Complex Interpolation of Weighted Besov- and Lizorkin-Triebel Spaces

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Abstract

We study complex interpolation of weighted Besov and Lizorkin-Triebel spaces. The used weights w_0, w_1 are local Muckenhoupt weights in the sense of Rychkov. As a first step we calculate the Calderón products of associated sequence spaces. Finally, as a corollary of these investigations, we obtain results on complex interpolation of radial subspaces of Besov and Lizorkin-Triebel spaces on $\mathbb{R}^d$.

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1 Introduction

Nowadays interpolation theory has been established as an important tool in various branches of mathematics, in particular in analysis of PDE's. Within the known interpolation methods the complex interpolation method of Calderón, denoted by $[\,\cdot\,,\,\cdot\,]_\Theta$, is of particular importance and probably the most often used one.

Let $L_p(\mathbb{R}^d, w)$ denote the weighted Lebesgue space with weight w. Here in this paper we study generalizations of the following formula

$$\Big[L_{p_0}(\mathbb{R}^d, w_0), L_{p_1}(\mathbb{R}^d, w_1)\Big]_\Theta \quad = \quad L_p(\mathbb{R}^d, w)\,, \qquad 1 \le p_0, p_1 < \infty\,, \tag{1}$$

$$\frac{1}{p} \quad := \quad \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}\,, \qquad w := w_0^{(1-\Theta)\,p/p_0}\, w_1^{\Theta\, p/p_1}\,, \tag{2}$$

see, e.g., [1, Thm. 5.5.3] or [2, Thm. 1.18.5]. We shall replace the weighted Lebesgue spaces $L_p(\mathbb{R}^d, w)$ by weighted Besov and Lizorkin-Triebel spaces. There are already some contributions dealing with this problem. Let us mention here Bownik [3] and Wojciechowska [4]. But both authors only deal with the case $w = w_0 = w_1$. Bownik considers weights related to doubling measures and Wojciechowska is dealing with local Muckenhoupt weights (as we shall do in most of the cases). Whereas in (1) it will be enough that w_0 and w_1 are positive, in the generalizations, we have in mind, it is not clear what is the correct class of weights. It seems that necessary conditions concerning the weights are not known in this context.

To calculate

$$\Big[F^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0),\, F^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)\Big]_\Theta \qquad \text{and} \qquad \Big[B^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0),\, B^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)\Big]_\Theta \tag{3}$$

we shall apply a method which has been used by Bownik [3] and Wojciechowska [4] as well. First we shall calculate the Calderón products of associated sequence spaces. Afterwards we shall use the known coincidence of Calderón products and the complex method of interpolation (under certain extra conditions) to shift the results to the complex interpolation of these sequence spaces. Finally, these results are lifted by wavelet isomorphisms to the level of function spaces. This method has been developed by Calderón [5], Frazier, Jawerth [6], Mendez, Mitrea [7] and Kalton, Maybororda, Mitrea [8]. The latter two references are connected with the extension of the complex method to certain quasi-Banach spaces. Also in our paper we shall work with quasi-Banach spaces. In fact, we will allow the maximal range of the parameters in (3) with the exception of $F^s_{\infty,q}(\mathbb{R}^d, w)$. Of course, it would be interesting to incorporate these spaces as well but this requires additional effort.

The paper is organized as follows. In Section 2 we recall the basic notions for the complex method and also describe the state of the art in the unweighted case. The next section is devoted to the calculation of the Calderón products of some weighted sequence spaces. For $w_0, w_1 \in \mathcal{A}_\infty^{\ell oc}$ we define

$$w(x) := w_0(x)^{\frac{(1-\Theta)p}{p_0}}\, w_1(x)^{\frac{\Theta p}{p_1}}\,, \qquad x \in \mathbb{R}^d\,.$$

Then we will establish the formulas

$$f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} = f^{s}_{p,q}(\mathbb{R}^d, w)$$

as well as

$$b^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, b^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} = b^{s}_{p,q}(\mathbb{R}^d, w)$$

under rather general conditions on the parameters. This will be the most complicated part. In Section 4 we deal with complex interpolation of weighted Besov and Lizorkin-Triebel spaces. On the one side we simply shift here the results, obtained for Calderón products, to complex interpolation formulas, on the other side we apply the result of Shestakov [9] to calculate (3) also in some of those situations where both spaces are not separable. Finally, in Section 5 we apply the results obtained before to derive some complex interpolation formulas for radial subspaces of Besov and Lizorkin-Triebel spaces. This was actually the original motivation for our work.
Definitions and some properties of the classes of weights and classes of function spaces under consideration here are collected in the Appendix at the end of this paper.

Notation

As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ denotes the integers and $\mathbb{R}$ the real numbers. For the complex numbers we use the symbol $\mathbb{C}$, for the Euclidean d-space we use $\mathbb{R}^d$ and $\mathbb{Z}^d$ denotes the collection of all elements in $\mathbb{R}^d$ having integer components. At very few places we shall need the Fourier transform $\mathcal{F}$ as well as its inverse transformation $\mathcal{F}^{-1}$, always defined on the Schwartz space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions.
If X and Y are two quasi-Banach spaces, then the symbol $X \hookrightarrow Y$ indicates that the embedding is continuous. As usual, the symbol c denotes positive constants which depend only on the fixed parameters s, p, q and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line. Sometimes we will use the symbols "$\lesssim$" and "$\gtrsim$" instead of "$\le$" and "$\ge$", respectively. The meaning of $A \lesssim B$ is given by: there exists a constant $c > 0$ such that $A \le c\,B$. Similarly $\gtrsim$ is defined. The symbol $A \asymp B$ will be used as an abbreviation of $A \lesssim B \lesssim A$.
Inhomogeneous weighted Besov and Lizorkin-Triebel spaces are denoted by $B^s_{p,q}(\mathbb{R}^d, w)$

and $F^s_{p,q}(\mathbb{R}^d, w)$, respectively. If the weight is identically 1, we shall drop w in notation. In case there is no reason to distinguish between these two scales we will use the notation $A^s_{p,q}(\mathbb{R}^d, w)$. Definitions, properties as well as some references are given in the Appendix.

2 Complex interpolation of Besov and Lizorkin-Triebel spaces - the state of the art

For convenience of the reader we recall some notions from interpolation theory as well as some results in the framework of Besov and Lizorkin-Triebel spaces.
For the basics of interpolation theory we refer to the monographs [1, 2, 10, 11].

2.1 The complex method of interpolation

The complex method in case of interpolation couples of Banach spaces is a well-studied subject, see the quoted monographs above. Here we are interested in the complex method in case of interpolation couples of certain quasi-Banach spaces. For that reason we give some details. We follow [8], see also [7].

Definition 1 *A quasi-Banach space $(X, \| \cdot |X\|)$ is called analytically convex if there is a constant C such that for every polynomial $P : \mathbb{C} \to X$ we have*

$$\|P(0)|X\| \le C \max_{|z|=1} \|P(z)|X\|.$$

In the framework of analytically convex quasi-Banach spaces the Maximum Modulus Principle holds. Let

$$S_0 := \{z \in \mathbb{C} :\ 0 < \Re e\, z < 1\} \qquad \text{and} \qquad S := \{z \in \mathbb{C} :\ 0 \le \Re e\, z \le 1\}\,.$$

Proposition 2 *For a quasi-Banach space $(X, \| \cdot |X\|)$ the following conditions are equivalent:*
(i) *X is analytically convex.*
(ii) *There exists a constant C such that*

$$\max\{\|f(z)|X\| : z \in S_0\} \ \le\ C\ \max\{\|f(z)|X\| : z \in S \setminus S_0\}$$

for any function $f : S \to X$, analytic on S_0 and continuous and bounded on S.

We refer to [8, Thm. 7.4]. Based on this property the following definition makes sense.

Definition 3 *Let (X_0, X_1) be an interpolation couple of quasi-Banach spaces, i.e., X_0 and X_1 are continuously embedded into a larger topological vector space Y, and $X_0 \cap X_1$ is dense in X_j, $j = 0, 1$. In addition, let $X_0 + X_1$ be analytically convex. Let $\mathcal{A}$ be the set of all bounded and analytic functions $f : S_0 \to X_0 + X_1$, which extend continuously to the closure S of the strip s.t. the traces $t \mapsto f(j + it)$ are bounded continuous functions into X_j, $j = 0, 1$. We endow $\mathcal{A}$ with the quasi-norm*

$$\| f \,|\mathcal{A}\| := \max \Big\{ \sup_{t\in\mathbb{R}} \| f(it) \,|X_0\|, \sup_{t\in\mathbb{R}} \| f(1 + it) \,|X_0\| \Big\} .$$

Let $0 < \Theta < 1$. Further, we define $[X_0, X_1]_\Theta$ to be the set of all $x \in \mathcal{A}(\Theta) := \{f(\Theta) : f \in \mathcal{A}\}$ and

$$\| x \,|[X_0, X_1]_\Theta\| := \inf \Big\{ \| f \,|\mathcal{A}\| : \ f(\Theta) = x \Big\} .$$

Remark 1 Any Banach space is analytically convex. Hence, if (X_0, X_1) is an interpolation couple of Banach spaces, this reduces to the standard definition of $[X_0, X_1]_\Theta$.

Lemma 4 *Let $0 < q \le \infty$, $s \in \mathbb{R}$ and $w \in A_\infty^{\ell oc}$.*
(i) Let $0 < p < \infty$. Then $F^s_{p,q}(\mathbb{R}^d, w)$ is analytically convex.
(ii) Let $0 < p \le \infty$. Then $B^s_{p,q}(\mathbb{R}^d, w)$ is analytically convex.

Proof. If X is an analytically convex quasi-Banach space and Y is a closed subspace of Y then Y is analytically convex, see Prop. 7.5 in [8]. By means of Prop. 37 it will be enough to prove analytic convexity for the sequence spaces $f^s_{p,q}(\mathbb{R}^d, w)$ and $b^s_{p,q}(\mathbb{R}^d, w)$. In contrast to the function spaces the sequence spaces are quasi-Banach lattices. We need a further notion. A quasi-Banach lattice of functions $(X, \| \cdot |X\|)$ is called *lattice r-convex* if

$$\Big\| \Big(\sum_{j=1}^m |f_j|^r\Big)^{1/r} |X\Big\| \le \Big(\sum_{j=1}^m \|f_j|X\|^r\Big)^{1/r}$$

for any finite family $\{f_j\}_{1\le j\le m}$ of functions from X.
There are simple criteria for a quasi-Banach lattice of functions to be analytically convex, see [8, Thm. 7.8]: X is analytically convex if, and only if, X is lattice r-convex for some $r > 0$.
It remains to show that the sequence spaces $a^s_{p,q}(\mathbb{R}^d, w)$, $a \in \{b, f\}$, are lattice r-convex. This holds with $r \le \min(p, q, 1)$ by standard arguments (use the generalized Minkowski inequality). ■

Remark 2 The unweighted case was considered in Mendez and Mitrea [7], see also Kalton, Mayboroda and Mitrea [8, Prop. 7.7]. For weights related to doubling measures the statement has been settled by Bownik [3].

2.2 The state of the art

For convenience of the reader we recall what is known in the unweighted situation. Comments to the weighted case will be given within the text.

Proposition 5 *Let* $0 < p_0, p_1 \le \infty$, $0 < q_0,\, q_1 \le \infty$, $s_0, s_1 \in \mathbb{R}$, *and* $0 < \Theta < 1$. *Define*

$$s := (1-\Theta)\, s_0 + \Theta\, s_1\, , \qquad \frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \qquad \textit{and} \qquad \frac{1}{q} := \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}\, . \tag{4}$$

(i) *Let* $\min(q_0, q_1) < \infty$. *Then we have*

$$B^s_{p,q}(\mathbb{R}^d) = \Big[B^{s_0}_{p_0,q_0}(\mathbb{R}^d), B^{s_1}_{p_1,q_1}(\mathbb{R}^d)\Big]_\Theta\, . \tag{5}$$

(ii) *Let* $\max(p_0, p_1) < \infty$ *and* $\min(q_0, q_1) < \infty$. *Then we have*

$$F^s_{p,q}(\mathbb{R}^d) = \Big[F^{s_0}_{p_0,q_0}(\mathbb{R}^d), F^{s_1}_{p_1,q_1}(\mathbb{R}^d)\Big]_\Theta\, . \tag{6}$$

Remark 3 (i) Prop. 5 in this generality can be found in Frazier and Jawerth [6] (the F-case) and in Kalton, Mayboroda, Mitrea [8, Thm. 9.1](F- and B-case). With the extra condition $s_0 \neq s_1$ one can find (5) also in Mendez, Mitrea [7]. However, Prop. 5 has many forerunners in case $\min(p_0, p_1, q_0, q_1) \ge 1$, e.g., Calderón, J.L. Lions, Magenes, Taibleson, Grisvard, Schechter, Peetre and Triebel. We refer to [1, 12, 2] and the references given there.
(ii) Let us mention, that formula (6) remains true in case if either $\max(p_0, q_0) < \infty$ or $\max(p_1, q_1) < \infty$, see [6] and [8]. However, in our paper we shall not deal with the spaces $F^s_{\infty,q}(\mathbb{R}^d)$ and its weighted counterparts.
(iii) The counterpart of (6) for anisotropic Lizorkin-Triebel spaces (more exactly, the generalization to) has been proved by Bownik [3].
(iv) It is of certain interest to notice that in some cases complex interpolation of pairs of Besov spaces does not result in a Besov space. More exactly, if $1 < p < \infty$, $s_0, s_1 \in \mathbb{R}$, $s_0 \neq s_1$, $s := (1-\Theta)\, s_0 + \Theta\, s_1$, then

$$\mathring{B}^s_{p,\infty}(\mathbb{R}^d) = \Big[B^{s_0}_{p,\infty}(\mathbb{R}^d), B^{s_1}_{p,\infty}(\mathbb{R}^d)\Big]_\Theta\, , \tag{7}$$

where $\mathring{B}^s_{p,\infty}(\mathbb{R}^d)$ denotes the closure of the set of test functions in $B^s_{p,\infty}(\mathbb{R}^d)$, a space strictly smaller than $B^s_{p,\infty}(\mathbb{R}^d)$. We refer to [2, Thm. 2.4.1]. Later on, see Subsection 4.3, we shall supplement this formula.
(v) In [1, Thm. 6.4.5] the following formula is claimed to be true:

$$B^s_{p,q}(\mathbb{R}^d) = \Big[B^{s_0}_{p_0,q_0}(\mathbb{R}^d), B^{s_1}_{p_1,q_1}(\mathbb{R}^d)\Big]_\Theta\, , \tag{8}$$

where $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, $s_0 \neq s_1$ and s, p, q as in (4). Of course, this is in contradiction with part (iv) in case $q = q_0 = q_1 = \infty$. We do not believe in this formula in cases, when both spaces $B^{s_0}_{p_0,q_0}(\mathbb{R}^d)$ and $B^{s_1}_{p_1,q_1}(\mathbb{R}^d)$ are not separable, see Subsection 4.3 for more information.
(vi) There is a number of further methods of interpolation where the outcome is known in case of pairs of either Besov or Lizorkin-Triebel spaces. Most prominent is the real method of interpolation. For corresponding results we refer to [1, Thm. 6.4.5] and [13, 2.4]. Triebel [13, 2.4] also had invented a certain modification of the complex method and has been able to prove the counterparts of (5), (6) for this modified complex method. However, it is not known whether this modified method has the interpolation property. Frazier, Jawerth [6] and Bownik [3] also studied the $\pm$-method of Gustaffson and Peetre, denoted by $\langle X_0, X_1, \theta\rangle$, and the method $\langle X_0, X_1\rangle_\theta$, due to Gagliardo. Then Prop. 5(ii) remains true also for these methods.

3 Calderón products of sequence spaces associated to weighted Besov and Lizorkin-Triebel spaces

After having introduced the necessary definitions in Subsection 3.1 we shall first deal with the Calderón products of the sequence spaces $f^s_{p,q}(\mathbb{R}^d, w)$ (oriented on the ingenious proof of Frazier and Jawerth in the unweighted situation). In the third subsection we shall investigate Calderón products of the sequence spaces $b^s_{p,q}(\mathbb{R}^d, w)$ by employing a totally different method.

3.1 Definition and basic properties of the Calderón product

Let $(\mathfrak{X}, \mathcal{S}, \mu)$ be a $\sigma-$finite measure space and let $\mathfrak{M}$ be the class of all complex–valued, $\mu-$measurable functions on $\mathfrak{X}$. Then a quasi-Banach space $X \subset \mathfrak{M}$ is called a *quasi-Banach lattice of functions* if for every $f \in X$ and $g \in \mathfrak{M}$ with $|g(x)| \leq |f(x)|$ for $\mu-$a.e. $x \in \mathfrak{X}$ one has $g \in X$ and $\|g|X\| \leq \|f|X\|$.

Definition 6 *Let $(\mathfrak{X}, \mathcal{S}, \mu)$ be a $\sigma-$finite measure space and let $\mathfrak{M}$ be the class of all complex–valued, $\mu-$measurable functions on $\mathfrak{X}$. Let $X_j \subset \mathfrak{M}$, $j = 0, 1$, be quasi-Banach lattices of functions. Let $0 < \Theta < 1$. Then the Calderón product $X_0^{1-\Theta} X_1^{\Theta}$ of X_0 and X_1 is the collection of all functions $f \in \mathfrak{M}$ s.t. the quasi-norm*

$$\|f|X_0^{1-\Theta}X_1^{\Theta}\| \ := \ \inf\Big\{\|f_0|X_0\|^{1-\Theta}\|f_1|X_1\|^{\Theta} : \, |f| \leq |f_0|^{1-\Theta}|f_1|^{\Theta} \quad \mu - a.e., \\ f_j \in X_j, \, j = 0, 1\Big\}$$

is finite.

Remark 4 (i) Calderón products have been introduced by Calderón [5, 13.5] (in a little bit different form which coincides with the above one). The usefulness of this method and its limitations have been perfectly described by Frazier and Jawerth [6] which we quote now: *Although restricted to the case of a lattice, the Calderón product has the advantage of being defined in the quasi-Banach case, and, frequently, of being easy to compute. It has the disadvantage that the interpolation property (i.e., the property that a linear transformation T bounded on X_0 and X_1 should be bounded on the spaces in between) is not clear in general.*
(ii) A further remark to the literature. Calderón products are not investigated in the most often quoted books on interpolation theory: Bergh and Löfström [1] (except a short remark on page 129), Triebel [2] and Bennett and Sharpley [11]. However, in the monographs of Kreĭn, Petunin and Semënov [10, pp. 242-246], Brudnyi and Kruglyak [14, 4.3] and in the lecture note of Maligranda [15] a few informations about Calderón products can be found, sometimes in the more general framework of Calderón-Lozanovskii constructions. All these references are concerned with Banach spaces. Since we need this concept in quasi-Banach spaces as well, we refer in addition to Nilsson [16], Frazier and Jawerth [6], Mendez and Mitrea [7], Kalton, Mayboroda and Mitrea [8] and Yang, Yuan and Zhuo [17].

We collect a few useful properties for later use, see [17].

Lemma 7 *Let $(\mathfrak{X}, \mathcal{S}, \mu)$ be a $\sigma-$finite measure space and let $\mathfrak{M}$ be the class of all complex–valued, $\mu-$measurable functions on $\mathfrak{X}$. Let $X_j \subset \mathfrak{M}$, $j = 0, 1$, be quasi-Banach lattices of functions. Let $0 < \Theta < 1$.*
(i) *Then the Calderón product $X_0^{1-\Theta} X_1^{\Theta}$ is a quasi-Banach space.*
(ii) *Define $\widetilde{X_0^{1-\Theta} X_1^{\Theta}}$ as the collection of all f s.t. there exist a positive real number λ and elements $g \in X_0$ and $h \in X_1$ satisfying*

$$|f| \le \lambda |g|^{1-\theta} \, |h|^{\theta} \, , \qquad \| \, g \, |X_0\| \le 1 \, , \qquad \| \, h \, |X_1\| \le 1 \, .$$

We put

$$\| \, f \, |\widetilde{X_0^{1-\Theta} X_1^{\Theta}}\| := \inf \Big\{ \lambda > 0 : \quad |f| \le \lambda |g|^{1-\theta} \, |h|^{\theta} \, , \qquad \| \, g \, |X_0\| \le 1 \, , \qquad \| \, h \, |X_1\| \le 1 \Big\} .$$

Then $\widetilde{X_0^{1-\Theta} \, X_1^{\Theta}} = X_0^{1-\Theta} \, X_1^{\Theta}$ follows with equality of quasi-norms.

Here is one well-known example of a Calderón product which can be easily calculated. Let $w : \mathbb{R}^d \to [0, \infty)$ be measurable and positive a.e.. The weighted Lebesgue space $L_p(\mathbb{R}^d, w)$ is the collection of all measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that

$$\| f \,| L_p(\mathbb{R}^d, w)\| := \Big(\int_{\mathbb{R}^d} |f(x)|^p \, w(x)\, dx \Big)^{1/p} < \infty\, .$$

In case $p = \infty$ we shall use the convention $L_\infty(\mathbb{R}^d, w) := L_\infty(\mathbb{R}^d)$, i.e., we always take $w \equiv 1$.

Lemma 8 *Let $0 < \Theta < 1$, $0 < p_0, p_1 \le \infty$ and let $w_j : \ \mathbb{R}^d \to [0,\infty)$, $j = 0, 1$, be measurable and positive a.e.. We define*

$$\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \qquad \textit{and} \qquad w := w_0^{\frac{(1-\Theta)p}{p_0}} \, w_1^{\frac{\Theta p}{p_1}}\, . \tag{9}$$

Then

$$L_{p_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, L_{p_1}(\mathbb{R}^d, w_1)^{\Theta} = L_p(\mathbb{R}^d, w)$$

with coincidence of the quasi-norms.

Proof. *Step 1.* Let $\max(p_0, p_1) < \infty$.
Substep 1. We prove $L_{p_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, L_{p_1}(\mathbb{R}^d, w_1)^{\Theta} \subset L_p(\mathbb{R}^d, w)$. Let $f \in L_{p_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, L_{p_1}(\mathbb{R}^d, w_1)^{\Theta}$. Then there exist f_0, f_1 s.t.

$$|f(x)| \le |f_0(x)|^{1-\Theta}\, |f_1(x)|^{\Theta} \qquad \text{a.e. in } \mathbb{R}^d$$

and $f_j \in L_{p_j}(\mathbb{R}^d, w_j)$, $j = 0, 1$. We employ this inequality together with Hölder's inequality and obtain

$$\begin{aligned} \Big(\int_{\mathbb{R}^d} |f(x)|^p \, w(x)\, dx \Big)^{1/p} &\le \Big(\int_{\mathbb{R}^d} |f_0(x)|^{(1-\Theta)p} \, w_0(x)^{\frac{(1-\Theta)p}{p_0}} \, |f_1(x)|^{\Theta p} \, w_1(x)^{\frac{\Theta p}{p_1}} \, dx \Big)^{1/p} \\ &\le \ \Big(\int_{\mathbb{R}^d} |f_0(x)|^{p_0} \, w_0(x)\, dx \Big)^{(1-\Theta)/p_0} \ \Big(\int_{\mathbb{R}^d} |f_1(x)|^{p_1} \, w_1(x)\, dx \Big)^{\Theta/p_1}\, . \end{aligned}$$

Hence, $f \in L_p(\mathbb{R}^d, w)$ and $\| f \,| L_p(\mathbb{R}^d, w)\| \le \| f \,| L_{p_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, L_{p_1}(\mathbb{R}^d, w_1)^{\Theta}\|$.
Substep 1.2. We prove $L_p(\mathbb{R}^d, w) \subset L_{p_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, L_{p_1}(\mathbb{R}^d, w_1)^{\Theta}$. For given $f \in L_p(\mathbb{R}^d, w)$ we define

$$f_0(x) := |f(x)|^{p/p_0} \Big(\frac{w(x)}{w_0(x)}\Big)^{1/p_0} \qquad \text{and} \qquad f_1(x) := |f(x)|^{p/p_1} \Big(\frac{w(x)}{w_1(x)}\Big)^{1/p_1}\, .$$

Then $f_j \in L_{p_j}(\mathbb{R}^d, w_j)$, $j = 0, 1$, which implies $f \in L_{p_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, L_{p_1}(\mathbb{R}^d, w_1)^{\Theta}$ and

$$\begin{aligned} \| f \,| L_{p_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, L_{p_1}(\mathbb{R}^d, w_1)^{\Theta}\| &\le \| f \,| L_p(\mathbb{R}^d, w)\|^{(1-\Theta)p/p_0}\, \| f \,| L_p(\mathbb{R}^d, w)\|^{\Theta)p/p_1} \\ &= \| f \,| L_p(\mathbb{R}^d, w)\|\, . \end{aligned}$$

This proves the claim.
Step 2. Let $\min(p_0, p_1) < \max(p_0, p_1) = \infty$. We shall concentrate on the case

$0 < p_0 < p_1 = \infty$. Then, by our convention, $w_1 := 1$. The modifications, needed in Substep 1.1, are obvious. The function f_1, used in Substep 1.2, is now given by $f_1 = 1$. With this choice the needed arguments are the same.
Step 3. Let $p_0 = p_1 = \infty$. The proof of $L_\infty(\mathbb{R}^d)^{1-\Theta}\, L_\infty(\mathbb{R}^d)^{\Theta} = L_\infty(\mathbb{R}^d)$ is obvious. ■

Remark 5 In case $1 \le p_0, p_1 \le \infty$ this result can be found in [14, Ex. 4.3.8]. For the unweighted case we also refer to [10, formula 1.6.1 on page 2.4.6] and [15, Ex. 3 on page 179].

Weighted L_p-spaces are lattice r-convex with $r \le \min(1,p)$, hence analytically convex, see the proof of Lemma 4 for an explanation of this notion and [8, Thm. 7.8] for a proof. Hence, complex interpolation of pairs of weighted L_p-spaces makes sense. There are nice connections between complex interpolation spaces and the corresponding Calderón product, see the original paper of Calderón [5] or Thm. 7.9 in [8].

Proposition 9 *Let $(\mathfrak{X}, \mathcal{S}, \mu)$ be a complete separable metric space, let μ be a $\sigma-$finite Borel measure on $\mathfrak{X}$, and let X_0, X_1 be a pair of quasi-Banach lattices of functions on $(\mathfrak{X}, \mu)$. Then, if both X_0 and X_1 are analytically convex and separable, it follows that $X_0 + X_1$ is analytically convex and*

$$[X_0, X_1]_\Theta = X_0^{1-\Theta}\, X_1^{\Theta}\,, \qquad 0 < \Theta < 1. \tag{10}$$

Lemma 8 and Prop. 9 immediately imply the following extension of (1).

Corollary 10 *Let $0 < \Theta < 1$, $0 < p_0, p_1 < \infty$ and let $w_j :\ \mathbb{R}^d \to [0,\infty)$, $j = 0,1$, be measurable and positive a.e.. Let p and w be defined as in (9). Then*

$$\Big[L_{p_0}(\mathbb{R}^d, w_0),\ L_{p_1}(\mathbb{R}^d, w_1)\Big]_\Theta = L_p(\mathbb{R}^d, w)$$

in the sense of equivalence of quasi-norms.

Remark 6 Also in Gustavsson [18] and Nilsson [16] interpolation of $L_{p_0}(\mathbb{R}^d, w_0)$ and $L_{p_1}(\mathbb{R}^d, w_1)$ is discussed for the full range of p_0 and p_1. They considered $\langle L_{p_0}(\mathbb{R}^d, w_0),\, L_{p_1}(\mathbb{R}^d, w_1)\rangle$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes an interpolation method introduced by Gagliardo and which coincides with the Calderón product under certain conditions, see [16].

3.2 Calderón products of $f^s_{p,q}(\mathbb{R}^d,w)$ spaces

In case of weighted Besov or Lizorkin-Triebel spaces there exist wavelet isomorphisms which relate these spaces to weighted sequence spaces, see the Appendix for more details. We first study Calderón products of these sequence spaces.
Here we are going to use the following abbreviations. By

$$Q_{j,k} := \{x \in \mathbb{R}^d : \quad 2^{-j}k_\ell \le x_\ell < 2^{-j}(k_\ell+1)\,, \quad \ell = 1, \ldots , d\}\,, \quad j \in \mathbb{N}_0\,,\ k \in \mathbb{Z}^d\,,$$

we denote the dyadic cubes in $\mathbb{R}^d$ (with volume ≤ 1). The symbol $\mathcal{X}_{j,k}$ is used for the characteristic function of the cube $Q_{j,k}$.

Definition 11 *Let $0 < q \le \infty$, $s \in \mathbb{R}$ and let $w : \mathbb{R}^d \to [0,\infty)$ be a nonnegative measurable function. In case $0 < p < \infty$ we define*

$$\begin{aligned} f^s_{p,q}(\mathbb{R}^d,w) := \Bigg\{ & \{\lambda_{j,k}\}_{j,k} : \quad \lambda_{j,k} \in \mathbb{C}\,, \\ & \|\,(\lambda_{j,k})\,|f^s_{p,q}(\mathbb{R}^d,w)\| := \Bigg\| \Big(\sum_{j=0}^\infty \sum_{k\in\mathbb{Z}^d} 2^{sjq}\,|\lambda_{j,k}|^q\,\mathcal{X}_{j,k}(\cdot)\Big)^{1/q} \Bigg| L_p(\mathbb{R}^d,w)\Bigg\| < \infty \Bigg\}. \end{aligned} \tag{11}$$

Obviously the spaces $f^s_{p,q}(\mathbb{R}^d,w)$ are quasi-Banach lattices. For us only those weights w will be of interest which are locally integrable and satisfy

$$0 < w(Q_{j,k}) := \int_{Q_{j,k}} w(x)\,dx < \infty \qquad \text{for all} \quad j \in \mathbb{N}_0,\ k \in \mathbb{Z}^d\,. \tag{12}$$

Remark 7 (i) In case $w(x) = 1$ for all $x \in \mathbb{R}^d$ we are back in the unweighted situation. The associated sequence spaces are denoted simply by $f^s_{p,q}(\mathbb{R}^d)$.
(ii) Let w satisfy (12). Let $\mathring{f}^s_{p,q}(\mathbb{R}^d,w)$ denote the closure of the finite sequences in $f^s_{p,q}(\mathbb{R}^d,w)$. Then

$$\mathring{f}^s_{p,q}(\mathbb{R}^d,w) = f^s_{p,q}(\mathbb{R}^d,w) \qquad \Longleftrightarrow \qquad q < \infty.$$

Especially, if $q = \infty$, then $\mathring{f}^s_{p,\infty}(\mathbb{R}^d,w)$ is a proper subspace of $f^s_{p,\infty}(\mathbb{R}^d,w)$.
(iii) Let w satisfy (12). It is easily checked that $f^s_{p,q}(\mathbb{R}^d,w)$ is separable if, and only if, $q < \infty$.
(iv) Frazier and Jawerth have introduced also the spaces $f^s_{\infty,q}(\mathbb{R}^d)$. Here in this paper we shall not deal with these classes.

Now we turn to the investigation of the Calderón products of these sequence spaces. As mentioned above we are interested in the most general situation (except the use of $f^s_{\infty,q}(\mathbb{R}^d,w)$). The class of weights, we are dealing with, is $\mathcal{A}^{\ell oc}_\infty$, see the Appendix for the definition. With certain care we shall study also the limiting situations $\max(q_0,q_1) = \infty$.

Theorem 12 *Let $0 < \Theta < 1$. Let $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \le \infty$ and $s_0, s_1 \in \mathbb{R}$. We put*

$$\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}\,, \qquad \frac{1}{q} := \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1} \qquad \text{and} \qquad s := (1-\Theta)\, s_0 + \Theta\, s_1\,. \tag{13}$$

Let $w_0, w_1 \in \mathcal{A}_\infty^{\ell oc}$ and define w by the formula

$$w := w_0^{\frac{(1-\Theta)p}{p_0}}\, w_1^{\frac{\Theta p}{p_1}}\,. \tag{14}$$

Then

$$f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} = f^{s}_{p,q}(\mathbb{R}^d, w) \tag{15}$$

holds in the sense of equivalent quasi-norms.

Proof. By Lemma 35 the weight w belongs to $\mathcal{A}_\infty^{\ell oc}$, i.e., our sequence spaces $f^s_{p,q}(\mathbb{R}^d, w)$ are well-defined. Muckenhoupt weights and therefore also local Muckenhoupt weights can not vanish on a set of positive measure. Hence, (12) holds for w_0, w_1 and w.
Step 1. We shall prove

$$f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} \hookrightarrow f^{s}_{p,q}(\mathbb{R}^d, w)\,.$$

We suppose, that sequences $\lambda := (\lambda_{j,k})_{j,k}$, $\lambda^\ell := (\lambda^\ell_{j,k})_{j,k}$, $\ell = 0, 1$, are given and that

$$|\lambda_{j,k}| \le |\lambda^0_{j,k}|^{1-\Theta} \cdot |\lambda^1_{j,k}|^{\Theta}$$

holds for all $j \in \mathbb{N}_0$ and all $k \in \mathbb{Z}^d$. We have to show that there exists a constant c s.t.

$$\|\lambda | f^s_{p,q}(\mathbb{R}^d, w)\| \le c\, \|\lambda^0 | f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)\|^{1-\Theta} \cdot \|\lambda^1 | f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)\|^{\Theta}$$

holds for all such $\lambda, \lambda^0, \lambda^1$. But this follows directly by Hölder's inequality (with $c = 1$).
Step 2. Now we turn to the proof of

$$f^{s}_{p,q}(\mathbb{R}^d, w) \hookrightarrow f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta}\,.$$

We assume in addition $\max(q_0, q_1) < \infty$. Let the sequence $\lambda \in f^s_{p,q}(\mathbb{R}^d, w)$ be given. We have to find sequences λ^0 and λ^1 such that $|\lambda_{j,k}| \le |\lambda^0_{j,k}|^{1-\Theta} \cdot |\lambda^1_{j,k}|^{\Theta}$ for every $j \in \mathbb{N}_0$ and $k \in \mathbb{Z}^d$ and

$$\|\lambda^0 | f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)\|^{1-\Theta} \cdot \|\lambda^1 | f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)\|^{\Theta} \le c\, \|\lambda | f^s_{p,q}(\mathbb{R}^d, w)\| \tag{16}$$

with some constant c independent of λ. We follow ideas of the proof of Theorem 8.2 in Frazier and Jawerth [6], see also Bownik [3]. Since w_0, w_1 are local Muckenhoupt weights, they are positive and finite a.e., hence, also w is positive and finite a.e.. Let

$$A := \left\{ x \in \mathbb{R}^d :\ 0 < \frac{w(x)}{w_0(x)} < \infty \quad \text{and} \quad 0 < \frac{w(x)}{w_1(x)} < \infty \right\}.$$

The functions w, w_0 and w_1 are locally integrable and positive a.e., therefore $\mathbb{R}^d \setminus A$ is a set of measure zero. We put

$$A_\ell := \left\{ x \in A : \left(\sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} 2^{jsq} |\lambda_{j,k}|^q \mathcal{X}_{j,k}(x) \right)^{1/q} \cdot \left(\frac{w(x)}{w_0(x)} \right)^{\frac{1}{p_0} \cdot \frac{1}{\frac{p}{p_0} - \frac{q}{q_0}}} > 2^\ell \right\},$$

$\ell \in \mathbb{Z}$. Obviously $A_{\ell+1} \subset A_\ell$, $\ell \in \mathbb{Z}$. Now we introduce a (partial) decomposition of $\mathbb{N}_0 \times \mathbb{Z}^d$ by taking

$$C_\ell := \left\{ (j,k) : \ |Q_{j,k} \cap A_\ell| > \frac{|Q_{j,k}|}{2} \quad \text{and} \quad |Q_{j,k} \cap A_{\ell+1}| \le \frac{|Q_{j,k}|}{2} \right\}, \quad \ell \in \mathbb{Z}.$$

The sets C_ℓ are pairwise disjoint, i.e., $C_\ell \cap C_m = \emptyset$ if $\ell \neq m$.
Substep 2.1. We claim that $\lambda_{j,k} = 0$ holds for all tuples $(j,k) \notin \bigcup_\ell C_\ell$. Let us consider one such tuple (j_0,k_0) and let us choose $l_0 \in \mathbb{Z}$ arbitrarily. As $(j_0,k_0) \notin C_{\ell_0}$, then either

$$|Q_{j_0,k_0} \cap A_{\ell_0}| \le \frac{|Q_{j_0,k_0}|}{2} \qquad \text{or} \qquad |Q_{j_0,k_0} \cap A_{\ell_0+1}| > \frac{|Q_{j_0,k_0}|}{2}. \tag{17}$$

Let us assume for the moment that the second condition is satisfied. By induction on ℓ it follows

$$|Q_{j_0,k_0} \cap A_{\ell+1}| > \frac{|Q_{j_0,k_0}|}{2} \qquad \text{for all} \quad \ell \ge \ell_0. \tag{18}$$

Let $D := \bigcap_\ell Q_{j_0,k_0} \cap A_\ell$. The family $\{Q_{j_0,k_0} \cap A_\ell\}_\ell$ is a decreasing family of sets, i.e., $Q_{j_0,k_0} \cap A_{\ell+1} \subset Q_{j_0,k_0} \cap A_\ell$. Therefore, in view of (18), the measure of the set D is larger than or equal to $\frac{|Q_{j_0,k_0}|}{2}$. We obtain

$$\begin{aligned}
\| \lambda \,| f^s_{p,q}(\mathbb{R}^d, w)\|^p \quad &:= \quad \left\| \left(\sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} 2^{sjq} \, |\lambda_{j,k}|^q \, \mathcal{X}_{j,k}(\cdot) \right)^{1/q} \right| L_p(\mathbb{R}^d, w) \Bigg\|^p \\
&> \quad \int_{Q_{j_0,k_0} \cap A_\ell} \left(2^\ell \left(\frac{w_0(x)}{w(x)} \right)^{\frac{1}{p_0} \cdot \frac{1}{\frac{p}{p_0} - \frac{q}{q_0}}} \right)^p w(x) \, dx \\
&> \quad 2^{\ell p} \int_D w_0(x)^{\frac{p}{p_0} \cdot \frac{1}{\frac{p}{p_0} - \frac{q}{q_0}}} \, w(x)^{\frac{q}{q_0} \cdot \frac{1}{\frac{q}{q_0} - \frac{p}{p_0}}} \, dx \,.
\end{aligned}$$

The norm $\| \lambda \,| f^s_{p,q}(\mathbb{R}^d, w)\|^p$ is finite since $\lambda \in f^s_{p,q}(\mathbb{R}^d, w)$. In consequence the integral over D is a finite positive number. We recall that the function we integrate is positive a.e. and $|D| \ge \frac{|Q_{j_0,k_0}|}{2}$. Letting ℓ tend to infinity we get a contradiction. Hence, we have to turn in (17) to the situation where the first condition is satisfied. We claim

$$|Q_{j_0,k_0} \cap A_\ell| \le \frac{|Q_{j_0,k_0}|}{2} \qquad \text{for all} \qquad \ell \le \ell_0 \,.$$

Again this follows by induction on ℓ using $(j_0,k_0) \notin \bigcup_\ell C_\ell$. Obviously this yields

$$|Q_{j_0,k_0} \cap A^c_\ell| \ge 2^{-j_0 d - 1} \qquad \text{for all} \qquad \ell \le \ell_0 \,. \tag{19}$$

Let now $E := \bigcap_\ell Q_{j_0,k_0} \cap A \cap A_\ell^c$. The family $\{Q_{j_0,k_0} \cap A \cap A_\ell^c\}_\ell$ satisfies

$$(Q_{j_0,k_0} \cap A \cap A_{\ell-1}) \quad \subset \quad (Q_{j_0,k_0} \cap A \cap A_\ell)\,.$$

Therefore, in view of (19), the measure of the set E is larger than or equal to $\frac{|Q_{j_0,k_0}|}{2}$. By selecting a point $x \in E$ we conclude that

$$2^{j_0 s}\,|\lambda_{j_0,k_0}| \le \left(\sum_{j=0}^\infty \sum_{k\in\mathbb{Z}^d} 2^{jsq}|\lambda_{j,k}|^q \mathcal{X}_{j,k}(x)\right)^{1/q} \le 2^\ell \, \left(\frac{w_0(x)}{w(x)}\right)^{\frac{1}{p_0}\cdot\frac{1}{\frac{p}{p_0}-\frac{q}{q_0}}} ,$$

for any $\ell \le \ell_0$. Now, for ℓ tending to $-\infty$ the claim, namely $\lambda_{j_0,k_0} = 0$, follows.
Substep 2.2. If $(j,k) \not\in \bigcup_{\ell\in\mathbb{Z}} C_\ell$, then we define $\lambda^0_{j,k} := \lambda^1_{j,k} := 0$. If $(j,k) \in C_\ell$, we put

$$\lambda^0_{j,k} := 2^{\ell\gamma}\, 2^{ju}\, |\lambda_{j,k}|^{q/q_0} \quad \text{and} \quad \lambda^1_{j,k} := 2^{\ell\delta}\, 2^{jv}\, |\lambda_{j,k}|^{q/q_1},$$

where

$$\gamma := \frac{p}{p_0} - \frac{q}{q_0}\,, \quad \delta := \frac{p}{p_1} - \frac{q}{q_1}\,, \quad u := q\,\Theta\,\left[\frac{s_1}{q_0} - \frac{s_0}{q_1}\right]\,, \quad v := q\,(1-\Theta)\,\left[\frac{s_0}{q_1} - \frac{s_1}{q_0}\right].$$

We observe, that

$$(\lambda^0_{j,k})^{1-\Theta} \cdot (\lambda^1_{j,k})^{\Theta} = 2^{l[\gamma(1-\Theta)+\delta\Theta]} \cdot 2^{j[u(1-\Theta)+v\Theta]} \cdot |\lambda_{j,k}| = |\lambda_{j,k}|\,,$$

which holds now for all pairs (j,k). To prove (16), it will be sufficient to establish the following two inequalities

$$\|\lambda^0|f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)\| \;\lesssim\; \|\lambda|f^s_{p,q}(\mathbb{R}^d, w)\|^{p/p_0} \tag{20}$$

$$\|\lambda^1|f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)\| \;\lesssim\; \|\lambda|f^s_{p,q}(\mathbb{R}^d, w)\|^{p/p_1}\,. \tag{21}$$

Substep 2.3. First we deal with (20) under the condition $\gamma \ge 0$. Our restrictions on p_0, p_1 and q_0, q_1 are symmetric. It follows from (13) that

$$\min\left(\frac{p_0}{q_0}, \frac{p_1}{q_1}\right) \le \frac{p}{q} \le \max\left(\frac{p_0}{q_0}, \frac{p_1}{q_1}\right).$$

As $\gamma \ge 0$, we get also $p/p_0 \ge q/q_0$ and $\delta \le 0$. By employing the sets C_ℓ we derive

$$\begin{aligned}
\|\lambda^0|f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)\| &= \Big\|\Big(\sum_{j=0}^\infty \sum_{k\in\mathbb{Z}^d} 2^{js_0q_0}(\lambda^0_{j,k})^{q_0}\mathcal{X}_{j,k}(\cdot)\Big)^{1/q_0}\Big|L_{p_0}(\mathbb{R}^d, w_0)\Big\| \\
&= \Big\|\Big(\sum_{\ell=-\infty}^\infty \sum_{(j,k)\in C_\ell} 2^{js_0q_0}\,2^{\ell\gamma q_0}\,2^{juq_0}\,|\lambda_{j,k}|^q\mathcal{X}_{j,k}(\cdot)\Big)^{1/q_0}\Big|L_{p_0}(\mathbb{R}^d, w_0)\Big\| \\
&= \Big\|\Big(\sum_{\ell=-\infty}^\infty \sum_{(j,k)\in C_\ell} f_{j,k}(\,\cdot\,)^Q\Big)^{1/Q}\Big|L_P(\mathbb{R}^d, w_0)\Big\|^{P/p_0},
\end{aligned}$$

where

$$f_{j,k}(\,\cdot\,) := \Big(2^{js_0q_0}\, 2^{\ell\gamma q_0}\, 2^{juq_0}\, |\lambda_{j,k}|^q\, \mathcal{X}_{j,k}(\,\cdot\,)\Big)^{\frac{p_0}{q_0P}}, \qquad (j,k)\in C_\ell\,,$$

and P and $Q = \frac{q_0P}{p_0}$ are chosen such that $w_0 \in \mathcal{A}_P, 1 < P < \infty, 1 < Q \le \infty$. Next we apply the weighted vector-valued maximal inequality (41) for the local Hardy-Littlewood maximal function from the Appendix together with the estimate

$$\mathcal{X}_{j,k}(x) \le 2\,(M^{\ell oc}\mathcal{X}_{Q_{j,k}\cap A_\ell})(x)\,, \qquad x\in\mathbb{R}^d, \quad (j,k)\in C_\ell\,.$$

Using $u + s_0 = \frac{sq}{q_0}$, $\gamma \ge 0$ and

$$\bigcup_{L=-\infty}^{\infty} (A_L \setminus A_{L+1}) = \bigcup_{\ell=-\infty}^{\infty} A_\ell$$

we may further proceed

$$\begin{aligned}
\|\lambda^0|f^{s_0}_{p_0,q_0}(\mathbb{R}^d,w_0)\| &\lesssim \Big\|\Big(\sum_{\ell=-\infty}^{\infty}\sum_{(j,k)\in C_\ell} 2^{\ell\gamma q_0}\, 2^{juq_0}\, 2^{js_0q_0}\, |\lambda_{j,k}|^q\, \mathcal{X}_{Q_{j,k}\cap A_\ell}(\cdot)\Big)^{1/q_0}\Big|L_{p_0}(\mathbb{R}^d,w_0)\Big\| \\
&= \Big\|\Big(\sum_{\ell=-\infty}^{\infty}\sum_{(j,k)\in C_\ell} 2^{\ell\gamma q_0}\, 2^{jsq}\, |\lambda_{j,k}|^q \mathcal{X}_{Q_{j,k}\cap A_\ell}(\cdot)\Big)^{1/q_0}\Big|L_{p_0}(\mathbb{R}^d,w_0)\Big\| \\
&\le \Big\|\sum_{L=-\infty}^{\infty} \mathcal{X}_{A_L\setminus A_{L+1}}(\cdot)\Big(\sum_{\ell=-\infty}^{\infty}\sum_{(j,k)\in C_\ell} 2^{\ell\gamma q_0}\, 2^{jsq}\, |\lambda_{j,k}|^q \mathcal{X}_{Q_{j,k}\cap A_\ell}(\cdot)\Big)^{1/q_0}\Big|L_{p_0}(\mathbb{R}^d,w_0)\Big\| \\
&\le \Big\|\sum_{L=-\infty}^{\infty} \mathcal{X}_{A_L\setminus A_{L+1}}(\cdot)\Big(\sum_{\ell=-\infty}^{L}\sum_{(j,k)\in C_\ell} 2^{\ell\gamma q_0}\, 2^{jsq}\, |\lambda_{j,k}|^q \mathcal{X}_{j,k}(\cdot)\Big)^{1/q_0}\Big|L_{p_0}(\mathbb{R}^d,w_0)\Big\|.
\end{aligned}$$

Introducing the abbreviation $D_L := \bigcup\limits_{m\le L} C_m$ and using that $\gamma\ge 0$, we obtain

$$\begin{aligned}
\|\lambda^0|f^{s_0}_{p_0,q_0}(\mathbb{R}^d,w_0)\| &\lesssim \Big\|\sum_{L=-\infty}^{\infty} \mathcal{X}_{A_L\setminus A_{L+1}}(\cdot)\, 2^{L\gamma}\Big(\sum_{(j,k)\in D_L} 2^{jsq}\, |\lambda_{j,k}|^q \mathcal{X}_{j,k}(\cdot)\Big)^{1/q_0}\Big|L_{p_0}(\mathbb{R}^d,w_0)\Big\| \\
&\le \Big\|\sum_{L=-\infty}^{\infty} \mathcal{X}_{A_L\setminus A_{L+1}}(\cdot)\, 2^{L\gamma}\Big(\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} 2^{jsq}\, |\lambda_{j,k}|^q \mathcal{X}_{j,k}(\cdot)\Big)^{1/q_0}\Big|L_{p_0}(\mathbb{R}^d,w_0)\Big\|.
\end{aligned}$$

Let

$$f(\cdot) := \Big(\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} 2^{jsq}\, |\lambda_{j,k}|^q \mathcal{X}_{j,k}(\cdot)\Big)^{1/q}.$$

We employ the definition of A_L and find

$$\begin{aligned}
\|\lambda^0|f^{s_0}_{p_0,q_0}(\mathbb{R}^d,w_0)\| &\lesssim \Big\|\sum_{L=-\infty}^{\infty} \mathcal{X}_{A_L\setminus A_{L+1}}(\cdot)\, f^\gamma(\cdot)\,\Big(\frac{w(\cdot)}{w_0(\cdot)}\Big)^{\frac{1}{p_0}\cdot\frac{1}{\frac{p}{p_0}-\frac{q}{q_0}}\cdot\gamma} f^{\frac{q}{q_0}}(\cdot)\Big|L_{p_0}(\mathbb{R}^d,w_0)\Big\| \\
&= \Big\| f^{\gamma+q/q_0}\,\Big(\frac{w}{w_0}\Big)^{\frac{1}{p_0}}\Big|L_{p_0}(\mathbb{R}^d,w_0)\Big\| \\
&= \|\, f^{\frac{p}{p_0}}\,(w/w_0)^{\frac{1}{p_0}}\,|L_{p_0}(\mathbb{R}^d,w_0)\| \\
&= \|f|L_p(\mathbb{R}^d,w)\|^{p/p_0} = \|\lambda|f^s_{p,q}(\mathbb{R}^d,w)\|^{p/p_0}.
\end{aligned}$$

Substep 2.4. Now we prove (21). We only make some comments concerning necessary modifications in comparison with Step 2.3. We first point out, that the identity

$$\left(\frac{w(x)}{w_0(x)}\right)^{\frac{1-\Theta}{p_0}} = \left(\frac{w(x)}{w_1(x)}\right)^{-\frac{\Theta}{p_1}}, \qquad x \in A$$

raised to the appropriate power gives

$$\left(\frac{w(x)}{w_0(x)}\right)^{\frac{1}{p_0}\cdot\frac{1}{\frac{p}{p_0}-\frac{q}{q_0}}} = \left(\frac{w(x)}{w_1(x)}\right)^{\frac{1}{p_1}\cdot\frac{1}{\frac{p}{p_1}-\frac{q}{q_1}}}, \qquad x \in A. \tag{22}$$

This means, that the definition of the sets A_ℓ and C_ℓ does not change, if we replace (w_0, p_0, q_0) by (w_1, p_1, q_1). As $\delta \le 0$ in this case, we are forced to replace the sets $Q_{j,k} \cap A_\ell$ by $Q_{j,k} \cap A^c_{\ell+1}$. Observe that $|Q_{j,k} \cap A^c_{\ell+1}| \ge \frac{|Q_{j,k}|}{2}$ and hence

$$\mathcal{X}_{j,k}(x) \le 2\,(M^{\ell oc}\mathcal{X}_{Q_{j,k}\cap A^c_{\ell+1}})(x)\,, \qquad x \in \mathbb{R}^d, \quad (j,k) \in C_\ell\,.$$

This, together with $v + s_1 = sq/q_1$ and the maximal inequality (41), leads to

$$\begin{aligned}
\|\lambda^1|f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)\| &\lesssim \Big\|\Big(\sum_{\ell=-\infty}^{\infty}\sum_{(j,k)\in C_\ell} 2^{\ell\delta q_1}\,2^{jvq_1}\,2^{js_1q_1}\,|\lambda_{j,k}|^q\,\mathcal{X}_{Q_{j,k}\cap A^c_{\ell+1}}(\cdot)\Big)^{1/q_1}\Big|L_{p_1}(\mathbb{R}^d, w_1)\Big\| \\
&= \Big\|\Big(\sum_{\ell=-\infty}^{\infty}\sum_{(j,k)\in C_\ell} 2^{\ell\delta q_1}\,2^{jsq}\,|\lambda_{j,k}|^q\mathcal{X}_{Q_{j,k}\cap A^c_{\ell+1}}(\cdot)\Big)^{1/q_1}\Big|L_{p_1}(\mathbb{R}^d, w_1)\Big\| \\
&\le \Big\|\sum_{L=-\infty}^{\infty}\mathcal{X}_{A^c_{L+1}\setminus A^c_L}(\cdot)\Big(\sum_{\ell=-\infty}^{\infty}\sum_{(j,k)\in C_\ell} 2^{\ell\delta q_1}\,2^{jsq}\,|\lambda_{j,k}|^q\mathcal{X}_{Q_{j,k}\cap A^c_{\ell+1}}(\cdot)\Big)^{1/q_1}\Big|L_{p_1}(\mathbb{R}^d, w_1)\Big\| \\
&\le \Big\|\sum_{L=-\infty}^{\infty}\mathcal{X}_{A^c_{L+1}\setminus A^c_L}(\cdot)\Big(\sum_{\ell=L-1}^{\infty}\sum_{(j,k)\in C_\ell} 2^{\ell\delta q_1}\,2^{jsq}\,|\lambda_{j,k}|^q\mathcal{X}_{j,k}(\cdot)\Big)^{1/q_1}\Big|L_{p_1}(\mathbb{R}^d, w_1)\Big\|.
\end{aligned}$$

Defining $E_L := \bigcup\limits_{m \ge L-1} C_m$ and making use of $\delta \le 0$ we obtain

$$\|\lambda^1|f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)\| \lesssim \Big\|\sum_{L=-\infty}^{\infty}\mathcal{X}_{A^c_{L+1}\setminus A^c_L}(\cdot)\,2^{L\delta}\Big(\sum_{(j,k)\in E_L} 2^{jsq}\,|\lambda_{j,k}|^q\mathcal{X}_{j,k}(\cdot)\Big)^{1/q_1}\Big|L_{p_1}(\mathbb{R}^d, w_1)\Big\|.$$

As above we put

$$f(\cdot) := \Big(\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} 2^{jsq}\,|\lambda_{j,k}|^q\mathcal{X}_{j,k}(\cdot)\Big)^{1/q}.$$

The definition of A_L yields

$$2^L < f(x)\,\left(\frac{w(x)}{w_1(x)}\right)^{\frac{1}{p_1}\cdot\frac{1}{\frac{p}{p_1}-\frac{q}{q_1}}} \le 2^{L+1}, \qquad x \in A^c_{L+1}\setminus A^c_L\,,$$

see (22). Inserting this in the previous inequality we get

$$\begin{aligned}\|\lambda^1|f^{s_1}_{p_1,q_1}(\mathbb{R}^d,w_1)\| &\lesssim \Big\| \sum_{L=-\infty}^{\infty} \mathcal{X}_{A^c_{L+1}\setminus A^c_L}(\cdot)\, f^{\delta+\frac{q}{q_1}}(\cdot) \left(\frac{w(\cdot)}{w_1(\cdot)}\right)^{\frac{1}{p_1}\cdot\frac{1}{\frac{p}{p_1}-\frac{q}{q_1}}\cdot\delta} \Big| L_{p_1}(\mathbb{R}^d,w_1)\Big\| \\ &\lesssim \|\, f^{\frac{p}{p_1}}\,(w/w_1)^{\frac{1}{p_1}}\,|L_{p_1}(\mathbb{R}^d,w_1)\| \\ &= \|f|L_p(\mathbb{R}^d,w)\|^{p/p_1} = \|\lambda|f^s_{p,q}(\mathbb{R}^d,w)\|^{p/p_1}.\end{aligned}$$

This proves (21).

Step 3. Let $\max(q_0,q_1)=\infty$.

Substep 3.1. First we consider $0<q_1<q_0=\infty$. We shall discuss the needed modifications only. As in Step 2, if $(j,k)\notin\bigcup_{\ell\in\mathbb{Z}} C_\ell$, then we define $\lambda^0_{j,k}=\lambda^1_{j,k}=0$. If $(j,k)\in C_\ell$, we put

$$\lambda^0_{j,k} := 2^{\ell\gamma}\, 2^{ju} \quad \text{and} \quad \lambda^1_{j,k} := 2^{\ell\delta}\, 2^{jv}\, |\lambda_{j,k}|^{q/q_1}, \tag{23}$$

where

$$\gamma := \frac{p}{p_0}\,, \quad \delta := \frac{p}{p_1}-\frac{q}{q_1}\,, \quad u := -q\,\Theta\,\frac{s_0}{q_1}\,, \quad v := q\,(1-\Theta)\,\frac{s_0}{q_1}\,.$$

This implies $(\lambda^0_{j,k})^{1-\Theta}\cdot(\lambda^1_{j,k})^{\Theta}=|\lambda_{j,k}|$. Again we are going to establish the inequalities (20) and (21). Since there is nothing changed with respect to (21) we deal with (20). Obviously $\gamma>0$. Formally it looks like that we lost the influence of λ. However, this is not true. By employing the same arguments as in Step 2.3 and $u+s_0=0$ we obtain

$$\begin{aligned}\|\lambda^0|f^{s_0}_{p_0,\infty}(\mathbb{R}^d,w_0)\| &\lesssim \|\sup_{\ell\in\mathbb{Z}}\ \sup_{(j,k)\in C_\ell} 2^{\ell\gamma}\,\mathcal{X}_{Q_{j,k}\cap A_\ell}(\cdot)\,|L_{p_0}(\mathbb{R}^d,w_0)\| \\ &\le \Big\| \sum_{L=-\infty}^{\infty} \mathcal{X}_{A_L\setminus A_{L+1}}(\cdot) \sup_{-\infty<\ell\le L}\ \sup_{(j,k)\in C_\ell} 2^{\ell\gamma}\,\mathcal{X}_{j,k}(\cdot)\,\Big|L_{p_0}(\mathbb{R}^d,w_0)\Big\| \\ &\le \Big\| \sum_{L=-\infty}^{\infty} \mathcal{X}_{A_L\setminus A_{L+1}}(\cdot)\, 2^{L\gamma} \sup_{(j,k)\in D_L} \mathcal{X}_{j,k}(\cdot)\Big|L_{p_0}(\mathbb{R}^d,w_0)\Big\|\end{aligned}$$

Next we use the definition of the set A_L. In case $x\in\mathcal{X}_{A_L\setminus A_{L+1}}$ we conclude

$$2^L < \Big(\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} 2^{jsq}\,|\lambda_{j,k}|^q\,\mathcal{X}_{j,k}(x)\Big)^{1/q}\Big(\frac{w(x)}{w_0(x)}\Big)^{1/p} \le 2^{L+1}\,.$$

We insert this into the previous inequality and find

$$\begin{aligned}&\|\lambda^0|f^{s_0}_{p_0,\infty}(\mathbb{R}^d,w_0)\| \\ &\quad\lesssim \Big\| \sum_{L=-\infty}^{\infty} \mathcal{X}_{A_L\setminus A_{L+1}}(\cdot) \Big(\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} 2^{jsq}\,|\lambda_{j,k}|^q\,\mathcal{X}_{j,k}(x)\Big)^{\gamma/q}\Big(\frac{w(x)}{w_0(x)}\Big)^{\gamma/p}\Big|L_{p_0}(\mathbb{R}^d,w_0)\Big\| \\ &\quad\le \Big\|\Big(\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} 2^{jsq}\,|\lambda_{j,k}|^q\,\mathcal{X}_{j,k}(x)\Big)^{\gamma/q}\Big(\frac{w(x)}{w_0(x)}\Big)^{1/p_0}\Big|L_{p_0}(\mathbb{R}^d,w_0)\Big\| \\ &\quad= \Big\|\Big(\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} 2^{jsq}\,|\lambda_{j,k}|^q\,\mathcal{X}_{j,k}(x)\Big)^{1/q}\Big|L_p(\mathbb{R}^d,w)\Big\|^{p/p_0},\end{aligned}$$

as we wanted to prove.

Substep 3.2. Observe that the case $0 < q_0 < q_1 = \infty$ follows by symmetry.

Substep 3.3. It remains to study the case $q_0 = q_1 = \infty$. If $(j,k) \in C_\ell$, we choose $\lambda^i_{j,k}$, $i = 1, 2$, s.t.

$$\lambda^0_{j,k} := 2^{\ell\gamma}\, 2^{ju}\, |\lambda_{j,k}| \quad \text{and} \quad \lambda^1_{j,k} := 2^{\ell\delta}\, 2^{jv}\, |\lambda_{j,k}|, \tag{24}$$

where

$$\gamma := \frac{p}{p_0} - 1\,, \quad \delta := \frac{p}{p_1} - 1\,, \quad u := \Theta\,(s_1 - s_0)\,, \quad v := (1-\Theta)\,(s_0 - s_1)\,.$$

Again this implies $(\lambda^0_{j,k})^{1-\Theta} \cdot (\lambda^1_{j,k})^{\Theta} = |\lambda_{j,k}|$. Without loss of generality we assume $p_0 \le p \le p_1$, i.e., $\gamma \ge 0$. Now, using $u + s_0 = s$, we may proceed as in Substep 3.1 and obtain

$$\|\lambda^0|f^{s_0}_{p_0,\infty}(\mathbb{R}^d, w_0)\| \lesssim \Big\| \sum_{L=-\infty}^{\infty} \mathcal{X}_{A_L \setminus A_{L+1}}(\cdot)\, 2^{L\gamma} \sup_{(j,k)\in D_L} 2^{js}\, |\lambda_{j,k}|\, \mathcal{X}_{j,k}(\cdot) \Big| L_{p_0}(\mathbb{R}^d, w_0) \Big\|\,.$$

Employing the definition of the set A_L we conclude

$$2^L < \sup_{j=0,1,\ldots} \sup_{k\in\mathbb{Z}^d} 2^{js}\, |\lambda_{j,k}|\, \mathcal{X}_{j,k}(x) \Big(\frac{w(x)}{w_0(x)}\Big)^{\frac{1}{p_0}\frac{1}{\frac{p}{p_0}-1}} \le 2^{L+1}\,, \qquad x \in \mathcal{X}_{A_L \setminus A_{L+1}}\,.$$

We insert this into the previous inequality and find

$$\begin{aligned}
&\|\lambda^0|f^{s_0}_{p_0,\infty}(\mathbb{R}^d, w_0)\| \\
&\quad \lesssim \Big\| \sum_{L=-\infty}^{\infty} \mathcal{X}_{A_L \setminus A_{L+1}}(\cdot) \Big(\sup_{j=0,1,\ldots} \sup_{k\in\mathbb{Z}^d} 2^{js}\, |\lambda_{j,k}|\, \mathcal{X}_{j,k}(x)\Big)^{\gamma+1} \Big(\frac{w(x)}{w_0(x)}\Big)^{\gamma\frac{1}{p_0}\frac{1}{\frac{p}{p_0}-1}} \Big| L_{p_0}(\mathbb{R}^d, w_0)\Big\| \\
&\quad \le \Big\| \Big(\sup_{j=0,1,\ldots} \sup_{k\in\mathbb{Z}^d} 2^{js}\, |\lambda_{j,k}|\, \mathcal{X}_{j,k}(x)\Big)^{p/p_0} \Big(\frac{w(x)}{w_0(x)}\Big)^{\frac{1}{p_0}} \Big| L_{p_0}(\mathbb{R}^d, w_0)\Big\| \\
&\quad = \Big\| \sup_{j=0,1,\ldots} \sup_{k\in\mathbb{Z}^d} 2^{js}\, |\lambda_{j,k}|\, \mathcal{X}_{j,k}(x) \Big| L_p(\mathbb{R}^d, w)\Big\|^{p/p_0}\,,
\end{aligned}$$

as we wanted to prove. The estimate of $\|\lambda^1|f^{s_1}_{p_1,\infty}(\mathbb{R}^d, w_1)\|$ follows by similar arguments. ■

Remark 8 (i) The identity

$$f^{s_0}_{p_0,q_0}(\mathbb{R}^d)^{1-\Theta}\, f^{s_1}_{p_1,q_1}(\mathbb{R}^d)^{\Theta} = f^s_{p,q}(\mathbb{R}^d)$$

i.e., formula (15) with $w_0 = w_1 \equiv 1$, has been proved in Frazier and Jawerth [6]. Our proof, given here, is just an adaptation to the weighted situation. However, let us mention, that we had some advantage from Bownik's proof in [3], in particular in case $\max(q_0, q_1) = \infty$. In fact, Bownik had considered the situation

$$f^{s_0}_{p_0,q_0}(\mathbb{R}^d, \mu)^{1-\Theta}\, f^{s_1}_{p_1,q_1}(\mathbb{R}^d, \mu)^{\Theta} = f^s_{p,q}(\mathbb{R}^d, \mu)\,,$$

where μ is a doubling measure. In Substep 2.1 we also used some ideas from Yang, Yuan and Zhuo [17]. These three authors dealt with extensions to sequence spaces related to Lizorkin-Triebel spaces built on Morrey spaces. Further we would like to mention that Wojciechowska [4] recently proved

$$f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w)^{1-\Theta}\, f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w)^{\Theta} = f^{s}_{p,q}(\mathbb{R}^d, w)\,,$$

where $w \in \mathcal{A}^{\ell oc}_\infty$. The class $\mathcal{A}^{\ell oc}_\infty$ and the set of doubling measures are incomparable (more exactly, there exists weights in $\mathcal{A}^{\ell oc}_\infty$ which are exponentially growing and therefore do not induce a doubling measure, vice versa there are doubling measures which do not induce a weight in $\mathcal{A}_\infty$). An example of a doubling measure such that the associated weight does not belong to $\mathcal{A}_\infty$ can be found in Wik's paper [19]. Furthermore, we refer to [20, §I.8.8] for an example of a doubling measure, which is singular with respect to the Lebesgue measure, i.e. without any associated weight.
Thm. 12 with $w_0 \neq w_1$ seems to be a novelty. However, as the previously mentioned results of Bownik indicate, there is some hope to extend it (as well as Cor. 22) to larger classes of weights.
(ii) Frazier, Jawerth [6] and Bownik [3] have also treated extensions of Thm. 12 to $\max(p_0, p_1) = \infty$.

At the end of this subsection we would like to discuss the class of admissible weights. We had concentrated on the class $\mathcal{A}^{\ell oc}_\infty$ in Thm. 12. We do not believe that this is the end of the story and expect, that Thm. 12 holds also for more general weights. Let us assume for the moment that Prop. 37 extends to some weights w_0, w_1 (not necessarily belonging to $\mathcal{A}^{\ell oc}_\infty$). In addition we need the identification of $L_{p_j}(\mathbb{R}^d, w_j)$ and $F^0_{p_j,2}(\mathbb{R}^d, w_j)$, $1 \le p_j < \infty$, $j = 0, 1$, which is known to be true only for the class $\mathcal{A}^{\ell oc}_\infty$, see Rychkov [21]. These two properties are also needed for the associated weight w. Then Lemma 8 implies

$$F^0_{p_0,2}(\mathbb{R}^d, w_0)^{1-\Theta}\, F^0_{p_1,2}(\mathbb{R}^d, w_1)^{\Theta} = F^0_{p,2}(\mathbb{R}^d, w)$$

and therefore

$$f^{d/2}_{p_0,2}(\mathbb{R}^d, w_0)^{1-\Theta}\, f^{d/2}_{p_1,2}(\mathbb{R}^d, w_1)^{\Theta} = f^{d/2}_{p,2}(\mathbb{R}^d, w)\,.$$

3.3 Calderón products of $b^s_{p,q}(\mathbb{R}^d,w)$ spaces

Definition 13 *Let $0<p,q\le\infty$, $s\in\mathbb{R}$ and let $w:\ \mathbb{R}^d\to[0,\infty)$ be a nonnegative and locally integrable function. We put*

$$b^s_{p,q}(\mathbb{R}^d,w) := \Big\{ \{\lambda_{j,k}\}_{j,k}:\quad \lambda_{j,k}\in\mathbb{C}\,,$$
$$\|\,(\lambda_{j,k})\,|b^s_{p,q}(\mathbb{R}^d,w)\| := \Big(\sum_{j=0}^\infty \Big\| \sum_{k\in\mathbb{Z}^d} 2^{sj}\,|\lambda_{j,k}|\,\mathcal{X}_{j,k}(\cdot)\,\Big|L_p(\mathbb{R}^d,w)\Big\|^q\Big)^{1/q} <\infty\Big\}\,. \tag{25}$$

Remark 9 (i) In case $w(x)=1$ for all $x\in\mathbb{R}^d$ we are back in the unweighted situation. The associated sequence spaces are denoted simply by $b^s_{p,q}(\mathbb{R}^d)$.
(ii) Let w satisfy (12). Let $\mathring{b}^s_{p,q}(\mathbb{R}^d,w)$ denote the closure of the finite sequences in $b^s_{p,q}(\mathbb{R}^d,w)$. We have

$$\mathring{b}^s_{p,q}(\mathbb{R}^d,w) = b^s_{p,q}(\mathbb{R}^d,w) \qquad \Longleftrightarrow \qquad \max(p,q)<\infty\,.$$

If $\max(p,q)=\infty$, then $\mathring{b}^s_{p,q}(\mathbb{R}^d,w)$ is a proper subspace of $b^s_{p,q}(\mathbb{R}^d,w)$.
(iii) Let w satisfy (12). It is easily checked that $b^s_{p,q}(\mathbb{R}^d,w)$ is separable if, and only if, $\max(p,q)<\infty$.

Before we are turning to a description of the associated Calderón products we would like to introduce a second type of sequence spaces. Observe

$$\Big\| \sum_{k\in\mathbb{Z}^d} 2^{js}\,|\lambda_{j,k}|\,\mathcal{X}_{j,k}(\cdot)\,\Big|L_p(\mathbb{R}^d,w)\Big\| = \Big(\sum_{k\in\mathbb{Z}^d} 2^{jsp}\,|\lambda_{j,k}|^p \int_{Q_{j,k}} w(x)\,dx\Big)^{1/p}\,.$$

Now we shall replace $\int_{Q_{j,k}} w(x)\,dx$ by the positive real number $y_{j,k}$, i.e., instead of a weight function we are using a sequence of positive real numbers.

Definition 14 *Let $0<p,q\le\infty$, $s\in\mathbb{R}$ and let $y:=(y_{j,k})_{j,k}$ be a sequence of positive real numbers. We put*

$$b^s_{p,q}(\mathbb{R}^d,s-y) := \Big\{ \{\lambda_{j,k}\}_{j,k}:\quad \lambda_{j,k}\in\mathbb{C}\,,$$
$$\|\,(\lambda_{j,k})\,|b^s_{p,q}(\mathbb{R}^d,s-y)\| := \Big(\sum_{j=0}^\infty \Big(\sum_{k\in\mathbb{Z}^d} 2^{jsp}\,|\lambda_{j,k}|^p\,y_{j,k}\Big)^{q/p}\Big)^{1/q} <\infty\Big\}\,. \tag{26}$$

Remark 10 Each space $b^s_{p,q}(\mathbb{R}^d,w)$, $w\in\mathcal{A}^{\ell oc}_\infty$, can be interpreted as a space $b^s_{p,q}(\mathbb{R}^d,s-y)$ by taking

$$y_{j,k} := \int_{Q_{j,k}} w(x)\,dx\,,\qquad j\in\mathbb{N}_0\,,\quad k\in\mathbb{Z}^d.$$

As a consequence of the formula

$$b^{s_0}_{p_0,q_0}(\mathbb{R}^d)^{1-\Theta}\, b^{s_1}_{p_1,q_1}(\mathbb{R}^d)^{\Theta} = b^s_{p,q}(\mathbb{R}^d)\,,$$

due to Kalton, Mayboroda, Mitrea [8, Prop. 9.3], we derive the following.

Corollary 15 *Let $0 < \Theta < 1$. Let $0 < p_0, p_1 \le \infty$, $0 < q_0, q_1 \le \infty$ and $s_0, s_1 \in \mathbb{R}$. Let p, q and s be as in (13). Let $y^0 := (y^0_{j,k})_{j,k}$, $y^1 := (y^1_{j,k})_{j,k}$ be sequences of positive real numbers. We put*

$$y_{j,k} := (y^0_{j,k})^{\frac{(1-\Theta)p}{p_0}}\, (y^1_{j,k})^{\frac{\Theta p}{p_1}}\,.$$

Then

$$b^{s_0}_{p_0,q_0}(\mathbb{R}^d, s-y^0)^{1-\Theta}\, b^{s_1}_{p_1,q_1}(\mathbb{R}^d, s-y^1)^{\Theta} = b^s_{p,q}(\mathbb{R}^d, s-y) \tag{27}$$

holds in the sense of equivalent quasi-norms.

Proof. Of course, $y := (y_{j,k})_{j,k}$ is a sequence of positive real numbers.
Step 1. A preparation. Let $\varrho := (\varrho_{j,k})_{j,k}$ be a sequence of positive real numbers. We introduce the associated family of mappings

$$T_{\varrho,p}:\ (\lambda_{j,k})_{j,k} \mapsto (\lambda_{j,k}\,\cdot\,\varrho^{1/p}_{j,k})_{j,k}\,,$$

where $0 < p \le \infty$ is fixed. Obviously, $T_{\varrho,p}$ is an isomorphism considered as a mapping of $b^s_{p,q}(\mathbb{R}^d, s-\varrho)$ onto $b^s_{p,q}(\mathbb{R}^d)$ for all s and all q.
Step 2. Again the embedding

$$b^{s_0}_{p_0,q_0}(\mathbb{R}^d, s-y^0)^{1-\Theta}\, b^{s_1}_{p_1,q_1}(\mathbb{R}^d, s-y^1)^{\Theta} \hookrightarrow b^s_{p,q}(\mathbb{R}^d, s-y)$$

follows by repeated use of Hölder's inequality.
Step 3. We deal with

$$b^s_{p,q}(\mathbb{R}^d, s-y) \hookrightarrow b^{s_0}_{p_0,q_0}(\mathbb{R}^d, s-y^0)^{1-\Theta}\, b^{s_1}_{p_1,q_1}(\mathbb{R}^d, s-y^1)^{\Theta}\,.$$

Let the sequence $\lambda \in b^s_{p,q}(\mathbb{R}^d, s-y)$ be given. We have to find sequences λ^0 and λ^1 such that $|\lambda_{j,k}| \le |\lambda^0_{j,k}|^{1-\Theta}\cdot|\lambda^1_{j,k}|^{\Theta}$ for every $j \in \mathbb{N}_0$ and $k \in \mathbb{Z}^d$ and

$$\|\lambda^0|b^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)\|^{1-\Theta}\cdot\|\lambda^1|b^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)\|^{\Theta} \le c\,\|\lambda|b^s_{p,q}(\mathbb{R}^d, w)\|$$

with some constant c independent of λ. Now we are going to use the unweighted case

$$b^s_{p,q}(\mathbb{R}^d) = b^{s_0}_{p_0,q_0}(\mathbb{R}^d)^{1-\Theta}\, b^{s_1}_{p_1,q_1}(\mathbb{R}^d)^{\Theta}\,,$$

see Kalton, Mayboroda, Mitrea [8, Prop. 9.3]. Let $\gamma_{j,k} := \lambda_{j,k}\,\cdot\,y^{1/p}_{j,k}$, $j \in \mathbb{N}_0$, $k \in \mathbb{Z}^d$. Since $\gamma := (\gamma_{j,k})_{j,k} \in b^s_{p,q}(\mathbb{R}^d)$ this implies the existence of sequences $\gamma^0 := (\gamma^0_{j,k})_{j,k} \in$

$b^{s_0}_{p_0,q_0}(\mathbb{R}^d)$, $\gamma^1 := (\gamma^1_{j,k})_{j,k} \in b^{s_1}_{p_1,q_1}(\mathbb{R}^d)$ s.t. $|\gamma_{j,k}| \le |\gamma^0_{j,k}|^{1-\Theta} \cdot |\gamma^1_{j,k}|^\Theta$ for all $j \in \mathbb{N}_0$ and all $k \in \mathbb{Z}^d$ and

$$\|\gamma^0|b^{s_0}_{p_0,q_0}(\mathbb{R}^d)\|^{1-\Theta} \cdot \|\gamma^1|b^{s_1}_{p_1,q_1}(\mathbb{R}^d)\|^\Theta \le c\,\|\gamma|b^s_{p,q}(\mathbb{R}^d)\|$$

with some constant c independent of γ. We define

$$\lambda^0_{j,k} := \frac{\gamma^0_{j,k}}{(y^0_{j,k})^{1/p_0}} \qquad \text{and} \qquad \lambda^1_{j,k} := \frac{\gamma^1_{j,k}}{(y^1_{j,k})^{1/p_1}}\,.$$

The sequences λ^0, λ^1 obviously meet the above requirements. ■

For nonnegative and locally integrable functions w_0, w_1, satisfying (12), we define

$$y^i_{j,k} := \int_{Q_{j,k}} w_i(x)\,dx\,, \qquad j \in \mathbb{N}_0\,, \quad k \in \mathbb{Z}^d\,, \quad i = 0,1\,.$$

Then

$$\begin{aligned} b^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, b^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^\Theta &= b^{s_0}_{p_0,q_0}(\mathbb{R}^d, s-y^0)^{1-\Theta}\, b^{s_1}_{p_1,q_1}(\mathbb{R}^d, s-y^1)^\Theta \\ &= b^s_{p,q}(\mathbb{R}^d, s-y) \end{aligned}$$

follows by using the natural interpretation, see (27). However, without extra conditions on the functions w^0 and w^1 we can not interpret $b^s_{p,q}(\mathbb{R}^d, s-y)$ as the space $b^s_{p,q}(\mathbb{R}^d, w)$, where the function w is defined as in (14). A sufficient condition consists in

$$\int_{Q_{j,k}} w(x)\,dx \asymp \Big(\int_{Q_{j,k}} w^0(x)\,dx\Big)^{\frac{(1-\Theta)p}{p_0}} \Big(\int_{Q_{j,k}} w^1(x)\,dx\Big)^{\frac{\Theta p}{p_1}}$$

for all j and all k. This gives rise to the following definition.

Definition 16 *Let $0 < p_0, p_1 \le \infty$ and $0 < \Theta < 1$. Define p by $1/p := (1-\Theta)/p_0 + \Theta/p_1$. Let $w_j : \mathbb{R}^d \to [0,\infty)$, $j = 0,1$, be nonnegative and locally integrable functions s.t. (12) is satisfied. We say that the pair (w_0, w_1) belongs to the class $\mathfrak{W}(\Theta, p_0, p_1)$ if*

$$\int_{Q_{j,k}} w_0(x)^{\frac{(1-\Theta)p}{p_0}}\, w_1(x)^{\frac{\Theta p}{p_1}}(x)\,dx \asymp \Big(\int_{Q_{j,k}} w_0(x)\,dx\Big)^{\frac{(1-\Theta)p}{p_0}} \Big(\int_{Q_{j,k}} w_1(x)\,dx\Big)^{\frac{\Theta p}{p_1}} \tag{28}$$

holds for all $j \in \mathbb{N}_0$ and all $k \in \mathbb{Z}^d$.

Lemma 17 *Suppose $w_0, w_1 \in \mathcal{A}^{\ell oc}_\infty$. Then the pair (w_0, w_1) belongs to all classes $\mathfrak{W}(\Theta, p_0, p_1)$.*

Proof. The inequality

$$\int_{Q_{j,k}} w(x)\,dx \le \Big(\int_{Q_{j,k}} w_0(x)\,dx\Big)^{\frac{(1-\Theta)p}{p_0}} \Big(\int_{Q_{j,k}} w_1(x)\,dx\Big)^{\frac{\Theta p}{p_1}}$$

is a consequence of Hölder's inequality. To prove the opposite inequality we shall use the following characterization of $\mathcal{A}_\infty^{\ell oc}$: a weight v belongs to $\mathcal{A}_\infty^{\ell oc}$ if, and only if, there exists a constant $C > 0$ s.t.

$$\frac{1}{|Q|}\int_Q v(x)dx \,\le\, C\,\exp\Big(\frac{1}{|Q|}\int_Q \log v(x)dx\Big)$$

holds for all cubes (with sides parallel to the axes) and volume ≤ 1, cf. Thm. 2.15, p. 407, in [22]. Using this inequality we obtain for all such cubes Q

$$\Big(\frac{1}{|Q|}\int_Q w_0(x)dx\Big)^{\frac{(1-\Theta)p}{p_0}} \;\le\; C_0\,\exp\Big(\frac{(1-\Theta)p}{p_0}\,\frac{1}{|Q|}\int_Q \log w_0(x)\,dx\Big)$$

and

$$\Big(\frac{1}{|Q|}\int_Q w_1(x)dx\Big)^{\frac{\Theta p}{p_1}} \;\le\; C_1\,\exp\Big(\frac{\Theta p}{p_1}\,\frac{1}{|Q|}\int_Q \log w_1(x)\,dx\Big)\,.$$

Multiplying these inequalities and applying Jensen's inequality yields

$$\begin{aligned}
&\Big(\int_{Q_{j,k}} w_0(x)\,dx\Big)^{\frac{(1-\Theta)p}{p_0}}\,\Big(\int_{Q_{j,k}} w_1(x)\,dx\Big)^{\frac{\Theta p}{p_1}}\\
&\quad\le\; C_0\,C_1\,|Q|\exp\Big(\frac{1}{|Q|}\int_Q \log(w_0(x))^{\frac{(1-\Theta)p}{p_0}}\,dx + \frac{1}{|Q|}\int_Q \log(w_1(x))^{\frac{\Theta p}{p_1}}\,dx\Big)\\
&\quad=\; C\,|Q|\exp\Big(\frac{1}{|Q|}\int_Q \log\big((w_0(x))^{\frac{(1-\Theta)p}{p_0}}(w_1(x))^{\frac{\Theta p}{p_1}}\big)\,dx\Big)\\
&\quad\le\; C\int_Q (w_0(x))^{\frac{(1-\Theta)p}{p_0}}(w_1(x))^{\frac{\Theta p}{p_1}}\,dx\\
&\quad=\; C\int_Q w(x)\,dx\,,
\end{aligned}$$

which completes the proof of the equivalence. ∎

Corollary 18 *Let $0 < \Theta < 1$. Let $0 < p_0, p_1 \le \infty$, $0 < q_0, q_1 \le \infty$ and $s_0, s_1 \in \mathbb{R}$. Let p, q and s be as in (13). Let $w_0, w_1 \in \mathcal{A}_\infty^{\ell oc}$ and w as in (14). Then*

$$b^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, b^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} = b^{s}_{p,q}(\mathbb{R}^d, w) \tag{29}$$

holds in the sense of equivalent quasi-norms.

Proof. This is an immediate consequence of Lemma 17 and Cor. 15. ∎

Remark 11 (i) Any extension of Lemma 17 yields an extension of Cor. 18. From this point of view it would be of interest to characterize the classes $\mathfrak{W}(\Theta, p_0, p_1)$.
(ii) The formula

$$b^{s_0}_{p_0,q_0}(\mathbb{R}^d)^{1-\Theta}\, b^{s_1}_{p_1,q_1}(\mathbb{R}^d)^{\Theta} = b^{s}_{p,q}(\mathbb{R}^d)$$

has been proved by Mendez and Mitrea [7] under the additional restriction $s_0 \neq s_1$. Later on this restriction has been removed by Kalton, Mayboroda and Mitrea [8]. Also in the situation of the b-spaces the case of different weights seems to be new.

3.4 Calderón products of $\mathring{a}^s_{p,q}(\mathbb{R}^d, w)$ spaces

It is of certain use to study Calderón products of the spaces $\mathring{f}^s_{p,q}(\mathbb{R}^d, w)$ and $\mathring{b}^s_{p,q}(\mathbb{R}^d, w)$ separately.

Theorem 19 *Let $0 < \Theta < 1$. Let $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \leq \infty$ and $s_0, s_1 \in \mathbb{R}$. Let p, q and s be defined as in (13). Let $w_0, w_1 \in \mathcal{A}^{\ell oc}_\infty$ and define w by the formula (14). Then*

$$\mathring{f}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, \mathring{f}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} = \mathring{f}^{s}_{p,q}(\mathbb{R}^d, w)$$

and

$$\mathring{f}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} = f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, \mathring{f}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} = \mathring{f}^{s}_{p,q}(\mathbb{R}^d, w)$$

hold in the sense of equivalent quasi-norms.

Proof. The cases $\max(q_0, q_1) < \infty$ are already covered by Thm. 12. We may concentrate on $\max(q_0, q_1) = \infty$. It will be enough to make some comments to the needed modifications in the proof of Thm. 12.
Step 1. We shall prove

$$\mathring{f}^{s_0}_{p_0,\infty}(\mathbb{R}^d, w_0)^{1-\Theta}\, f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} \hookrightarrow \mathring{f}^{s}_{p,q}(\mathbb{R}^d, w)\,.$$

We suppose, that sequences $\lambda := (\lambda_{j,k})_{j,k}$, $\lambda^\ell := (\lambda^\ell_{j,k})_{j,k}$, $\ell = 0, 1$, are given and that

$$|\lambda_{j,k}| \leq |\lambda^0_{j,k}|^{1-\Theta} \cdot |\lambda^1_{j,k}|^{\Theta} \tag{30}$$

holds for all $j \in \mathbb{N}_0$ and all $k \in \mathbb{Z}^d$. Now the essential observation is that if $\lambda^0 := (\lambda^0_{j,k})_{j,k}$ is a finite sequence (i.e. only a finite number of the $\lambda^0_{j,k}$ is not vanishing), then λ has the same property. Employing Step 1 of the proof of Thm. 12 we know

$$\|\lambda|f^s_{p,q}(\mathbb{R}^d, w)\| \leq \|\lambda^0|f^{s_0}_{p_0,\infty}(\mathbb{R}^d, w_0)\|^{1-\Theta} \cdot \|\lambda^1|f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)\|^{\Theta}$$

holds for all such $\lambda, \lambda^0, \lambda^1$.

For $\lambda \in f^s_{p,q}(\mathbb{R}^d, w)$ we define the M-cutoff sequence $\lambda^{(M)}$, $M \in \mathbb{N}$, by putting $\lambda^{(M)}_{j,k} = 0$ if $j > M$ or $\sup_i |k_i| > M$ and $\lambda^{(M)}_{j.k} = \lambda_{j,k}$ otherwise. Then $\lambda \in \mathring{f}^s_{p,q}(\mathbb{R}^d, w)$ if and only if $\lambda^{(M)}$ converge to λ in $f^s_{p,q}(\mathbb{R}^d, w)$ if $M \to \infty$, cf. [17]. Now (30) implies

$$|\lambda_{j,k} - \lambda^{(M)}_{j,k}| \leq |\lambda^0_{j,k} - \lambda^{0\,(M)}_{j,k}|^{1-\Theta} \cdot |\lambda^1_{j,k}|^{\Theta}\,,$$

thus

$$\| \lambda - \lambda^{(M)} \, | f^s_{p,q}(\mathbb{R}^d, w) \| \le \| \lambda^0 - \lambda^{0\,(M)} \, | f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0) \|^{1-\Theta} \cdot \| \lambda^1 | f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1) \|^{\Theta} \, .$$

Hence, $\lambda \in \mathring{f}^s_{p,q}(\mathbb{R}^d, w)$ and from this the claim follows. The embedding

$$f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta} \, \mathring{f}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} \hookrightarrow \mathring{f}^s_{p,q}(\mathbb{R}^d, w)$$

follows by symmetry. Finally, observe

$$\mathring{f}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta} \, \mathring{f}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} \hookrightarrow \mathring{f}^s_{p,q}(\mathbb{R}^d, w) \, .$$

Step 2. Now we turn to the proof of

$$\mathring{f}^s_{p,q}(\mathbb{R}^d, w) \hookrightarrow \mathring{f}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta} \, \mathring{f}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} \, .$$

Let the sequence $\lambda \in \mathring{f}^s_{p,q}(\mathbb{R}^d, w)$ be given. We have to find sequences $\lambda^0 \in \mathring{f}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)$ and $\lambda^1 \in \mathring{f}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)$ such that $|\lambda_{j,k}| \le |\lambda^0_{j,k}|^{1-\Theta} \cdot |\lambda^1_{j,k}|^{\Theta}$ for every $j \in \mathbb{N}_0$ and $k \in \mathbb{Z}^d$ and

$$\| \lambda^0 | f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0) \|^{1-\Theta} \cdot \| \lambda^1 | f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1) \|^{\Theta} \le c \, \| \lambda | f^s_{p,q}(\mathbb{R}^d, w) \| \qquad (31)$$

with some constant c independent of λ. For the moment we suppose that λ is a finite sequence. Since $\max(q_0, q_1) = \infty$, we have to use the formulas (23) and (24) from Step 3 of the proof of Thm. 12. In case $q_0 = q_1 = \infty$ it is immediate that λ^0 and λ^1 are also finite sequences. For the case $0 < q_1 < q_0 = \infty$ we need a simple modification (and similarly in case $0 < q_0 < q_1 = \infty$). Obviously we may assume that λ^1 is a finite sequence. We define

$$\lambda^0_{j,k} := \begin{cases} 2^{\ell\gamma} \, 2^{ju} & \text{if} \quad \lambda^1_{j,k} \neq 0, \\ 0 & \text{otherwise} \, . \end{cases}$$

Then also λ^0 is a finite sequence and (31) is guaranteed by Step 3 of the proof of Thm. 12. Now we turn to a Cauchy sequence $(\lambda^M)_M$ of finite sequences, convergence in $f^s_{p,q}(\mathbb{R}^d, w)$ with limit λ. Using the formulas (23) and (24) we obtain a sequence $\lambda^{1,M}$ of finite sequences which is convergent in $f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)$ with limit λ^1. Similarly, taking into account the above modification, we get a sequence $\lambda^{0,M}$ of finite sequences which is convergent in $f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)$ with limit λ^0. In both cases convergence is derived by using the inequalities

$$\begin{aligned} \| \lambda^0 | f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0) \| \; &\lesssim \; \| \lambda | f^s_{p,q}(\mathbb{R}^d, w) \|^{p/p_0} \\ \| \lambda^1 | f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1) \| \; &\lesssim \; \| \lambda | f^s_{p,q}(\mathbb{R}^d, w) \|^{p/p_1} \, , \end{aligned}$$

see again Step 3 of the proof of Thm. 12. Since

$$\mathring{f}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, \mathring{f}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} \hookrightarrow f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, \mathring{f}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta}$$

as well as

$$\mathring{f}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, \mathring{f}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} \hookrightarrow \mathring{f}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta}$$

the proof is complete. ■

Remark 12 In case $w = w_0 = w_1 \equiv 1$ we refer to Yang, Yuan and Zhuo [17] for a partially different proof.

To derive the counterpart for the b-spaces we will not use the method of Kalton, Mayboroda and Mitrea [8]. These authors reduced the proof of

$$b^s_{p,q}(\mathbb{R}^d) = b^{s_0}_{p_0,q_0}(\mathbb{R}^d)^{1-\Theta}\, b^{s_1}_{p_1,q_1}(\mathbb{R}^d)^{\Theta}$$

to the complex interpolation formula

$$B^s_{p,q}(\mathbb{R}^d) = [B^{s_0}_{p_0,q_0}(\mathbb{R}^d),\, B^{s_1}_{p_1,q_1}(\mathbb{R}^d)]_{\Theta}\,.$$

This time our aim will consists in proving complex interpolation formulas based on assertions on Calderón products. Our proof will be based on the results in the unweighted case, i.e.,

$$\mathring{b}^{s_0}_{p_0,q_0}(\mathbb{R}^d)^{1-\Theta}\, \mathring{b}^{s_1}_{p_1,q_1}(\mathbb{R}^d)^{\Theta} = \mathring{b}^{s}_{p,q}(\mathbb{R}^d)$$

and

$$\mathring{b}^{s_0}_{p_0,q_0}(\mathbb{R}^d)^{1-\Theta}\, b^{s_1}_{p_1,q_1}(\mathbb{R}^d)^{\Theta} = b^{s_0}_{p_0,q_0}(\mathbb{R}^d)^{1-\Theta}\, \mathring{b}^{s_1}_{p_1,q_1}(\mathbb{R}^d)^{\Theta} = \mathring{b}^{s}_{p,q}(\mathbb{R}^d)$$

in the sense of equivalent quasi-norms. For this we refer to the recent paper of Yang, Yuan and Zhuo [17].

Theorem 20 *Let $0 < \Theta < 1$. Let $0 < p_0, p_1, q_0, q_1 \le \infty$ and $s_0, s_1 \in \mathbb{R}$. Let p, q and s be defined as in (13). Let $w_0, w_1 \in \mathcal{A}^{\ell oc}_{\infty}$ and define w by the formula (14). Then*

$$\mathring{b}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, \mathring{b}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} = \mathring{b}^{s}_{p,q}(\mathbb{R}^d, w) \tag{32}$$

and

$$\mathring{b}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, b^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} = b^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, \mathring{b}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} = \mathring{b}^{s}_{p,q}(\mathbb{R}^d, w) \tag{33}$$

hold in the sense of equivalent quasi-norms.

Proof. *Step 1.* Concerning the embedding

$$\mathring{b}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, b^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} \hookrightarrow \mathring{b}^{s}_{p,q}(\mathbb{R}^d, w)$$

the arguments from Step 1 of the proof of Thm. 19 carry over to the present situation. The embedding

$$b^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, \mathring{b}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} \hookrightarrow \mathring{b}^{s}_{p,q}(\mathbb{R}^d, w)$$

follows by symmetry.

Step 2. It remains to prove

$$\mathring{b}^{s}_{p,q}(\mathbb{R}^d, w) \hookrightarrow \mathring{b}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta}\, \mathring{b}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta}\,.$$

Here we proceed as in Step 3 of the proof of Cor. 15 taking into account Lemma 17, and the formulas (32), (33). ■

4 Complex interpolation of weighted Besov and Lizorkin-Triebel spaces

Now we transfer our results on Calderón products into results on complex interpolation.

4.1 Complex interpolation of the spaces $a^s_{p,q}(\mathbb{R}^d, w)$

We have to take into account the following supplement to Prop. 9.

Lemma 21 *Let X_0, X_1 be a pair of quasi-Banach sequence lattices. Then, if both X_0 and X_1 are analytically convex and at least one is separable, it follows that $X_0 + X_1$ is analytically convex and*

$$[X_0, X_1]_\Theta = X_0^{1-\Theta}\, X_1^{\Theta}\,, \qquad 0 < \Theta < 1. \tag{34}$$

Proof. This is the contents of the Remark in front of Thm. 7.10 in [8], see also [7]. ■

We only need to summarize what we did before.

Corollary 22 *Let $0 < \Theta < 1$, $0 < p_0, p_1 \le \infty$, $0 < q_0, q_1 \le \infty$ and $s_0, s_1 \in \mathbb{R}$. Let p, q and s be as in (13). Let $w_0, w_1 \in \mathcal{A}^{\ell oc}_\infty$ and define w by the formula (14).*

(i) *If* $\max(p_0, p_1) < \infty$ *and* $\min(q_0, q_1) < \infty$, *then*

$$[f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0),\, f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)]_\Theta = f^{s}_{p,q}(\mathbb{R}^d, w)$$

holds in the sense of equivalent quasi-norms.
(ii) *If either* $\max(p_0, q_0) < \infty$ *or* $\max(p_1, q_1) < \infty$*, then*

$$[b^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0),\ b^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)]_\Theta = b^s_{p,q}(\mathbb{R}^d, w)$$

holds in the sense of equivalent quasi-norms.

Proof. *Step 1.* Proof of (i). We have to combine Thm. 12, Lemma 4 and Lemma 21. Because of $f^s_{p,q}(\mathbb{R}^d, w)$ is separable if, and only if, $q < \infty$ the claim follows under the condition $\max(q_0, q_1) < \infty$. Taking into account Lemma 21 we replace $\max(q_0, q_1) < \infty$ by $\min(q_0, q_1) < \infty$.
Step 2. Proof of (ii). This time we combine Cor. 18, Lemma 4 and Lemma 21. Because of $b^s_{p,q}(\mathbb{R}^d, w)$ is separable if, and only if, $\max(p, q) < \infty$ the claim follows under the condition $\max(p_0, p_1, q_0, q_1) < \infty$. By means of Lemma 21 we may replace $\max(p_0, p_1, q_0, q_1) < \infty$ by either $\max(p_0, q_0) < \infty$ or $\max(p_1, q_1) < \infty$. ■

Remark 13 In case $w_0 = w_1 \equiv 1$ this has been proved in Frazier, Jawerth [6] (f-case), Mendez, Mitrea [7] (b-case with $s_0 \neq s_1$) and Kalton, Mayboroda and Mitrea [8] (b-case). The formula

$$[f^{s_0}_{p_0,q_0}(\mathbb{R}^d, \mu),\ f^{s_1}_{p_1,q_1}(\mathbb{R}^d, \mu)]_\Theta = f^s_{p,q}(\mathbb{R}^d, \mu)$$

with μ being a doubling measure has been established by Bownik [3]. In case $w_0 = w_1 = w \in \mathcal{A}^{\ell oc}_\infty$

$$[f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0),\ f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)]_\Theta = f^s_{p,q}(\mathbb{R}^d, w)$$

has been proved in Wojciechowska [4].

The same type of arguments, this time applied with Thm. 19 instead of Thm. 12 and with Thm. 20 instead of Cor. 18 yields the next interesting result. Observe, that all spaces $\mathring{a}^s_{p,q}(\mathbb{R}^d, w)$, $a \in \{b, f\}$, are separable and analytically convex (use Lemma 4).

Corollary 23 *Let* $0 < \Theta < 1$*,* $0 < p_0, p_1 \leq \infty$*,* $0 < q_0, q_1 \leq \infty$ *and* $s_0, s_1 \in \mathbb{R}$*. Let* p, q *and* s *be as in (13). Let* $w_0, w_1 \in \mathcal{A}^{\ell oc}_\infty$ *and define* w *by the formula (14).*
(i) *If* $\max(p_0, p_1) < \infty$*, then*

$$\begin{aligned}[\mathring{f}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0),\ \mathring{f}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)]_\Theta &= [\mathring{f}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0),\ f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)]_\Theta \\ &= [f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0),\ \mathring{f}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)]_\Theta = \mathring{f}^s_{p,q}(\mathbb{R}^d, w)\end{aligned}$$

holds in the sense of equivalent quasi-norms.
(ii) *Always we have*

$$\begin{aligned}[\mathring{b}^{s_0}_{p_0,q_0}(\mathbb{R}^d,w_0),\, \mathring{b}^{s_1}_{p_1,q_1}(\mathbb{R}^d,w_1)]_\Theta &= [\mathring{b}^{s_0}_{p_0,q_0}(\mathbb{R}^d,w_0),\, b^{s_1}_{p_1,q_1}(\mathbb{R}^d,w_1)]_\Theta \\ &= [b^{s_0}_{p_0,q_0}(\mathbb{R}^d,w_0),\, \mathring{b}^{s_1}_{p_1,q_1}(\mathbb{R}^d,w_1)]_\Theta = \mathring{b}^{s}_{p,q}(\mathbb{R}^d,w)\end{aligned}$$

in the sense of equivalent quasi-norms.

Remark 14 In case $w_0 = w_1 \equiv 1$, Cor. 23 can be found in Yang, Yuan and Zhuo [17].

4.2 Complex interpolation of weighted Besov and Lizorkin-Triebel spaces

Now we turn to the complex interpolation of the distribution spaces $F^s_{p,q}(\mathbb{R}^d,w)$ and $B^s_{p,q}(\mathbb{R}^d,w)$. For a definition of these classes we refer to the Appendix. Observe that neither $F^s_{p,q}(\mathbb{R}^d,w)$ nor $B^s_{p,q}(\mathbb{R}^d,w)$ are quasi-Banach lattices in general. Here our main result is as follows.

Theorem 24 *Let* $0 < p_0, p_1 \le \infty$, $0 < q_0, q_1 \le \infty$, $s_0, s_1 \in \mathbb{R}$, $0 < \Theta < 1$ *and define* p, q *and* s *according to (13). Let* $w_0, w_1 \in \mathcal{A}^{\ell oc}_\infty$ *be local Muckenhoupt weights and* w *defined as in (14).*
(i) *If* $\max(p_0,p_1) < \infty$ *and* $\min(q_0,q_1) < \infty$, *then*

$$[F^{s_0}_{p_0,q_0}(\mathbb{R}^d,w_0),\, F^{s_1}_{p_1,q_1}(\mathbb{R}^d,w_1)]_\Theta = F^s_{p,q}(\mathbb{R}^d,w)$$

holds in the sense of equivalent quasi-norms.
(ii) *If either* $\max(p_0,q_0) < \infty$ *or* $\max(p_1,q_1) < \infty$, *then*

$$[B^{s_0}_{p_0,q_0}(\mathbb{R}^d,w_0),\, B^{s_1}_{p_1,q_1}(\mathbb{R}^d,w_1)]_\Theta = B^s_{p,q}(\mathbb{R}^d,w)$$

holds in the sense of equivalent quasi-norms.

Proof. It is enough to combine Cor. 22 and Prop. 37 (in the Appendix). If either $p_0 = \infty$ or $p_1 = \infty$, then one has to take into account also Remark 22. ■

Remark 15 (i) We do not know whether the condition $\max(p_0,q_0) < \infty$ or $\max(p_1,q_1) < \infty$ in part (ii) of the theorem can be replaced by the weaker condition $\min(q_0,q_1) < \infty$ (compare with Prop. 5).

(ii) The above results complement the knowledge on interpolation of weighted Besov and Lizorkin-Triebel spaces with Muckenhoupt weights. Bownik [3] has proved

$$[F^{s_0}_{p_0,q_0}(\mathbb{R}^d,\mu),\, F^{s_1}_{p_1,q_1}(\mathbb{R}^d,\mu)]_\Theta = F^s_{p,q}(\mathbb{R}^d,\mu)\,,$$

where μ is a doubling measure. Furthermore, Wojciechowska [4] recently proved

$$[F^{s_0}_{p_0,q_0}(\mathbb{R}^d,w),\, F^{s_1}_{p_1,q_1}(\mathbb{R}^d,w)]_\Theta = F^s_{p,q}(\mathbb{R}^d,w)\,,$$

where $w \in \mathcal{A}^{\ell oc}_\infty$. For various interpolation formulas for the real method we refer to Bui [23] ($w \in \mathcal{A}_\infty$) and Rychkov [21] ($w \in \mathcal{A}^{\ell oc}_\infty$).

We finish this subsection by formulating a consequence of Cor. 23. Let $\mathring{A}^s_{p,q}(\mathbb{R}^d,w)$ denote the closure of the test functions in $A^s_{p,q}(\mathbb{R}^d,w)$. Using the same arguments as above, but replacing Prop. 37 by Prop. 38, we obtain the following.

Theorem 25 *Let $0<\Theta<1$, $0<p_0,p_1\le\infty$, $0<q_0,q_1\le\infty$ and $s_0,s_1\in\mathbb{R}$. Let p,q and s be as in (13). Let $w_0,w_1\in\mathcal{A}^{\ell oc}_\infty$ and define w by the formula (14).*
(i) *If $\max(p_0,p_1)<\infty$, then*

$$\begin{aligned}[]
[\mathring{F}^{s_0}_{p_0,q_0}(\mathbb{R}^d,w_0),\, \mathring{F}^{s_1}_{p_1,q_1}(\mathbb{R}^d,w_1)]_\Theta &= [\mathring{F}^{s_0}_{p_0,q_0}(\mathbb{R}^d,w_0),\, F^{s_1}_{p_1,q_1}(\mathbb{R}^d,w_1)]_\Theta \\
&= [F^{s_0}_{p_0,q_0}(\mathbb{R}^d,w_0),\, \mathring{F}^{s_1}_{p_1,q_1}(\mathbb{R}^d,w_1)]_\Theta = \mathring{F}^s_{p,q}(\mathbb{R}^d,w)
\end{aligned}$$

holds in the sense of equivalent quasi-norms.
(ii) *Always we have*

$$\begin{aligned}[]
[\mathring{B}^{s_0}_{p_0,q_0}(\mathbb{R}^d,w_0),\, \mathring{B}^{s_1}_{p_1,q_1}(\mathbb{R}^d,w_1)]_\Theta &= [\mathring{B}^{s_0}_{p_0,q_0}(\mathbb{R}^d,w_0),\, B^{s_1}_{p_1,q_1}(\mathbb{R}^d,w_1)]_\Theta \\
&= [B^{s_0}_{p_0,q_0}(\mathbb{R}^d,w_0),\, \mathring{B}^{s_1}_{p_1,q_1}(\mathbb{R}^d,w_1)]_\Theta = \mathring{B}^s_{p,q}(\mathbb{R}^d,w)
\end{aligned}$$

in the sense of equivalent quasi-norms.

Remark 16 (i) Of course, Thm. 25 is only of interest in the cases $\max(p_0,p_1,q_0,q_1)=\infty$. All other cases are covered by Thm. 24.
(ii) Thm. 25 in case $w_0=w_1\equiv 1$ has been proved in Yang, Yuan, Zhuo [17]. Since a long time for Besov spaces the case $w_0=w_1\equiv 1$, $1<p_0,p_1<\infty$, has been known. We refer to Triebel [2, Rem. 2.4.1/3].

4.3 Shrinking some gaps

The results in Subsection 4.2 do not cover all possible interpolation couples of the form

$$[A^{s_0}_{p_0,q_0}(\mathbb{R}^d,w_0),\, A^{s_1}_{p_1,q_1}(\mathbb{R}^d,w_1)]_\Theta\,, \qquad A\in\{B,F\}\,.$$

Those cases, where both spaces are not separable, are not covered. This means that a description of

$$
\begin{aligned}
&[F^{s_0}_{p_0,\infty}(\mathbb{R}^d, w_0),\, F^{s_1}_{p_1,\infty}(\mathbb{R}^d, w_1)]_\Theta && 0 < p_0, p_1 < \infty\,, \\
&[B^{s_0}_{p_0,\infty}(\mathbb{R}^d, w_0),\, B^{s_1}_{p_1,\infty}(\mathbb{R}^d, w_1)]_\Theta && 0 < p_0, p_1 \le \infty\,, \\
&[B^{s_0}_{\infty,q_0}(\mathbb{R}^d, w_0),\, B^{s_1}_{\infty,q_1}(\mathbb{R}^d, w_1)]_\Theta && 0 < q_0, q_1 \le \infty\,, \quad \min(q_0, q_1) < \infty\,,
\end{aligned}
$$

is still open. We can not fill this gap. However, we can make it smaller. The method we will apply is based on a result of Shestakov [9] (see also [14, Rem. 4.3.5, pp. 557]): for Banach lattices X_0, X_1 and $0 < \Theta < 1$ we have the identity

$$
[X_0, X_1]_\Theta = \overline{X_0 \cap X_1}^{Y_\Theta}\,, \qquad Y_\Theta := X_0^{1-\Theta}\, X_1^\Theta\,. \tag{35}
$$

To use these results we have to switch to sequence spaces for a moment. Thm. 12 and Cor. 18 combined with (35) yield

$$
[a^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0),\, a^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)]_\Theta \hookrightarrow a^s_{p,q}(\mathbb{R}^d, w)\,, \qquad a \in \{b, f\}\,,
$$

since

$$
\overline{a^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0) \,\cap\, a^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)}^{a^s_{p,q}(\mathbb{R}^d,w)} \hookrightarrow a^s_{p,q}(\mathbb{R}^d, w)\,.
$$

An application of the trivial embedding

$$
[\mathring{A}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0),\, \mathring{A}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)]_\Theta \hookrightarrow [A^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0),\, A^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)]_\Theta\,, \qquad A \in \{B, F\}\,,
$$

and of Prop. 37 yield the following lemma.

Lemma 26 *Let $1 \le p_0, p_1 \le \infty$, $1 \le q_0, q_1 \le \infty$, $s_0, s_1 \in \mathbb{R}$, $0 < \Theta < 1$ and define p, q and s according to (13). Let $w_0, w_1 \in \mathcal{A}^{\ell oc}_\infty$ be local Muckenhoupt weights and w defined as in (14).*
(i) *If $\max(p_0, p_1) < \infty$, then*

$$
\mathring{F}^s_{p,q}(\mathbb{R}^d, w) \hookrightarrow [F^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0),\, F^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)]_\Theta \hookrightarrow F^s_{p,q}(\mathbb{R}^d, w)
$$

holds.
(ii) *Always we have*

$$
\mathring{B}^s_{p,q}(\mathbb{R}^d, w) \hookrightarrow [B^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0),\, B^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)]_\Theta \hookrightarrow B^s_{p,q}(\mathbb{R}^d, w)\,.
$$

Remark 17 Let $w \in \mathcal{A}^{\ell oc}_\infty$. The restriction to Banach spaces in Lemma 26 is caused by the use of (35). Recently, Yang, Yuan and Zhuo [17] have proved

$$
[a^{s_0}_{p_0,q_0}(\mathbb{R}^d, w),\, a^{s_1}_{p_1,q_1}(\mathbb{R}^d, w)]_\Theta \hookrightarrow a^s_{p,q}(\mathbb{R}^d, w)\,, \qquad a \in \{b, f\}\,,
$$

without restrictions on the parameters. In fact, they did it in the unweighted case, but it can be immediately lifted to the present situation. Hence, Lemma 26 remains true for values of $p_0, p_1, q_0, q_1 < 1$ if restricted to the case of one weight.

As mentioned above, in some situations we can go one step further.

Theorem 27 *Let $s_0, s_1 \in \mathbb{R}$ and $0 < \Theta < 1$. Further, let either $1 \le p_0 = p_1 < \infty$ and $s_0 > s_1$ or $1 \le p_0 < p_1 < \infty$ and*

$$s_0 - \frac{d}{p_0} \ge s_1 - \frac{d}{p_1.} \tag{36}$$

As always p, q and s are defined according to (13). Let $w \in \mathcal{A}_\infty^{\ell oc}$ be a local Muckenhoupt weight. Then

$$\mathring{F}^s_{p,\infty}(\mathbb{R}^d, w) = [F^{s_0}_{p_0,\infty}(\mathbb{R}^d, w),\, F^{s_1}_{p_1,\infty}(\mathbb{R}^d, w)]_\Theta \,.$$

holds.

Proof. The conditions guarantee

$$F^{s_0}_{p_0,\infty}(\mathbb{R}^d) \hookrightarrow F^s_{p,1}(\mathbb{R}^d) \hookrightarrow F^{s_1}_{p_1,\infty}(\mathbb{R}^d)\,,$$

see [13, Thm. 2.7.1]. By means of Prop. 37 this yields

$$f^{s_0}_{p_0,\infty}(\mathbb{R}^d) \hookrightarrow f^s_{p,1}(\mathbb{R}^d) \hookrightarrow f^{s_1}_{p_1,\infty}(\mathbb{R}^d)\,,$$

which can be lifted to the weighted case

$$f^{s_0}_{p_0,\infty}(\mathbb{R}^d, w) \hookrightarrow f^s_{p,1}(\mathbb{R}^d, w) \hookrightarrow f^{s_1}_{p_1,1}(\mathbb{R}^d, w)$$

by using an appropriate isomorphism. This implies

$$f^{s_0}_{p_0,\infty}(\mathbb{R}^d, w) \cap f^{s_1}_{p_1,\infty}(\mathbb{R}^d, w) = f^{s_0}_{p_0,\infty}(\mathbb{R}^d, w)\,.$$

We claim

$$\overline{f^{s_0}_{p_0,\infty}(\mathbb{R}^d, w)}^{f^s_{p,\infty}(\mathbb{R}^d,w)} = \mathring{f}^s_{p,\infty}(\mathbb{R}^d, w)\,.$$

To prove this, observe $\mathring{f}^s_{p,1}(\mathbb{R}^d, w) = f^s_{p,1}(\mathbb{R}^d, w)$. Hence, finite sequences are dense in the set $f^{s_0}_{p_0,\infty}(\mathbb{R}^d, w)$ when equipped with the quasi-norm of $f^s_{p,1}(\mathbb{R}^d, w)$. From the trivial embedding $f^s_{p,1}(\mathbb{R}^d, w) \hookrightarrow f^s_{p,\infty}(\mathbb{R}^d, w)$. We derive the density of the finite sequences in the set $f^{s_0}_{p_0,\infty}(\mathbb{R}^d, w)$ when equipped with the quasi-norm of $f^s_{p,\infty}(\mathbb{R}^d, w)$. Shestakov's identity (35) yields

$$\mathring{f}^s_{p,\infty}(\mathbb{R}^d, w) = [f^{s_0}_{p_0,\infty}(\mathbb{R}^d, w),\, f^{s_1}_{p_1,\infty}(\mathbb{R}^d, w)]_\Theta \,.$$

By means of Prop. 37 we complete the proof. ∎

Arguing as before we can prove the following counterpart for Besov spaces. For the embedding relations we refer to [13, 2.7.1].

Theorem 28 *Let $1 \leq q_0, q_1 \leq \infty$, $s_0, s_1 \in \mathbb{R}$ and $0 < \Theta < 1$. Further, let $1 \leq p_0 \leq p_1 \leq \infty$ and*

$$s_0 - \frac{d}{p_0} > s_1 - \frac{d}{p_1} \tag{37}$$

As always p, q and s are defined according to (13). Let $w \in \mathcal{A}_\infty^{\ell oc}$ be a local Muckenhoupt weight. Then

$$\mathring{B}^s_{p,q}(\mathbb{R}^d, w) = [B^{s_0}_{p_0,q_0}(\mathbb{R}^d, w),\, B^{s_1}_{p_1,q_1}(\mathbb{R}^d, w)]_\Theta \,.$$

holds.

Remark 18 (i) The case $w \equiv 1$, $1 < p_0 = p_1 < \infty$, $q_0 = q_1 = q = \infty$ has been known before, we refer to Triebel [2, Thm. 2.4.1].
(ii) Let us comment on the conditions (36) and (37). Let $w \equiv 1, 1 \leq p_0 < p_1 \leq \infty$ and

$$s_0 - \frac{d}{p_0} \leq s_1 - \frac{d}{p_1}. \tag{38}$$

Furthermore, for $j \in \mathbb{N}_0$ let K_j be a subset of $\mathbb{Z}^d$ with cardinality

$$\# K_j = \lceil 2^{-j\{(s_1 - s_0)\cdot \frac{1}{1/p_1 - 1/p_0} - d\}} \rceil \,,$$

where $\lceil t \rceil$ denotes the smallest integer larger than or equal to $t \in \mathbb{R}$. We define a sequence $\lambda := \{\lambda_{j,k}\}_{j,k}$ by

$$\lambda_{j,k} = \begin{cases} 2^{j\cdot \frac{p_1 s_1 - p_0 s_0}{p_0 - p_1}} & \text{if} \quad k \in K_j, \\ 0 & \text{otherwise.} \end{cases}$$

A simple calculation shows that $\lambda \in b^s_{p,\infty}(\mathbb{R}^d) \setminus \mathring{b}^s_{p,\infty}(\mathbb{R}^d)$ as well as $\lambda \in b^{s_0}_{p_0,\infty}(\mathbb{R}^d) \cap b^{s_1}_{p_1,\infty}(\mathbb{R}^d)$. The result of Shestakov (35) then yields

$$\lambda \in \overline{b^{s_0}_{p_0,\infty}(\mathbb{R}^d) \cap b^{s_1}_{p_1,\infty}(\mathbb{R}^d)}^{b^s_{p,\infty}(\mathbb{R}^d)} = [b^{s_0}_{p_0,\infty}(\mathbb{R}^d), b^{s_1}_{p_1,\infty}(\mathbb{R}^d)]_\theta.$$

Hence, the embedding $\mathring{b}^s_{p,q}(\mathbb{R}^d) \hookrightarrow [b^{s_0}_{p_0,q_0}(\mathbb{R}^d),\, b^{s_1}_{p_1,q_1}(\mathbb{R}^d)]_\Theta$ is strict in this case. With other words, (37) is also necessary for the validity of the interpolation formula in Thm. 28.

5 Complex interpolation of radial subspaces of Besov and Lizorkin-Triebel spaces

In a series of papers the authors have studied radial subspaces of Besov and Lizorkin-Triebel spaces, see [24, 25, 26]. The motivation came from the interesting interplay of decay and smoothness properties of radial functions as expressed in its simplest form in the radial Lemma of Strauss [27] and with important applications for the compactness of some embeddings. We refer also to Lions [28] and Cho and Ozawa [29] in this connection.

5.1 The main result for radial subspaces

In [30] one of the authors has proved that in case $p,q \geq 1$ the spaces $RB^s_{p,q}(\mathbb{R}^d)$, $RF^s_{p,q}(\mathbb{R}^d)$ are complemented subspaces of $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$, respectively. By means of the method of retraction and coretraction, see, e.g., Theorem 1.2.4 in [2], this allows to transfer the interpolation formulas for the original spaces $B^s_{p,q}(\mathbb{R}^d)$, $F^s_{p,q}(\mathbb{R}^d)$ to its radial subspaces.
Such simple arguments do not seem to be available in case of quasi-Banach spaces. In [25] we announced the following result concerning complex interpolation of radial subspaces of Besov and Lizorkin-Triebel spaces.

Theorem 29 *Let* $0 < p_0, p_1 < \infty$, $0 < q_0,\, q_1 \leq \infty$, $s_0, s_1 \in \mathbb{R}$, *and* $0 < \Theta < 1$. *Define* $s := (1-\Theta)\, s_0 + \Theta\, s_1$,

$$\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \qquad \textit{and} \qquad \frac{1}{q} := \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}\, .$$

(i) *Let* $\min(q_0,q_1) < \infty$. *Then we have*

$$RB^s_{p,q}(\mathbb{R}^d) = \Big[RB^{s_0}_{p_0,q_0}(\mathbb{R}^d), RB^{s_1}_{p_1,q_1}(\mathbb{R}^d)\Big]_\Theta\, .$$

(ii) *Let* $\min(q_0,q_1) < \infty$. *Then we have*

$$RF^s_{p,q}(\mathbb{R}^d) = \Big[RF^{s_0}_{p_0,q_0}(\mathbb{R}^d), RF^{s_1}_{p_1,q_1}(\mathbb{R}^d)\Big]_\Theta\, .$$

5.2 The proof of Thm. 29

We are going to prove Thm. 29. Our main idea consists in a reduction of this problem to the weighted case. Therefore we need to recall some results from [25].
For a real number s we denote by $[s]$ the integer part, i.e. the largest integer m such that $m \leq s$. We put $w_{d-1}(t) := |t|^{d-1}$, $t \in \mathbb{R}$, $d \geq 2$. Finally, if $f : \mathbb{R}^d \to \mathbb{C}$ is a radial function, we denote by

$$(\mathrm{tr}\ f)(t) = f(t,0,\ldots,0), \quad t \in \mathbb{R}$$

its trace. The corresponding extension operator is then given by

$$(\mathrm{ext}\ g)(x) = g(|x|), \quad x \in \mathbb{R}^d,$$

where g is an even function on $\mathbb{R}$.

Proposition 30 *Let* $d \geq 2$, $0 < p < \infty$ *and* $0 < q \leq \infty$.
(i) *Suppose either* $s > d(\frac{1}{p} - \frac{1}{d})$ *or* $s = d(\frac{1}{p} - \frac{1}{d})$ *and* $q \leq 1$. *Then the mapping* tr *is a linear isomorphism of* $RB^s_{p,q}(\mathbb{R}^d)$ *onto* $RB^s_{p,q}(\mathbb{R}, w_{d-1})$ *with inverse* ext .
(ii) *Suppose either* $s > d(\frac{1}{p} - \frac{1}{d})$ *or* $s = d(\frac{1}{p} - \frac{1}{d})$ *and* $0 < p \leq 1$. *Then the mapping* tr *is a linear isomorphism of* $RF^s_{p,q}(\mathbb{R}^d)$ *onto* $RF^s_{p,q}(\mathbb{R}, w_{d-1})$ *with inverse* ext .

Remark 19 Let $p > 1$. As it is well-known, the weight $w(x) := |x|^\alpha$ belongs to $\mathcal{A}_p(\mathbb{R})$ if, and only if, $-1 < \alpha < p-1$. This implies $w_{d-1} \in \mathcal{A}_p(\mathbb{R})$ for any $p > d$, see [20, pp. 218].

We would like to apply Thm. 24 with respect to $RB^s_{p,q}(\mathbb{R}, w_{d-1})$ and $RF^s_{p,q}(\mathbb{R}, w_{d-1})$, respectively. Therefore we consider the following mapping:

$$Tf(x) := \frac{1}{2}\left(f(x) + f(-x)\right), \qquad x \in \mathbb{R}\,.$$

Of course,

$$T \in \mathcal{L}(A^s_{p,q}(\mathbb{R}, w_{d-1}), RA^s_{p,q}(\mathbb{R}, w_{d-1}))\,, \qquad A \in \{B, F\}\,.$$

Furthermore, $Tf = f$ if $f \in RA^s_{p,q}(\mathbb{R}, w_{d-1})$, i.e., $RA^s_{p,q}(\mathbb{R}, w_{d-1})$ is a retract of $A^s_{p,q}(\mathbb{R}, w_{d-1})$. By the standard method of retraction and coretraction, see [2, 1.2.4] and Lemma 7.11 in [8], we obtain the following.

Lemma 31 *Let $d \ge 2$, $0 < \Theta < 1$, $0 < p_0, p_1 \le \infty$, $0 < q_0, q_1 \le \infty$, $s_0, s_1 \in \mathbb{R}$ and define p, q and s according to (13).*
(i) *If* $\max(p_0, p_1) < \infty$ *and* $\min(q_0, q_1) < \infty$, *then*

$$[RF^{s_0}_{p_0,q_0}(\mathbb{R}, w_{d-1}),\, RF^{s_1}_{p_1,q_1}(\mathbb{R}, w_{d-1})]_\Theta = RF^s_{p,q}(\mathbb{R}, w_{d-1})$$

holds in the sense of equivalent quasi-norms.
(ii) *If* $\max(p_0, q_0) < \infty$ *or* $\max(p_1, q_1) < \infty$, *then*

$$[RB^{s_0}_{p_0,q_0}(\mathbb{R}, w_{d-1}),\, RB^{s_1}_{p_1,q_1}(\mathbb{R}, w_{d-1})]_\Theta = RB^s_{p,q}(\mathbb{R}, w_{d-1})$$

holds in the sense of equivalent quasi-norms.

The next step consists in combining Lemma 31 and Prop. 30.

Lemma 32 *Let $d \ge 2$, $0 < \Theta < 1$, $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \le \infty$, $s_0, s_1 \in \mathbb{R}$ and define p, q and s according to (13). Furthermore, we suppose*

$$s_i > d\,(\frac{1}{p_i} - \frac{1}{d})\,, \qquad i = 0, 1\,, \tag{39}$$

If $\min(q_0, q_1) < \infty$, *then*

$$[RF^{s_0}_{p_0,q_0}(\mathbb{R}^d),\, RF^{s_1}_{p_1,q_1}(\mathbb{R}^d)]_\Theta = RF^s_{p,q}(\mathbb{R}^d)$$

and

$$[RB^{s_0}_{p_0,q_0}(\mathbb{R}^d),\, RB^{s_1}_{p_1,q_1}(\mathbb{R}^d)]_\Theta = RB^s_{p,q}(\mathbb{R}^d)$$

hold in the sense of equivalent quasi-norms.

Proof of Thm. 29. It is enough to remove the restrictions for s_0 and s_1 in Lemma 32, see (39). But this is an easy task. Let $\sigma \in \mathbb{R}$. We consider the family of lifting operators

$$I_\sigma f := \mathcal{F}^{-1}\,[(1+|\xi|^2)^{\sigma/2}\,\mathcal{F}\,f(\xi)](\,\cdot\,)\,, \qquad f \in \mathcal{S}'(\mathbb{R}^d)\,.$$

As it is well-known, see, e.g., [13, Thm. 2.3.8], I_σ is an isomorphism, which maps $A^s_{p,q}(\mathbb{R}^d)$ onto $A^{s-\sigma}_{p,q}(\mathbb{R}^d)$. By standard properties of the Fourier transform we deduce that $f \in RA^s_{p,q}(\mathbb{R}^d)$ implies $I_\sigma f \in RA^{s-\sigma}_{p,q}(\mathbb{R}^d)$. By the same argument, I_σ is an isomorphism, which maps $RA^s_{p,q}(\mathbb{R}^d)$ onto $RA^{s-\sigma}_{p,q}(\mathbb{R}^d)$. Hence,

$$[I_\sigma(RF^{s_0}_{p_0,q_0}(\mathbb{R}^d)),\, I_\sigma(RF^{s_1}_{p_1,q_1}(\mathbb{R}^d))]_\Theta = I_\sigma(RF^s_{p,q}(\mathbb{R}^d))\,.$$

Now, Thm. 29 follows from Lemma 32 by choosing σ appropriate.

6 Appendix – Muckenhoupt weights and function spaces

For convenience of the reader we collect some definitions and properties around Muckenhoupt weights and associated weighted function spaces.

6.1 Muckenhoupt weights

A weight function (or simply a weight) is a nonnegative and measurable function on $\mathbb{R}^d$. We collect a few facts including the definition of Muckenhoupt and local Muckenhoupt weights. As usual, p' is related to p via the formula $1/p + 1/p' = 1$.

Definition 33 *Let $1 < p < \infty$. Let w be a nonnegative, locally integrable function on $\mathbb{R}^d$.*
(i) Then w belongs to the Muckenhoupt class $\mathcal{A}_p$, if

$$A_p(w) := \sup_B \Big(\frac{1}{|B|}\int_B w(x)\,dx\Big)^{1/p} \cdot \Big(\frac{1}{|B|}\int_B w(x)^{-p'/p}\,dx\Big)^{1/p'} < \infty\,,$$

where the supremum is taken with respect to all balls B in $\mathbb{R}^d$.
(ii) The weight w belongs to the local Muckenhoupt class $\mathcal{A}^{\ell oc}_p$, if we restrict the set of admissible balls in the supremum in (i) to those with volume ≤ 1. We put

$$A^{\ell oc}_p(w) := \sup_{|B|\le 1} \Big(\frac{1}{|B|}\int_B w(x)\,dx\Big)^{1/p} \cdot \Big(\frac{1}{|B|}\int_B w(x)^{-p'/p}\,dx\Big)^{1/p'} < \infty\,.$$

The classes $\mathcal{A}_\infty$ and $\mathcal{A}_\infty^{\ell oc}$ are defined as

$$\mathcal{A}_\infty := \bigcup_{p>1} \mathcal{A}_p \qquad \text{and} \qquad \mathcal{A}_\infty^{\ell oc} := \bigcup_{p>1} \mathcal{A}_p^{\ell oc}\,,$$

respectively.

Remark 20 (i) Good sources for Muckenhoupt weights are Stein [20], Garcia-Cuerva and Rubio de Francia [22] and the graduate text [31] of Duoandikoetxea.
(ii) The classes of local Muckenhoupt weights $\mathcal{A}_p^{\ell oc}$ have been introduced by Rychkov [21].

The following Lemma of Rychkov [21] will be of some use.

Lemma 34 *Let $1 < p \le \infty$, $w \in \mathcal{A}_p^{\ell oc}$, and Q a cube with sides parallel to the axes and volume 1. Then there exists a weight $\overline{w} \in \mathcal{A}_p$ s.t.*

$$\overline{w} = w \qquad \textit{on } Q \textit{ and} \qquad A_p(\overline{w}) \le c\, A_p^{\ell oc}(w)\,.$$

Here c does not depend on w and Q.

By Mf we denote the Hardy-Littlewood maximal function of f, given by

$$Mf(x) := \sup_B \frac{1}{|B|} \int_B |f(y)|\, dy\,,$$

where the supremum is taken with respect to all balls in $\mathbb{R}^d$ with center x. Furthermore, by $M^{\ell oc}$ we denote the following local counterpart of M, namely

$$M^{\ell oc} f(x) := \sup_Q \frac{1}{|Q|} \int_Q |f(y)|\, dy\,,$$

where the supremum is taken with respect to all cubes Q containing x, with sides parallel to the axes and volume ≤ 1. The weighted Lebesgue space $L_p(\mathbb{R}^d, w)$ is the collection of all measurable functions $f:\, \mathbb{R}^d \to \mathbb{C}$ such that

$$\|\, f\, |L_p(\mathbb{R}^d, w)\| := \Big(\int_{\mathbb{R}^d} |f(x)|^p\, w(x)\, dx \Big)^{1/p} < \infty\,.$$

In case $p = \infty$ we are back in the unweighted situation, i.e., $w \equiv 1$. We shall also need the following maximal inequality. Let $1 < p < \infty$, $1 < q \le \infty$ and $w \in \mathcal{A}_p$. Then there exists a constant c such that

$$\Big(\int_{\mathbb{R}^d} \Big(\sum_{j=0}^\infty |Mf_j(x)|^q \Big)^{p/q} w(x)\, dx \Big)^{1/q} \le c \Big(\int_{\mathbb{R}^d} \Big(\sum_{j=0}^\infty |f_j(x)|^q \Big)^{p/q} w(x)\, dx \Big)^{1/q}, \quad (40)$$

holds for all sequences $(f_j)_j \subset L_p(\mathbb{R}^d, w)$, see [32], [33] or [20, Thm. V.3.1]. Rychkov [21] proved the local version: for $1 < p < \infty$, $1 < q \le \infty$ and $w \in \mathcal{A}_p^{\ell oc}$ there exists a constant c s.t.

$$\Big(\int_{\mathbb{R}^d} \Big(\sum_{j=0}^{\infty} |M^{\ell oc} f_j(x)|^q \Big)^{p/q} w(x)\, dx \Big)^{1/q} \le c \Big(\int_{\mathbb{R}^d} \Big(\sum_{j=0}^{\infty} |f_j(x)|^q \Big)^{p/q} w(x)\, dx \Big)^{1/q}, \quad (41)$$

holds for all sequences $(f_j)_j \subset L_p(\mathbb{R}^d, w)$.

We need one further property of Muckenhoupt weights.

Lemma 35 *Let* $0 < \Theta < 1$ *and* $0 < p_0, p_1 \le \infty$. *We put*

$$\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}\,.$$

(i) Let $w_0, w_1 \in \mathcal{A}_\infty$ *and define*

$$w := w_0^{\frac{(1-\Theta)p}{p_0}}\, w_1^{\frac{\Theta p}{p_1}}\,.$$

Then $w \in \mathcal{A}_\infty$.

(ii) If w_0, w_1 *belong to* $\mathcal{A}_\infty^{\ell oc}$, *then* $w \in \mathcal{A}_\infty^{\ell oc}$.

Proof. *Step 1.* We prove (i). If $w_0, w_1 \in \mathcal{A}_\infty$ then there exist $r_0, r_1 \in (1, \infty)$ such that $w_i \in \mathcal{A}_{r_i}$, $i = 0, 1$. First, let $\max(p_0, p_1) < \infty$. If $r_i \le p_i$, then by the monotonicity of Muckenhoupt classes $w_i \in \mathcal{A}_{p_i}$ and this implies that $w \in \mathcal{A}_p$, see Stein [20, V.6.1(a), pp. 218]. If $\max\big(\frac{r_0}{p_0}, \frac{r_1}{p_1}\big) > 1$, then we choose $\alpha > \max\big(\frac{r_0}{p_0}, \frac{r_1}{p_1}\big)$. Now $w_i \in \mathcal{A}_{\alpha p_i}$, $i = 0, 1$, follows because of

$$\frac{1}{\alpha\, p} = \frac{1-\Theta}{\alpha\, p_0} + \frac{\Theta}{\alpha\, p_1} \qquad \text{and} \qquad w = w_0^{\frac{(1-\Theta)\alpha\, p}{\alpha\, p_0}}\, w_1^{\frac{\Theta\,\alpha\, p}{\alpha\, p_1}}\,.$$

So applying the same argument as before we get $w \in \mathcal{A}_{\alpha p} \subset \mathcal{A}_\infty$. For the remaining cases $\max(p_0, p_1) = \infty$ it is enough to observe that the function $w(x) = 1$, $x \in \mathbb{R}^d$, belongs to $\mathcal{A}_\infty$ ($p_0 = p_1 = \infty$) and in case $0 < p_0 < \infty = p_1$ we have $(1-\theta)p/p_0 = 1$, i.e., $w = w_0$.

Step 2. The monotonicity of the local Muckenhoupt classes $\mathcal{A}_p^{\ell oc}$ has been proved in [21]. The above used result from [20, V.6.1(a), pp. 218] is based on Hölder's inequality. For that reason it carries over to the local situation. Alternatively one could argue with Lemma 34. ∎

6.2 Weighted Besov and Lizorkin-Triebel spaces

Now we introduce weighted Besov and Lizorkin-Triebel spaces. Since we work with local Muckenhoupt weights we need larger space of distributions than the spaces of tempered distributions. Recall that the class $\mathcal{A}_\infty^{loc}$ contains weights of exponential growth. We follows the ideas of Rychkov that are based on local reproducing formula, cf. [21].

Let $S_e(\mathbb{R}^d)$ denote the set of all $\psi \in C^\infty(\mathbb{R}^d)$ such that

$$q_N(\psi) = \sup_{x\in\mathbb{R}^d} e^{N|x|} \sum_{|\alpha|\le N} |\partial^\alpha \psi(x)| \, < \, \infty \qquad \text{for all} \quad N \in \mathbb{N}_0 \, .$$

Then $S_e(\mathbb{R}^d)$, equipped with the topology generated by the system of semi-norms q_N, is a locally convex space. Its dual space $S_e'(\mathbb{R}^d)$ can be identified with a subspace of the space of distributions $\mathcal{D}'(\mathbb{R}^d)$ in the obvious way.

Let $\varphi_0 \in C_0^\infty(\mathbb{R}^d)$ and $\varphi(x) = \varphi_0(x) - 2^{-d}\varphi_0(\frac{x}{2})$ be functions such that

$$\int_{\mathbb{R}^d} \varphi_0(x)dx \, = \, 0 \qquad \text{and} \qquad \int_{\mathbb{R}^d} x^\beta \varphi(x)dx \, = \, 0$$

for any multiindex $\beta \in \mathbb{N}_0^d$, $|\beta| \le B$, where B is a fixed natural number.

Definition 36 *Let $0 < q \le \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}_\infty^{\ell oc}$. Moreover let $B > [s]$.*
(i) *Let $0 < p < \infty$. Then the weighted Besov space $B^s_{p,q}(\mathbb{R}^d, w)$ is the collection of all $f \in S_e'(\mathbb{R}^d)$ such that*

$$\| \, f \, | B^s_{p,q}(\mathbb{R}^d, w)\| := \Big(\sum_{j=0}^\infty 2^{jsq} \, \| \, \varphi_j * f \, | L_p(\mathbb{R}^d, w)\|^q \Big)^{1/q} < \infty \, .$$

(ii) *Let $0 < p < \infty$. Then the weighted Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^d, w)$ is the collection of all $f \in S_e'(\mathbb{R}^d)$ such that*

$$\| \, f \, | F^s_{p,q}(\mathbb{R}^d, w)\| := \Big\| \, \Big(\sum_{j=0}^\infty 2^{jsq} \, | \, \varphi_j * f(\,\cdot\,) \, |^q \Big)^{1/q} \Big| L_p(\mathbb{R}^d, w) \Big\| < \infty \, .$$

Remark 21 (i) For $w \equiv 1$ we are in the unweighted case. The associated spaces are denoted by $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$. The above definition coincides with characterization of $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$ by so called local means, cf. [39] and [36].
(ii) Observe, that we did not define weighted spaces with $p = \infty$. However, it will be convenient for us to use the convention $B^s_{\infty,q}(\mathbb{R}^d, w) := B^s_{\infty,q}(\mathbb{R}^d)$.
(iii) Weighted Besov and Lizorkin-Triebel spaces with $w \in \mathcal{A}_\infty$ have been first studied systematically by Bui [23, 34], cf. also [35] and [36]. In addition we refer to Haroske, Piotrowska [37] and [38]. Standard references for unweighted spaces are the monograph's [12, 13, 39, 40] as well as [6]. These classes with $\mathcal{A}_\infty^{\ell oc}$ weights have been treated by Rychkov [21], Izuki and Sawano [41], and Wojciechowska [42, 4]. Different types of measure have been considered by Bownik and Ho [43] and Bownik [3].

Wavelet characterizations of weighted spaces

Here we need the following result.

Proposition 37 *Let* $0 < p < \infty$, $0 < q \le \infty$ *and* $s \in \mathbb{R}$. *Let* $w \in \mathcal{A}_\infty^{\ell oc}$. *Then there exists a linear isomorphism* T *which maps* $B^s_{p,q}(\mathbb{R}^d, w)$ *onto* $b^{s+d/2}_{p,q}(\mathbb{R}^d, w)$ *and* $F^s_{p,q}(\mathbb{R}^d, w)$ *onto* $f^{s+d/2}_{p,q}(\mathbb{R}^d, w)$.

Remark 22 (i) The mapping T is generated by an appropriate wavelet system. A proof of Prop. 37 can be found in Izuki and Sawano [41], cf. also Wojciechowska [42] for a different proof. In case of Besov spaces and $w \in \mathcal{A}_\infty$ we also refer to [38] and Bownik, Ho [43] in this context.
(ii) Prop. 37 extends to $p = \infty$ for Besov spaces, see Remark 21. Wavelet characterizations of unweighted Besov spaces are proved at various places, we refer to Meyer [44], Kahane and Lemarie-Rieusset [45], Triebel [40, 3.1.3] and Wojtaszczyk [46].

There is a little supplement to the previous proposition dealing with the classes $\mathring{A}^s_{p,q}(\mathbb{R}^d, w)$ and $\mathring{a}^s_{p,q}(\mathbb{R}^d, w)$ ($A \in \{B, F\}$, $a \in \{b, f\}$).

Proposition 38 *Let* $0 < q \le \infty$, $s \in \mathbb{R}$ *and* $w \in \mathcal{A}_\infty^{\ell oc}$.
(i) Let $0 < p < \infty$. *Then there exists a linear isomorphism* T *which maps* $\mathring{B}^s_{p,q}(\mathbb{R}^d, w)$ *onto* $\mathring{b}^{s+d/2}_{p,q}(\mathbb{R}^d, w)$ *and* $\mathring{F}^s_{p,q}(\mathbb{R}^d, w)$ *onto* $\mathring{f}^{s+d/2}_{p,q}(\mathbb{R}^d, w)$.
(ii) There exists a linear isomorphism T *which maps* $\mathring{B}^s_{\infty,q}(\mathbb{R}^d, w)$ *onto* $\mathring{b}^{s+d/2}_{\infty,q}(\mathbb{R}^d, w)$.

The proof of Prop. 38 follows the same pattern as the proof of Prop. 37. We leave out the details but see [17, Prop. 2.1] for the unweighted case.

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