Carl's inequality for quasi-Banach spaces

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December 12, 2015

Abstract

We prove that for any two quasi-Banach spaces X and Y and any $\alpha > 0$ there exists a constant $c_\alpha > 0$ such that

$$\sup_{1\le k\le n} k^\alpha e_k(T) \le c_\alpha \sup_{1\le k\le n} k^\alpha c_k(T)$$

holds for all linear and bounded operators $T : X \to Y$. Here $e_k(T)$ is the k-th entropy number of T and $c_k(T)$ is the k-th Gelfand number of T. For Banach spaces X and Y this inequality is widely used and well-known as Carl's inequality. For general quasi-Banach spaces it is a new result, which closes a gap in the argument of Donoho in his seminal paper on compressed sensing.

1 Introduction

The theory of s-numbers [7, 27, 29] (sometimes also called n-widths) emerged from the studies of geometry of Banach spaces and of operators between them but found many applications in numerical analysis as well as linear and non-linear approximation theory [9, 10, 11, 26, 24]. It turned out to be also useful in estimates of eigenvalues of operators [5, 8, 22, 28].

Recently, the s-numbers were used in the area of compressed sensing [4, 12], cf. also [3, 15], to provide general lower bounds for the performance of sparse recovery methods. In its basic setting, compressed sensing studies pairs (A, Δ) of linear measurement maps $A \in \mathbb{R}^{n\times N}$ and (non-linear) recovery maps $\Delta : \mathbb{R}^n \to \mathbb{R}^N$, such that the recovery error $x-\Delta(Ax)$ is small for k-sparse vectors $x \in \Sigma_k = \{x \in \mathbb{R}^N : \#\{i : x_i \ne 0\} \le k\}$. To allow for stability needed in applications, it is also necessary that the methods of compressed sensing are extendable to compressible vectors, i.e. to vectors which can be very well approximated by sparse vectors. The performance of a pair (A, Δ) in recovery of vectors from some set $K \subset \mathbb{R}^N$ is measured in the worst case by

$$\varepsilon(A, \Delta, K, Y) = \sup_{x\in K} \|x - \Delta(Ax)\|_Y,$$

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where Y is a (quasi-)norm on $\mathbb{R}^N$. The search for the optimal recovery pair is then expressed in the so-called compressive n-widths

$$E^n(K,Y) = \inf\Big\{\varepsilon(A,\Delta,K,Y) : A \in \mathbb{R}^{n\times N}, \Delta : \mathbb{R}^n \to \mathbb{R}^N\Big\}.$$

Based on previous work in approximation theory and information based complexity [25, 26, 30] it was observed in [9, 12, 21] that the compressive n-widths of a symmetric and subadditive set K (i.e. a set K with $K = -K$ and $K + K \subset aK$ for some $a > 0$) are equivalent to Gelfand numbers of K, which are defined as

$$c_n(K) = \inf_{\substack{M \subset\subset \mathbb{R}^N \\ \operatorname{codim} M < n}} \sup_{x \in K \cap M} \|x\|_Y.$$

Here, the infimum is taken over all linear subspaces M of $\mathbb{R}^N$ with codimension smaller than n. Any lower bound on Gelfand numbers of K therefore immediately translates into lower bounds on recovery errors of vectors from K. Especially, if an algorithm achieves the same recovery rate as the corresponding lower bound obtained by estimates of Gelfand numbers, we know that this algorithm is asymptotically optimal.

In the frame of compressed sensing, the unit balls of ℓ_p^N for $0 < p \le 1$ are typically used as a good model for compressible vectors and the error of recovery is mostly measured in the Euclidean norm of ℓ_2^N. Consequently, Donoho [12] investigated the decay of $E^n(B_p^N, \ell_2^N)$ and, consequently, the decay of Gelfand numbers of B_p^N in ℓ_2^N, which will be denoted by $c_n(id : \ell_p^N \to \ell_2^N)$ later on. The first estimates of these quantities for $p = 1$ were obtained by Garnaev, Gluskin, and Kashin [16, 18, 20]. For $p < 1$, the estimate from above appeared first in [12] and using the approach of [24] it was proved also in [34].

One of the most useful tools in the study of s-numbers is Carl's inequality [5], which relates the behavior of several of the most important scales of s-numbers to their entropy numbers (see below for the exact definitions). If X and Y are Banach spaces and if $T : X \to Y$ is a bounded linear operator between them, then Carl's inequality states that for every natural number $n \in \mathbb{N}$

$$\sup_{1\le k\le n} k^\alpha e_k(T) \le c_\alpha \sup_{1\le k\le n} k^\alpha s_k(T). \tag{1.1}$$

Here, $e_k(T)$ denotes the entropy numbers of T and $s_k(T)$ stands for approximation, Gelfand, or Kolmogorov numbers, respectively. For the definition of these quantities, let $T : X \to Y$ be a bounded linear operator between quasi-Banach spaces X and Y. Then we define the Gelfand numbers $c_n(T)$, the Kolmogorov numbers $d_n(T)$, the approximation numbers $a_n(T)$ and the entropy numbers $e_n(T)$, respectively, by

$$\begin{aligned}
c_n(T) &= \inf_{\substack{M \subset\subset X \\ \operatorname{codim} M < n}} \sup_{\substack{x \in M \\ \|x\|_X \le 1}} \|Tx\|_Y \\
d_n(T) &= \inf_{\substack{N \subset\subset Y \\ \dim N < n}} \sup_{\|x\|_X \le 1} \inf_{z \in N} \|Tx - z\|_Y \\
a_n(T) &= \inf\{\|T - L\| : L : X \to Y, \operatorname{rank}(L) < n\} \\
e_n(T) &= \inf\Big\{\varepsilon > 0 : T(B_X) \subset \bigcup_{j=1}^{2^{n-1}} (y_j + \varepsilon B_Y)\Big\}.
\end{aligned}$$

In the last definition, B_X can denote either the open or the closed unit ball in X. While usually the closed unit ball is used, for technical reasons we prefer to work with the open unit ball $B_X = \{x \in X : \|x\|_X < 1\}$.

By applying (1.1) to $T = id : \ell_p^N \to \ell_2^N$ and using the known results [23, 33] on entropy numbers $e_n(id : \ell_p^N \to \ell_2^N)$, Donoho obtained a lower bound for $c_n(id : \ell_p^N \to \ell_2^N)$ and, consequently, also for $E^n(B_p^N, \ell_2^N)$. The results obtained can be summarized as

$$c_p \min\Big\{1, \frac{1+\log(N/n)}{n}\Big\}^{1/p-1/2} \le c_n(id : \ell_p^N \to \ell_2^N) \le C_p \min\Big\{1, \frac{1+\log(N/n)}{n}\Big\}^{1/p-1/2}, \tag{1.2}$$

where $1 \le n \le N$ are natural numbers and the positive numbers c_p, C_p do not depend on n and N. The use of (1.1) in the proof of the lower estimate of (1.2) appeared for the first time in [6] and we give a sketch of this argument in Section 4 for readers convenience.

Unfortunately, the argument just presented contains one crucial flaw, which was overlooked by Donoho. Carl's inequality (1.1) is proven in [5] only when X and Y are Banach spaces. This gap was observed by S. Foucart and H. Rauhut during the preparation of [15]. Furthermore, the arguments of Carl use implicitly the Hahn-Banach theorem (which is of course not available for general quasi-Banach spaces) and it was not clear if this approach can be somehow adapted to quasi-Banach spaces. The solution was found in [14], where the authors used a completely different approach to prove the lower bound in (1.2) for all $0 < p < 2$. They avoided the use of Carl's inequality and proved (1.2) directly, using techniques from compressed sensing. The question if Carl's inequality allows for an extension to quasi-Banach spaces and Gelfand numbers remained open. Indeed, the authors of [14] expressed their belief that "Carl's theorem actually fails for Gelfand widths of general quasi-norm balls". Let us also mention, that Carl's inequality extends easily to quasi-Banach spaces and Kolmogorov numbers or approximation numbers with only minor modifications necessary, cf. [2, 17] or [13, Section 1.3.3].

The main result of this note is that Carl's inequality also holds for quasi-Banach spaces and Gelfand numbers. Consequently, (1.1) is true also for quasi-Banach spaces with $s_k(T)$ standing again for any of the scales of approximation, Gelfand, or Kolmogorov numbers, respectively. As an application, this also provides an alternative proof for the lower bound in (1.2).

Theorem 1.1. *Let X and Y be quasi-Banach spaces. Then for any $\alpha > 0$ there exists a constant $c_\alpha > 0$ such that*

$$\sup_{1\le k\le n} k^\alpha e_k(T) \le c_\alpha \sup_{1\le k\le n} k^\alpha c_k(T) \tag{1.3}$$

holds for all linear and bounded operators $T : X \to Y$.

We now explain the original proof of Carl's inequality (1.3) for Gelfand numbers in the case that X and Y are Banach spaces (cf. [5], [7, Theorem 3.1.1], or [31, Theorem 5.2]) and show why this somehow indirect approach completely fails in the quasi-Banach case. The proof proceeds in the following way.

First, (1.3) is shown for approximation numbers $a_k(T)$ instead of Gelfand numbers. As noted before, this is easily extended to the quasi-Banach case. Afterwards, an isometric embedding $j : Y \to \ell_\infty(S)$ for some set S is used. Such an embedding exists for any Banach space Y and can be constructed with the Hahn-Banach theorem, e.g. with S being the unit sphere or the unit ball in the dual space Y^*. Already such an isometric embedding obviously does not exist if Y is not a Banach space. Now, making use of the isometric embedding j, the following two properties of entropy and Gelfand numbers, namely

(i) $e_n(T) \le 2e_n(j \circ T)$ for every $n \in \mathbb{N}$, and

(ii) $c_n(T) = a_n(j \circ T)$ for every $n \in \mathbb{N}$

are essential. Equipped with these tools, (1.3) then follows simply by

$$\sup_{1\le k\le n} k^\alpha e_k(T) \le 2 \sup_{1\le k\le n} k^\alpha e_k(j\circ T) \le 2c_\alpha \sup_{1\le k\le n} a_k(j\circ T) \le 2c_\alpha \sup_{1\le k\le n} c_k(T). \quad (1.4)$$

Let us comment on (i) and (ii) - and point out, why (ii) fails completely in the quasi-Banach case.

The proof of (i) is easy. Let $(j\circ T)(B_X)$ be covered by 2^{n-1} balls of radius ε in $\ell_\infty(S)$. Then (by just the triangle inequality) it can be covered by 2^{n-1} balls of radius 2ε in the same space with centers in $(j\circ T)(X)$. Finally, as j is isometric, this can be translated into a covering of $T(B_X)$ by 2^{n-1} balls of radius 2ε in Y.

The proof of (ii) is more involved - and makes heavy use of the Hahn-Banach theorem. We will only comment on the more difficult inequality $a_n(j\circ T) \le c_n(T)$, which was used in (1.4). The essential property of the space $\ell_\infty(S)$ here is the extension property: any linear bounded operator $U : M \to \ell_\infty(S)$ from a closed linear subspace M of a Banach space X can be extended to an operator $\tilde{U} : X \to \ell_\infty(S)$ with $\|\tilde{U}\| = \|U\|$. Again, the proof of this extension property needs the Hahn-Banach Theorem now for the Banach space X. So, this step is in general not possible if X is not a Banach space.

With the extension property the proof of (ii) is finished as follows. Given a subspace M of X with $\operatorname{codim} M < n$, we can extend $U = j\circ T|_M$ to an operator $\tilde{U} : X \to \ell_\infty(S)$ with $\|\tilde{U}\| = \|U\| = \|T|_M\|$ and, letting $L = j\circ T - \tilde{U}$ we conclude that $\operatorname{rank}(L) < n$ and $a_n(j\circ T) \le \|j\circ T - L\| = \|\tilde{U}\| = \|T|_M\|$. Finally, (ii) follows by taking the infimum over all such M.

This discussion makes clear, that in this approach to Carl's inequality via the approximation numbers, the property that both X and Y are Banach spaces (and not merely quasi-Banach spaces) is essential.

The structure of the paper is as follows. In Section 2 we collect some notation and basic facts about quasi-Banach spaces. Section 3 gives the proof of our main result, Theorem 1.1. With standard arguments, we derive the version of Carl's inequality for Lorentz norms, Theorem 3.4. Finally, in Section 4 we show how to use this new result and the upper bound of (1.2) to show the lower bound in (1.2).

2 Quasi-Banach spaces

This section collects some basic facts about quasi-Banach spaces. We restrict ourselves to the minimum needed later on and refer to [19] and the references therein for an extensive overview. If X is a (real) vector space, we say that $\|\cdot\|_X : X \to [0,\infty)$ is a quasi-norm if

(i) $\|x\|_X = 0$ if, and only if $x = 0$,

(ii) $\|\alpha x\|_X = |\alpha| \cdot \|x\|_X$ for all $\alpha \in \mathbb{R}$ and $x \in X$,

(iii) there is a constant $C \ge 1$, such that $\|x+y\|_X \le C(\|x\|_X + \|y\|_X)$ for all $x, y \in X$.

If X is complete with respect to the metric induced by $\|\cdot\|_X$, it is called a quasi-Banach space. By the fundamental Aoki-Rolewicz theorem [1, 32], every quasi-norm is equivalent to some p-norm, i.e. there exists a mapping $|||\cdot|||_X : X \to [0,\infty)$ and $0 < p \le 1$, such that $|||\cdot|||_X$ satisfies (i) and (ii) as above, (iii) gets replaced by $|||x+y|||_X^p \le |||x|||_X^p + |||y|||_X^p$ and $|||\cdot|||_X$ is equivalent to $\|\cdot\|_X$ on X. The expression $|||\cdot|||_X$ is then called a p-norm and X is a p-Banach space. As the validity of Carl's inequality does not change if we replace the quasi-norms on X and Y by equivalent quasi-norms, we shall always assume that X and Y are equipped with a p-norm and a q-norm, respectively.

2.1 Quotients of quasi-Banach spaces

If X and Y are quasi-Banach spaces and T is a bounded linear operator between them, we can still define Gelfand numbers $c_n(T)$ as before, as the notion of codimension is algebraic. Furthermore, if X is a p-Banach space and $M \subset\subset X$ is a subspace, we can also define the quotient space X/M and the usual definition makes it again a p-Banach space. Indeed, let $[x],[y] \in X/M$. Then there are (for every $\varepsilon > 0$) $z^x, z^y \in M$, such that

$$\|x - z^x\|_X \le (1+\varepsilon)\|[x]\|_{X/M} \quad \text{and} \quad \|y - z^y\|_X \le (1+\varepsilon)\|[y]\|_{X/M}.$$

We then obtain

$$\begin{aligned}\|[x+y]\|_{X/M}^p &\le \|x+y-z^x-z^y\|_X^p \le \|x-z^x\|_X^p + \|y-z^y\|_X^p\\ &\le (1+\varepsilon)^p(\|[x]\|_{X/M}^p + \|[y]\|_{X/M}^p)\end{aligned}$$

and the statement follows by letting $\varepsilon \to 0$.

2.2 Entropy numbers of identity mappings

We give an analogue of [27, (12.1.13)] for quasi-Banach spaces.

Lemma 2.1. *Let X be a real m-dimensional p-Banach space, where $m \in \mathbb{N}$ and $0 < p \le 1$. Then*

$$e_n(id : X \to X) \le 4^{1/p} 2^{-\frac{n-1}{m}} \tag{2.1}$$

for all $n \in \mathbb{N}$.

Proof. The inequality $e_1(id : X \to X) \le 1$ holds also for quasi-Banach spaces. If $(n-1) \le 2m/p$, then $2^{\frac{n-1}{m}} \le 4^{1/p}$ and (2.1) follows.

We assume therefore that $(n-1) > 2m/p$. We choose $\varepsilon > 0$ by

$$\left[\frac{(1+\varepsilon^p/2)^{1/p}}{\varepsilon/2^{1/p}}\right]^m = 2^{n-1}, \quad \text{i.e.} \quad \varepsilon = \left[\frac{2}{2^{\frac{p(n-1)}{m}} - 1}\right]^{1/p} < 1.$$

Let now $x_1, \dots, x_N \in B_X$ be a maximal subset of B_X with mutual distances $\|x_i - x_j\|_X \ge \varepsilon$. Then B_X can be covered by the balls $x_i + \varepsilon B_X$ and the balls $x_i + \frac{\varepsilon}{2^{1/p}} B_X$ are mutually disjoint. Indeed, if there would be a $z \in X$ with $\|x_i - z\|_X < \varepsilon/2^{1/p}$ and $\|x_j - z\|_X < \varepsilon/2^{1/p}$, then $\|x_i - x_j\|_X^p \le \|x_i - z\|_X^p + \|x_j - z\|_X^p < \varepsilon^p$. Furthermore, if $y \in x_i + \frac{\varepsilon}{2^{1/p}} B_X$, then $y = x_i + z$ with $\|z\|_X < \frac{\varepsilon}{2^{1/p}}$ and

$$\|y\|_X^p \le \|x_i\|_X^p + \|z\|_X^p < 1 + \varepsilon^p/2.$$

Hence, $x_i + \frac{\varepsilon}{2^{1/p}} B_X$ are mutually disjoint, all included in $(1+\varepsilon^p/2)^{1/p} B_X$. Comparing the volumes (with respect to any translation invariant normalized measure on X), we get

$$N \cdot \left(\frac{\varepsilon}{2^{1/p}}\right)^m \le (1+\varepsilon^p/2)^{m/p}, \quad \text{i.e.} \quad N \le \left[\frac{(1+\varepsilon^p/2)^{1/p}}{\varepsilon/2^{1/p}}\right]^m = 2^{n-1}.$$

We therefore obtain that

$$e_n(id : X \to X) \le \varepsilon = \left[\frac{2}{2^{\frac{p(n-1)}{m}} - 1}\right]^{1/p} \le \left[4 \cdot 2^{-\frac{p(n-1)}{m}}\right]^{1/p}$$

and (2.1) follows again.

□

3 Proof of Carl's inequality

In this section we prove our main result, Theorem 1.1, as well as its Lorentz space counterpart, Theorem 3.4.

3.1 Proof of Theorem 1.1

Let X be a p-Banach space and let Y be a q-Banach space. We fix a sequence $(M_n)_{n\in\mathbb{N}}$ of finite codimensional subspaces of X and let $\gamma_n = \|T|_{M_n}\|$. We also fix a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive numbers with $\varepsilon_n \le 1$. Let $\mathcal{M}_n \subset B_{X/M_n}$ be an ε_n-net of the unit ball of X/M_n, i.e. for any $[x] \in B_{X/M_n}$ there exist $[x_n] \in \mathcal{M}_n$ such that

$$\big\|[x] - [x_n]\big\|_{X/M_n} = \inf_{z\in M_n} \|x - x_n - z\|_X < \varepsilon_n.$$

Let $\mathcal{N}_n \subset 2^{1/p} B_X$ be a lifting of $\mathcal{M}_n$, so for any $x \in B_X$ there exist $x_n \in \mathcal{N}_n$ and $z_n \in M_n$ with

$$\|x - (x_n + z_n)\|_X < \varepsilon_n.$$

Finally, let $\delta_0 = 1$ and

$$\delta_n = \prod_{j=1}^n \varepsilon_j \qquad \text{for } n \in \mathbb{N}.$$

The proof of Theorem 1.1 relies on an iterative construction. The single steps are based on the lifting just described and the details are given in Lemma 3.1. Its inductive use is then the subject of Lemma 3.2.

Lemma 3.1. *For any $x \in X$ with $\|x\|_X < \delta$ for some $0 < \delta \le 1$ there exist $x_n \in \mathcal{N}_n$ and $z_n \in M_n$ such that*

$$\|z_n\|_X < 4^{1/p} \qquad \textit{and} \qquad \|x - \delta(x_n + z_n)\|_X < \delta \cdot \varepsilon_n.$$

Proof. Since $\|x/\delta\|_X < 1$, we find $x_n \in \mathcal{N}_n$ and $z_n \in M_n$ such that

$$\|x/\delta - (x_n + z_n)\|_X < \varepsilon_n.$$

This shows the second inequality. The bound on z_n follows from the p-triangle inequality

$$\|z_n\|_X^p \le \|x/\delta - (x_n + z_n)\|_X^p + \|x/\delta\|_X^p + \|x_n\|_X^p < 4.$$

□

Lemma 3.2. *For any $x \in B_X$, there exist sequences $(x_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}$ with $x_n \in \mathcal{N}_n$ and $z_n \in M_n$ such that*

(i) $\|z_n\|_X < 4^{1/p}$ *for* $n \in \mathbb{N}$,

(ii) $\|x - \sum_{k=1}^n \delta_{k-1}(x_k + z_k)\|_X < \delta_n$ *for* $n \in \mathbb{N}$,

(iii) $\|Tx - \sum_{k=1}^n \delta_{k-1} T x_k\|_Y^q \le (\|T\|\delta_n)^q + 4^{q/p} \sum_{k=1}^n \delta_{k-1}^q \gamma_k^q$ *for* $n \in \mathbb{N}$.

Proof. The existence of sequences $(x_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}$ with $x_n \in \mathcal{N}_n$ and $z_n \in M_n$ satisfying (i) and (ii) follows inductively from Lemma 3.1. It remains to prove (iii). We use the q-triangle inequality, (i),(ii) and $\gamma_n = \left\|T|_{M_n}\right\|$ to conclude that

$$\begin{aligned}\left\|Tx - \sum_{k=1}^n \delta_{k-1} T x_k\right\|_Y^q &\le \left\|Tx - \sum_{k=1}^n \delta_{k-1} T(x_k + z_k)\right\|_Y^q + \sum_{k=1}^n \delta_{k-1}^q \|Tz_k\|_Y^q \\ &\le \left(\|T\|\delta_n\right)^q + 4^{q/p} \sum_{k=1}^n \delta_{k-1}^q \gamma_k^q.\end{aligned}$$

□

The next theorem now follows using the optimal subspaces M_n and the inequality (2.1) for entropy numbers for the m_n-dimensional p-Banach space X/M_n.

Theorem 3.3. *Let $T : X \to Y$ be a bounded linear operator from the p-Banach space X to the q-Banach space Y, where $0 < p, q \le 1$. Let $(k_n)_{n\in\mathbb{N}}$ and $(m_n)_{n\in\mathbb{N}}$ be sequences of positive integers. Then*

$$e_{k_1+\cdots+k_n+1-n}(T)^q \le 2^{2nq/p - \sum_{j=1}^n \frac{k_j-1}{m_j}\cdot q} \|T\|^q + 4^{q/p} \sum_{\ell=1}^n 2^{2(\ell-1)q/p - \sum_{j=1}^{\ell-1} \frac{k_j-1}{m_j}\cdot q} c_{m_\ell+1}(T)^q.$$

Proof. Using the inequality for entropy numbers for the m_j-dimensional subspaces X/M_{m_j} with

$$\varepsilon_j = 4^{1/p} 2^{-\frac{k_j-1}{m_j}}, \quad \delta_k = \prod_{j=1}^k \varepsilon_j = 2^{2k/p - \sum\limits_{j=1}^{k} \frac{k_j-1}{m_j}}, \qquad j,k \in \mathbb{N},$$

Lemma 3.2 yields

$$e_{k_1+\cdots+k_n+1-n}(T)^q \le 2^{2nq/p - \sum_{j=1}^n \frac{k_j-1}{m_j}\cdot q} \|T\|^q + 4^{q/p} \sum_{\ell=1}^n 2^{2(\ell-1)q/p - \sum_{j=1}^{\ell-1} \frac{k_j-1}{m_j}\cdot q} \gamma_{m_\ell}(T)^q$$

and the claim follows by taking the infimum over all subspaces $(M_{m_j})_{j=1}^n$ with $\operatorname{codim} M_{m_j} \le m_j$. □

We are now ready to prove Theorem 1.1. It is enough to show

$$n^\alpha e_n(T) \le \gamma_\alpha \sup_{1\le k\le n} k^\alpha c_k(T)$$

for every $n \in \mathbb{N}$. By homogeneity, we may assume that $c_k(T) \le k^{-\alpha}$ for $1 \le k \le n$, in particular $c_1(T) = \|T\| \le 1$. By monotonicity, it is also enough to prove the statement for $n = C\,2^N$, where $N \in \mathbb{N}$ and C is a universal natural number. Choose $\beta > \alpha > 0$, $m_j = 2^{N-j}$, $j = 1, \ldots, N$, and

$$k_j = \lceil 2^{N-j}(2/p + \beta) + 1 \rceil, \quad j = 1, \ldots, N.$$

Then $(k_j - 1)/m_j \geq 2/p + \beta$ and $\varepsilon_j := 4^{1/p} 2^{-\frac{k_j - 1}{m_j}} \leq 2^{-\beta}$. By Theorem 3.3 we get

$$\begin{aligned} e_{k_1 + \cdots + k_N + 1 - N}(T)^q &\leq 2^{-\beta N q} + 4^{q/p} \sum_{l=1}^{N} 2^{-\beta(l-1)q} (2^{N-l} + 1)^{-\alpha q} \\ &\leq 2^{-\beta N q} + 4^{q/p} 2^{-\alpha N q} 2^{\beta q} \sum_{l=1}^{N} 2^{l(\alpha - \beta) q} \leq c_{\alpha,\beta} 2^{-N\alpha q}. \end{aligned}$$

Furthermore, let $C \geq 1$ be a natural number with $C \geq 2/p + \beta + 1$. Then

$$\begin{aligned} 1 - N + \sum_{j=1}^{N} k_j &\leq 1 - N + \sum_{j=1}^{N} [2^{N-j}(2/p + \beta) + 2] = 1 + N + (2/p + \beta) \sum_{j=1}^{N} 2^{N-j} \\ &\leq 1 + N + (2/p + \beta) 2^N \leq C 2^N. \end{aligned}$$

Putting these estimates together, we obtain

$$e_{C2^N}(T)^q \leq c_{\alpha,\beta} 2^{-N\alpha q} \leq c' (C 2^N)^{-\alpha q},$$

which gives the desired statement.

3.2 Lorentz space version

Using standard techniques, cf. [7, Theorem 3.1.2], Carl's inequality can be easily extended to compare the Lorentz quasi-norms of $(e_k(T))_{k \in \mathbb{N}}$ and $(c_k(T))_{k \in \mathbb{N}}$.

Theorem 3.4. *Let X and Y be quasi-Banach spaces and let $T : X \to Y$ be a bounded linear operator. Then for every $0 < s \leq \infty$ and every $0 < t < \infty$ there exists a constant $c_{s,t}$ such that for every $m \in \mathbb{N}$*

$$\left(\sum_{k=1}^{m} k^{t/s - 1} e_k(T)^t \right)^{1/t} \leq c_{s,t} \left(\sum_{k=1}^{m} k^{t/s - 1} c_k(T)^t \right)^{1/t}. \tag{3.1}$$

Proof. Let $\alpha > \max(1/s, 1/t)$ be fixed. By Theorem 1.1 and Hardy's inequality [7, Lemma 1.5.3] we get

$$\begin{aligned} \sum_{k=1}^{m} k^{t/s-1} e_k(T)^t &= \sum_{k=1}^{m} k^{t/s-1-\alpha t} (k^\alpha e_k(T))^t \leq \sum_{k=1}^{m} k^{t/s-1-\alpha t} (\sup_{1 \leq l \leq k} l^\alpha e_l(T))^t \\ &\leq c_\alpha \sum_{k=1}^{m} k^{t/s-1-\alpha t} (\sup_{1 \leq l \leq k} l^\alpha c_l(T))^t \\ &\leq c_\alpha \sum_{k=1}^{m} k^{t/s-1-\alpha t} \Big(\sup_{1 \leq l \leq k} \Big(\sum_{j=1}^{l} c_j(T)^{1/\alpha} \Big)^\alpha \Big)^t \\ &= c_\alpha \sum_{k=1}^{m} k^{t/s-1-\alpha t} \Big(\sum_{j=1}^{k} c_j(T)^{1/\alpha} \Big)^{\alpha t} = c_\alpha \sum_{k=1}^{m} k^{t/s-1} \Big(\frac{1}{k} \sum_{j=1}^{k} c_j(T)^{1/\alpha} \Big)^{\alpha t} \\ &\leq c_{\alpha,s,t} \sum_{k=1}^{m} k^{t/s-1} c_k(T)^t. \end{aligned}$$

□

4 Lower bound on Gelfand numbers from Carl's inequality

In this section, we sketch the use of Carl's inequality (1.3) in the proof of the lower bound in (1.2). This argument appeared first in [6] and we reproduce it (with only minor modifications) here for reader's convenience.

Theorem 4.1. *For $N \in \mathbb{N}$, $1 \le n \le N$ and $0 < p \le 2$ it holds*

$$c_p \min\left\{1, \frac{1+\log(N/n)}{n}\right\}^{\frac{1}{p}-\frac{1}{2}} \le c_n(id\colon \ell_p^N \to \ell_2^N) \le C_p \min\left\{1, \frac{1+\log(N/n)}{n}\right\}^{\frac{1}{p}-\frac{1}{2}} \tag{4.1}$$

for some constants c_p, C_p not depending on n and N.

Proof. The upper bound of this inequality was already provided in [34], so it only remains to prove the lower bound. We follow the ideas of [6, Corollary 2.6]. By [23, 33], it is known that for $0 < p \le q \le \infty$

$$e_n(id : \ell_p^N \to \ell_q^N) \approx \begin{cases} 1 & 1 \le n \le \log N \\ \left(\frac{1+\log(N/n)}{n}\right)^{1/p-1/q} & \log N \le n \le N \\ 2^{-n/N} N^{1/q-1/p} & N \le n, \end{cases} \tag{4.2}$$

where the constants of equivalence do not depend on the natural numbers n and N. Using Carl's inequality (1.3), the results on entropy numbers (4.2) and the upper bound in (1.2), we deduce the lower bound in (1.2).

For $p = 2$, (4.1) is very well known and follows by basic properties of s-numbers. We shall therefore assume that $p < 2$ and for brevity let us set $\alpha = 1/p - 1/2 > 0$. Using Carl's inequality we obtain for any natural number n with $\log N \le n \le N$

$$\begin{aligned} C_1(n(1+\log(N/n)))^\alpha &\le n^{2\alpha} e_n(id\colon \ell_p^N \to \ell_2^N) \le \sup_{1\le j\le n} j^{2\alpha} e_j(id\colon \ell_p^N \to \ell_2^N) \\ &\le c_{2\alpha} \sup_{1\le j\le n} j^{2\alpha} c_j(id\colon \ell_p^N \to \ell_2^N). \end{aligned}$$

For some $\lambda > 1$, which we shall fix later on, let us split up this supremum into two parts to get

$$\sup_{1\le j\le n} j^{2\alpha} c_j(id\colon \ell_p^N \to \ell_2^N) \le \sup_{1\le j\le \lfloor\frac{n}{\lambda}\rfloor} j^{2\alpha} c_j(id\colon \ell_p^N \to \ell_2^N) + \sup_{\lceil\frac{n}{\lambda}\rceil\le j\le n} j^{2\alpha} c_j(id\colon \ell_p^N \to \ell_2^N). \tag{4.3}$$

We estimate the first summand on the right hand side by the upper bound in (4.1)

$$\sup_{1\le j\le \lfloor\frac{n}{\lambda}\rfloor} j^{2\alpha} c_j(id\colon \ell_p^N \to \ell_2^N) \le C_p \sup_{1\le j\le \lfloor\frac{n}{\lambda}\rfloor} j^{2\alpha}\left(\frac{1+\log(N/j)}{j}\right)^\alpha \le C_p \left(\frac{n(1+\log(\lambda N/n))}{\lambda}\right)^\alpha,$$

where we used that the function $x \to x\cdot(1+\log(N/x))$ is increasing for $1 \le x \le N$. Using $\lambda > 1$ we end up with

$$\begin{aligned} \sup_{1\le j\le \lfloor\frac{n}{\lambda}\rfloor} j^{2\alpha} c_j(id\colon \ell_p^N \to \ell_2^N) &\le C_p \left(\frac{n(1+\log\lambda+\log(N/n))}{\lambda}\right)^\alpha \\ &\le C_p \left(\frac{1+\log\lambda}{\lambda}\cdot n(1+\log(N/n))\right)^\alpha. \end{aligned}$$

The second summand in (4.3) can easily be estimated by monotonicity of Gelfand numbers

$$\sup_{\lceil \frac{n}{\lambda} \rceil \le j \le n} j^{2\alpha} c_j(id\colon \ell_p^N \to \ell_2^N) \le n^{2\alpha} c_{\lceil \frac{n}{\lambda} \rceil}(id\colon \ell_p^N \to \ell_2^N).$$

Putting both estimates together we arrive at

$$c_{2\alpha} c_{\lceil \frac{n}{\lambda} \rceil}(id\colon \ell_p^N \to \ell_2^N) \ge \left(C_1 - c_{2\alpha} C_p \left(\frac{1+\log\lambda}{\lambda} \right)^\alpha \right) \left(\frac{1+\log(N/n)}{n} \right)^\alpha .$$

Observing that $(1+\log\lambda)/\lambda \to 0$ for $\lambda \to \infty$, there exists some $\lambda_0 > 1$ such that

$$c_{\lceil \frac{n}{\lambda_0} \rceil}(id\colon \ell_p^N \to \ell_2^N) \ge C_2 \left(\frac{1+\log(N/n)}{n} \right)^\alpha$$

holds for all $n \in \mathbb{N}$ with $\log N \le n \le N$ and some constant $C_2 > 0$ independent of n and N.

Let now k be a natural number with $\log N \le k \le N/\lambda_0$ and put $n = \lfloor \lambda_0(k-1)+1 \rfloor$. Then $n \le \lambda_0 k \le N$ and $\lceil n/\lambda_0 \rceil = k$. By monotonicity of the function $x \to (1+\log(N/x))/x$ we therefore obtain

$$\begin{aligned} c_k(id\colon \ell_p^N \to \ell_2^N) &\ge C_2 \left(\frac{1+\log(N/n)}{n} \right)^\alpha \ge C_2 \left(\frac{1+\log(N/(\lambda_0 k))}{\lambda_0 k} \right)^\alpha \\ &\ge \frac{C_2}{\lambda_0^\alpha (1+\log\lambda_0)^\alpha} \left(\frac{1+\log(N/k)}{k} \right)^\alpha = C_3 \left(\frac{1+\log(N/k)}{k} \right)^\alpha . \end{aligned}$$

This proves the lower bound in (4.1) for all $k \in \mathbb{N}$ with $\log N \le k \le N/\lambda_0$.

It remains to prove (4.1) for $n < \log N$ and for $N/\lambda_0 \le n \le N$. If $n < \log N$, then the claim follows from $c_n(id\colon \ell_p^N \to \ell_2^N) \ge c_{\lceil \log N \rceil}(id\colon \ell_p^N \to \ell_2^N)$. Finally, for $N/\lambda_0 \le n \le N$ we use

$$\begin{aligned} c_n(id\colon \ell_p^N \to \ell_2^N) &\ge c_N(id\colon \ell_p^N \to \ell_2^N) = \inf_{\substack{M \subset\subset \ell_p^N \\ \mathrm{codim} M < N}} \sup_{\substack{x \in M \\ \|x\|_p \le 1}} \|x\|_2 = \inf_{\substack{M' \subset\subset \ell_p^N \\ \dim M' = 1}} \sup_{\substack{x \in M' \\ \|x\|_p \le 1}} \|x\|_2 \\ &= \inf_{x \in \ell_p^N, x \ne 0} \frac{\|x\|_2}{\|x\|_p} = \|id\colon \ell_2^N \to \ell_p^N\|^{-1} = \left(\frac{1}{N} \right)^{\frac{1}{p} - \frac{1}{2}} . \end{aligned}$$

□

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