{\mathcal{O}}$ *becomes a continuous mapping of* $S_2^{r_1,r_2,r_3}W(\mathbb{R}^3)$ *into*

$$S_2^{r_1,r_2}W(\mathcal{R}_3^{-1},\mathbb{R}^2)+S_2^{r_1,r_3}W(\mathcal{R}_2^{-1},\mathbb{R}^2)+S_2^{r_2,r_3}W(\mathcal{R}_1^{-1},\mathbb{R}^2) \tag{2.14}$$

Proof. The boundedness of $\mathrm{tr}_{\mathcal{O}}$ follows from the identity

$$(\mathrm{tr}_{\mathcal{O}}\,f)(\vec{z}) = \sum_{i=1}^{3}(\mathrm{tr}_{\mathcal{O}}\,f_i)(\vec{z}) = \sum_{i=1}^{3}(\mathrm{tr}_i\,f_i)(\mathcal{R}_i\vec{z})\,,$$

cf. (2.7), (2.9), the definition of the spaces $S_2^{r_1,r_2}W(\mathcal{M},\mathbb{R}^2)$ and the inequality (2.13) and its counterparts for tr_2 and tr_3. □

The restriction $\min(r_1,r_2,r_3)>1/2$ has been convenient but is by no means necessary. Moreover, as we shall see by the next theorem the operator $\mathrm{tr}_{\mathcal{O}}$ is surjective in Lemma 2.8. The description of the trace class becomes more complicated than in Lemma 2.8 if $\min(r_1,r_2,r_3)<1/2$.

Theorem 2.9. *Let $\mathcal{O}$ be an orthogonal basis of Γ and let $\mathcal{R}_i, i=1,2,3$ be matrices associated with $\mathcal{O}$ by (1.1), (2.8) and (2.10).*
Let $\overline{r}=(r_1,r_2,r_3)\in\mathbb{R}^3$ with $r_i\neq 1/2, i=1,2,3$ and

$$\min\left(r_1,r_2,r_3,r_1+r_2-\frac{1}{2},r_1+r_3-\frac{1}{2},r_2+r_3-\frac{1}{2}\right)>0\,. \tag{2.15}$$

Then

$$\mathrm{tr}_{\mathcal{O}}\in\mathcal{L}\Big(S_2^{\overline{r}}W(\mathbb{R}^3),S^1(\mathbb{R}^2)+S^2(\mathbb{R}^2)+S^3(\mathbb{R}^2)\Big), \tag{2.16}$$

where

$$S^1(\mathbb{R}^2):=\begin{cases} S_2^{r_2,r_3}W(\mathcal{R}_1^{-1},\mathbb{R}^2), & \text{if } r_1>\frac{1}{2}\,,\\ S_2^{r_2,r_3+r_1-\frac{1}{2}}W(\mathcal{R}_1^{-1},\mathbb{R}^2)\cap S_2^{r_2+r_1-\frac{1}{2},r_3}W(\mathcal{R}_1^{-1},\mathbb{R}^2), & \text{if } r_1<\frac{1}{2}\,,\end{cases}$$

and similarly for S^2 and S^3.
Conversely, to each function $g\in S^1(\mathbb{R}^2)+S^2(\mathbb{R}^2)+S^3(\mathbb{R}^2)$ there exists a function $f\in S_2^{\overline{r}}W(\mathbb{R}^3)$ such that $\mathrm{tr}_{\mathcal{O}}f=g$.

Proof. Step 1. Preparations. For $\alpha,\beta,t\in\mathbb{R}$ we define

$$I(\alpha,\beta,t):=\int_{-\infty}^{\infty}(1+(t+\tau)^2)^{-\alpha}\,(1+\tau^2)^{-\beta}\,d\tau\,.$$

In case $\alpha+\beta>1/2$, $\beta<1/2$, elementary calculations yield

$$I(\alpha,\beta,t)\le c\begin{cases}(1+t^2)^{-\beta} & \text{if}\quad \alpha>1/2\,,\\ (1+t^2)^{-\beta}\,(1+\log(1+|t|)) & \text{if}\quad \alpha=1/2\,,\\ (1+t^2)^{-(\alpha+\beta)+1/2} & \text{if}\quad \alpha<1/2\,,\end{cases} \tag{2.17}$$

for some c independent of t.
Step 2. The boundedness of $\mathrm{tr}_{\mathcal{O}}$ in case $\min(r_1,r_2,r_3)>1/2$ has been proven before.
Now we suppose $0<r_1<1/2$. We proceed as at the beginning of this subsection and obtain

$$\begin{aligned}\int_{\mathbb{R}^2}&(1+\xi_2^2)^{r_2}(1+\xi_3^2)^{r_3+r_1-\frac{1}{2}}\Big|\int_{\mathbb{R}}\mathcal{F}_3f_1(\tau_1,\xi_2+\tau_1,\xi_3+\tau_1)\,d\tau_1\Big|^2\,d\xi_1\,d\xi_2\\ &\le\int_{\mathbb{R}^3}(1+\xi_2^2)^{r_2}(1+\xi_3^2)^{r_3+r_1-\frac{1}{2}}\,I(\alpha,r_1,\xi_3)\,(1+\tau_1^2)^{r_1}\,(1+(\tau_1+\xi_3)^2)^{\alpha}\\ &\qquad\qquad\times\;|\mathcal{F}_3f_1(\tau_1,\xi_2+\tau_1,\xi_3+\tau_1)|^2\,d\tau_1\,d\xi_1\,d\xi_2\\ &\le c_1\int_{\mathbb{R}^3}[1+(\tau_2-\tau_1)^2]^{r_2}[1+(\tau_3-\tau_1)^2]^{r_3-\alpha}(1+\tau_1^2)^{r_1}\,(1+\tau_3^2)^{\alpha}\,|\mathcal{F}_3f_1(\tau_1,\tau_2,\tau_3)|^2\,d\vec{\tau}\,,\end{aligned}$$

where we have used (2.17) with some α satisfying $\frac{1}{2}-r_1<\alpha<\frac{1}{2}$. Choosing α sufficiently close to $\frac{1}{2}-r_1$ our restriction $r_1+r_3>1/2$, see (2.15), guarantees $r_3-\alpha\ge 0$. Furthermore, taking into account the information on the support of $\mathcal{F}f_1$ we arrive at

$$\|\;\mathrm{tr}_1\,f_1\,|S_2^{r_2,r_3+r_1-\frac{1}{2}}W(\mathbb{R}^2)\|\le c_2\,\|\,f_1\,|S_2^{\overline{r}}W(\mathbb{R}^2)\|\le c_2\,\|\,f\,|S_2^{\overline{r}}W(\mathbb{R}^2)\|$$

with some c independent of f. Interchanging the roles of ξ_1 and ξ_2 also

$$\| \operatorname{tr}_1 f_1 \,|S_2^{r_2+r_1-\frac{1}{2},r_3}W(\mathbb{R}^2)\| \le c_3 \,\| f \,|S_2^{\bar{r}}W(\mathbb{R}^2)\|$$

follows. Moreover, by symmetry we obtain the needed estimates of $\operatorname{tr}_i f_1$, $i = 2,3$, as well. This completes the proof of the boundedness.

Step 3. Construction of an extension operator.

Substep 3.1. Construction of a linear extension operator for $S_2^{r_1,r_2}W(\mathbb{R}^2)$. Let $\varphi \in C_0^\infty(\mathbb{R})$ be a function such that $\int \varphi(t)\,dt = \sqrt{2\pi}$. Then, for $g \in C_0^\infty(\mathbb{R}^2)$ and $x \in \mathbb{R}^3$, we define

$$\begin{aligned}
f_1(x) &= \operatorname{ext}_1^* g(x) := \mathcal{F}_3^{-1}\big[\varphi(\xi_1)\,\mathcal{F}_2 g(\xi_2-\xi_1,\xi_3-\xi_1)\big](x)\\
f_2(x) &= \operatorname{ext}_2^* g(x) := \mathcal{F}_3^{-1}\big[\varphi(\xi_2)\,\mathcal{F}_2 g(\xi_1-\xi_2,\xi_3-\xi_2)\big](x)\\
f_3(x) &= \operatorname{ext}_3^* g(x) := \mathcal{F}_3^{-1}\big[\varphi(\xi_3)\,\mathcal{F}_2 g(\xi_1-\xi_3,\xi_2-\xi_3)\big](x)\,.
\end{aligned}$$

Hence, e.g. for f_3, we conclude

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathcal{F}_3 f_3(\xi_1+\tau_3,\xi_2+\tau_3,\tau_3)\,d\tau_3 = \mathcal{F}_2 g(\xi_1,\xi_2)$$

and from this identity we derive

$$g(x_1,x_2) = (\operatorname{tr}_3 f_3)(x_1,x_2) = f_3(x_1,x_2,-x_1-x_2)\,, \qquad (x_1,x_2)\in\mathbb{R}^2\,.$$

Similarly

$$g = \operatorname{tr}_1 f_1 \qquad \text{and} \qquad g = \operatorname{tr}_2 f_2\,.$$

The regularity of $\operatorname{ext}_3^* g$ is easily checked in view of

$$\begin{aligned}
&\int_{\mathbb{R}^3} (1+\xi_1^2)^{r_1}\,(1+\xi_2^2)^{r_2}\,(1+\xi_3^2)^{r_3}\,|\mathcal{F}_3 \operatorname{ext}^* g(\xi)|^2\,d\vec{\xi}\\
&\quad = \int_{\mathbb{R}^3} (1+|\xi_1+\tau_3|^2)^{r_1}\,(1+|\xi_2+\tau_3|^2)^{r_2}\,(1+\tau_3^2)^{r_3}\,|\varphi(\tau_3)\,\mathcal{F}_2 g(\xi_1,\xi_2)|^2\,d\xi_1\,d\xi_2\,d\tau_3\\
&\quad \le c_1 \int_{\mathbb{R}} (1+|\tau_3|^2)^{r_1+r_2+r_3}\,|\varphi(\tau_3)|^2\,d\tau_3\;\| g\,|S_2^{r_1,r_2}W(\mathbb{R}^2)\|^2\\
&\quad \le c_2\,\| g\,|S_2^{r_1,r_2}W(\mathbb{R}^2)\|^2\,,
\end{aligned}$$

where we also used the fact that φ has compact support. This proves $\operatorname{ext}_3^* \in \mathcal{L}(S_2^{r_1,r_2}W(\mathbb{R}^2), S_2^{r_1,r_2,r_3}W(\mathbb{R}^3))$ for any $r_3\in\mathbb{R}$. Similarly, $\operatorname{ext}_1^* \in \mathcal{L}(S_2^{r_2,r_3}W(\mathbb{R}^2), S_2^{r_1,r_2,r_3}W(\mathbb{R}^3))$ for any r_1 and $\operatorname{ext}_2^* \in \mathcal{L}(S_2^{r_1,r_3}W(\mathbb{R}^2), S_2^{r_1,r_2,r_3}W(\mathbb{R}^3))$ for any r_2, respectively.

Substep 3.2. Construction of an extension operator in case $\min(r_1,r_2,r_3) > 1/2$. We shall use the abbreviations $A_1 = S_2^{r_2,r_3}W(\mathcal{R}_1^{-1},\mathbb{R}^2)$, $A_2 = S_2^{r_1,r_3}W(\mathcal{R}_2^{-1},\mathbb{R}^2)$ and $A_3 = S_2^{r_1,r_2}W(\mathcal{R}_3^{-1},\mathbb{R}^2)$. Let $g\in A_1+A_2+A_3$. Further, let $g = g_1+g_2+g_3$, where

$$g_i \in A_i\,, \quad i=1,2,3 \qquad \text{and} \qquad \| g\,|A_1+A_2+A_3\| \le 2\sum_{i=1}^{3} \| g_i\,|A_i\|\,.$$

By definition $g_1(\mathcal{R}_1^{-1}\cdot) \in S_2^{r_2,r_3}W(\mathbb{R}^2)$ and consequently, by Step 3.1, $f_1 := \operatorname{ext}_1^* g_1(\mathcal{R}_1^{-1}\cdot) \in S_2^{r_1,r_2,r_3}W(\mathbb{R}^3)$. Similarly, $f_2 := \operatorname{ext}_2^* g_2(\mathcal{R}_2^{-1}\cdot)$, $f_3 := \operatorname{ext}_3^* g_3(\mathcal{R}_3^{-1}\cdot) \in S_2^{r_1,r_2,r_3}W(\mathbb{R}^3)$. We put

$f := f_1 + f_2 + f_3$. Because of

$$\begin{aligned}\mathrm{tr}_{\mathcal{O}} f &= \sum_{i=1}^{3} \mathrm{tr}_{\mathcal{O}} f_i = \sum_{i=1}^{3} (\mathrm{tr}_i\, f_i)(\mathcal{R}_i\, \cdot) \\ &= \sum_{i=1}^{3} \Big(\mathrm{tr}_i\, \mathrm{ext}_i^*\, g_i(\mathcal{R}_i^{-1} \cdot)\Big)(\mathcal{R}_i\, \cdot) \\ &= \sum_{i=1}^{3} g_i = g\,,\end{aligned}$$

see Substep 3.1, this proves the existence of a bounded extension of g if $\min(r_1, r_2, r_3) > 1/2$.

Substep 3.3. Let $0 < r_1 < 1/2$. We shall use the abbreviations $A_1 = S_2^{r_2, r_3 + r_1 - \frac12} W(\mathbb{R}^2)$, $A_2 = S_2^{r_2 + r_1 - \frac12, r_3} W(\mathbb{R}^2)$. By the arguments from the previous substep (and by symmetry) it will be sufficient to construct a function $f_1 \in S_2^{r_1, r_2, r_3} W(\mathbb{R}^3)$ such that $\mathrm{tr}_1 f_1 = g_1(\mathcal{R}_1^{-1} \cdot) \in A_1 \cap A_2$. To shorten notation we write h_1 instead of $g_1(\mathcal{R}_1^{-1} \cdot)$. To begin with we define two subsets of $\mathbb{R}^3$

$$\begin{aligned}\Omega_1 := \Big\{ (\xi_1, \xi_2, \xi_3) \quad : \quad & |\xi_2 - \xi_1| \le |\xi_3 - \xi_1|\,, \\ & \frac{|\xi_2 - \xi_1|}{4} \le |\xi_1| \le \frac{|\xi_2 - \xi_1|}{2} \qquad \text{if} \quad |\xi_2 - \xi_1| \ge 1\,, \\ & \qquad |\xi_1| \le 1 \qquad \text{if} \qquad |\xi_2 - \xi_1| < 1 \Big\}\,, \\ \Omega_2 := \Big\{ (\xi_1, \xi_2, \xi_3) \quad : \quad & |\xi_3 - \xi_1| < |\xi_2 - \xi_1|\,, \\ & \frac{|\xi_3 - \xi_1|}{4} \le |\xi_1| \le \frac{|\xi_3 - \xi_1|}{2} \qquad \text{if} \quad |\xi_3 - \xi_1| \ge 1\,, \\ & \qquad |\xi_1| \le 1 \qquad \text{if} \qquad |\xi_3 - \xi_1| < 1 \Big\}\,.\end{aligned}$$

Obviously, these sets are disjoint. Let $\mathcal{X}_i$ denote the characteristic function of Ω_i, $i = 1, 2$. Then we define

$$\begin{aligned} f_1(x) \quad := \quad & \int e^{ix\xi}\, \mathcal{F}_2 h_1(\xi_2 - \xi_1, \xi_3 - \xi_1) \\ \times \quad & \Big(\mathcal{X}_1(\xi)\, \frac{(1 + (\xi_2 - \xi_1)^2)^{r_2 + r_1 - 1/2}\, (1 + (\xi_3 - \xi_1)^2)^{r_3}}{(1 + \xi_1^2)^{r_1}\, (1 + \xi_2^2)^{r_2}\, (1 + \xi_3^2)^{r_3}} H^1(\xi_2 - \xi_1, \xi_3 - \xi_1) \\ & + \quad \mathcal{X}_2(\xi)\, \frac{(1 + (\xi_2 - \xi_1)^2)^{r_2}\, (1 + (\xi_3 - \xi_1)^2)^{r_3 + r_1 - 1/2}}{(1 + \xi_1^2)^{r_1}\, (1 + \xi_2^2)^{r_2}\, (1 + \xi_3^2)^{r_3}} H^2(\xi_2 - \xi_1, \xi_3 - \xi_1) \Big)\, d\vec{\xi}\,,\end{aligned}$$

where the functions H^1, H^2 will be chosen later. First we prove $\mathrm{tr}_1 f_1 = h_1$. It is sufficient to assume $h_1 \in C_0^\infty(\mathbb{R}^2)$. Setting $\tau_2 = \xi_2 - \xi_1$ and $\tau_3 = \xi_3 - \xi_1$ we find

$$\begin{aligned} f_1(-x_2 - x_3, x_2, x_3) \quad = \quad & \int_{|\tau_2| \le |\tau_3|} e^{i(x_2\tau_2 + x_3\tau_3)}\, \mathcal{F}_2 h_1(\tau_2, \tau_3)\, (1 + \tau_2^2)^{r_2 + r_1 - 1/2}\, (1 + \tau_3^2)^{r_3}\, H^1(\tau_2, \tau_3) \\ & \times \int_{I(\tau_2)} \frac{1}{(1 + \xi_1^2)^{r_1}\, (1 + (\tau_2 + \xi_1)^2)^{r_2}\, (1 + (\tau_3 + \xi_1)^2)^{r_3}}\, d\xi_1\, d\tau_2\, d\tau_3 \\ + \quad & \int_{|\tau_3| < |\tau_2|} e^{i(x_2\tau_2 + x_3\tau_3)}\, \mathcal{F}_2 h_1(\tau_2, \tau_3)\, (1 + \tau_2^2)^{r_2}\, (1 + \tau_3^2)^{r_3 + r_1 - 1/2}\, H^2(\tau_2, \tau_3) \\ & \times \int_{I(\tau_3)} \frac{1}{(1 + \xi_1^2)^{r_1}\, (1 + (\tau_2 + \xi_1)^2)^{r_2}\, (1 + (\tau_3 + \xi_1)^2)^{r_3}}\, d\xi_1\, d\tau_2\, d\tau_3 \end{aligned}$$

with $I(\tau_2)$ and $I(\tau_3)$ being appropriate subsets of $\mathbb{R}$. The functions H^1 and H^2 are determined through the identities

$$
\begin{aligned}
H^1(\tau_2,\tau_3) &= \frac{1}{2\pi}\Big(\int_{I(\tau_2)} \frac{(1+\tau_2^2)^{r_2+r_1-1/2}\,(1+\tau_3^2)^{r_3}}{(1+\xi_1^2)^{r_1}\,(1+(\tau_2+\xi_1)^2)^{r_2}\,(1+(\tau_3+\xi_1)^2)^{r_3}}\,d\xi_1\Big)^{-1},\\
H^2(\tau_2,\tau_3) &= \frac{1}{2\pi}\Big(\int_{I(\tau_3)} \frac{(1+\tau_2^2)^{r_2}\,(1+\tau_3^2)^{r_3+r_1-1/2}}{(1+\xi_1^2)^{r_1}\,(1+(\tau_2+\xi_1)^2)^{r_2}\,(1+(\tau_3+\xi_1)^2)^{r_3}}\,d\xi_1\Big)^{-1}.
\end{aligned}
$$

As a consequence we obtain

$$
f_1(-x_2-x_3,x_2,x_3) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i(x_2\tau_2+x_3\tau_3}\,\mathcal{F}_2h_1(\tau_2,\tau_3)\,d\tau_2\,d\tau_3 = h_1(x_2,x_3)
$$

as claimed. ¿From the definition of the sets Ω_i we derive the existence of two positive constants c_1 and c_2 such that for all τ_2,τ_3

$$
c_1 \le H^1(\tau_2,\tau_3) \le c_2
$$

as well as

$$
c_1 \le H^2(\tau_2,\tau_3) \le c_2\,.
$$

This will be used to prove that f_1 is sufficiently regular. Indeed, we have

$$
\begin{aligned}
\int_{\mathbb{R}^3} &\quad (1+\xi_1^2)^{r_1}\,(1+\xi_2^2)^{r_2}\,(1+\xi_3^2)^{r_3}|\mathcal{F}_3f_1(\xi)|^2\,d\vec{\xi}\\
&= \int_{\mathbb{R}^3}(1+\xi_1^2)^{-r_1}\,(1+\xi_2^2)^{-r_2}\,(1+\xi_3^2)^{-r_3}|\mathcal{F}_2h_1(\xi_2-\xi_1,\xi_3-\xi_1)|^2\\
&\times \Big(\mathcal{X}_1(\xi)\;|H^1(\xi_2-\xi_1,\xi_3-\xi_1)|^2\,(1+(\xi_2-\xi_1)^2)^{2r_2+2r_1-1}\,(1+(\xi_3-\xi_1)^2)^{2r_3}\\
&\quad + \mathcal{X}_2(\xi)\;|H^2(\xi_2-\xi_1,\xi_3-\xi_1)|^2\,(1+(\xi_2-\xi_1)^2)^{2r_2}\,(1+(\xi_3-\xi_1)^2)^{2r_3+2r_1-1}\Big)\,d\vec{\xi}\\
&=: J_1+J_2\,.
\end{aligned}
$$

A change of coordinates, the boundedness of H^1 and the definition of Ω_1 yield

$$
\begin{aligned}
J_1 &\le c_2^2\int_{|\tau_2|\le|\tau_3|}|\mathcal{F}_2h_1(\tau_2,\tau_3)|^2\,(1+\tau_2^2)^{2r_2+2r_1-1}\,(1+\tau_3^2)^{2r_3}\\
&\qquad\times\int_{I(\tau_2)}(1+\xi_1^2)^{-r_1}\,(1+(\tau_2+\xi_1)^2)^{-r_2}\,(1+(\tau_3+\xi_1)^2)^{-r_3}\,d\xi_1\,d\tau_2\,d\tau_3\\
&\le c_3\int_{\mathbb{R}^2}|\mathcal{F}_2h_1(\tau_2,\tau_3)|^2\,(1+\tau_2^2)^{r_2+r_1-1/2}\,(1+\tau_3^2)^{r_3}\,d\tau_2\,d\tau_3\,.
\end{aligned}
$$

The estimate of J_2 is similar. Hence

$$
\|\,f_1\,|S_2^{r_1,r_2,r_3}W(\mathbb{R}^3)\| \le c_4\,\Big(\|\,h_1\,|S_2^{r_2,r_3+r_1-\frac12}W(\mathbb{R}^2)\| + \|\,h_1\,|S_2^{r_2+r_1-\frac12,r_3}W(\mathbb{R}^2)\|\Big)
$$

with some constant c_4 independent of h_1. This proves the boundedness of the extension. □

Remark 2.10. Let us mention that we have not shown the existence of a linear and continuous extension operator. The step in which g is splitted into the three functions g_1, g_2 and g_3 need not be linear. This problem will be investigated in the next subsection.

2.4.2 A Description of the Trace Classes on the Fourier Side

For simplicity we concentrate on the situation $\min(r_1, r_2, r_3) > 1/2$. The sum $S_2^{r_1,r_2}W(\mathcal{R}_3^{-1}, \mathbb{R}^2) + S_2^{r_1,r_3}W(\mathcal{R}_2^{-1}, \mathbb{R}^2) + S_2^{r_2,r_3}W(\mathcal{R}_1^{-1}, \mathbb{R}^2)$ is not direct. It is obvious that

$$C_0^\infty(\mathbb{R}^3) \subset \Big(S_2^{r_1,r_2}W(\mathcal{R}_3^{-1}, \mathbb{R}^2) \cap S_2^{r_1,r_3}W(\mathcal{R}_2^{-1}, \mathbb{R}^2) \cap S_2^{r_2,r_3}W(\mathcal{R}_1^{-1}, \mathbb{R}^2)\Big) .$$

At this moment it is not clear whether the connection between g and its optimal decomposition $g_1 + g_2 + g_3$ can be realized in a linear way. But that can be seen easily by the Fourier-analytic description of the trace space.

Let $\mathcal{O}$ be an orthogonal basis of $\Gamma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$ and let $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ be the matrices associated with $\mathcal{O}$. First, we notice that $g_3 \in S_2^{r_1,r_2}W(\mathcal{R}_3^{-1}, \mathbb{R}^2)$ if, and only if,

$$\underbrace{\Big[1 + (\sigma_{2,2}\xi_1 - \sigma_{1,2}\xi_2)^2\Big]^{r_1/2} \Big[1 + (\sigma_{2,1}\xi_1 - \sigma_{1,1}\xi_2)^2\Big]^{r_2/2}}_{m_3(\xi_1,\xi_2)} \mathcal{F}g_3(\xi_1, \xi_2) \in L_2(\mathbb{R}^2) , \tag{2.18}$$

cf. Lemma 2.2(iii). Similarly, $g_1 \in S_2^{r_2,r_3}W(\mathcal{R}_1^{-1}, \mathbb{R}^2)$ if, and only if,

$$\underbrace{\Big[1 + \Big(\sigma_{2,3}\xi_1 - \sigma_{1,3}\xi_2\Big)^2\Big]^{r_2/2} \Big[1 + \Big(\sigma_{2,2}\xi_1 - \sigma_{1,2}\xi_2\Big)^2\Big]^{r_3/2}}_{m_1(\xi_1,\xi_2)} \mathcal{F}g_1(\xi_1, \xi_2) \in L_2(\mathbb{R}^2). \tag{2.19}$$

and $g_2 \in S_2^{r_1,r_3}W(\mathcal{R}_2^{-1}, \mathbb{R}^2)$ if, and only if,

$$\underbrace{\Big[1 + \Big(\sigma_{2,3}\xi_1 - \sigma_{1,3}\xi_2\Big)^2\Big]^{r_1/2} \Big[1 + (\sigma_{2,1}\xi_1 - \sigma_{1,1}\xi_2)^2\Big]^{r_3/2}}_{m_2(\xi_1,\xi_2)} \mathcal{F}g_2(\xi_1, \xi_2) \in L_2(\mathbb{R}^2) . \tag{2.20}$$

In view of these characterizations we define

$$m(\xi_1, \xi_2) := \min\Big(m_1(\xi_1, \xi_2), m_2(\xi_1, \xi_2), m_3(\xi_1, \xi_2)\Big) . \tag{2.21}$$

and

$$L_2(\mathbb{R}^2, m) := \Big\{g \in L_2(\mathbb{R}^2) : \quad m\,\mathcal{F}g \in L_2(\mathbb{R}^2)\Big\} \tag{2.22}$$

equipped with the natural norm

$$\|\, g\,|L_2(\mathbb{R}^2, m)\| := \|\, m\,\mathcal{F}g\,|L_2(\mathbb{R}^2)\| .$$

Now we arrive at the main result of this section.

Theorem 2.11. *Let $\mathcal{O}$ be an orthogonal basis of Γ and let $\mathcal{R}_i, i = 1, 2, 3$ be matrices associated with $\mathcal{O}$ by* (1.1), (2.8) *and* (2.10). *Suppose (2.15) and $r_i \neq 1/2$, $i = 1, 2, 3$. Then there exists a continuous function m such that* $\mathrm{tr}_{\mathcal{O}}$ *becomes a retraction of $S_2^{r_1,r_2,r_3}W(\mathbb{R}^3)$ onto $L_2(\mathbb{R}^2, m)$. There is a bounded linear extension operator* $\mathrm{ext} \in \mathcal{L}(L_2(\mathbb{R}^2, m), S_2^{r_1,r_2,r_3}W(\mathbb{R}^3))$ *such that* $\mathrm{tr}_{\mathcal{O}} \circ \mathrm{ext} = I$ *(identity on $L_2(\mathbb{R}^2, m)$).*

Proof. We concentrate on the case $\min(r_1, r_2, r_3) > 1/2$. Then the function m is given by (2.21). The modifications which have to be made for the general situation are obvious. We omit the details.

Step 1. Boundedness. Again we shall use the abbreviations $A_1 = S_2^{r_2,r_3}W(\mathcal{R}_1^{-1},\mathbb{R}^2)$, $A_2 = S_2^{r_1,r_3}W(\mathcal{R}_2^{-1},\mathbb{R}^2)$ and $A_3 = S_2^{r_1,r_2}W(\mathcal{R}_3^{-1},\mathbb{R}^2)$. Let $g \in A_1 + A_2 + A_3$ and let $g = g_1 + g_2 + g_3$ be an optimal decomposition of g with $g_i \in A_i$. Then

$$m(\xi)\,|\mathcal{F}g(\xi)| \le \sum_{i=1}^{3} m_i(\xi)\,|\mathcal{F}g_i(\xi)|\,, \qquad \xi \in \mathbb{R}^2\,.$$

But this implies

$$\|\,g\,|L_2(\mathbb{R}^2,m)\| \le \sum_{i=1}^{3} \|\,m_i\,\mathcal{F}g_i\,|L_2(\mathbb{R}^2)\| \le c\,\sum_{i=1}^{3} \|\,g_i\,|A_i\|\,,$$

with some c independent of g.
Vice versa, if $g \in L_2(\mathbb{R}^2,m)$, then we define

$$\Omega_i := \Big\{(\xi_1,\xi_2):\quad m_i(\xi_1,\xi_2) = m(\xi_1,\xi_2)\Big\}\,, \tag{2.23}$$

$\mathcal{X}_i$ denotes its characteristic function, and

$$g_i(x) := \mathcal{F}^{-1}[\mathcal{X}_i(\xi)\,\mathcal{F}g(\xi)](x)\,, \quad i = 1,2,3. \tag{2.24}$$

Thanks to

$$|\Omega_i \cap \Omega_j| = 0\,, \quad i \neq j\,, \qquad \text{and} \qquad \bigcup_{i=1}^{3}\Omega_i = \mathbb{R}^2\,,$$

($|\cdot|$ Lebesgue measure in $\mathbb{R}^2$) this implies $g = g_1 + g_2 + g_3$ and

$$\|\,m_i\,\mathcal{F}g_i\,|L_2(\mathbb{R}^2)\| \le \|\,m\,\mathcal{F}g\,|L_2(\mathbb{R}^2)\|\,, \qquad i = 1,2,3\,.$$

Summarizing we have proved the coincidence of $S_2^{r_1,r_2}W(\mathcal{R}_3^{-1},\mathbb{R}^2) + S_2^{r_1,r_3}W(\mathcal{R}_2^{-1},\mathbb{R}^2) + S_2^{r_2,r_3}W(\mathcal{R}_1^{-1},\mathbb{R}^2)$ and $L_2(\mathbb{R}^2,m)$ in the sense of equivalent norms. Hence $\mathrm{tr}_{\mathcal{O}} \in \mathcal{L}(S_2^{r_1,r_2,r_3}W(\mathbb{R}^3), L_2(\mathbb{R}^2,m))$.
Step 2. The linear extension. Since the mappings $g \to g_i$, $i = 1,2,3$, cf. (2.24), are linear and continuous, the extension operator constructed in the proof of Theorem 2.9 is linear and bounded as well. $\square$

2.4.3 The Trace Space for a Dominating Direction

This subsection contains an additional observation of minor importance. So we concentrate on $\min(r_1,r_2,r_3) > 1/2$.
A simplified description of the trace spaces can be given in case that one of the parameters r_1,r_2,r_3 is dominating the sum of the other.

Lemma 2.12. *Let $\mathcal{O}$ be an orthogonal basis of Γ and let $\mathcal{R}_i, i = 1,2,3$ be matrices associated with $\mathcal{O}$ by* (1.1), (2.8) *and* (2.10)*. Then the embeddings*

$$S_2^{r_1,r_3}W(\mathcal{R}_2^{-1},\mathbb{R}^2) \hookrightarrow S_2^{r_1,r_2}W(\mathcal{R}_3^{-1},\mathbb{R}^2) \qquad \textit{and} \qquad S_2^{r_2,r_3}W(\mathcal{R}_1^{-1},\mathbb{R}^2) \hookrightarrow S_2^{r_1,r_2}W(\mathcal{R}_3^{-1},\mathbb{R}^2)$$

exists if, and only if, $r_3 \ge r_1 + r_2$.

Proof. Again we work in the Fourier image. Let m_1, m_2 and m_3 be the functions defined in (2.18)-(2.20). Then the first embedding is equivalent to the boundedness of m_3/m_2 and the second is equivalent to the boundedness of m_3/m_1, respectively.
Let us turn to the boundedness of the first quotient. By a change of coordinates

$$y_1 := \sigma_{2,2}\xi_1 - \sigma_{1,2}\xi_2 \qquad \text{and} \qquad y_2 := \sigma_{2,3}\xi_1 - \sigma_{1,3}\xi_2$$

and taking care of $\vec{\sigma}_1, \vec{\sigma}_2 \in \Gamma$ the boundedness of m_3/m_2 is equivalent to

$$\sup_{y_1,y_2\in\mathbb{R}} \frac{(1+y_1^2)^{r_1}\,[1+(y_1+y_2)^2]^{r_2}}{(1+y_1^2)^{r_3}(1+y_2^2)^{r_2}} < \infty\,.$$

With $y_2 = 0$ the necessity of $r_3 \geq r_1 + r_2$ follows. Sufficiency can be derived from

$$1 + (y_1+y_2)^2 \leq 2(1+y_1^2)(1+y_2^2)\,.$$

□

Theorem 2.13. *Let $\mathcal{O}$ be an orthogonal basis of $\Gamma = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1+x_2+x_3 = 0\}$ and let $\mathcal{R}_3$ be the matrix associated with $\mathcal{O}$ by* (1.1) *and* (2.10). *Let $\min(r_1,r_2,r_3) > 1/2$ and suppose $r_3 \geq r_1 + r_2$. Then $\mathrm{tr}_{\mathcal{O}}$ becomes a retraction of $S_2^{r_1,r_2,r_3}W(\mathbb{R}^3)$ onto $S_2^{r_1,r_2}W(\mathcal{R}_3^{-1},\mathbb{R}^2))$ and*

$$S_2^{r_1,r_2}W(\mathcal{R}_3^{-1},\mathbb{R}^2)) = L_2(\mathbb{R}^2, m_3) \qquad \textit{(equivalent norms).}$$

Proof. ¿From Lemma 2.12 we derive

$$S_2^{r_1,r_2}W(\mathcal{R}_3,\mathbb{R}^2) + S_2^{r_2,r_3}W(\mathcal{R}_1,\mathbb{R}^2) + S_2^{r_1,r_3}W(\mathcal{R}_2,\mathbb{R}^2) = S_2^{r_1,r_2}W(\mathcal{R}_3,\mathbb{R}^2)$$

with equivalent norms. Now the statement follows from Theorems 2.9 and 2.11. The last identity has been derived in (2.18). □

Also $\mathrm{tr}_1, \mathrm{tr}_2$ and tr_3 have additional properties if one of the smoothness parameters dominates the sum of the other.

Theorem 2.14. *Let $\mathcal{O}$ be an orthogonal basis of $\Gamma = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1+x_2+x_3 = 0\}$ and let $\mathcal{R}_3$ be the matrix associated with $\mathcal{O}$ by* (1.1) *and* (2.10). *Let $\min(r_1,r_2,r_3) > 1/2$ and suppose $r_3 \geq r_1 + r_2$.*
Then tr_3 becomes a retraction of $S_2^{r_1,r_2,r_3}W(\mathbb{R}^3)$ onto $S_2^{r_1,r_2}W(\mathbb{R}^2)$, i.e. there exists a linear extension operator $\mathrm{ext}^* \in \mathcal{L}(S_2^{r_1,r_2}W(\mathbb{R}^2), S_2^{r_1,r_2,r_3}W(\mathbb{R}^3))$ *s.t.* $\mathrm{tr}_3 \circ \mathrm{ext}^* = I$.

Proof. Step 1. Boundedness of tr_3. To show that, we use again (2.12). Furthermore, taking $h(\xi_1,\xi_2,\tau_3) := \mathcal{F}f(\xi_1+\tau_3, \xi_2+\tau_3, \tau_3)$, it will be enough to show the existence of some positive constant c such that

$$\begin{aligned} \int_{\mathbb{R}^2} (1+y_1^2)^{r_1}\,(1+y_2^2)^{r_2} \left|\int_{\mathbb{R}} h(y_1,y_2,y_3)\,dy_3\right|^2 dy_1\,\mathrm{d}y_2 \qquad (2.25)\\ \leq c \int_{\mathbb{R}^3} [1+(y_1+y_3)^2]^{r_1}\,[1+(y_2+y_3)^2]^{r_2}\,[1+y_3^2]^{r_3}\,|h(y_1,y_2,y_3)|^2\,dy_1\,dy_2\,dy_3\,. \end{aligned}$$

Let us denote

$$\Theta_1(y_1,y_2) := (1+y_1^2)^{r_1}\,(1+y_2^2)^{r_2}$$

and

$$\Theta_2(y_1,y_2,y_3) := [1+(y_1+y_3)^2]^{r_1}\,[1+(y_2+y_3)^2]^{r_2}\,(1+y_3^2)^{r_3}\,,$$

respectively. Then Hölder's inequality leads to

$$\begin{aligned}(1+y_1^2)^{r_1}\,(1+y_2^2)^{r_2} \quad & \Big(\int_{\mathbb{R}} |h(y_1,y_2,y_3)|\,dy_3\Big)^2 \\ = \quad & \Big(\int_{\mathbb{R}} \frac{\sqrt{\Theta_1(y_1,y_2)}}{\sqrt{\Theta_2(y_1,y_2,y_3)}}\sqrt{\Theta_2(y_1,y_2,y_3)}|h(y_1,y_2,y_3)|\,dy_3\Big)^2 \\ \le \quad & \underbrace{\Big(\sup_{y_1,y_2\in\mathbb{R}} \int_{\mathbb{R}} \frac{\Theta_1(y_1,y_2)}{\Theta_2(y_1,y_2,y_3)}\,dy_3\Big)}_{:=\,\Theta(r_1,r_2,r_3)} \int_{\mathbb{R}} \Theta_2(y_1,y_2,y_3)|h(y_1,y_2,y_3)|^2\,dy_3\,.\end{aligned}$$

If $\Theta(r_1,r_2,r_3)<\infty$, then it is enough to integrate this inequality with respect to $y_1,y_2\in\mathbb{R}$ to obtain (2.25). To prove finiteness of $\Theta(r_1,r_2,r_3)$ under the given restrictions is elementary.

Step 2. Surjectivity of tr_3. Here we make use of the operator ext_3^*, defined in the proof of Theorem 2.9, Substep 3.1. □

Remark 2.15. By symmetry we have similar statements with respect to tr_1 as well as to tr_2, e.g. if $\min(r_1,r_2,r_3)>1/2$ and $r_2\ge r_1+r_3$ then tr_2 becomes a retraction of $S_2^{r_1,r_2,r_3}W(\mathbb{R}^3)$ onto $S_2^{r_1,r_3}W(\mathbb{R}^2)$.

2.4.4 An Example

We consider the orthogonal basis $\vec{\sigma}_1 := (1,-1,0)$, and $\vec{\sigma}_2 := (1,1,-2)$ of Γ. Then the functions m_i, $i=1,2,3$, defined in (2.18)-(2.20), are given by

$$\begin{aligned} m_1^2(\xi_1,\xi_2) &= \Big[1+(2\xi_1)^2\Big]^{r_2}\Big[1+(\xi_1+\xi_2)^2\Big]^{r_3}, \\ m_2^2(\xi_1,\xi_2) &= \Big[1+(2\xi_1)^2\Big]^{r_1}\Big[1+(\xi_1-\xi_2)^2\Big]^{r_3}, \\ m_3^2(\xi_1,\xi_2) &= \Big[1+(\xi_1+\xi_2)^2\Big]^{r_1}\Big[1+(\xi_1-\xi_2)^2\Big]^{r_2}. \end{aligned}$$

Let $r_1=r_2=r_3=1$ and define

$$\begin{aligned} w(\xi_1,\xi_2) := \min\Big(&1+5\xi_1^2+\xi_2^2+2\xi_1\xi_2+4\xi_1^4+4\xi_1^2\xi_2^2+8\xi_1^3\xi_2, \\ &1+5\xi_1^2+\xi_2^2-2\xi_1\xi_2+4\xi_1^4+4\xi_1^2\xi_2^2-8\xi_1^3\xi_2, 1+2\xi_1^2+2\xi_2^2+\xi_1^4-2\xi_1^2\xi_2^2+\xi_2^4\Big) \end{aligned}$$

cf. (2.21). Hence, the trace space of the Sobolev space $S_2^{1,1,1}W(\mathbb{R}^3)$ with respect to this orthogonal basis is the collection of all functions $g\in L_2(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} w(\xi_1,\xi_2)\,|\mathcal{F}g(\xi_1,\xi_2)|^2 d\xi < \infty\,.$$

Furthermore, the trace space of the Sobolev space $S_2^{1,1,2}W(\mathbb{R}^3)$ with respect to this orthogonal basis is the collection of all functions $g\in L_2(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} \Big(1+2\xi_1^2+2\xi_2^2+\xi_1^4-2\xi_1^2\xi_2^2+\xi_2^4\Big)\,|\mathcal{F}g(\xi_1,\xi_2)|^2 d\xi < \infty\,.$$

3 Besov and Triebel-Lizorkin Spaces

Now we turn to the general case of Besov and Triebel-Lizorkin spaces. To begin with we recall the Fourier-analytic definition as well as the characterization by atoms of these classes. Since we shall need the spaces for $d = 3$ and for $d = 2$ we shall work for a while with the general d-dimensional case.

3.1 Notation

As usual, $\mathbb{R}^d$ denotes the $d-$dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integers and $\mathbb{C}$ denotes the complex numbers. If $x, y \in \mathbb{R}^d$, we write $x > y$ if, and only if, $x_i > y_i$ for every $i = 1, \ldots, d$. Similarly, we define the relations $x \geq y, x < y, x \leq y$. Finally, in slight abuse of notation, we write $x > \lambda$ for $x \in \mathbb{R}^d, \lambda \in \mathbb{R}$ if $x_i > \lambda, i = 1, \ldots, d$. For a real number x we denote by $x_+ := \max(x, 0)$ the nonnegative part.
Let $S(\mathbb{R}^d)$ be the Schwartz space of all complex–valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$.

3.2 The Fourier-analytic Approach

Let $\varphi \in S(\mathbb{R})$ with

$$\varphi(t) = 1 \quad \text{if} \quad |t| \leq 1 \quad \text{and} \quad \varphi(t) = 0 \quad \text{if} \quad |t| \geq \frac{3}{2}. \tag{3.1}$$

We put $\varphi_0 = \varphi$, $\varphi_1(t) = \varphi(t/2) - \varphi(t)$ and

$$\varphi_j(t) := \varphi_1(2^{-j+1}t), \quad t \in \mathbb{R}, \quad j \in \mathbb{N}.$$

Hence we have $\sum_{j=0}^{\infty} \varphi_j(t) = 1$ for all $t \in \mathbb{R}$. For $\overline{k} = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we define $\varphi_{\overline{k}}(x) := \varphi_{k_1}(x_1) \cdot \ldots \cdot \varphi_{k_d}(x_d)$. Then, since

$$\sum_{\overline{k} \in \mathbb{N}_0^d} \varphi_{\overline{k}}(x) = 1 \quad \text{for every} \quad x \in \mathbb{R}^d, \tag{3.2}$$

the system $\{\varphi_{\overline{k}}\}_{\overline{k} \in \mathbb{N}_0^d}$ forms a smooth dyadic resolution of unity. This will be used to define the classes of functions we are interested in.

Definition 3.1. Let $\overline{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$, and $0 < q \leq \infty$.
(i) Let $0 < p \leq \infty$. Then the Besov space of dominating mixed smoothness $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|S^{\overline{r}}_{p,q}B(\mathbb{R}^d)||_\varphi = \Big(\sum_{\overline{k} \in \mathbb{N}_0^d} 2^{q\overline{k}\cdot\overline{r}} || \mathcal{F}^{-1}[\varphi_{\overline{k}} \mathcal{F}f] \, |L_p(\mathbb{R}^d)||^q \Big)^{1/q} = ||2^{\overline{k}\cdot\overline{r}} \mathcal{F}^{-1}[\varphi_{\overline{k}} \mathcal{F}f] \, |\ell_q(L_p)|| \tag{3.3}$$

is finite.
(ii) Let $0 < p < \infty$. Then the Triebel-Lizorkin space of dominating mixed smoothness $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)||_\varphi = \Big|\Big| \Big(\sum_{\overline{k} \in \mathbb{N}_0^d} |2^{\overline{k}\cdot\overline{r}} \mathcal{F}^{-1}[\varphi_{\overline{k}} \mathcal{F}f](\cdot) \, |^q \Big)^{1/q} |L_p(\mathbb{R}^d) \Big|\Big| = ||2^{\overline{k}\cdot\overline{r}} \mathcal{F}^{-1}[\varphi_{\overline{k}} \mathcal{F}f] \, |L_p(\ell_q)|| \tag{3.4}$$

is finite.

Remark 3.2. 1. Sometimes, we shall write $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ instead of $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ or $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$.

2. Different functions φ (with properties described above) lead to equivalent quasi-norms on $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$. We shall write $||f|S^{\overline{r}}_{p,q}A(\mathbb{R}^d)||$ meaning one of these quasi-norms (which one is in general with no importance in our context). For details see [ST, Section 2.2.3].

3. For a systematic investigation of these classes we refer to the monographs [Am] and [ST]. More recent developments may be found in [Ba], [Ho] and [Vy1, Vy2, Vy3].

4. For $1 < p < \infty$ we have the coincidence of $S^{\overline{r}}_{p,2}F(\mathbb{R}^d)$ and the Sobolev space $S^{\overline{r}}_{p}W(\mathbb{R}^d)$ in the sense of equivalent norms, cf. [LN] and [ST, 2.3.1].

3.3 Atomic Decomposition

In the mid-eighties Frazier and Jawerth [FJ1] have been the first who studied atomic decompositions of Besov spaces. One of the applications has been a description of the solution of the trace problem with respect to hyperplanes in the isotropic situation. Here we follow the same philosophy. We shall make use of the characterization of Besov and Lizorkin-Triebel spaces by means of atoms for studiing the properties of $\mathrm{tr}_{\mathcal{O}}$.
Atomic decomposition techniques allow a certain discretization. Function spaces are replaced by sequence spaces. This method has been studied in various situations by now, cf. [FJ1, FJ2, AH, Tr2] for isotropic spaces of Besov and Lizorkin-Triebel type and [HN] for some generalizations in various directions. Besov and Lizorkin-Triebel spaces of dominating mixed smoothness have been characterized in such a way in [Vy2].

3.3.1 Sequence Spaces

For $\overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d$ we denote by $Q_{\overline{\nu}\,\overline{m}}$ the cube with the centre at the point $2^{-\overline{\nu}}\overline{m} = (2^{-\nu_1}m_1, \ldots, 2^{-\nu_d}m_d)$, sides parallel to the coordinate axes and of lengths $2^{-\nu_1}, \ldots, 2^{-\nu_d}$. We denote by $\chi_{\overline{\nu}\,\overline{m}} = \chi_{Q_{\overline{\nu}\,\overline{m}}}$ the characteristic function of $Q_{\overline{\nu}\,\overline{m}}$ and by $c\,Q_{\overline{\nu}\,\overline{m}}$ we mean a cube concentric with $Q_{\overline{\nu}\,\overline{m}}$ with sides c times larger.

Definition 3.3. If $0 < p, q \leq \infty$, $\overline{r} \in \mathbb{R}^d$ and

$$\lambda = \{\lambda_{\overline{\nu}\,\overline{m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d\}\,, \tag{3.5}$$

then we define

$$s^{\overline{r}}_{pq}b := \left\{\lambda : \quad ||\lambda|s^{\overline{r}}_{pq}b|| = \Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\Big(\sum_{\overline{m}\in\mathbb{Z}^d} |\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\Big)^{1/q} < \infty\right\} \tag{3.6}$$

and

$$s^{\overline{r}}_{pq}f := \left\{\lambda : \quad ||\lambda|s^{\overline{r}}_{pq}f|| = \left|\left|\Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} |2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}}(\cdot)|^q\Big)^{1/q} |L_p(\mathbb{R}^d)\right|\right| < \infty\right\} \tag{3.7}$$

with the usual modification for p and/or q equal to ∞.

Remark 3.4. We shall use the same convention as in case of the distribution spaces: from time to time we shall write $\|\lambda|s^{\overline{r}}_{pq}a\|$ instead of $\|\lambda|s^{\overline{r}}_{pq}b\|$ or $\|\lambda|s^{\overline{r}}_{pq}f\|$, respectively.

3.3.2 Atomic Decompositions

We will be very brief and refer for details to [Vy1] and [Vy2]. Here we concentrate on the "regular" case, i.e.

$$\overline{r} > \begin{cases} \sigma_p = \max\Big(\frac{1}{p} - 1, 0\Big) & \text{in the B-case} \\ \sigma_{pq} = \max\Big(\frac{1}{\min(p,q)} - 1, 0\Big) & \text{in the F-case.} \end{cases} \tag{3.8}$$

The phrase "regular" indicates that only those distribution spaces are considered which consists of regular distributions. Then, compared with the general case, no moment conditions have to be satisfied by the elementary building blocks called atoms. As usual, $[x]$ denotes the integer part of the real number x. If Q is a cube and δ is a positive real number then δQ denotes the cube with the same center as Q, sides parallel to those of Q and sidelength multiplied by δ.

Definition 3.5. Let $\overline{K} = (K_1, \dots, K_d) \in \mathbb{N}_0^d$ and $\delta > 1$. A $\overline{K}$-times differentiable complex-valued function $a(x)$ is called $\overline{K}$-atom related to $Q_{\overline{\nu}\,\overline{m}}$ if

$$\operatorname{supp} a \subset \delta\, Q_{\overline{\nu}\,\overline{m}}, \tag{3.9}$$

and

$$\sup_{x \in \mathbb{R}^d} |D^\alpha a(x)| \le 2^{\alpha \cdot \overline{\nu}} \qquad \text{for} \quad 0 \le \alpha \le \overline{K} \tag{3.10}$$

Theorem 3.6. *Let $0 < p, q \le \infty$, ($p < \infty$ in the $F-$case) and $\overline{r} \in \mathbb{R}^d$ with (3.8). Fix $\overline{K} \in \mathbb{N}_0^d$ with*

$$K_i \ge (1 + [r_i])_+ \qquad i = 1, \dots, d\,, \tag{3.11}$$

and δ sufficiently large.
Then $f \in S'(\mathbb{R}^d)$ belongs to $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ if, and only if, it can be represented as

$$f = \sum_{\overline{\nu} \in \mathbb{N}_0^d} \sum_{\overline{m} \in \mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}} a_{\overline{\nu}\,\overline{m}}(x), \qquad (\textit{convergence in} \quad S'(\mathbb{R}^d)), \tag{3.12}$$

where $\{a_{\overline{\nu}\,\overline{m}}(x)\}_{\overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d}$ are $\overline{K}$-atoms related to $Q_{\overline{\nu}\,\overline{m}}$ and $\lambda \in s^{\overline{r}}_{pq}a$. Furthermore,

$$\inf ||\lambda| s^{\overline{r}}_{pq}a||,$$

where the infimum is taken over all admissible representations (3.12), yields an equivalent quasi-norm in $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$.

Remark 3.7. To explain our philosophy, let the function a be a $\overline{K} = (K_1, K_2, K_3)$-atom related to $Q_{\overline{\nu}\,\overline{m}}$, where $\nu = (\nu_1, \nu_2, \nu_3)$ and $m = (m_1, m_2, m_3)$. Then

$$(\operatorname{tr}_3 a)(x_1, x_2) = a(x_1, x_2, -(x_1 + x_2))$$

becomes a (K_1, K_2)-atom with respect to $Q_{(\nu_1,\nu_2),(m_1,m_2)}$ if $K_3 \ge K_1 + K_2$ and $\nu_3 \le \min(\nu_1, \nu_2)$. Similarly $\operatorname{tr}_2 a$ ($\operatorname{tr}_1 a$) becomes a (K_1, K_3)-atom ((K_2, K_3)-atom) with respect to $Q_{(\nu_1,\nu_3),(m_1,m_3)}$ ($Q_{(\nu_2,\nu_3),(m_2,m_3)}$) if $K_2 \ge K_1 + K_3$ ($K_1 \ge K_2 + K_3$) and $\nu_2 \le \min(\nu_1, \nu_3)$ ($\nu_1 \le \min(\nu_2, \nu_3)$). This simple observation will motivate an appropriate decomposition of the atomic decomposition of a function which turns out to be a basic step in our proof of the boundedness of $\operatorname{tr}_{\mathcal{O}}$.

3.4 Traces of Besov Spaces of Dominating Mixed Smoothness

For a better comparison we recall the properties of the mapping $f(x_1, x_2, x_3) \mapsto f(x_1, x_2, 0)$ in this general context, cf. e.g. Amanov [Am, 9.5] and Schmeißer, Triebel [ST, 2.4.2] (further references are given in [ST, Remark 2.4.2]).

Proposition 3.8. *Let $0 < q \leq \infty$.*
(i) *Let $0 < p \leq \infty$ and $r_3 > 1/p$. Then the mapping*

$$T: \ f(x_1, x_2, x_3) \to f(x_1, x_2, 0)$$

extends to a retraction from $S^{r_1,r_2,r_3}_{p,q}B(\mathbb{R}^3)$ onto $S^{r_1,r_2}_{p,q}B(\mathbb{R}^2)$.
(ii) *Let $0 < p < \infty$ and $r_3 > \frac{1}{p}$. Then the mapping T extends to a retraction from $S^{r_1,r_2,r_3}_{p,q}F(\mathbb{R}^3)$ onto $S^{r_1,r_2}_{p,q}F(\mathbb{R}^2)$.*

As mentioned in Introduction, to reflect the underlying geometry of our problem, we have to define some new spaces with dominating mixed smoothness, cf. Subsection 2.2 for $p = 2$.

Definition 3.9. Let $0 < q \leq \infty$, $0 < p \leq \infty$ in the B-case and $0 < p < \infty$ in the F-case. Let $\mathcal{R}$ be a $(2, 2)$-matrix with $\det \mathcal{R} \neq 0$. Then we put

$$\begin{aligned} S^{\overline{r}}_{p,q}A(\mathcal{R}, \mathbb{R}^2) &:= \Big\{ f \in S'(\mathbb{R}^2) : f \circ \mathcal{R} \in S^{\overline{r}}_{p,q}A(\mathbb{R}^2) \Big\} , \\ ||f|S^{\overline{r}}_{p,q}A(\mathcal{R}, \mathbb{R}^2)|| &:= ||\, f \circ \mathcal{R}\, |S^{\overline{r}}_{p,q}A(\mathbb{R}^2)||\, . \end{aligned}$$

Recall that for $p = q = 2$ we have coincidence of $S^{\overline{r}}_{2,2}B(\mathcal{R}, \mathbb{R}^2)$ with $S^{\overline{r}}_{2}W(\mathcal{R}, \mathbb{R}^2)$ in the sense of equivalent norms, cf. [LN] or [ST, Thm. 2.3.1]. By means of these classes we are able to describe the trace classes for Besov as well as for Lizorkin-Triebel classes.

The counterpart of Theorem 2.9 for Besov spaces is as follows.

Theorem 3.10. *Let $\mathcal{O}$ be an orthogonal basis of Γ and let $\mathcal{R}_i, i = 1, 2, 3$ be matrices associated with $\mathcal{O}$ by (1.1), (2.8) and (2.10).*
Let $0 < p, q \leq \infty$ and $\overline{r} = (r_1, r_2, r_3) \in \mathbb{R}^3$ with $r_i \neq \frac{1}{p}, i = 1, 2, 3$ and

$$\min\left(r_1, r_2, r_3, r_1 + r_2 - \frac{1}{p}, r_1 + r_3 - \frac{1}{p}, r_2 + r_3 - \frac{1}{p}\right) > \sigma_p. \tag{3.13}$$

Then

$$\mathrm{tr}_{\mathcal{O}} \in \mathcal{L}\Big(S^{\overline{r}}_{p,q}B(\mathbb{R}^3), S^1(\mathbb{R}^2) + S^2(\mathbb{R}^2) + S^3(\mathbb{R}^2)\Big), \tag{3.14}$$

where

$$S^1(\mathbb{R}^2) := \begin{cases} S^{r_2,r_3}_{p,q}B(\mathcal{R}_1^{-1}, \mathbb{R}^2), & \text{if } r_1 > \frac{1}{p}\, , \\ S^{r_2,r_3+r_1-\frac{1}{p}}_{p,q}B(\mathcal{R}_1^{-1}, \mathbb{R}^2) \cap S^{r_2+r_1-\frac{1}{p},r_3}_{p,q}B(\mathcal{R}_1^{-1}, \mathbb{R}^2), & \text{if } r_1 < \frac{1}{p}\, , \end{cases}$$

and similarly for S^2 and S^3.
Conversely, to each function $g \in S^1(\mathbb{R}^2) + S^2(\mathbb{R}^2) + S^3(\mathbb{R}^2)$ there exists a function $f \in S^{\overline{r}}_{p,q}B(\mathbb{R}^3)$ such that $\mathrm{tr}_{\mathcal{O}} f = g$.

Proof. The restrictions in (3.13) are guaranteeing that we may apply Theorem 3.6 for $S^{\overline{r}}_{p,q}B(\mathbb{R}^3)$ as well as for all spaces appearing in the definition of the target spaces but taken with the identity matrix instead of $\mathcal{R}_i^{-1}$, $i \in \{1,2,3\}$.
Step 1. According to Theorem 3.6, each $f \in S^{\overline{r}}_{p,q}B(\mathbb{R}^3)$ may be decomposed into

$$f = \sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}} a_{\overline{\nu}\,\overline{m}}(x), \tag{3.15}$$

with

$$\|\, \lambda \,| s^{\overline{r}}_{p,q} b\| \le c\, \| f | S^{\overline{r}}_{p,q}B(\mathbb{R}^3)\| \tag{3.16}$$

with some constant c independent of f. We require some additional regularity of the atoms, cf. Definition 3.5:

$$K_i \ge \max\Big([r_1]+[r_2]+2, [r_1]+[r_3]+2, [r_2]+[r_3]+2\Big), \qquad i=1,2,3\,. \tag{3.17}$$

In view of Remark 3.7 we decompose f into three parts $f_i, i=1,2,3$, where

$$\begin{aligned}
f_1(x) &:= \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=\nu_1}^{\infty} \sum_{\nu_3=\nu_1}^{\infty} \sum_{\overline{m}\in\mathbb{Z}^3} \lambda_{\overline{\nu}\,\overline{m}}\, a_{\overline{\nu}\,\overline{m}}(x), &(3.18)\\
f_2(x) &:= \sum_{\nu_2=0}^{\infty} \sum_{\nu_1=\nu_2+1}^{\infty} \sum_{\nu_3=\nu_2}^{\infty} \sum_{\overline{m}\in\mathbb{Z}^3} \lambda_{\overline{\nu}\,\overline{m}}\, a_{\overline{\nu}\,\overline{m}}(x), &(3.19)\\
f_3(x) &:= \sum_{\nu_3=0}^{\infty} \sum_{\nu_1=\nu_3+1}^{\infty} \sum_{\nu_2=\nu_3+1}^{\infty} \sum_{\overline{m}\in\mathbb{Z}^3} \lambda_{\overline{\nu}\,\overline{m}}\, a_{\overline{\nu}\,\overline{m}}(x). &(3.20)
\end{aligned}$$

This allows us to decompose $\mathrm{tr}_{\mathcal{O}}\, f$ into (see (2.7))

$$(\mathrm{tr}_{\mathcal{O}}\, f)(z_1,z_2) = \sum_{i=1}^{3} (\mathrm{tr}_i\, f_i)(\mathcal{R}_i \vec{z})\,. \tag{3.21}$$

So, to establish (3.14) it is enough to prove the existence of a constant c independent of f such that

$$\|\, \mathrm{tr}_1\, f_1 \,| S^{r_2,r_3}_{p,q} B(\mathbb{R}^2)\| \le c\, \|\, f \,| S^{\overline{r}}_{p,q}B(\mathbb{R}^3)\| \tag{3.22}$$

if $r_1 > \frac{1}{p}$ and

$$\begin{aligned}
\|\, \mathrm{tr}_1\, f_1 \,| S^{r_2,r_3+r_1-\frac{1}{p}}_{p,q} B(\mathbb{R}^2)\| &\le c\, \|\, f \,| S^{\overline{r}}_{p,q}B(\mathbb{R}^3)\|\,, &(3.23)\\
\|\, \mathrm{tr}_1\, f_1 \,| S^{r_2+r_1-\frac{1}{p},r_3}_{p,q} B(\mathbb{R}^2)\| &\le c\, \| f | S^{\overline{r}}_{p,q}B(\mathbb{R}^3)\| &(3.24)
\end{aligned}$$

if $r_1 < \frac{1}{p}$ and corresponding analoga for $\mathrm{tr}_i\, f_i$, $i = 2,3$.
Step 2. Proof of (3.22)–(3.24). We proceed similar to [Vy3]. For brevity we put

$$\begin{aligned}
\Upsilon_1 &:= \{\overline{\nu} \in \mathbb{N}_0^3 : \nu_1 \le \min(\nu_2,\nu_3)\,\}\,,\\
\Upsilon_2 &:= \{\overline{\nu} \in \mathbb{N}_0^3 : \nu_2 \le \min(\nu_1,\nu_3)\,\}\,,\\
\Upsilon_3 &:= \{\overline{\nu} \in \mathbb{N}_0^3 : \nu_3 \le \min(\nu_1,\nu_2)\,\}\,.
\end{aligned}$$

Then

$$\operatorname{tr}_1 f_1(x_2,x_3) = \sum_{\overline{\nu}\in\Upsilon_1}\sum_{\overline{m}\in B_{\overline{\nu}}} \lambda_{\overline{\nu}\,\overline{m}}\, a_{\overline{\nu}\,\overline{m}}(-x_2-x_3,x_2,x_3)\,, \tag{3.25}$$

where

$$B_{\overline{\nu}} := \{\overline{m}\in\mathbb{Z}^3 : \quad \operatorname{supp} a_{\overline{\nu}\,\overline{m}}\cap\Gamma\neq\emptyset\}\,. \tag{3.26}$$

Due to (3.9), for given $\overline{\nu}\in\Upsilon_1$ and $m_2,m_3\in\mathbb{Z}$, there are at most N integers $m_1\in\mathbb{Z}$, such that $\overline{m}=(m_1,m_2,m_3)\in B_{\overline{\nu}}$. The number N does not depend on $\overline{\nu}$ and m_2,m_3. To simplify notation we shall work only with one number m_1, denoted by $m_1(\overline{\nu},m_2,m_3)$ or simply by m_1 if the values of $\overline{\nu},m_2$ and m_3 are clear from context. Rewriting (3.25) this gives

$$\begin{aligned}\operatorname{tr}_1 f_1(x_2,x_3) &= \sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty}\sum_{(m_2,m_3)\in\mathbb{Z}^2}\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} \lambda_{\overline{\nu}\,(m_1,m_2,m_3)}\, a_{\overline{\nu}\,(m_1,m_2,m_3)}(-x_2-x_3,x_2,x_3)\\ &= \sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty}\sum_{(m_2,m_3)\in\mathbb{Z}^2} \gamma_{(\nu_2,\nu_3)\,(m_2,m_3)}\, b_{(\nu_2,\nu_3)\,(m_2,m_3)}(x_2,x_3),\end{aligned} \tag{3.27}$$

where

$$\gamma_{(\nu_2,\nu_3)\,(m_2,m_3)} = \sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} |\lambda_{\overline{\nu}\,(m_1,m_2,m_3)}|\,, \tag{3.28}$$

$b_{(\nu_2,\nu_3)\,(m_2,m_3)}(x_2,x_3)=0$ if $\gamma_{(\nu_2,\nu_3)\,(m_2,m_3)}=0$, and

$$b_{(\nu_2,\nu_3)\,(m_2,m_3)}(x_2,x_3) = \frac{1}{\gamma_{(\nu_2,\nu_3)\,(m_2,m_3)}}\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} \lambda_{\overline{\nu}\,(m_1,m_2,m_3)}\, a_{\overline{\nu}\,(m_1,m_2,m_3)}(-x_2-x_3,x_2,x_3)$$

if $\gamma_{(\nu_2,\nu_3)\,(m_2,m_3)}>0$. We recall, that in this sum m_1 is an abbreviation for $m_1(\overline{\nu},m_2,m_3)$.
Step 3. We claim

1. $b_{(\nu_2,\nu_3)\,(m_2,m_3)}$ are atoms in the sense of Definition 3.5 related to $(\nu_2,\nu_3),(m_2,m_3)$ up to a general constant.

2. $||\,\gamma\,|s^{r_2,r_3}_{p,q}b||\le c\,||\,\lambda\,|s^{\overline{r}}_{p,q}b||$ if $r_1>\frac{1}{p}$,

3. $||\,\gamma\,|s^{r_2,r_3+r_1-\frac{1}{p}}_{p,q}b||\le c\,||\,\lambda\,|s^{\overline{r}}_{p,q}b||$ and $||\,\gamma\,|s^{r_2+r_1-\frac{1}{p},r_3}_{p,q}b||\le c\,||\,\lambda\,|s^{\overline{r}}_{p,q}b||$ if $r_1<\frac{1}{p}$.

Substep 3.1. The proof of the first assertion is elementary, see Remark 3.7. Two comments are in order. The first one concerns regularity. If the components of $\overline{K}$ are large enough then b is sufficiently smooth to satisfy (3.10) for some $\widetilde{K}$ such that we can apply Theorem 3.6 with respect to the target space, cf. (3.17). The second comment concerns the estimate (3.10). As claimed this estimate is satisfied by the functions $b_{(\nu_2,\nu_3),(m_2,m_3)}$ up to a general constant c_α depending on α. Since we need to control a finite number of derivatives only we conclude that $C\,b_{(\nu_2,\nu_3),(m_2,m_3)}$ are atoms with $C^{-1}:=\max_\alpha c_\alpha$. This is enough for our purpose.
Substep 3.2. Let $r_1>\frac{1}{p}$. Let $r_1-1/p=\varepsilon_1+\varepsilon_2$, $\varepsilon_i>0$, $i=1,2$. Obviuosly, $\varepsilon_1>0$ guarantees

$$\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} |\lambda_{\overline{\nu}\,\overline{m}}| \le c_1\left(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} |\,2^{\nu_1\varepsilon_1}\,\lambda_{\overline{\nu}\,\overline{m}}|^p\right)^{1/p}\,.$$

Next we use $\varepsilon_2 > 0$ and obtain

$$
\begin{aligned}
\|\,\gamma\,|s^{r_2,r_3}_{p,q}b\|^q &= \sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty} 2^{[\nu_2(r_2-\frac{1}{p})+\nu_3(r_3-\frac{1}{p})]q}\Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}|\gamma_{(\nu_2,\nu_3)(m_2,m_3)}|^p\Big)^{q/p}\\
&\le c_2\sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty} 2^{[\nu_2(r_2-\frac{1}{p})+\nu_3(r_3-\frac{1}{p})]q}\Big(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} 2^{\nu_1\varepsilon_1 p}\sum_{(m_2,m_3)\in\mathbb{Z}^2}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\\
&\le c_3\sum_{\overline{\nu}\in\Upsilon_1} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\\
&\le c_3\,\|\,\lambda\,|s^{\overline{r}}_{p,q}b\|^q.
\end{aligned}
$$

Substep 3.3. Let $r_1 < \frac{1}{p}$. To begin with let $p \ge 1$. The triangle inequality yields

$$
\Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}\Big(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)}|\lambda_{\overline{\nu}\,\overline{m}}|\Big)^p\Big)^{1/p} \le \sum_{\nu_1=0}^{\min(\nu_2,\nu_3)}\Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{1/p}. \tag{3.29}
$$

If now $q \le 1$, we get

$$
\begin{aligned}
\|\,\gamma\,|s^{r_2,r_3+r_1-\frac{1}{p}}_{p,q}b\|^q &\le \sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty} 2^{[\nu_2(r_2-\frac{1}{p})+\nu_3(r_3+r_1-\frac{1}{p}-\frac{1}{p})]q}\left[\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)}\Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{1/p}\right]^q\\
&\le \sum_{\overline{\nu}\in\Upsilon_1} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p} \le \|\,\lambda\,|s^{\overline{r}}_{p,q}b\|^q.
\end{aligned}
$$

For $q > 1$, we denote

$$
\varrho_{\overline{\nu}} := \Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{1/p}
$$

and apply Hölder's inequality to obtain

$$
\begin{aligned}
&\|\,\gamma\,|s^{r_2,r_3+r_1-\frac{1}{p}}_{p,q}b\|^q\\
&\quad\le \sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty} 2^{[\nu_2(r_2-\frac{1}{p})+\nu_3(r_3+r_1-\frac{1}{p}-\frac{1}{p})]q}\left[\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} 2^{(\nu_3-\nu_1)(r_1-\frac{1}{p})}2^{(\nu_1-\nu_3)(r_1-\frac{1}{p})}\,\varrho_{\overline{\nu}}\right]^q\\
&\quad\le \left[\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} 2^{(\nu_3-\nu_1)(r_1-\frac{1}{p})q'}\right]^{q/q'}\sum_{\overline{\nu}\in\Upsilon_1} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\,\varrho^q_{\nu}\\
&\quad\le c\,\|\,\lambda\,|s^{\overline{r}}_{p,q}b\|^q.
\end{aligned}
$$

This proves our claims if $p \ge 1$. Now let $p < 1$. We substitute (3.29) by

$$
\Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}\Big(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)}|\lambda_{\overline{\nu}\,\overline{m}}|\Big)^p\Big)^{1/p} \le \Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{1/p} = \Big(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)}\varrho^p_{\overline{\nu}}\Big)^{1/p}. \tag{3.30}
$$

If $q \le p$ the monotonicity of the ℓ_r-quasinorms yields

$$
\begin{aligned}
\|\,\gamma\,|s^{r_2,r_3+r_1-\frac{1}{p}}_{p,q}b\|^q &\le \sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty} 2^{[\nu_2(r_2-\frac{1}{p})+\nu_3(r_3+r_1-\frac{1}{p}-\frac{1}{p})]q}\left[\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)}\varrho^p_{\overline{\nu}}\right]^{q/p}\\
&\le \sum_{\overline{\nu}\in\Upsilon_1} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\varrho^q_{\nu} \le \|\,\lambda\,|s^{\overline{r}}_{p,q}b\|^q.
\end{aligned} \tag{3.31}
$$

And for $q > p$, we combine (3.30) with Hölder's inequality

$$\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} 2^{(\nu_3-\nu_1)(r_1-\frac{1}{p})p} 2^{(\nu_1-\nu_3)(r_1-\frac{1}{p})p} \varrho_{\overline{\nu}}^{p} \leq c \left(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} 2^{(\nu_1-\nu_3)(r_1-\frac{1}{p})q} \varrho_{\overline{\nu}}^{q} \right)^{p/q}$$

to derive (3.31) again. Moreover, the second estimate in Claim 3 follows by interchanging the roles of r_2 and r_3. This completes the estimates claimed for γ.

Step 4. We shall prove the estimate for $\mathrm{tr}_1 f_1$. In case $r_1 > \frac{1}{p}$ we argue, by using Claim 2 and Theorem 3.6, first in $d = 2$ and later in $d = 3$, as follows

$$\| \mathrm{tr}_1 f_1 | S^{r_2,r_3}_{p,q} B(\mathbb{R}^2)\| \leq c_1 \| \gamma | s^{r_2,r_3}_{p,q} b\| \leq c_2 \| \lambda | s^{\overline{r}}_{p,q} b\| \leq c_3 \| f | S^{\overline{r}}_{p,q} B(\mathbb{R}^3)\| .$$

Mutatis mutandis the case $r_1 < \frac{1}{p}$ can be treated. The estimates of $\mathrm{tr}_i f_i$, $i = 2, 3$ follow by symmetry.

Step 5. Now we construct the (non-)linear extension operator. We start with a function $g \in S^1(\mathbb{R}^2) + S^2(\mathbb{R}^2) + S^3(\mathbb{R}^2)$. Then there are $g_i \in S^i(\mathbb{R}^2)$, $i = 1, 2, 3$, such that $g_i \in S^i(\mathbb{R}^2)$, $g = g_1 + g_2 + g_3$ and

$$\| g_i | S^i(\mathbb{R}^2)\| \leq 2 \| g | S^1(\mathbb{R}^2) + S^2(\mathbb{R}^2) + S^3(\mathbb{R}^2)\| .$$

We shall extend each g_i separately. It means, we construct three functions $f_1, f_2, f_3 \in S^{\overline{r}}_{p,q} B(\mathbb{R}^3)$ such that $\mathrm{tr}_{\mathcal{O}} f_i = g_i$, $i = 1, 2, 3$. The desirable extension will than be given by $f = f_1 + f_2 + f_3$.

Substep 5.1 We restrict ourselves to $i = 1$, the other cases follow by symmetry. To begin with we treat the case $r_1 > 1/p$. We put $h_1 := g_1 \circ \mathcal{R}_1^{-1}$. Then $h_1 \in S^{r_2,r_3}_{p,q} B(\mathbb{R}^2)$ and, according to (2.7), we get

$$g_1(z_1, z_2) = (\mathrm{tr}_{\mathcal{O}} f_1)(z_1, z_2) = (\mathrm{tr}_1 f_1)(\mathcal{R}_1 \vec{z})$$

for all $\vec{z} = (z_1, z_2) \in \mathbb{R}^2$ if, and only if,

$$g_1(\mathcal{R}_1^{-1} \vec{z}) = h_1(\vec{z}) = (\mathrm{tr}_1 f_1)(z_1, z_2) = f_1(-z_1 - z_2, z_1, z_2), \qquad \vec{z} = (z_1, z_2) \in \mathbb{R}^2.$$

Hence, our original task, namely to find f_1 such that $\mathrm{tr}_{\mathcal{O}} f_1 = g_1$, where $g_1 \in S^{r_2,r_3}_{p,q} B(\mathcal{R}_1^{-1}, \mathbb{R}^2)$ is given, can be replaced by searching for f_1 such that $\mathrm{tr}_1 f_1 = h_1$, where $h_1 \in S^{r_2,r_3}_{p,q} B(\mathbb{R}^2)$. Again we make use of atomic decompositions. According to Theorem 3.6 we can decompose

$$h_1(x_2, x_3) = \sum_{(\nu_2,\nu_3)\in\mathbb{N}_0^2} \sum_{(m_2,m_3)\in\mathbb{Z}^2} \gamma_{(\nu_2,\nu_3)\,(m_2,m_3)} b_{(\nu_2,\nu_3)\,(m_2,m_3)}(x_2, x_3),$$

where

$$c_1 \| \gamma | s^{r_2,r_3}_{p,q} b\| \leq \| h_1 | S^{r_2,r_3}_{p,q} B(\mathbb{R}^2)\| \leq c_2 \| \gamma | s^{r_2,r_3}_{p,q} b\|$$

for certain positive constants c_1 and c_2 independent of h_1. Now we choose an integer m_1 such that $|2^{-\nu_1} m_1 + 2^{-\nu_2} m_2 + 2^{-\nu_3} m_3| \leq 2^{-\nu_1}$ and define

$$a_{\overline{\nu}\,\overline{m}}(x_1, x_2, x_3) := \psi(2^{\nu_1} x_1 - m_1)\, b_{(\nu_2,\nu_3)\,(m_2,m_3)}(x_2, x_3),$$

where

$$\psi \in S(\mathbb{R}), \quad \mathrm{supp}\, \psi \subset [-2(1 + \delta), 2(1 + \delta)], \quad \psi(t) = 1 \text{ if } t \in [-(1 + \delta), (1 + \delta)]$$

and δ is the number from (3.9). For $\nu_1 \le \min(\nu_2,\nu_3)$ some easy calculations yield

$$a_{\overline{\nu}\,\overline{m}}(-x_2-x_3,x_2,x_3) = b_{(\nu_2,\nu_3)\,(m_2,m_3)}(x_2,x_3)\,, \qquad (x_2,x_3)\in\mathbb{R}^2\,.$$

If the first component of $\overline{m}$ differs from this specific m_1 then we define $a_{\overline{\nu}\,\overline{m}} \equiv 0$. Further, we put

$$\lambda_{(\nu_1,\nu_2,\nu_3)\,(m_1,m_2,m_3)} := \begin{cases} \gamma_{(\nu_2,\nu_3)\,(m_2,m_3)} & \text{if} \quad \nu_1 = 0\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{3.32}$$

and

$$f_1 := \operatorname{ext} h_1 = \sum_{\overline{\nu}\in\Upsilon_1}\sum_{\overline{m}\in\mathbb{Z}^3} \lambda_{\overline{\nu}\,\overline{m}}\, a_{\overline{\nu}\,\overline{m}}\,.$$

Then

$$\|\operatorname{ext} h_1\,|S^{\overline{r}}_{p,q}B(\mathbb{R}^3)\| \le C_1\,\|\lambda\,|s^{\overline{r}}_{p,q}b\| = C_1\,\|\gamma\,|s^{r_2,r_3}_{p,q}b\| \le C_2\,\|h_1\,|S^{r_2,r_3}_{p,q}B(\mathbb{R}^2)\|. \tag{3.33}$$

This showes that f_1 represents an appropriate extension of h_1 if $r_1 > \frac{1}{p}$.

Substep 5.2. Let $r_1 < 1/p$. First of all notice that this time $h_1 \in S^{r_2,r_3+r_1-\frac{1}{p}}_{p,q}B(\mathbb{R}^2) \cap S^{r_2+r_1-\frac{1}{p},r_3}_{p,q}B(\mathbb{R}^2)$. We have to modify the definition of λ, cf. (3.32). This time we use

$$\lambda_{(\nu_1,\nu_2,\nu_3)\,(m_1,m_2,m_3)} := \begin{cases} \gamma_{(\nu_2,\nu_3)\,(m_2,m_3)} & \text{if} \quad \nu_1 = \min(\nu_2,\nu_3)\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{3.34}$$

for the specific value of m_1 as chosen in Substep 5.1. In all other cases we put $\lambda_{\overline{\nu}\,\overline{m}} = 0$. Then

$$\begin{aligned}
&\|\operatorname{ext} h_1\,|S^{\overline{r}}_{p,q}B(\mathbb{R}^3)\|^q \le C_1\,\|\lambda\,|s^{\overline{r}}_{p,q}b\|^q \\
&= C_1 \sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty} 2^{[\min(\nu_2,\nu_3)(r_1-1/p)+\nu_2(r_2-\frac{1}{p})+\nu_3(r_3-\frac{1}{p})]q}\Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}|\gamma_{(\nu_2,\nu_3)(m_2,m_3)}|^p\Big)^{q/p} \\
&= C_1\Bigg(\sum_{\nu_2=0}^{\infty}\sum_{\nu_3=\nu_2}^{\infty} 2^{[\nu_2(r_1-1/p)+\nu_2(r_2-\frac{1}{p})+\nu_3(r_3-\frac{1}{p})]q}\Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}|\gamma_{(\nu_2,\nu_3)(m_2,m_3)}|^p\Big)^{q/p} \\
&\quad + \sum_{\nu_3=0}^{\infty}\sum_{\nu_2=\nu_3+1}^{\infty} 2^{[\nu_3(r_1-1/p)+\nu_2(r_2-\frac{1}{p})+\nu_3(r_3-\frac{1}{p})]q}\Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}|\gamma_{(\nu_2,\nu_3)(m_2,m_3)}|^p\Big)^{q/p}\Bigg) \\
&\le C_1\Big(\|\gamma\,|s^{r_2+r_1-\frac{1}{p},r_3}_{p,q}b\| + \|\gamma\,|s^{r_2,r_3+r_1-\frac{1}{p}}_{p,q}b\|\Big)^q \\
&\le C_2\Big(\|h_1\,|S^{r_2+r_1-\frac{1}{p},r_3}_{p,q}B(\mathbb{R}^2)\| + \|h_1\,|S^{r_2,r_3+r_1-\frac{1}{p}}_{p,q}B(\mathbb{R}^2)\|\Big)^q.
\end{aligned}$$

Hence, also in this situation we have an appropriate extension of g_1. The modifications for an extension of g_2 and g_3 are obvious. □

Remark 3.11. The reader may notice that the only possible failure of linearity of the extension operator comes from the (generally non-linear) decomposition of g into $g = g_1 + g_2 + g_3$.

It remains to consider the limiting cases where at least one of the r_i equals $1/p$. We concentrate on the more simple situation where $0 < p, q \le 1$.

Proposition 3.12. *Let $\mathcal{O}$ be an orthogonal basis of Γ and let $\mathcal{R}_i, i = 1,2,3$ be matrices associated with $\mathcal{O}$ by* (1.1), (2.8) *and* (2.10).
Let $0 < p,q \le 1$. Then the statement of Theorem 3.10 remains true without the assumption $r_i \neq 1/p$, $i = 1,2,3$.

Proof. The proof of Theorem 3.10 extends to the present situation since in Substep 3.2 one can work with $\varepsilon_1 = \varepsilon_2 = 0$. □

Remark 3.13. Proposition 3.12 does not extend to values of p larger than 1. In analogy to the two-dimensional situation, cf. [Vy3] for details, more complicated spaces occur. We omit details.

3.5 Traces of Lizorkin-Triebel Spaces

Now we turn to the Lizorkin-Triebel classes. To prove an analog of Theorem 3.10 for these spaces we can proceed in the same way as in case of the Besov spaces. We shall describe the needed modifications only.

Theorem 3.14. *Let $\mathcal{O}$ be an orthogonal basis of Γ and let $\mathcal{R}_i, i = 1,2,3$ be matrices associated with $\mathcal{O}$ by* (1.1), (2.8) *and* (2.10)*. Let $0 < p < \infty$ and $0 < q \le \infty$. Let $\overline{r} = (r_1, r_2, r_3) \in \mathbb{R}^3$ with*

$$\min(r_1, r_2, r_3) > \max\left(\frac{1}{p}, \sigma_{pq}\right). \tag{3.35}$$

Then

$$\mathrm{tr}_{\mathcal{O}} \in \mathcal{L}\Big(S^{\overline{r}}_{p,q}F(\mathbb{R}^3),\ S^{r_2,r_3}_{p,q}F(\mathcal{R}_1^{-1},\mathbb{R}^2) + S^{r_1,r_3}_{p,q}F(\mathcal{R}_2^{-1},\mathbb{R}^2) + S^{r_1,r_2}_{p,q}F(\mathcal{R}_3^{-1},\mathbb{R}^2)\Big). \tag{3.36}$$

Conversely, to each function $g \in S^{r_2,r_3}_{p,q}F(\mathcal{R}_1^{-1},\mathbb{R}^2) + S^{r_1,r_3}_{p,q}F(\mathcal{R}_2^{-1},\mathbb{R}^2) + S^{r_1,r_2}_{p,q}F(\mathcal{R}_3^{-1},\mathbb{R}^2)$ there exists a function $f \in S^{\overline{r}}_{p,q}F(\mathbb{R}^3)$ such that $\mathrm{tr}_{\mathcal{O}} f = g$.

Proof. We shall use the same notation as in the proof of Theorem 3.10.
Step 1. Boundedness. In Step 1 of the proof of Theorem 3.10 we simply change the letter B to F. In Step 2 nothing is to change and we concentrate on Step 3 now. We have to prove that

$$||\,\gamma\,|s^{r_2,r_3}_{p,q}f||\le c\,||\,\lambda\,|s^{r_1,r_2,r_3}_{p,q}f|| \tag{3.37}$$

with some c independent of λ.
Instead we shall prove a pointwise inequality. So, first we fix a point $(x_2, x_3) \in \mathbb{R}^2$. Then there is only one element $(m_2, m_3) \in \mathbb{Z}^2$ such that $\chi_{(\nu_2,\nu_3)\,(m_2,m_3)}(x_2,x_3) = 1$. We denote $\gamma_{(\nu_2,\nu_3)} = \gamma_{(\nu_2,\nu_3)\,(m_2,m_3)}$. Similarly, for each $\overline{\nu} = (\nu_1,\nu_2,\nu_3)$, there is a unique $\overline{m}(\overline{\nu}) = (m_1,m_2,m_3)$ such that $\chi_{(\nu_1,\nu_2,\nu_3)\,(m_1,m_2,m_3)}(x_1,x_2,x_3) = 1$ and $\overline{m} \in B_{\overline{\nu}}$. We denote $\lambda_{\overline{\nu}} = \lambda_{\overline{\nu}\,\overline{m}}$.
Substep 2.1. Let $r_1 > 1/p$ and $0 < q \le 1$. Then

$$|\gamma_{(\nu_2,\nu_3)}|^q = \Big(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} |\lambda_{(\nu_1,\nu_2,\nu_3)}|\Big)^q \le \sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} |\lambda_{(\nu_1,\nu_2,\nu_3)}|^q\,,$$

and

$$\left(\sum_{\nu_2,\nu_3=0}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q}|\gamma_{(\nu_2,\nu_3)}|^q\right)^{p/q} \le \left(\sum_{\nu_1=0}^{\infty}\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q}|\lambda_{\overline{\nu}}|^q\right)^{p/q}.$$

To continue we distinguish two cases. Let $0<p\le q$. Then

$$\begin{aligned}\left(\sum_{\nu_1=0}^{\infty}\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q}|\lambda_{\overline{\nu}}|^q\right)^{p/q} &\le \sum_{\nu_1=0}^{\infty}\left(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q}|\lambda_{\overline{\nu}}|^q\right)^{p/q}\\ &\le \sum_{\nu_1=0}^{\infty} 2^{-\nu_1 r_1 p}\left(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_1 r_1+\nu_2 r_2+\nu_3 r_3]q}|\lambda_{\overline{\nu}}|^q\right)^{p/q}.\end{aligned}$$

Now let $0<q<p<\infty$. With $0<\varepsilon<r_1p-1$ and applying Hölder's inequality we find

$$\begin{aligned}\left(\sum_{\nu_1=0}^{\infty}\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q}|\lambda_{\overline{\nu}}|^q\right)^{p/q} &\le c\sum_{\nu_1=0}^{\infty} 2^{-\nu_1(r_1p-\varepsilon)}\left(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_1 r_1+\nu_2 r_2+\nu_3 r_3]q}|\lambda_{\overline{\nu}}|^q\right)^{p/q}\\ &\le c\sum_{\nu_1=0}^{\infty} 2^{-\nu_1}\left(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_1 r_1+\nu_2 r_2+\nu_3 r_3]q}|\lambda_{\overline{\nu}}|^q\right)^{p/q}.\end{aligned}$$

Substep 2.2. If $q>1$ we use triangle inequality

$$\begin{aligned}\Big(\sum_{\nu_2,\nu_3=0}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q}\Big(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)}|\lambda_{\overline{\nu}}|\Big)^q\Big)^{1/q} &\le \sum_{\nu_1=0}^{\infty}\Big(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q}|\lambda_{\overline{\nu}}|^q\Big)^{1/q}\\ &\le \sum_{\nu_1=0}^{\infty} 2^{-\nu_1 r_1}\Big(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_1 r_1+\nu_2 r_2+\nu_3 r_3]q}|\lambda_{\overline{\nu}}|^q\Big)^{1/q}.\end{aligned}$$

If $0<p\le 1$

$$\Big(\sum_{\nu_2,\nu_3=0}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q}\Big(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)}|\lambda_{\overline{\nu}}|\Big)^q\Big)^{p/q} \le \sum_{\nu_1=0}^{\infty} 2^{-\nu_1 r_1 p}\Big(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_1 r_1+\nu_2 r_2+\nu_3 r_3]q}|\lambda_{\overline{\nu}}|^q\Big)^{p/q}$$

follows. If $p>1$ we apply again Hölder's inequality and find

$$\begin{aligned}\Big(\sum_{\nu_2,\nu_3=0}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q}\Big(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)}|\lambda_{\overline{\nu}}|\Big)^q\Big)^{p/q} &\le c\sum_{\nu_1=0}^{\infty} 2^{-\nu_1(r_1p-\varepsilon)}\Big(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_1 r_1+\nu_2 r_2+\nu_3 r_3]q}|\lambda_{\overline{\nu}}|^q\Big)^{p/q}\\ &\le c\sum_{\nu_1=0}^{\infty} 2^{-\nu_1}\Big(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_1 r_1+\nu_2 r_2+\nu_3 r_3]q}|\lambda_{\overline{\nu}}|^q\Big)^{p/q}.\end{aligned}$$

Substep 2.3. Summarizing in all situations we have found

$$\left(\sum_{\nu_2,\nu_3=0}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q}|\gamma_{(\nu_2,\nu_3)}|^q\right)^{p/q} \le c\sum_{\nu_1=0}^{\infty} 2^{-\nu_1}\left(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_1 r_1+\nu_2 r_2+\nu_3 r_3]q}|\lambda_{\overline{\nu}}|^q\right)^{p/q}, \qquad (3.38)$$

where c does not depend on λ. We have to show that this inequality implies (3.37). For fixed (x_2,x_3) we choose a sequence of intervals I_{ν_1} such that

$$I_{\nu_1}\cap I_{\nu_1'}=\emptyset, \quad \nu_1\ne\nu_1', \qquad |I_{\nu_1}|\ge c\,2^{-\nu_1}$$

for some $c > 0$ and

$$\Big\{(x_1, x_2, x_3): \quad x_1 \in I_{\nu_1}\Big\} \subset Q_{\overline{\nu}\,\overline{m}}\,, \qquad \overline{\nu} \in \Upsilon_1\,, \quad \overline{m} \in B_{\overline{\nu}}\,.$$

Then (3.38) implies

$$\Big(\sum_{\nu_2,\nu_3=0}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q} |\gamma_{(\nu_2,\nu_3)}\, \chi_{(\nu_2,\nu_3)\,(m_2,m_3)}(x_2,x_3)|^q\Big)^{p/q}$$
$$\leq \; c \sum_{\nu_1=0}^{\infty} \int_{I_{\nu_1}} \Big(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_1 r_1+\nu_2 r_2+\nu_3 r_3]q} |\lambda_{\overline{\nu}}\, \chi_{\overline{\nu}}(x_1,x_2,x_3)|^q\Big)^{p/q} \, dx_1 .$$

Integration with respect to x_2 and x_3 completes the proof of the boundedness of $\mathrm{tr}_1\, f_1$. The rest is the same as in the B-case.

Step 2. The extension. Here the same construction as in the B-case can be applied, cf. Substep 5.1 of the proof of Theorem 3.10. □

The above proof can be used also in case that some of the r_i coincide with $1/p$, at least under additional restrictions on p and q.

Proposition 3.15. *Let $\mathcal{O}$ be an orthogonal basis of Γ and let $\mathcal{R}_i, i = 1, 2, 3$ be matrices associated with $\mathcal{O}$ by* (1.1)*,* (2.8) *and* (2.10)*.*
Let $0 < p \leq \min(1, q)$. Then the statement of Theorem 3.14 remains true under the weaker restriction

$$\min(r_1, r_2, r_3) \geq \frac{1}{p} \qquad \textit{and} \qquad \min(r_1, r_2, r_3) > \sigma_{p,q}.$$

Remark 3.16. A final remark. In the general situation of the Besov-Lizorkin-Triebel spaces we have proved a full counterpart of Theorem 2.9. In fact, it is not only a counterpart. Based on the identities $S^{\overline{r}}_2 W(\mathbb{R}^3) = S^{\overline{r}}_{2,2} F(\mathbb{R}^3) = S^{\overline{r}}_{2,2} B(\mathbb{R}^3)$ (in the sense of equivalent norms) we have given a new proof of Theorem 2.9. Because of $S^{\overline{r}}_p W(\mathbb{R}^3) = S^{\overline{r}}_{p,2} F(\mathbb{R}^3)$, $1 < p < \infty$, (also in the sense of equivalent norms) Theorem 3.14 contains the extension to Sobolev spaces of dominating mixed smoothness with p different from 2. However, we do not have counterparts of Theorems 2.11 and 2.13, respectively. Here a good description of the spaces $S^{r_1,r_2}_{p,q} A(\mathcal{R}, \mathbb{R}^2)$ in terms of atoms would be desirable, see Lemma 2.2(iii) for the Fourier-analytic counterpart.

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