A diagonal embedding theorem for function spaces with dominating mixed smoothness

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Abstract

The aim of this paper is to study the diagonal embeddings of function spaces with dominating mixed smoothness. From certain point of view, this paper may be considered as a direct continuation of [8] and [6].

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1 Introduction

Spaces with dominating mixed smoothness were introduced by S. M. Nikol'skii ([4], [5]). The simplest case on the plane $\mathbb{R}^2$ are the spaces of Sobolev type

$$S_p^{\overline{r}}W(\mathbb{R}^2) = \Big\{f|f \in L_p(\mathbb{R}^2), ||f|S_p^{\overline{r}}W(\mathbb{R}^2)|| = ||f|L_p|| + \Big|\Big| \frac{\partial^{r_1} f}{\partial x_1^{r_1}} \mid L_p \Big|\Big| + \\ + \Big|\Big| \frac{\partial^{r_2} f}{\partial x_2^{r_2}} \mid L_p \Big|\Big| + \Big|\Big| \frac{\partial^{r_1+r_2} f}{\partial x_1^{r_1}\partial x_2^{r_2}} \mid L_p \Big|\Big| < \infty\Big\}, \tag{1.1}$$

where $1 < p < \infty, r_i = 0,1,2,\dots; (i = 1,2)$. The mixed derivative $\frac{\partial^{r_1+r_2} f}{\partial x_1^{r_1}\partial x_2^{r_2}}$ plays the dominant part here and gave the name to this class of spaces.

These spaces were studied extensively by many mathematicians. We quote Amanov ([1]), Schmeisser and Triebel ([7]) to mention at least some of them. We describe some aspects of this theory necessary in the sequel in Section 2. Sections 3 and 4 are devoted to the study of the trace operator

$$T : f(x_1, x_2) \to f(x_1, x_1). \tag{1.2}$$

In [8] Triebel proved that, for $1 \le p \le \infty$, the trace operator (1.2) is a retraction from $S^{(r_1,r_2)}_{p,1}B(\mathbb{R}^2)$ onto $B^{\varrho}_{p,1}(\mathbb{R})$, where $\varrho = \min(r_1, r_2, r_1 + r_2 - \frac{1}{p}) > 0$. The q-dependence was studied in [6]. Rodriguez proved that (1.2) is a retraction from $S^{(r_1,r_2)}_{p,q}B(\mathbb{R}^2)$ onto $B^{\varrho}_{p,q}(\mathbb{R})$, where

$$0 < p \le \infty, 0 < q < \infty, \varrho > \sigma_p = \max\Big(\frac{1}{p} - 1, 0\Big) \text{ and } \min(r_1, r_2) \ne \frac{1}{p}.$$

In the "limiting case" $\min(r_1, r_2) = \frac{1}{p}$ the same result is proven for $q \le \min(1, p)$.

We fill some of the minor gaps left open by Rodriguez in the B-case and study the trace operator in the context of F-spaces. As these include the spaces of dominating mixed smoothness of Sobolev type (1.1), we answer the question of their traces on the diagonal.

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2 Notation and Definitions

As usual, $\mathbb{R}^d$ denotes the $d-$dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers and $\mathbb{C}$ denotes the plain of complex numbers.

If $x, y \in \mathbb{R}^d$, we write $x > y$ if, and only if, $x_i > y_i$ for every $i = 1, \ldots, d$. Similarly, we define the relations $x \geq y, x < y, x \leq y$. Finally, in slight abuse of notation, we write $x > \lambda$ for $x \in \mathbb{R}^d, \lambda \in \mathbb{R}$ if $x_i > \lambda, i = 1, \ldots, d$.

When $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ is a multi-index, we denote its length by $|\alpha| = \sum_{j=1}^d \alpha_j$.

Let $S(\mathbb{R}^d)$ be the Schwartz space of all complex–valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$. We denote the $d-$dimensional Fourier transform of a function $\varphi \in S(\mathbb{R}^d)$ by $\hat{\varphi}$. Its inverse is denoted by $\varphi^\vee$. Both $\hat{}$ and $^\vee$ are extended to the dual Schwartz space $S'(\mathbb{R}^d)$ in the usual way.

We recall the basic aspects of the theory of function spaces used in the sequel. We don't mean to give some extensive survey on various decomposition techniques. Especially, as far as the standard Besov ($B^s_{p,q}(\mathbb{R}^d)$) and Triebel-Lizorkin ($F^s_{p,q}(\mathbb{R}^d)$) spaces are considered, we use the references [9] and [10]. Furthermore, we give the definition of function spaces with dominating mixed smoothness in general dimension. Setting $d = 1$, one gets the one-dimensional version $B^s_{p,q}(\mathbb{R})$ or $F^s_{p,q}(\mathbb{R})$, respectively.

Let $\varphi \in S(\mathbb{R})$ with

$$\varphi(t) = 1 \quad \text{if} \quad |t| \leq 1 \quad \text{and} \quad \varphi(t) = 0 \quad \text{if} \quad |t| \geq \frac{3}{2}. \tag{2.1}$$

We put $\varphi_0 = \varphi, \varphi_1(t) = \varphi(t/2) - \varphi(t)$ and

$$\varphi_j(t) = \varphi_1(2^{-j+1}t), \quad t \in \mathbb{R}, \quad j \in \mathbb{N}.$$

For $\overline{k} = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we define $\varphi_{\overline{k}}(x) = \varphi_{k_1}(x_1) \cdots \varphi_{k_d}(x_d)$. Then, since

$$\sum_{\overline{k} \in \mathbb{N}_0^d} \varphi_{\overline{k}}(x) = 1 \quad \text{for every} \quad x \in \mathbb{R}^d, \tag{2.2}$$

the system $\{\varphi_{\overline{k}}\}_{\overline{k} \in \mathbb{N}_0^d}$ forms a dyadic resolution of unity with the inner tensor product structure.

Definition 2.1. Let $\overline{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$, $0 < q \leq \infty$.
(i) Let $0 < p \leq \infty$. Then $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|S^{\overline{r}}_{p,q}B(\mathbb{R}^d)||_\varphi = \Big(\sum_{\overline{k} \in \mathbb{N}_0^d} 2^{q\overline{k}\cdot\overline{r}} ||(\varphi_{\overline{k}}\hat{f})^\vee|L_p(\mathbb{R}^d)||^q \Big)^{1/q} = ||2^{\overline{k}\cdot\overline{r}}(\varphi_{\overline{k}}\hat{f})^\vee|\ell_q(L_p)|| \tag{2.3}$$

is finite.
(ii) Let $0 < p < \infty$. Then $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)||_\varphi = \Big|\Big| \Big(\sum_{\overline{k} \in \mathbb{N}_0^d} |2^{\overline{k}\cdot\overline{r}}(\varphi_{\overline{k}}\hat{f})^\vee(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^d) \Big|\Big| = ||2^{\overline{k}\cdot\overline{r}}(\varphi_{\overline{k}}\hat{f})^\vee|L_p(\ell_q)|| \tag{2.4}$$

is finite.

Remark 2.2. Sometimes, we write $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ meaning one of spaces $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ or $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$. As mentioned above, by setting $d=1$, we get $B^s_{p,q}(\mathbb{R}) = S^{(s)}_{p,q}B(\mathbb{R})$ and $F^s_{p,q}(\mathbb{R}) = S^{(s)}_{p,q}F(\mathbb{R})$. If we replace in this case the factor 2^{ks} by $(k+1)^{\alpha}2^{ks}$, $\alpha \in \mathbb{R}$, we get the spaces of generalised smoothness $A^{(s,\alpha)}_{p,q}(\mathbb{R})$. We refer to [3] and references given there for details.

Our approach uses the full power of several decomposition techniques developed for these function spaces in [9], [3] and [12]. They all work with sequence spaces associated to these function spaces.

For $\overline{\nu} \in \mathbb{N}^d_0, \overline{m} \in \mathbb{Z}^d$ we denote by $Q_{\overline{\nu}\,\overline{m}}$ the cube with the centre at the point $2^{-\overline{\nu}}\overline{m} = (2^{-\nu_1}m_1, \ldots, 2^{-\nu_d}m_d)$ with sides parallel to the coordinate axes and of lengths $2^{-\nu_1}, \ldots, 2^{-\nu_d}$. We denote by $\chi_{\overline{\nu}\,\overline{m}} = \chi_{Q_{\overline{\nu}\,\overline{m}}}$ the characteristic function of $Q_{\overline{\nu}\,\overline{m}}$ and by $c\,Q_{\overline{\nu}\,\overline{m}}$ we mean a cube concentric with $Q_{\overline{\nu}\,\overline{m}}$ with sides c times longer.

Definition 2.3. If $0 < p, q \leq \infty$, $\overline{r} \in \mathbb{R}^d$ and

$$\lambda = \{\lambda_{\overline{\nu}\,\overline{m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}^d_0, \overline{m} \in \mathbb{Z}^d\} \tag{2.5}$$

then we define

$$s^{\overline{r}}_{pq}b = \left\{\lambda : ||\lambda|s^{\overline{r}}_{pq}b|| = \left(\sum_{\overline{\nu}\in\mathbb{N}^d_0} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\Big(\sum_{\overline{m}\in\mathbb{Z}^d} |\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\right)^{1/q} < \infty\right\} \tag{2.6}$$

and

$$s^{\overline{r}}_{pq}f = \left\{\lambda : ||\lambda|s^{\overline{r}}_{pq}f|| = \left\|\left(\sum_{\overline{\nu}\in\mathbb{N}^d_0}\sum_{\overline{m}\in\mathbb{Z}^d} |2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}}(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R}^d)\right\| < \infty\right\} \tag{2.7}$$

with the usual modification for p and/or q equal to ∞.

Remark 2.4. We point out that with λ given by (2.5) and $g_{\overline{\nu}}(x) = \sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}}(x)$, we obtain

$$||\lambda|s^{\overline{r}}_{pq}b|| = ||2^{\overline{\nu}\cdot\overline{r}}g_{\overline{\nu}}|\ell_q(L_p)||, \qquad ||\lambda|s^{\overline{r}}_{pq}f|| = ||2^{\overline{\nu}\cdot\overline{r}}g_{\overline{\nu}}|L_p(\ell_q)||.$$

Definition 2.5. If $0 < p, q \leq \infty$, $r, \alpha \in \mathbb{R}$ and

$$\lambda = \{\lambda_{\mu n} \in \mathbb{C} : \mu \in \mathbb{N}_0, n \in \mathbb{Z}\} \tag{2.8}$$

then we define

$$b^{(r,\alpha)}_{pq} = \left\{\lambda : ||\lambda|b^{(r,\alpha)}_{pq}|| = \left(\sum_{\mu\in\mathbb{N}_0} (\mu+1)^{\alpha q}2^{\mu(r-\frac{1}{p})q}\Big(\sum_{n\in\mathbb{Z}} |\lambda_{\mu n}|^p\Big)^{q/p}\right)^{1/q} < \infty\right\} \tag{2.9}$$

and

$$f^{(r,\alpha)}_{pq} = \left\{\lambda : ||\lambda|f^{(r,\alpha)}_{pq}|| = \left\|\left(\sum_{\mu\in\mathbb{N}_0}\sum_{n\in\mathbb{Z}} |(\mu+1)^{\alpha}2^{\mu r}\lambda_{\mu n}\chi_{\mu n}(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R})\right\| < \infty\right\} \tag{2.10}$$

with the usual modification for p and/or q equal to ∞.

Next we briefly describe the atomic and subatomic decomposition. We refer to [11] and [12] for details. Compared to the situation there, we now concentrate on the "regular" case,

$$\overline{r} > \begin{cases} \sigma_p = \max\Big(\frac{1}{p} - 1, 0\Big) & \text{in the B-case} \\ \sigma_{pq} = \max\Big(\frac{1}{\min(p,q)} - 1, 0\Big) & \text{in the F-case.} \end{cases} \tag{2.11}$$

Definition 2.6. Let $\overline{K} \in \mathbb{N}_0^d$ and $\gamma > 1$. A $\overline{K}$-times differentiable complex-valued function $a(x)$ is called $\overline{K}$-atom related to $Q_{\overline{\nu}\,\overline{m}}$ if

$$\operatorname{supp} a \subset \gamma Q_{\overline{\nu}\,\overline{m}}, \tag{2.12}$$

and

$$|D^\alpha a(x)| \le 2^{\alpha\cdot\overline{\nu}} \qquad \text{for} \quad 0 \le \alpha \le \overline{K} \tag{2.13}$$

Theorem 2.7. *Let $0 < p, q \le \infty$, ($p < \infty$ in the $F-$case) and $\overline{r} \in \mathbb{R}^d$ with (2.11). Fix $\overline{K} \in \mathbb{N}_0^d$ with*

$$K_i \ge (1 + [r_i])_+ \qquad i = 1, \dots, d. \tag{2.14}$$

Then $f \in S'(\mathbb{R}^d)$ belongs to $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ if, and only if, it can be represented as

$$f = \sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}} a_{\overline{\nu}\,\overline{m}}(x), \quad \textit{convergence being in} \quad S'(\mathbb{R}^d), \tag{2.15}$$

where $\{a_{\overline{\nu}\,\overline{m}}(x)\}_{\overline{\nu}\in\mathbb{N}_0^d, \overline{m}\in\mathbb{Z}^d}$ are $\overline{K}$-atoms related to $Q_{\overline{\nu}\,\overline{m}}$ and $\lambda \in s^{\overline{r}}_{pq}a$. Furthermore,

$$\inf ||\lambda | s^{\overline{r}}_{pq}a||,$$

where the infimum runs over all admissible representations (2.15), is an equivalent quasi-norm in $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$.

Definition 2.8. Let $\psi \in S(\mathbb{R})$ be a non-negative function with

$$\operatorname{supp} \psi \subset \{t \in \mathbb{R} : |t| < 2^\phi\} \tag{2.16}$$

for some $\phi \ge 0$ and

$$\sum_{n\in\mathbb{Z}} \psi(t - n) = 1, \quad t \in \mathbb{R}. \tag{2.17}$$

We define $\Psi(x) = \psi(x_1) \cdot \ldots \cdot \psi(x_d)$ and $\Psi^\beta(x) = x^\beta \Psi(x)$ for $x = (x_1, \dots, x_d)$ and $\beta \in \mathbb{N}_0^d$. Further let $\overline{r} \in \mathbb{R}^d$ and $0 < p \le \infty$. Then

$$(\beta qu)_{\overline{\nu}\,\overline{m}}(x) = \Psi^\beta(2^{\overline{\nu}}x - \overline{m}), \quad \overline{\nu} \in \mathbb{N}_0^d,\ \overline{m} \in \mathbb{Z}^d \tag{2.18}$$

is called an β-quark related to $Q_{\overline{\nu}\,\overline{m}}$.

Theorem 2.9. *Let $0 < p, q \le \infty$ (with $p < \infty$ in the F-case) and $\overline{r} \in \mathbb{R}^d$ with (2.11).*

(i) *Let*

$$\lambda = \{\lambda^\beta : \beta \in \mathbb{N}_0^d\} \quad \textit{with} \quad \lambda^\beta = \{\lambda^\beta_{\overline{\nu}\,\overline{m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d\}$$

and let $\varrho > \phi$, where ϕ is the number from (2.16). Then $f \in S'(\mathbb{R}^d)$ belongs to $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ if, and only if, it can be represented as

$$f = \sum_{\beta\in\mathbb{N}_0^d} \sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in\mathbb{Z}^d} \lambda^\beta_{\overline{\nu}\,\overline{m}} (\beta qu)_{\overline{\nu}\,\overline{m}}(x), \quad \textit{convergence being in} \quad S'(\mathbb{R}^d), \tag{2.19}$$

where $(\beta qu)_{\overline{\nu}\,\overline{m}}(x)$ *are* β*-quarks related to* $Q_{\overline{\nu}\,\overline{m}}$ *and*

$$\sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^\beta|s^{\overline{r}}_{pq}a||<\infty.$$

Furthermore,

$$\inf\sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^\beta|s^{\overline{r}}_{pq}a||<\infty,$$

where the infimum runs over all admissible representations (2.19)*, is an equivalent quasi-norm in* $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$*.*

Remark 2.10. According to [9], [10] and [3], similar decomposition theorems are available also for spaces $A^{(s,\alpha)}_{p,q}(\mathbb{R})$. They may be obtained from Theorem 2.7 and Theorem 2.9 by setting $d=1$ and replacing $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ with $A^{(s,\alpha)}_{p,q}(\mathbb{R})$ and $s^{\overline{r}}_{p,q}a$ with $a^{(s,\alpha)}_{p,q}$.

Lemma 2.11. *Let* $0<p<\infty, 0<q\le\infty, \overline{r}\in\mathbb{R}^d$ *and* $\gamma_1,\gamma_2>0$*. Let*

$$E_{\overline{\nu}\,\overline{m}}\subset\gamma_1 Q_{\overline{\nu}\,\overline{m}},\quad \frac{|E_{\overline{\nu}\,\overline{m}}|}{|Q_{\overline{\nu}\,\overline{m}}|}\ge\gamma_2,\quad \overline{\nu}\in\mathbb{N}_0^d,\quad \overline{m}\in\mathbb{Z}^d. \tag{2.20}$$

Then

$$\Big|\Big|\ 2^{\overline{\nu}\cdot\overline{r}}|\lambda_{\overline{\nu}\,\overline{m}}|\chi_{E_{\overline{\nu}\,\overline{m}}}(\cdot)\ \Big|L_p(\ell_q)\Big|\Big|\approx||\lambda|s^{\overline{r}}_{p,q}f||$$

with constants of equivalence independent of λ*.*

Proof. We follow closely [2]. Namely, from (2.20) we see that

$$\chi_{E_{\overline{\nu}\,\overline{m}}}(x)\le c\,\overline{M}\chi_{Q_{\overline{\nu}\,\overline{m}}}(x),\quad x\in\mathbb{R}^d$$

and

$$\chi_{Q_{\overline{\nu}\,\overline{m}}}(x)\le c\,\overline{M}\chi_{E_{\overline{\nu}\,\overline{m}}}(x),\quad x\in\mathbb{R}^d.$$

Here $\overline{M}=M_2\circ M_1$, where

$$(M_1f)(x)=\sup_{s>0}\frac{1}{2s}\int_{x_1-s}^{x_1+s}|f(t,x_2)|\mathrm{d}t,\quad x=(x_1,x_2)\in\mathbb{R}^2, \tag{2.21}$$

and similar for M_2.
Then we take $\omega>0$ such that $\omega<\min(1,p,q)$ and observe

$$\Big|\Big|\ 2^{\overline{\nu}\cdot\overline{r}}|\lambda_{\overline{\nu}\,\overline{m}}|\chi_{E_{\overline{\nu}\,\overline{m}}}(\cdot)\ \Big|L_p(\ell_q)\Big|\Big|=\Big|\Big|\ 2^{\overline{\nu}\cdot\overline{r}\omega}|\lambda_{\overline{\nu}\,\overline{m}}|^\omega\chi_{E_{\overline{\nu}\,\overline{m}}}(\cdot)\ \Big|L_{\frac{p}{\omega}}(\ell_{\frac{q}{\omega}})\Big|\Big|^{\frac{1}{\omega}}$$

with a direct counterpart for $||\lambda|s^{\overline{r}}_{pq}f||$. This, together with the boundedness of the maximal operator $\overline{M}$ (see [7] or [12] for details) finishes the proof. □

By $\Gamma=\{(t,t)\in\mathbb{R}^2:t\in\mathbb{R}\}$ we denote the diagonal of $\mathbb{R}^2$. As Γ is isomorphic to $\mathbb{R}$, all the function spaces considered so far may be taken over from the real line to Γ. In the natural sense, we get $A^{(r,\alpha)}_{p,q}(\mathbb{R})=A^{(r,\alpha)}_{p,q}(\Gamma)$ for all admissible α,p,q and r.
Finally, we discuss the notion of the trace. The trace operator Tf, as it is described in (1.2), makes sense only when the function f satisfies some regularity conditions, especially, if it is continuous. This is satisfied for $f\in S^{\overline{r}}_{p,q}A(\mathbb{R}^2)$ with $\overline{r}>\frac{1}{p}$. To avoid this restriction, we

use the following general definition of the trace. It is well known that $S^0_{\infty,1}B(\mathbb{R}^2) \hookrightarrow C(\mathbb{R}^2)$. So, for $f \in S^0_{\infty,1}B(\mathbb{R}^2)$, we may define $(\mathrm{tr}_\Gamma f)(t) = f(t,t)$. If $S(\mathbb{R}^2)$ is a dense subspace of $S^{\overline{r}}_{p,q}A(\mathbb{R}^2)$ and tr_Γ satisfies the inequality

$$||\,\mathrm{tr}_\Gamma f|X(\Gamma)|| \leq c||f|S^{\overline{r}}_{p,q}A(\mathbb{R}^2)||, \quad f \in S(\mathbb{R}^2), \tag{2.22}$$

for some quasi-Banach space $X(\Gamma) \hookrightarrow S'(\mathbb{R})$, then there is a unique extension operator $\mathrm{tr}_\Gamma : S^{\overline{r}}_{p,q}A(\mathbb{R}^2) \to X(\Gamma)$. It turns out that this defines the $\mathrm{tr}_\Gamma f$ for all $f \in S^{\overline{r}}_{p,q}A(\mathbb{R}^2)$ with $\max(p,q) < \infty$ and $\overline{r} = (r_1, r_2)$ with $\overline{r}$ large enough and this definition does not depend on $X(\Gamma)$. In the last case, $q = \infty$, we use the embedding $S^{\overline{r}}_{p,\infty}A(\mathbb{R}^2) \hookrightarrow S^{\overline{r}-\epsilon}_{p,1}A(\mathbb{R}^2)$, with $\epsilon > 0$ small, which defines $\mathrm{tr}_\Gamma f$ as soon as the trace operator is defined on $S^{\overline{r}-\epsilon}_{p,1}A(\mathbb{R}^2)$.
We write $\mathrm{tr}_\Gamma : S^{\overline{r}}_{p,q}A(\mathbb{R}^2) \to X(\Gamma)$, if (2.22) is satisfied for all $f \in S^{\overline{r}}_{p,q}A(\mathbb{R}^2)$. The symbol $\mathrm{tr}_\Gamma\, S^{\overline{r}}_{p,q}A(\mathbb{R}^2) = X(\Gamma)$ is used to denote that $\mathrm{tr}_\Gamma : S^{\overline{r}}_{p,q}A(\mathbb{R}^2) \to X(\Gamma)$ and, moreover, there is an (linear, bounded) extension operator $\mathrm{ext} : X(\Gamma) \to S^{\overline{r}}_{p,q}A(\mathbb{R}^2)$ such that $\mathrm{tr}_\Gamma \circ \mathrm{ext} = id$.
Hence $\mathrm{tr}_\Gamma\, S^{\overline{r}}_{p,q}A(\mathbb{R}^2) = X(\Gamma)$ if, and only if, tr_Γ is a retraction from $S^{\overline{r}}_{p,q}A(\mathbb{R}^2)$ onto $X(\Gamma)$.

3 Traces of B-spaces

Theorem 3.1. *Let $0 < p, q \leq \infty$, and $\overline{r} = (r_1, r_2) \in \mathbb{R}^2$ with*

$$0 < r_1 \leq r_2, \varrho = \min\Big(r_1, r_1 + r_2 - \frac{1}{p}\Big) > \sigma_p.$$

If $r_2 \neq \frac{1}{p}$ or $r_2 = \frac{1}{p}$ and $q \leq \min(1,p)$ then

$$\mathrm{tr}_\Gamma\, S^{\overline{r}}_{p,q}B(\mathbb{R}^2) = B^{\varrho}_{p,q}(\Gamma).$$

If $r_2 = \frac{1}{p}$, $1 \leq \min(p,q)$ then

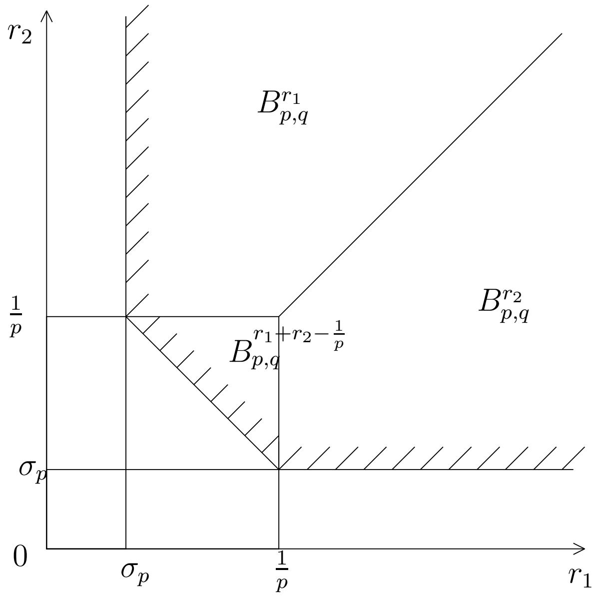
$$\mathrm{tr}_\Gamma\, S^{\overline{r}}_{p,q}B(\mathbb{R}^2) = B^{(r_1, \frac{1}{q}-1)}_{p,q}(\Gamma).$$

Finally, if $r_2 = \frac{1}{p}$, $p \leq \min(1,q)$ then

$$\mathrm{tr}_\Gamma : S^{\overline{r}}_{p,q}B(\mathbb{R}^2) \to B^{(r_1, \frac{1}{q}-\frac{1}{p})}_{p,q}(\Gamma)$$

and

$$\mathrm{ext} : B^{(r_1, \min(\frac{1}{q}-1, 0))}_{p,q}(\Gamma) \to S^{\overline{r}}_{p,q}B(\mathbb{R}^2).$$



Proof. Step 1. - quarkonial decomposition, definition of $\operatorname{tr}_\Gamma f$
Let $f \in S^{\overline{r}}_{p,q}B(\mathbb{R}^2)$. According to Theorem 2.9, f may be decomposed as

$$f = \sum_{\beta \in \mathbb{N}_0^2} f^\beta, \quad f^\beta(x) = \sum_{\overline{\nu} \in \mathbb{N}_0^2} \sum_{\overline{m} \in \mathbb{Z}^2} \lambda^\beta_{\overline{\nu}\,\overline{m}} (\beta qu)_{\overline{\nu}\,\overline{m}}(x) \tag{3.1}$$

with

$$\sup_{\beta \in \mathbb{N}_0^2} 2^{\varrho|\beta|} ||\lambda^\beta | s^{\overline{r}}_{p,q}b|| \approx ||f|S^{\overline{r}}_{p,q}B(\mathbb{R}^2)||. \tag{3.2}$$

We point out that we may assume that the coefficients λ of the optimal quarkonial decomposition (3.1) depend *linearly* on f. We refer again to [10] and [12] for detailed discussion of this effect.
Naturally, we define

$$\operatorname{tr}_\Gamma f = \sum_{\beta \in \mathbb{N}_0^2} (\operatorname{tr}_\Gamma f)_\beta, \quad (\operatorname{tr}_\Gamma f)_\beta(t) = \sum_{\overline{\nu} \in \mathbb{N}_0^2} \sum_{\overline{m} \in \mathbb{Z}^2} \lambda^\beta_{\overline{\nu}\,\overline{m}} (\beta qu)_{\overline{\nu}\,\overline{m}}(t,t). \tag{3.3}$$

In (3.3) we may restrict to $\overline{m}$ from

$$B_{\overline{\nu}} = \{\overline{m} \in \mathbb{Z}^2 : \operatorname{supp}(\beta qu)_{\overline{\nu}\,\overline{m}} \cap \Gamma \neq \emptyset\}.$$

Next we split

$$B_{\overline{\nu}} = \bigcup_{n \in \mathbb{Z}} B_{\overline{\nu}\,n} \tag{3.4}$$

such that

$$\sup_{\overline{\nu},n} |B_{\overline{\nu}\,n}| < \infty \tag{3.5}$$

and, for $\mu = \max(\nu_1, \nu_2)$,

$$\{t : (\beta qu)_{\overline{\nu}\,\overline{m}}(t,t) \neq 0\} \subset (2^{-\mu}(n-c), 2^{-\mu}(n+c)), \quad \overline{m} \in B_{\overline{\nu}\,n}, \tag{3.6}$$

for some fixed constant $c > 0$.

Using this new notation, we rewrite (3.3).

$$(\mathrm{tr}_\Gamma f)_\beta(t) = \sum_{\mu=0}^{\infty}\sum_{n\in\mathbb{Z}} \sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2\\ \max(\nu_1,\nu_2)=\mu}} \sum_{\overline{m}\in B_{\overline{\nu}n}} \lambda^\beta_{\overline{\nu}\,\overline{m}}(\beta q u)_{\overline{\nu}\,\overline{m}}(t,t) = \sum_{\mu=0}^{\infty}\sum_{n\in\mathbb{Z}} \gamma^\beta_{\mu n} a^\beta_{\mu n}(t), \tag{3.7}$$

where

$$\gamma^\beta_{\mu n} = 2^{\phi|\beta|} \sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2\\ \max(\nu_1,\nu_2)=\mu}} \sum_{\overline{m}\in B_{\overline{\nu}n}} |\lambda^\beta_{\overline{\nu}\,\overline{m}}|.$$

We have to prove that

1. $a^\beta_{\mu n}$ are atoms according to Definition 2.6, for $d=1$, related to (μ,n).
2. $||\gamma^\beta|b^\varrho_{p,q}|| \le c2^{\phi|\beta|}||\lambda^\beta|s^{\overline{r}}_{p,q}b||$, resp. $||\gamma^\beta|b^{(r_1,\alpha)}_{p,q}|| \le c2^{\phi|\beta|}||\lambda^\beta|s^{\overline{r}}_{p,q}b||$ $\qquad(\clubsuit)$
3. $\mathrm{tr}_\Gamma f$ defined by (3.3) coincides with the trace operator introduced in Section 2.

It is easy to prove the first statement. The support property (2.12) follows directly from (3.6). Also the second property (2.13) is satisfied (up to some constant which depends only on ψ from Definition 2.8). To prove the third statement, consider $f \in S^0_{\infty,1}B(\mathbb{R}^2)$. Then $\lambda^\beta \in s^0_{\infty 1}b$ for every $\beta \in \mathbb{N}_0^d$ and the series in (3.1) both converge uniformly on $\mathbb{R}^2$. So, for $f \in S^0_{\infty,1}B(\mathbb{R}^2)$, $\mathrm{tr}_\Gamma f$ defined by (3.3) coincides with the trace operator of Section 2. Using density arguments, this may be extended to all $f \in S^{\overline{r}}_{p,q}B(\mathbb{R}^2)$.
So, in the following we concentrate on the proof of $(\clubsuit)$.
This will finish the first part of the proof, namely the existence and boundedness of the trace operator $\mathrm{tr}_\Gamma : S^{\overline{r}}_{p,q}B(\mathbb{R}^2) \to B^\varrho_{p,q}(\Gamma)$. To see that, denote $\omega = \min(1,p,q)$ and write

$$\begin{aligned} ||\,\mathrm{tr}_\Gamma f|B^\varrho_{p,q}(\Gamma)||^\omega &\le \sum_{\beta\in\mathbb{N}_0^2} ||(\mathrm{tr}_\Gamma f)_\beta|B^\varrho_{p,q}(\Gamma)||^\omega \le c\sum_{\beta\in\mathbb{N}_0^2} ||\gamma^\beta|b^\varrho_{p,q}||^\omega \\ &\le c\sum_{\beta\in\mathbb{N}_0^2} 2^{\phi|\beta|\omega}||\lambda^\beta|s^{\overline{r}}_{p,q}b||^\omega \le c\sup_{\beta\in\mathbb{N}_0^2} 2^{\varrho|\beta|\omega}||\lambda^\beta|s^{\overline{r}}_{p,q}b||^\omega \le c\,||f|S^{\overline{r}}_{p,q}B(\mathbb{R}^2)||^\omega. \end{aligned}$$

Step 2. - Proof of $(\clubsuit)$. We take $\beta\in\mathbb{N}_0^2$ fixed and suppose, that the sequence

$$\lambda^\beta = \lambda = \{\lambda_{\overline{\nu},\overline{m}} : \overline{\nu}\in\mathbb{N}_0^2, \overline{m}\in B_{\overline{\nu}}\}$$

is given. Then we set

$$\gamma_{\mu n} = \sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2\\ \max(\nu_1,\nu_2)=\mu}} \sum_{\overline{m}\in B_{\overline{\nu}n}} |\lambda_{\overline{\nu}\,\overline{m}}|, \quad \mu\in\mathbb{N}_0, \quad n\in\mathbb{Z}.$$

We recall (3.4) for the relation of $B_{\overline{\nu}n}$ and $B_{\overline{\nu}}$.
Finally, we denote

$$\alpha(\overline{\nu}) = \max(\nu_1,\nu_2)\Big(\varrho - \frac{1}{p}\Big) - \overline{\nu}\cdot\Big(\overline{r} - \frac{1}{p}\Big) \tag{3.8}$$

and

$$\beta = \begin{cases} \frac{1}{q} - \frac{1}{\min(1,p)}, & \text{if} \quad r_2 = \frac{1}{p} \quad \text{and} \quad q \ge \min(1,p), \\ 0 & \text{in other cases.} \end{cases} \tag{3.9}$$

Next, we point out that, if $\varrho = r_1$,

$$\alpha(\overline{\nu}) = \begin{cases} \nu_2(r_1 - r_2) - \nu_1(r_1 - \frac{1}{p}) \le -\nu_1(r_2 - \frac{1}{p}) & \text{for} \quad \nu_1 \le \nu_2, \\ -\nu_2(r_2 - \frac{1}{p}) & \text{for} \quad \nu_1 \ge \nu_2. \end{cases} \tag{3.10}$$

and, for $\varrho = r_1 + r_2 - \frac{1}{p}$,

$$\alpha(\overline{\nu}) = \begin{cases} (\nu_2 - \nu_1)(r_1 - \frac{1}{p}) \le 0 & \text{for} \quad \nu_1 \le \nu_2, \\ (\nu_1 - \nu_2)(r_2 - \frac{1}{p}) \le 0 & \text{for} \quad \nu_1 \ge \nu_2. \end{cases} \tag{3.11}$$

The estimates (3.10) and (3.11) play a crucial role in the following calculations.
We need to prove that

$$||\{\gamma_{\mu n}\}|\ell_q((\mu+1)^\beta 2^{\mu(\varrho-\frac{1}{p})}\ell_p)|| \le c\,||\{\lambda_{\overline{\nu}\,\overline{m}}\}|\ell_q(2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})}\ell_p)||, \tag{3.12}$$

where ℓ_p and ℓ_q on the left-hand side denotes sequence spaces with one-dimensional summation and the same symbols stand for sequence spaces with two-dimensional summation on the right hand side.
If $p \le 1$, then

$$\sum_{n\in\mathbb{Z}} \gamma_{\mu n}^p \le \sum_{n\in\mathbb{Z}} \sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2 \\ \max(\nu_1,\nu_2)=\mu}} \sum_{\overline{m}\in B_{\overline{\nu}n}} |\lambda_{\overline{\nu}\,\overline{m}}^\beta|^p = \sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2 \\ \max(\nu_1,\nu_2)=\mu}} \sum_{\overline{m}\in B_{\overline{\nu}}} |\lambda_{\overline{\nu}\,\overline{m}}^\beta|^p. \tag{3.13}$$

And if $\frac{q}{p} \le 1 (\implies \beta = 0)$, we get immediately,

$$\sum_{\mu=0}^\infty 2^{\mu(\varrho-\frac{1}{p})q}\Big(\sum_{n\in\mathbb{Z}} \gamma_{\mu n}^p\Big)^{\frac{q}{p}} \le \sum_{\mu=0}^\infty 2^{\mu(\varrho-\frac{1}{p})q} \sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2 \\ \max(\nu_1,\nu_2)=\mu}} \Big(\sum_{\overline{m}\in B_{\overline{\nu}}} |\lambda_{\overline{\nu}\,\overline{m}}^\beta|^p\Big)^{\frac{q}{p}}.$$

This, together with (3.8)–(3.11), finishes the proof of (3.12) for $0 < q \le p \le 1$.
If $p \le 1$ and $\frac{q}{p} > 1$, we get by (3.13) and Hölder's inequality

$$\sum_{\mu=0}^\infty (\mu+1)^{\beta q} 2^{\mu(\varrho-\frac{1}{p})q}\Big(\sum_{n\in\mathbb{Z}} \gamma_{\mu n}^p\Big)^{\frac{q}{p}} \le \sum_{\mu=0}^\infty (\mu+1)^{\beta q}\Big(\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2 \\ \max(\nu_1,\nu_2)=\mu}} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})p+\alpha(\overline{\nu})p} \sum_{\overline{m}\in B_{\overline{\nu}}} |\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{\frac{q}{p}} \le$$

$$\le \sum_{\mu=0}^\infty (\mu+1)^{\beta q}\Big(\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2 \\ \max(\nu_1,\nu_2)=\mu}} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})p\frac{q}{p}}\big(\sum_{\overline{m}\in B_{\overline{\nu}}} |\lambda_{\overline{\nu}\,\overline{m}}|^p\big)^{\frac{q}{p}}\Big) \cdot \Big(\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2 \\ \max(\nu_1,\nu_2)=\mu}} 2^{\alpha(\overline{\nu})p(\frac{q}{p})'}\Big)^{\frac{\frac{q}{p}}{(\frac{q}{p})'}}.$$

Here $(\frac{q}{p})' = \frac{q}{q-p}$ is the conjugated index to $\frac{q}{p}$.
So, if $r_2 \ne \frac{1}{p}$, then $\beta = 0$ and, according to (3.10) and (3.11), the last sum is uniformly bounded and the result follows. If $r_2 = \frac{1}{p}$, the last sum is $\le c\,(\mu+1)^{\frac{q/p}{(\frac{q}{p})'}} = c\,(\mu+1)^{\frac{q-p}{p}} = (\mu+1)^{-\beta q}$.

Next we consider $p>1$. From (3.5) we get

$$\sum_{\overline{m}\in B_{\overline{\nu}\,n}} |\lambda_{\overline{\nu}\,\overline{m}}| \le c\,\underbrace{\Big(\sum_{\overline{m}\in B_{\overline{\nu}\,n}} |\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{\frac{1}{p}}}_{a_{\overline{\nu}\,n}}, \qquad n\in\mathbb{Z},\quad \overline{\nu}\in\mathbb{N}_0^2. \tag{3.14}$$

By this notation, we get

$$\begin{aligned}\sum_{\mu=0}^{\infty}(\mu+1)^{\beta q}2^{\mu(\varrho-\frac{1}{p})q}\Big(\sum_{n\in\mathbb{Z}}\gamma^p_{\mu\,n}\Big)^{\frac{q}{p}} &\le \sum_{\mu=0}^{\infty}(\mu+1)^{\beta q}2^{\mu(\varrho-\frac{1}{p})q}\Big(\sum_{n\in\mathbb{Z}}\Big(\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2\\ \max(\nu_1,\nu_2)=\mu}} a_{\overline{\nu}\,n}\Big)^p\Big)^{\frac{q}{p}} \le\\ &\le \sum_{\mu=0}^{\infty}(\mu+1)^{\beta q}2^{\mu(\varrho-\frac{1}{p})q}\Big(\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2\\ \max(\nu_1,\nu_2)=\mu}}\Big(\sum_{n\in\mathbb{Z}} a^p_{\overline{\nu}\,n}\Big)^{\frac{1}{p}}\Big)^q,\end{aligned} \tag{3.15}$$

where in the last step we have used the Minkowski's inequality ($p>1$).
If $q\le 1(\implies \beta=0)$, we may estimate the last expression from above by

$$\sum_{\mu=0}^{\infty}2^{\mu(\varrho-\frac{1}{p})q}\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2\\ \max(\nu_1,\nu_2)=\mu}}\Big(\sum_{n\in\mathbb{Z}} a^p_{\overline{\nu}\,n}\Big)^{\frac{q}{p}} = \sum_{\overline{\nu}\in\mathbb{N}_0^2} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})}2^{\alpha(\overline{\nu})}\Big(\sum_{\overline{m}\in B_{\overline{\nu}}}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{\frac{q}{p}}.$$

As $\alpha(\overline{\nu})\le 0$ for all $\overline{\nu}\in\mathbb{N}_0^2$, this finishes the proof.
If $q>1$, we continue in (3.15) using Hölder's inequality.

$$\begin{aligned}\mathrm{LHS}(3.15) &\le \sum_{\mu=0}^{\infty}(\mu+1)^{\beta q}\Big(\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2\\ \max(\nu_1,\nu_2)=\mu}} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})+\alpha(\overline{\nu})}\Big(\sum_{n\in\mathbb{Z}} a^p_{\overline{\nu}\,n}\Big)^{\frac{1}{p}}\Big)^q\\ &\le \sum_{\mu=0}^{\infty}(\mu+1)^{\beta q}\Big(\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2\\ \max(\nu_1,\nu_2)=\mu}} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\Big(\sum_{n\in\mathbb{Z}} a^p_{\overline{\nu}\,n}\Big)^{\frac{q}{p}}\Big)\cdot\Big(\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2\\ \max(\nu_1,\nu_2)=\mu}} 2^{\alpha(\overline{\nu})q'}\Big)^{\frac{q}{q'}}\end{aligned}$$

If now $r_2\neq\frac{1}{p}$, then the last sum is uniformly bounded for all $\mu\in\mathbb{N}_0$ and we get the desired estimate. If $r_2=\frac{1}{p}$ we get the same estimate with additional factor $(\mu+1)^{q-1}=(\mu+1)^{-\beta q}$.
Step 2. - extension operators
In this step we prove the boundedness of the corresponding extension operators.
We fix $f\in B^{\varrho}_{p,q}(\Gamma)$ (or $f\in B^{(\varrho,\frac{1}{q}-1)}_{p,q}(\Gamma)$, respectively). Then it may be decomposed into quarks

$$f=\sum_{\beta=0}^{\infty}f^{\beta}=\sum_{\beta=0}^{\infty}\sum_{\mu=0}^{\infty}\sum_{n\in\mathbb{Z}}\lambda^{\beta}_{\mu,n}(\beta q u)_{\mu,n},$$

where the coefficients $\{\lambda^{\beta}_{\mu,n}\}$ depend linearly on f and belong to the corresponding sequence space $b^{\varrho}_{p,q}$ or $b^{(\varrho,\frac{1}{q}-1)}_{p,q}$. Moreover,

$$\sup_{\beta\in\mathbb{N}}2^{\rho\beta}||\lambda^{\beta}|b^{(\varrho,\alpha)}_{p,q}||\approx||f|B^{(\varrho,\alpha)}_{p,q}(\mathbb{R})||$$

with constants independent of f.
We define

$$a^{\beta}_{(\nu_1,\nu_2)(m_1,m_2)}(x_1,x_2) = \begin{cases} (\beta q u)_{\nu_1,m_1}(x_1)h(2^{\nu_2}(x_2-2^{-\nu_1}m_1)), & \nu_2\le\nu_1,\ m_2=[2^{\nu_2-\nu_1}m_1+\frac{1}{2}] \\ (\beta q u)_{\nu_2,m_2}(x_2)h(2^{\nu_1}(x_1-2^{-\nu_2}m_2)), & \nu_1\le\nu_2,\ m_1=[2^{\nu_1-\nu_2}m_2+\frac{1}{2}], \end{cases}$$

where $h\in S(\mathbb{R})$ with $h(t)=1$ for $|t|\le 2^{\phi}$ and $h(t)=0$ for $|t|\ge 2^{\phi+1}$ and ϕ is the constant in (2.16). This definition ensures that $2^{-\phi\beta}a^{\beta}_{\overline{\nu}\,\overline{m}}$ are $\overline{K}$-atoms for every fixed $\overline{K}\in\mathbb{N}_0^2$ up to some constant which depends only on the function ψ involved in the definition of quarks and $\overline{K}$.
If now $r_2>\frac{1}{p}$ or $r_2=\frac{1}{p}$ and $q\le\min(1,p)$ then $\{\lambda^{\beta}_{\mu,n}\}\in b^{r_1}_{p,q}$ with $\sup_{\beta\in\mathbb{N}_0}2^{\rho\beta}||\lambda|b^{r_1}_{p,q}||\le c||f|B^{r_1}_{p,q}(\mathbb{R})||$. We define

$$\gamma^{\beta}_{(\mu,0)(n,[2^{-\mu}n+\frac{1}{2}])}=\lambda^{\beta}_{\mu,n},\quad \mu\in\mathbb{N}_0,\ n\in\mathbb{Z} \tag{3.16}$$

and zero otherwise. Finally we set

$$\operatorname{ext} f=\sum_{\beta=0}^{\infty}\operatorname{ext} f^{\beta}=\sum_{\beta=0}^{\infty}\sum_{\overline{\nu}\in\mathbb{N}_0^2}\sum_{\overline{m}\in\mathbb{Z}^2}\gamma^{\beta}_{\overline{\nu}\,\overline{m}}a^{\beta}_{\overline{\nu}\,\overline{m}} \tag{3.17}$$

and observe that for $\omega=\min(1,p,q)$

$$\begin{aligned}
||\operatorname{ext} f|S^{\overline{r}}_{p,q}B(\mathbb{R}^2)||^{\omega} &\le \sum_{\beta=0}^{\infty}||\operatorname{ext} f^{\beta}|S^{\overline{r}}_{p,q}B(\mathbb{R}^2)||^{\omega}\le c\sum_{\beta=0}^{\infty}2^{\phi\beta\omega}||\gamma^{\beta}|s^{\overline{r}}_{p,q}b||^{\omega} \\
&\le c\sup_{\beta\in\mathbb{N}_0}2^{\rho\beta\omega}||\gamma^{\beta}|s^{\overline{r}}_{p,q}b||^{\omega}=c\sup_{\beta\in\mathbb{N}_0}2^{\rho\beta\omega}\Bigg(\sum_{\nu\in\mathbb{N}_0^2}2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\Big(\sum_{\overline{m}\in\mathbb{Z}^2}|\gamma^{\beta}_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\Bigg)^{\omega/q} && (3.18) \\
&= c\sup_{\beta\in\mathbb{N}_0}2^{\rho\beta\omega}\Bigg(\sum_{\mu=0}^{\infty}2^{\mu(r_1-\frac{1}{p})q}\Big(\sum_{n\in\mathbb{Z}}|\lambda^{\beta}_{\mu\,n}|^p\Big)^{q/p}\Bigg)^{\omega/q} && (3.19) \\
&= c\sup_{\beta\in\mathbb{N}_0}2^{\rho\beta\omega}||\lambda^{\beta}|b^{r_1}_{p,q}||^{\omega}\le c||f|B^{r_1}_{p,q}(\mathbb{R})||^{\omega}.
\end{aligned}$$

Furthermore, the definition of $a^{\beta}_{\overline{\nu}\,\overline{m}}$ ensures that $\operatorname{tr}\circ\operatorname{ext} f=f$
The case $r_2<\frac{1}{p}$ follows the same scheme. We define

$$\gamma^{\beta}_{(\mu,\mu)(n,n)}=\lambda^{\beta}_{\mu,n},\quad \mu\in\mathbb{N},\quad n\in\mathbb{Z} \tag{3.20}$$

and $\gamma^{\beta}_{\overline{\nu}\,\overline{m}}=0$ otherwise. We get now similarly to (3.18)

$$\begin{aligned}
||\gamma^{\beta}|s^{\overline{r}}_{p,q}b|| &= \Bigg(\sum_{\overline{\nu}\in\mathbb{N}_0^2}2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\Big(\sum_{\overline{m}\in\mathbb{Z}^2}|\gamma^{\beta}_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\Bigg)^{1/q} \\
&= \Bigg(\sum_{\mu=0}^{\infty}2^{\mu(r_1+r_2-\frac{2}{p})q}\Big(\sum_{n\in\mathbb{Z}}|\lambda^{\beta}_{\mu\,n}|^p\Big)^{q/p}\Bigg)^{1/q}=||\lambda^{\beta}|b^{\varrho}_{p,q}||.
\end{aligned}$$

Finally, in the case $r_2 = \frac{1}{p}, q > 1$ and $q > p$ we set for $0 \le \nu_2 \le \mu$

$$\gamma^{\beta}_{(\mu,\nu_2)(n,\tilde{n})} = (\mu+1)^{-1}\lambda^{\beta}_{\mu,n}, \quad \tilde{n} = [2^{\nu_2}2^{-\mu}n + \frac{1}{2}]$$

and zero otherwise. Then we get for $\beta = \frac{1}{q} - 1$

$$\begin{aligned} ||\gamma^{\beta}|s^{\overline{r}}_{p,q}b|| &= \Bigg(\sum_{\nu\in\mathbb{N}_0^2} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\Big(\sum_{\overline{m}\in\mathbb{Z}^2} |\gamma^{\beta}_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\Bigg)^{1/q} \\ &= \Bigg(\sum_{\mu=0}^{\infty} 2^{\mu(r_1-\frac{1}{p})q}(\mu+1)\Big(\sum_{n\in\mathbb{Z}} |(\mu+1)^{-1}\lambda^{\beta}_{\mu\, n}|^p\Big)^{q/p}\Bigg)^{1/q} = ||\lambda^{\beta}|b^{(\varrho,\beta)}_{p,q}||. \end{aligned}$$

□

4 Traces of F spaces

Theorem 4.1. *Let*

$$0 < p < \infty, \quad 0 < q \le \infty, \quad 0 < r_1 \le r_2$$

with

$$\varrho = \min\Big(r_1, r_1 + r_2 - \frac{1}{p}\Big) > \sigma_{p,q}.$$

If $r_2 > \frac{1}{p}$ *then*

$$\mathrm{tr}_{\Gamma}\, S^{\overline{r}}_{p,q}F(\mathbb{R}^2) = F^{\varrho}_{p,q}(\Gamma). \tag{4.1}$$

If $r_2 < \frac{1}{p}$ *then*

$$\mathrm{tr}_{\Gamma}\, S^{\overline{r}}_{p,q}F(\mathbb{R}^2) = F^{\varrho}_{p,p}(\Gamma) = B^{\varrho}_{p,p}(\Gamma). \tag{4.2}$$

If $r_2 = \frac{1}{p}$ *and* $p \le \min(1,q)$ *then*

$$\mathrm{tr}_{\Gamma}\, S^{\overline{r}}_{p,q}F(\mathbb{R}^2) = F^{r_1}_{p,q}(\Gamma). \tag{4.3}$$

If $r_2 = \frac{1}{p}$ *and* $q < p \le 1$ *then*

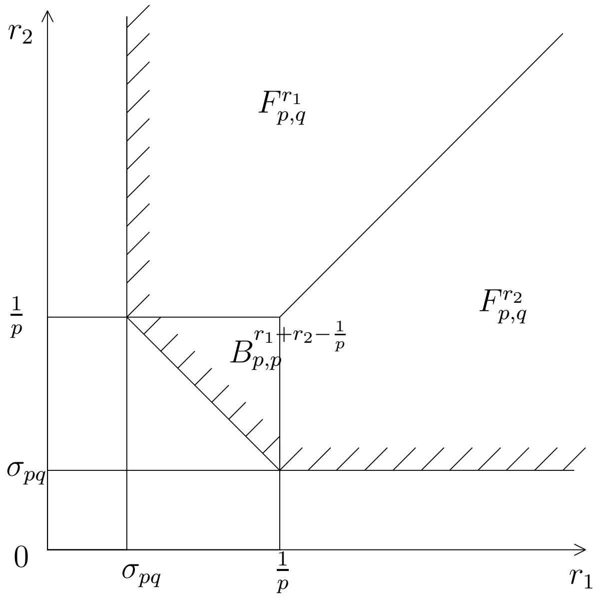
$$\mathrm{tr}_{\Gamma}\, S^{\overline{r}}_{p,q}F(\mathbb{R}^2) = F^{r_1}_{p,p}(\Gamma). \tag{4.4}$$

If $r_2 = \frac{1}{p}$ *and* $1 \le p \le q$ *then*

$$\mathrm{tr}_{\Gamma} : S^{\overline{r}}_{p,q}F(\mathbb{R}^2) \to F^{(r_1,\frac{1}{q}-1)}_{p,q}(\Gamma). \tag{4.5}$$

Finally, if $r_2 = \frac{1}{p}$ *and* $p \ge \max(1,q)$ *then*

$$\mathrm{tr}_{\Gamma} : S^{\overline{r}}_{p,q}F(\mathbb{R}^2) \to F^{(r_1,\frac{1}{p}-1)}_{p,p}(\Gamma). \tag{4.6}$$



Proof. We recall our task. We use again the notation (3.1)-(3.13).
We suppose, that the sequence

$$\lambda = \{\lambda_{\overline{\nu},\overline{m}} : \overline{\nu} \in \mathbb{N}_0^2, \overline{m} \in B_{\overline{\nu}}\}$$

is given. Then we set

$$\gamma_{\mu\, n} = \sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2 \\ \max(\nu_1,\nu_2)=\mu}} \sum_{\overline{m}\in B_{\overline{\nu}\, n}} |\lambda_{\overline{\nu}\,\overline{m}}|. \tag{4.7}$$

and recall (3.4) for the relation of $B_{\mu\, n}$ and $B_{\overline{\nu}}$. We need to prove that $(r_2 > \frac{1}{p})$

$$||\{\gamma_{\mu\, n}\}|f^{r_1}_{p,q}|| \leq c\, ||\{\lambda_{\overline{\nu}\,\overline{m}}\}|s^{\overline{r}}_{p,q}f|| \tag{4.8}$$

or $(r_2 < \frac{1}{p})$

$$||\{\gamma_{\mu\, n}\}|f^{r_1+r_2-\frac{1}{p}}_{p,p}|| \leq c\, ||\{\lambda_{\overline{\nu}\,\overline{m}}\}|s^{\overline{r}}_{p,q}f|| \tag{4.9}$$

respectively.
We split (4.7) into two parts,

$$\gamma^{(1)}_{\mu\, n} = \sum_{\nu_2=0}^{\mu} \sum_{\overline{m}\in B_{(\mu,\nu_2),n}} |\lambda_{(\mu,\nu_2),\overline{m}}|, \qquad \gamma^{(2)}_{\mu\, n} = \sum_{\nu_1=0}^{\mu} \sum_{\overline{m}\in B_{(\nu_1,\mu),n}} |\lambda_{(\nu_1,\mu),\overline{m}}| \tag{4.10}$$

and prove (4.8) and (4.9) for both these parts separately.
Step 1.
We start with the case $r_2 > \frac{1}{p}$. We recall the definitions of sequence spaces involved in (4.8) and obtain

$$||\{\gamma_{\mu\, n}\}|f^{r_1}_{p,q}||^p = \int_{-\infty}^{\infty} \Big(\sum_{\mu=0}^{\infty}\sum_{n\in\mathbb{Z}} |2^{\mu r_1}\gamma_{\mu\, n}\chi_{\mu\, n}(x_1)|^q\Big)^{\frac{p}{q}} \mathrm{d}x_1$$

and

$$||\{\lambda_{\overline{\nu}\,\overline{m}}\}|s^{\overline{r}}_{p,q}f||^p \geq c \int_{-\infty}^{\infty}\int_{x_1-1}^{x_1+1} \Big(\sum_{\overline{\nu}\in\mathbb{N}_0^2}\sum_{\overline{m}\in B_{\overline{\nu}}} |2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}}(x_1,x_2)|^q\Big)^{\frac{p}{q}} \mathrm{d}x_2\mathrm{d}x_1.$$

So, to prove (4.8) for $\gamma^{(1)}$, it is enough to prove

$$\Big(\sum_{\mu=0}^{\infty}\sum_{n\in\mathbb{Z}}|2^{\mu r_1}\gamma^{(1)}_{\mu n}\chi_{\mu n}(x_1)|^q\Big)^{\frac{p}{q}} \le c\int_{-1}^{1}\Big(\sum_{\overline{\nu}\in\mathbb{N}_0^2}\sum_{\overline{m}\in B_{\overline{\nu}}}|2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}}(x_1,x_1+x_2)|^q\Big)^{\frac{p}{q}}\mathrm{d}x_2 \quad (4.11)$$

for every fixed x_1.
Finally, we try to change the notation in such a way that we could switch from integrals to sums. With x_1 being fixed, there is only one $n=n(\mu)$ such that $\chi_{\mu n}(x_1)=1$. We denote $\gamma^{(1)}_{\mu}=\gamma^{(1)}_{\mu\, n(\mu)}$. So, the left hand side of (4.11) reduces to

$$\Big(\sum_{\mu=0}^{\infty}|2^{\mu r_1}\gamma^{(1)}_{\mu}|^q\Big)^{\frac{p}{q}}.$$

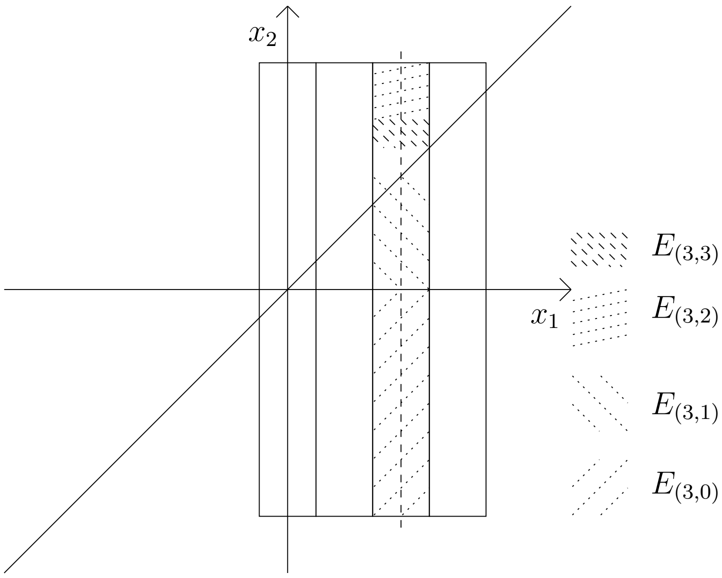
Finally, as a direct corollary of (3.5), we may suppose, that each $B_{\overline{\nu}n}$ contains only one element. So, to every $\mu\in\mathbb{N}_0$ and every $\nu_1\le\mu$ there is a unique $\overline{m}=\overline{m}(\mu,\nu_2)\in B_{(\mu,\nu_2)\,n(\mu)}$. We denote $\lambda_{(\mu,\nu_2)}=\lambda_{(\mu,\nu_2)\,\overline{m}(\mu,\nu_2)}$.
We reformulate once more our task. We start with a given sequence

$$\lambda=\{\lambda_{\overline{\nu}}:\overline{\nu}\in\mathbb{N}_0^2,\nu_1\ge\nu_2\},$$

and define

$$\gamma_\mu=\sum_{\nu_2=0}^{\mu}|\lambda_{(\mu,\nu_2)}|.$$

Finally, we use the Lemma 2.11 and choose the sets $E_{\overline{\nu}\,\overline{m}}$ such that $E_{(\mu,\nu_2),\overline{m}(\mu,\nu_2)}$ and $E_{(\mu,\nu_2'),\overline{m}(\mu,\nu_2')}$ are disjoint for $\nu_2\ne\nu_2'$.



It turns out, that it is enough to prove that

$$\Big(\sum_{\mu=0}^{\infty}|2^{\mu r_1}\gamma_\mu|^q\Big)^{\frac{p}{q}}\le c\sum_{j=0}^{\infty}2^{-j}\Big(\sum_{\mu=j}^{\infty}|2^{\mu r_1+jr_2}\lambda_{(\mu,j)}|^q\Big)^{\frac{p}{q}} \quad (4.12)$$

with c independent on the starting sequence λ. We just mention, that the $j-$sum comes from decomposition of the integral in (4.11) according to the supports of $\chi_{\overline{\nu}\,\overline{m}}$ involved.
First we discuss the case $q \le 1$. In that case,

$$\gamma_\mu^q \le \sum_{\nu_2=0}^{\mu} |\lambda_{(\mu,\nu_2)}|^q \le \sum_{\nu_2=0}^{\mu} 2^{\nu_2(r_2-\frac{1}{p})q}|\lambda_{(\mu,\nu_2)}|^q.$$

If moreover $\frac{p}{q} \le 1$,

$$\begin{aligned}\Big(\sum_{\mu=0}^{\infty} 2^{\mu r_1 q}\gamma_\mu^q\Big)^{\frac{p}{q}} &\le \Big(\sum_{\mu=0}^{\infty} 2^{\mu r_1 q}\sum_{\nu_2=0}^{\mu} 2^{\nu_2(r_2-\frac{1}{p})q}|\lambda_{(\mu,\nu_2)}|^q\Big)^{\frac{p}{q}}\\ &= \Big(\sum_{\nu_2=0}^{\infty} 2^{-\nu_2\frac{q}{p}}\sum_{\mu=\nu_2}^{\infty}|2^{\mu r_1+\nu_2 r_2}\lambda_{(\mu,\nu_2)}|^q\Big)^{\frac{p}{q}}\\ &\le c\sum_{\nu_2=0}^{\infty} 2^{-\nu_2}\Big(\sum_{\mu=\nu_2}^{\infty}|2^{\mu r_1+\nu_2 r_2}\lambda_{(\mu,\nu_2)}|^q\Big)^{\frac{p}{q}}\end{aligned}$$

This proves (4.12) for $p \le q \le 1$ and $r_2 \ge \frac{1}{p}$.
In the case $q \le 1, q < p$ we denote

$$b_{\nu_2}^q = \sum_{\mu=\nu_2}^{\infty} |2^{\mu r_1}\lambda_{(\mu,\nu_2)}|^q.$$

By this notation, the right-hand side of (4.12) may be rewritten like

$$RHS(4.12) = \sum_{\nu_2=0}^{\infty} 2^{-\nu_2}\Big(2^{\nu_2 r_2 q} b_{\nu_2}^q\Big)^{\frac{p}{q}} = \sum_{\nu_2=0}^{\infty} 2^{\nu_2(r_2-\frac{1}{p})p} b_{\nu_2}^p$$

and the left-hand side may be estimated by

$$\Big(\sum_{\mu=0}^{\infty} |2^{\mu r_1}\gamma_\mu^{(1)}|^q\Big)^{\frac{p}{q}} \le \Big(\sum_{\nu_2=0}^{\infty} b_{\nu_2}^q\Big)^{\frac{p}{q}}.$$

This (and Hölder's inequality) finishes the proof of (4.12) for $r_2 > \frac{1}{p}$ and $q \le 1, q < p$.
Next, we take $q > 1$. We denote $\beta = -\frac{1}{q'} = \frac{1}{q} - 1$ if $r_2 = \frac{1}{p}$ and $\beta = 0$ if $r_2 > \frac{1}{p}$.
By Hölder's inequality we get

$$\gamma_\mu \le c\,(\mu+1)^{-\beta}\Big(\sum_{\nu_2=0}^{\mu} 2^{\nu_2(r_2-\frac{1}{p})q}|\lambda_{\mu,\nu_2}|^q\Big)^{\frac{1}{q}}, \quad \mu \in \mathbb{N}_0.$$

Hence, for $p \le q$,

$$\begin{aligned}\Big(\sum_{\mu=0}^{\infty}(\mu+1)^{\beta q}2^{\mu r_1 q}\gamma_\mu^q\Big)^{\frac{p}{q}} &\le c\Big(\sum_{\mu=0}^{\infty} 2^{\mu r_1 q}\sum_{\nu_2=0}^{\mu} 2^{\nu_2(r_2-\frac{1}{p})q}|\lambda_{\mu,\nu_2}|^q\Big)^{\frac{p}{q}}\\ &= c\Big(\sum_{\nu_2=0}^{\infty} 2^{-\nu_2\frac{q}{p}}\sum_{\mu=\nu_2}^{\infty} 2^{\mu r_1 q+\nu_2 r_2 q}|\lambda_{(\mu,\nu_2)}|^q\Big)^{\frac{p}{q}}\\ &\le c\sum_{\nu_2=0}^{\infty} 2^{-\nu_2}\Big(\sum_{\mu=\nu_2}^{\infty} 2^{\mu r_1 q+\nu_2 r_2 q}|\lambda_{(\mu,\nu_2)}|^q\Big)^{\frac{p}{q}}.\end{aligned}$$

This finishes the proof of (4.12) for $\max(p,1) \le q$ and $r_2 > \frac{1}{p}$. But for $r_2 = \frac{1}{p}$ this also proves the generalisation of (4.12), where $2^{\mu r_1}$ is replaced by $(\mu+1)^\beta 2^{\mu r_1}$ on the left-hand side. Hence, also the boundedness of the trace operator in (4.5) follows.
For $p > q > 1$ and $r_2 - \frac{1}{p} > \varepsilon > 0$ we get similarly

$$\gamma_\mu \le c\Big(\sum_{\nu_2=0}^{\mu} 2^{\nu_2(r_2-\frac{1}{p}-\varepsilon)q}|\lambda_{\mu,\nu_2}|^q\Big)^{\frac{1}{q}}$$

and

$$\Big(\sum_{\mu=0}^{\infty} 2^{\mu r_1 q}\gamma_\mu^q\Big)^{\frac{p}{q}} \le c\Big(\sum_{\nu_2=0}^{\infty} 2^{\nu_2(r_2-\frac{1}{p}-\varepsilon)q} b_{\nu_2}^q\Big)^{\frac{p}{q}} \le c\sum_{\nu_2=0}^{\infty}\Big(2^{\nu_2(r_2-\frac{1}{p})q} b_{\nu_2}^q\Big)^{\frac{p}{q}}$$

This finishes the boundedness of the trace operator for $r_2 > \frac{1}{p}$. In the case of $r_2 = \frac{1}{p}$, we have only discussed the cases $p \le q \le 1$ and $1 \le p \le q$. To complete the proof in those cases, where the result depends on q, we consider $p \le 1 \le q$. We get by Minkowski's inequality

$$\begin{aligned}\Big(\sum_{\mu=0}^{\infty} 2^{\mu r_1 q}\Big(\sum_{\nu_2=0}^{\mu}|\lambda_{(\mu,\nu_2)}|\Big)^q\Big)^{\frac{p}{q}} &\le \Big(\sum_{\nu_2=0}^{\infty}\Big(\sum_{\mu=\nu_2}^{\infty} 2^{\mu r_1 q}|\lambda_{(\mu,\nu_2)}|^q\Big)^{\frac{1}{q}}\Big)^p \\ &\le \sum_{\nu_2=0}^{\infty}\Big(\sum_{\mu=\nu_2}^{\infty} 2^{\mu r_1 q}|\lambda_{(\mu,\nu_2)}|^q\Big)^{\frac{p}{q}} = RHS(4.12).\end{aligned}$$

Finally, to prove the boundedness of the trace operator in (4.4) and (4.6) we use the embedding

$$S^{\overline{r}}_{p,q}F(\mathbb{R}^2) \hookrightarrow S^{\overline{r}}_{p,p}B(\mathbb{R}^2),$$

which holds for $q \le p$, and Theorem 3.1.
Step 2.
Next we discuss the remaining case $0 < r_1 \le r_2 < \frac{1}{p}$, $\varrho = r_1 + r_2 - \frac{1}{p} > \sigma_{p,q}$.
We now need to prove (4.9). We introduce again the same notation as in the Step 1. and replace (4.12) by

$$\sum_{\mu=0}^{\infty}|2^{\mu\varrho}\gamma_\mu^{(1)}|^p \le c\sum_{j=0}^{\infty} 2^{-j}\Big(\sum_{\mu=j}^{\infty}|2^{\mu r_1 + j r_2}\lambda_{(\mu,j)}|^q\Big)^{\frac{p}{q}}. \qquad (4.13)$$

Finally, we prove (4.13) for all $0 < q \le \infty$ if we prove it for $q = \infty$. We denote

$$a_{\nu_2} = \sup_{\mu \ge \nu_2} 2^{\mu r_1}|\lambda_{(\mu,\nu_2)}|, \quad \nu_2 \in \mathbb{N}_0.$$

Then the right-hand side of (4.13) may be (for $q = \infty$) rewritten as

$$RHS(4.13) = c\sum_{\nu_2=0}^{\infty} 2^{-\nu_2}\big(\sup_{\mu\ge\nu_2} 2^{\mu r_1 + \nu_2 r_2}|\lambda_{\mu,\nu_2}|\big)^p = c\sum_{\nu_2=0}^{\infty} 2^{-\nu_2+\nu_2 r_2 p} a_{\nu_2}^p.$$

As for the left-hand side in (4.13), we get for $p \le 1$

$$LHS(4.13) \le \sum_{\mu=0}^{\infty} 2^{\mu\varrho p}\sum_{\nu_2=0}^{\mu}|\lambda_{\mu,\nu_2}|^p = \sum_{\nu_2=0}^{\infty}\sum_{\mu=\nu_2}^{\infty} 2^{\mu\varrho p}|\lambda_{\mu,\nu_2}|^p \le c\sum_{\nu_2=0}^{\infty} 2^{\nu_2(r_2-\frac{1}{p})p} a_{\nu_2}^p.$$

For $p>1$ we denote $\epsilon = \frac{1}{p} - r_2 > 0$ and get

$$\begin{aligned} LHS(4.13) &= \sum_{\mu=0}^{\infty} 2^{\mu r_1 p - \mu\epsilon p} \Big(\sum_{\nu_2=0}^{\mu} 2^{(\mu-\nu_2)\epsilon/2-(\mu-\nu_2)\epsilon/2} |\lambda_{\mu,\nu_2}| \Big)^p \\ &\le \sum_{\mu=0}^{\infty} 2^{\mu r_1 p - \mu\epsilon p} \big(\sum_{\nu_2=0}^{\mu} 2^{(\mu-\nu_2)p\epsilon/2} |\lambda_{\mu,\nu_2}|^p \big) \big(\sum_{\nu_2=0}^{\mu} 2^{-(\mu-\nu_2)p'\epsilon/2} \big)^{p/p'} \\ &\le c \sum_{\nu_2=0}^{\infty} \sum_{\mu=\nu_2}^{\infty} 2^{\mu r_1 p - \mu\epsilon p + (\mu-\nu_2)p\epsilon/2} |\lambda_{\mu,\nu_2}|^p \\ &\le c \sum_{\nu_2=0}^{\infty} 2^{-\nu_2 p\epsilon/2} a_{\nu_2}^p \sum_{\mu=\nu_2}^{\infty} 2^{-\mu\epsilon p + \mu p\epsilon/2} \le c \sum_{\nu_2=0}^{\infty} 2^{-\nu_2 \epsilon p} a_{\nu_2}^p. \end{aligned}$$

This finishes the proof of (4.8) and (4.9) for $\gamma^{(1)}$. One could follow the same arguments also for $\gamma^{(2)}$. Alternatively, to a given sequence

$$\lambda = \{\lambda_{\overline{\nu}} : \overline{\nu} \in \mathbb{N}_0^2, \nu_1 \le \nu_2\}$$

we consider a sequence

$$\overline{\lambda} = \{\overline{\lambda}_{\overline{\nu}} : \overline{\nu} \in \mathbb{N}_0^2, \nu_1 \ge \nu_2\}$$

defined by $\overline{\lambda}_{(\nu_1,\nu_2)} = \lambda_{(\nu_2,\nu_1)}$ and use (4.12) for $\overline{\gamma}^{(1)}$ associated with $\overline{\lambda}$. In this way, we prove (4.8) and (4.9) for $\gamma^{(2)}$ and finish the proof of boundedness of the trace operator.

Step 3.

Next, we consider the corresponding extension operators. We use the same operators as in the B-case. The first one (given by (3.16) and (3.17)) gives an extension operator in the case $r_2 > \frac{1}{p}$. To prove the corresponding inequality on the sequence space level, we again fix x_1 and prove a pointwise inequality, which now reduces to trivial

$$\Big(\sum_{\mu=0}^{\infty} 2^{\mu r_1 q} \big(\sum_{\nu_2=0}^{\mu} |\gamma_{\mu,\nu_2}| \big)^q \Big)^{p/q} = \big(\sum_{\mu=0}^{\infty} 2^{\mu r_1 q} |\lambda_\mu|^q \big)^{p/q}.$$

The same operator proves also (4.3).

The second operator, characterised by (3.17) and (3.20) gives an extension operator for $r_2 < \frac{1}{p}$ and in (4.4). We omit the trivial calculation. □

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