# Weak and quasi-polynomial tractability of approximation of infinitely differentiable functions

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January 15, 2013

**Abstract**

We comment on recent results in the field of information based complexity, which state (in a number of different settings), that approximation of infinitely differentiable functions is intractable and suffers from the curse of dimensionality. We show that renorming the space of infinitely differentiable functions in a suitable way allows weakly tractable uniform approximation by using only function values. Moreover, the approximating algorithm is based on a simple application of Taylor's expansion at the center of the unit cube. We discuss also the approximation on the Euclidean ball and the approximation in the $L_1$-norm, which is closely related to the problem of numerical integration.

**Key words:** weak tractability, uniform approximation, infinitely differentiable functions, curse of dimensionality, numerical integration

## 1 Introduction

We consider different classes $F_d$ of infinitely-differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ and discuss algorithms using only function values of $f$ in order to approximate $f$ uniformly, or in the $L_1$-norm. We are especially interested in the case of large $d \gg 1$.

In the classical setting of approximation theory, the dimension of the Euclidean space $d$ is fixed. Furthermore, the decay of the minimal error $e(n)$ of approximation of smooth functions in the Lebesgue space norm is very well studied for both algorithms using $n$ arbitrary linear functionals and for algorithms using only $n$ function evaluations. We refer to [5, 6, 7, 12, 14, 16, 17] and references therein. The decay is usually polynomial, speeds up with increasing smoothness and slows down with increasing dimension. Furthermore, this terminology typically hides the dependence of the constants on the dimension $d$, which might be even exponential. This motivates the question, what happens if both the dimension $d$ and the smoothness parameter $s$ tend to infinity.

If $n(\varepsilon, d)$ denotes the minimal number of function values needed to approximate all functions from $F_d$ up to the error $\varepsilon > 0$, we say, that the problem suffers by *curse of dimensionality*, if $n(\varepsilon, d)$ grows exponentially in $d$. This means, that there are positive numbers $c, \varepsilon_0$ and $\gamma$, such that

$$n(\varepsilon, d) \geq c(1+\gamma)^d \quad \text{for all} \quad 0 < \varepsilon \leq \varepsilon_0 \quad \text{and infinitely many} \quad d \in \mathbb{N}.$$

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†Research supported by the DFG Research Center Matheon "Mathematics for key technologies" in Berlin.

On the other hand, we say that the problem is *weakly tractable* if

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon,d)}{\varepsilon^{-1}+d} = 0.$$

Furthermore, the problem is *quasi-polynomially tractable* in the sense of [1], if there exist two constants $C, t > 0$, such that

$$n(\varepsilon,d) \le C \exp\{t(1+\ln(1/\varepsilon))(1+\ln(d))\} \tag{1}$$

for all $0 < \varepsilon < 1$ and all $d \in \mathbb{N}$. For the sake of completeness, we add that a problem is *polynomially tractable* if there exist non-negative numbers $C, p$ and $q$ such that

$$n(\varepsilon,d) \le C\varepsilon^{-p}d^q \quad \text{for all} \quad 0<\varepsilon<1 \quad \text{and} \quad d\in\mathbb{N}.$$

If $q = 0$ above then a problem is *strongly polynomially tractable*. We refer to the monographs [15, 8] and [11] for a detailed discussion of these and other arts of *(in)tractability* and the closely related field of *information based complexity*.

The $L_\infty$-approximation of infinitely differentiable functions was studied in [4], where the authors showed, that the problem is not strongly polynomially tractable. It was also discussed in [8], cf. Open Problem 2 therein. An essential breakthrough was achieved in [10] (which in turn is based on [9] and answers an open problem posed there), where uniform approximation of the functions from the class

$$\mathbb{F}_d = \{f : [0,1]^d \to \mathbb{R} : \sup_\alpha \|D^\alpha f\|_\infty \le 1\}$$

was shown to satisfy $n(\varepsilon,d) \ge 2^{\lfloor d/2 \rfloor}$ for all $0 < \varepsilon < 1$ and all $d \in \mathbb{N}$. Hence, the problem is intractable and suffers the curse of dimensionality. In the context of weighted spaces of infinitely differentiable functions, the problem was also discussed in [18].

Multivariate integration of infinitely differentiable functions from the class $\mathbb{F}_d$ was conjectured not to be polynomially tractable in [20] and was shown not to be strongly polynomially tractable in [19]. Furthermore, it is known (cf. [13] and [2]) that multivariate integration of functions from

$$C_d^k = \{f : [0,1]^d \to \mathbb{R} : \sup_{\alpha:|\alpha|\le k} \|D^\alpha f\|_\infty \le 1\}$$

suffers the curse of dimensionality for all $k \in \mathbb{N}$. Although multivariate integration of infinitely differentiable functions was also discussed in [3], it seems to be still an open problem if the curse of dimensionality holds also for multivariate integration and the class $\mathbb{F}_d$.

The main result of this paper is the following.

**Theorem 1.** *(i) Uniform approximation on the cube $[-1/2,1/2]^d$ of functions from the class*

$$F_d^1 = \Big\{f \in C^\infty([-1/2,1/2]^d) : \sup_{k\in\mathbb{N}_0} \sum_{|\beta|=k} \frac{\|D^\beta f\|_\infty}{\beta!} \le 1\Big\} \tag{2}$$

*is quasi-polynomially tractable.*

*(ii) Uniform approximation on the balls $B(0,r_d) = \{x \in \mathbb{R}^d : \|x\|_2 \le r_d\}$, where $r_d$ are chosen in such a way, that the volume of $B(0,r_d)$ is equal to one, and the functions are from the class*

$$F_d^2 = \{f \in C^\infty(B(0,r_d)) : \sup_{k\in\mathbb{N}_0} \|\partial_\nu^k f\|_\infty \le 1\} \tag{3}$$

*is weakly tractable. Here, $(\partial_\nu^k f)(x)$ denotes the $k$-th derivative of $f$ at $x \neq 0$ in the "normal" direction $x/\|x\|_2$.*

*(iii) Approximation in the $L_1$-norm on $B(0,r_d)$, where $r_d$ are as above and the functions are from the class*

$$F_d^3=\Big\{f\in C^\infty(B(0,r_d)):\sup_{k\in\mathbb{N}_0}\int_0^{r_d}S(\partial_\nu^k f,r)dr\le 1\Big\}\tag{4}$$

*is weakly tractable. Here $S(\partial_\nu^{k+1}f,r)$ are the averages of $|\partial_\nu^{k+1}f|$ on the sphere $r\mathbb{S}^{d-1}$.*

Our method is rather simple and involves only the Taylor's expansion of a smooth function at the center of the domain under consideration. The rest of the paper is devoted to the proof of this theorem.

# 2 Uniform approximation

We study first the uniform approximation of infinitely differentiable functions $f$ using function values of $f$. This goes in line with the paper of Novak and Woźniakowski [10]. It turns out that modifying the norm in a suitable way leads immediately to weak (and even quasi-polynomial) tractability, using only Taylor's theorem in the middle of the cube or ball, respectively.

Let us recall the standard multivariate notation, which we shall use with connection to the Taylor's theorem. If $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{N}_0^d$ is a multiindex, we denote

$$\begin{aligned}|\alpha|&=\alpha_1+\dots+\alpha_d,\\ D^\alpha f(x)&=\frac{\partial^{|\alpha|}f(x)}{\partial x_1^{\alpha_1}\dots\partial x_d^{\alpha_d}},\quad x\in\mathbb{R}^d,\\ \alpha!&=\alpha_1!\dots\alpha_d!,\\ x^\alpha&=x_1^{\alpha_1}\dots x_d^{\alpha_d},\quad x\in\mathbb{R}^d.\end{aligned}$$

## 2.1 Unit cube

In this part, we study uniform approximation of infinitely differentiable functions on $[-1/2,1/2]^d$. Our algorithm is based on Taylor's formula

$$f(x)=(T_kf)(x)+\sum_{|\beta|=k+1}(R_\beta f)(x)x^\beta,$$

where

$$\begin{aligned}(T_kf)(x)&=\sum_{|\alpha|\le k}\frac{(D^\alpha f)(0)}{\alpha!}x^\alpha,\\ (R_\beta f)(x)&=\frac{|\beta|}{\beta!}\int_0^1(1-t)^{|\beta|-1}(D^\beta f)(tx)dt.\end{aligned}$$

The approximation of $f$ by $T_kf$ at the point $x\in[-1/2,1/2]^d$ results into an error

$$\begin{aligned}|f(x)-T_kf(x)|&\le\sum_{|\beta|=k+1}|R_\beta f(x)|\cdot|x^\beta|\\ &\le\sum_{|\beta|=k+1}\frac{(k+1)\cdot|x^\beta|}{\beta!}\int_0^1(1-t)^k|(D^\beta f)(tx)|dt\\ &\le\Big(\frac12\Big)^{k+1}\sum_{|\beta|=k+1}\frac{(k+1)}{\beta!}\int_0^1(1-t)^kdt\cdot\|D^\beta f\|_\infty\end{aligned}$$

$$= \left(\frac{1}{2}\right)^{k+1} \sum_{|\beta|=k+1} \frac{\|D^\beta f\|_\infty}{\beta!}.$$

Hence,

$$\sup_{f \in F_d^1} \|f - T_k f\|_\infty \leq \left(\frac{1}{2}\right)^{k+1}$$

for every $k \in \mathbb{N}_0$, where $F_d^1$ was defined in (2). To discuss the tractability of the problem, we need to estimate also the number of points needed to recover $T_k f$. As we are allowed to take only samples of $f$, and not of its derivatives, we are actually not able to recover $T_k f$ exactly. But using finite order differences, we may approximate it to an arbitrary precision using a bounded number of points.

To estimate the number of sampling points needed to approximate all derivatives up to the order $k$, we use induction. To evaluate $f(0)$ we need to sample $f$ at the point $A_0 = \{0\}$. To calculate the first order differences, we need the points from the set

$$A_1 = A_0 \cup \bigcup_{j=1}^{d} (A_0 + h e_j),$$

which has $d+1$ points. For the second order differences, we need the values at

$$A_2 := A_1 \cup \bigcup_{j=1}^{d} (A_1 + h e_j) = \{h(\alpha_1, \dots, \alpha_d) : |\alpha| \leq 2\},$$

which has $\binom{d+2}{2}$ points. By induction we obtain, that to evaluate all finite order differences up to the order $k$, we need the values at

$$A_k = \{h(\alpha_1, \dots, \alpha_d) : |\alpha| \leq k\}$$

and that this set has

$$\sum_{j=0}^{k} \binom{d+j-1}{j} = \binom{d+k}{k}$$

points.

To show that the problem is weakly tractable, we proceed in the following way. Given an $1 > \varepsilon > 0$, we choose the smallest $k \in \mathbb{N}_0$, such that

$$\varepsilon \geq \left(\frac{1}{2}\right)^{k+1},$$

i.e.

$$k + 1 := \left\lceil \frac{\ln(1/\varepsilon)}{\ln 2} \right\rceil,$$

where $\lceil a \rceil$ denotes the smallest integer, which is larger then or equal to the real number $a \in \mathbb{R}$. Together with the estimate

$$n(\varepsilon, d) \leq \binom{d+k}{k} \leq \left(\frac{e(d+k)}{k}\right)^k,$$

this gives that there is $t > 0$, such that

$$\ln n(\varepsilon, d) \leq k(1 + \ln(d+k) - \ln k) \leq t(1 + \ln(1/\varepsilon))(1 + \ln d).$$

This immediately implies (1), i.e. the quasi-polynomial tractability.

## 2.2 Euclidean ball

In this section, we discuss the uniform approximation of an infinitely differentiable function $f$ on an Euclidean ball in $\mathbb{R}^n$ with radius $r_d > 0$. As all the sets of infinitely differentiable functions under consideration include all constant functions with values between -1 and 1, the initial error of approximation is always 1 for every sequence $(r_d)_{d\in\mathbb{N}}$. Nevertheless, we shall be interested at most in the case, when the Lebesgue measure of $B(0, r_d) := \{x \in \mathbb{R}^d : \|x\|_2 \le r_d\}$ is one. It is very well known, cf. [2], that $r_d \approx \sqrt{d}$ in this case, i.e. there are two absolute constants $C > c > 0$, such that $c\sqrt{d} \le r_d \le C\sqrt{d}$.

We start again with Taylor's expansion of an infinitely differentiable function $f$ at zero. Let $x \in B(0, r_d), x \neq 0$ and let $g_x(t) = f(tx), 0 \le t \le 1$. Then

$$f(x) = g_x(1) = \sum_{j=0}^{k} \frac{g_x^{(j)}(0)}{j!} + \frac{1}{k!}\int_0^1 (1-t)^k g_x^{(k+1)}(t)dt. \tag{5}$$

Iterating the formula

$$g_x'(t) = \langle (\nabla f)(tx), x\rangle = \langle (\nabla f)(tx), \frac{x}{\|x\|_2}\rangle \|x\|_2 = (\partial_\nu f)(tx)\cdot \|x\|_2$$

we obtain

$$g_x^{(k+1)}(t) = (\partial_\nu^{k+1} f)(tx)\cdot \|x\|_2^{k+1}, \qquad 0 < t < 1,$$

where $(\partial_\nu f)(x)$ denotes the derivative of $f$ at $x \neq 0$ in the direction $x/\|x\|_2$.

We denote

$$\widetilde{T}_k f(x) = \sum_{j=0}^{k} \frac{g_x^{(j)}(0)}{j!},$$

which allows to estimate the error of approximation of $f(x)$ by the Taylor's polynomial at $x$ by

$$\begin{aligned} |f(x) - \widetilde{T}_k f(x)| &\le \frac{\|x\|_2^{k+1}}{k!}\int_0^1 (1-t)^k \cdot |(\partial_\nu^{k+1} f)(tx)|dt \\ &\le \frac{r_d^{k+1}\|\partial_\nu^{k+1} f\|_\infty}{(k+1)!} \le \frac{r_d^{k+1}}{(k+1)!}, \end{aligned}$$

if $f \in F_d^2$, where $F_d^2$ is as in (3).

Finally, we observe that we need again $\binom{d+k}{k}$ points to approximate the derivatives $D^\alpha f(0)$ for every $|\alpha| \le k$, which again allows to approximate all the derivatives $g_x^{(j)}(0)$ for all $x \in B(0, r_d)$ and all $0 \le j \le k$.

Hence, if $d \in \mathbb{N}$ is fixed and $1 > \varepsilon > 0$ is given, we chose first the smallest $k$, for which

$$\varepsilon \ge \frac{r_d^{k+1}}{(k+1)!}. \tag{6}$$

This is always possible, as the right hand side goes to zero for $d$ fixed and $k \to \infty$. On the other side, let us mention that if (6) holds for $\varepsilon < 1$, then $k$ is at least of the order $r_d$. Using the estimate

$$\left(\frac{er_d}{k+1}\right)^{k+1} \ge \frac{r_d^{k+1}}{(k+1)!}$$

we obtain that if $r_d \approx \sqrt{d}$, then (6) is satisfied any time we have

$$k \ge \max(c_1\sqrt{d}, c_2 \ln(1/\varepsilon))$$

for some absolute constants $c_1, c_2 > 0$. This finally implies the weak tractability of the problem by

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon,d)}{\varepsilon^{-1}+d} \le \lim_{\varepsilon^{-1}+d\to\infty} \frac{k(1+\ln(d+k)-\ln k)}{\varepsilon^{-1}+d} \le \lim_{\varepsilon^{-1}+d\to\infty} \frac{k\ln(d+k)}{\varepsilon^{-1}+d} = 0.$$

Unfortunately, the calculation above does not give quasi-polynomial tractability in this case.

# 3 Numerical integration

Finally, we discuss a closely related problem of tractable integration of infinitely differentiable functions. Of course, on a domain with volume one, the error of approximation in $L_1$-norm may be bounded from above by the error of uniform approximation. Therefore, the problem is weakly tractable for the $F_d^2$ class considered above. Using again (5) for all $x \in B(0,r_d), x \neq 0$, we obtain

$$\begin{aligned}
\int_{B(0,r_d)} |f(x) - \widetilde{T}_k f(x)| dx &\le \int_{B(0,r_d)} \frac{1}{k!} \|x\|_2^{k+1} \int_0^1 (1-t)^k |(\partial_\nu^{k+1} f)(tx)| dt dx \\
&\le \frac{1}{k!} \int_{0<t<1} \int_{0<\|y\|_2<tr_d} \|y/t\|_2^{k+1} (1-t)^k |(\partial_\nu^{k+1} f)(y)| t^{-d} dy dt \\
&= \frac{1}{k!} \int_{B(0,r_d)} \|y\|_2^{k+1} \cdot |(\partial_\nu^{k+1} f)(y)| \cdot \int_{\|y\|_2/r_d}^1 t^{-(k+1)} (1-t)^k t^{-d} dt dy \\
&\le \frac{1}{k!} \int_{B(0,r_d)} \|y\|_2^{k+1} \cdot |(\partial_\nu^{k+1} f)(y)| \cdot \int_{\|y\|_2/r_d}^1 t^{-(k+d+1)} dt dy \\
&\le \frac{1}{k!} \int_{B(0,r_d)} \|y\|_2^{k+1} \cdot |(\partial_\nu^{k+1} f)(y)| \cdot \frac{1}{k+d} \cdot \Big(\frac{\|y\|_2}{r_d}\Big)^{-k-d} dy \\
&\le \frac{1}{k!} \cdot \frac{r_d^{k+d}}{k+d} \int_{B(0,r_d)} \|y\|_2^{1-d} \cdot |(\partial_\nu^{k+1} f)(y)| dy \\
&= \frac{1}{k!} \cdot \frac{r_d^{k+d}}{k+d} \int_0^{r_d} r^{1-d} \int_{r\mathbb{S}^{d-1}} |(\partial_\nu^{k+1} f)(y)| d\sigma(y) dr,
\end{aligned}$$

where $\sigma$ is the $d-1$ dimensional Hausdorff measure in $\mathbb{R}^d$ and $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$ is the unit sphere in $\mathbb{R}^d$. If we denote by $\omega_{d-1}$ the surface of $\mathbb{S}^{d-1}$, i.e. $\omega_{d-1} = \sigma(\mathbb{S}^{d-1})$, and by

$$S(\partial_\nu^{k+1} f, r) = \frac{1}{\omega_{d-1} r^{d-1}} \int_{r\mathbb{S}^{d-1}} |\partial_\nu^{k+1} f(y)| d\sigma(y)$$

the averages of $|\partial_\nu^{k+1} f|$ on the sphere $r\mathbb{S}^{d-1}$, we obtain

$$\int_{B(0,r_d)} |f(x) - \widetilde{T}_k f(x)| dx \le \frac{1}{k!} \cdot \frac{r_d^{k+d}}{k+d} \cdot \omega_{d-1} \int_0^{r_d} S(\partial_\nu^{k+1} f, r) dr.$$

Assuming finally, that the volume of the $B(0,r_d)$ is equal to $\omega_{d-1} r^d/d = 1$, we may further reduce this to

$$\int_{B(0,r_d)} |f(x) - \widetilde{T}_k f(x)| dx \le \frac{r_d^k}{k!} \int_0^{r_d} S(\partial_\nu^{k+1} f, r) dr,$$

which is smaller then $r_d^k/k!$ for every $f \in F_d^3$, cf. (4), or smaller then $r_d^{k+1}/k!$ for every $f \in F_d^4$, where

$$F_d^4 = \Big\{f \in C^\infty(B(0,r_d)) : \sup_{k\in\mathbb{N}_0} \sup_{0<r\le r_d} S(\partial_\nu^{k+1} f, r) \le 1\Big\}.$$

The proof of weak tractability follows in both cases from these estimates exactly as in Section 2.2.

**Remarks:** (i) We have provided only the upper bounds on $n(\varepsilon,d)$, which in turn led to tractability results for the classes $F_d^1, F_d^2$, and $F_d^3$, respectively. It would be interesting to know, if these results are optimal. For example, we do not know if uniform approximation of the class $F_d^1$ is polynomially tractable or not. We leave this as an open problem.
(ii) One could also modify the classes under consideration in a way used recently in [3]. This approach uses a sequence $L = (L_d)_{d\in\mathbb{N}}$ to define, for example,

$$F_d^2(L) = \{f \in C^\infty(B(0,r_d)) : \sup_{k\in\mathbb{N}_0} \|\partial_\nu^k f\|_\infty \le L_d\}.$$

Of course, the calculations given above could be to some extend transferred also to this setting, and we could really prove similar results for sequences which do not grow too quickly. Unfortunately, due to the lack of lower estimates, we could not hope for being able to *characterize* sequences $L$, for which weak (or quasi-polynomial) tractability still holds.

**Acknowledgment:** I would like to thank to Aicke Hinrichs, Erich Novak, Mario Ullrich, Markus Weimar and Henryk Woźniakowski for useful comments, remarks and hints to an earlier version of the manuscript.

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