Function spaces with dominating mixed smoothness

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Abstract

We study several techniques which are well known in the case of Besov and Triebel–Lizorkin spaces and extend them to spaces with dominating mixed smoothness. We use the ideas of Triebel to prove three important *decomposition theorems.* We deal with so–called *atomic, subatomic* and *wavelet* decompositions. All these theorems have much in common. Roughly speaking, they say that a function f belongs to some function space (say $S^{\bar{r}}_{p,q}A$) if, and only if, it can be decomposed as

$$f(x) = \sum_{\nu} \sum_{m} \lambda_{\nu\, m} a_{\nu\, m}(x), \qquad \text{convergence in } S',$$

with coefficients $\lambda = \{\lambda_{\nu\, m}\}$ in a corresponding sequence space (say $s^{\bar{r}}_{p,q}a$).

These decomposition theorems estabilish a very useful connection between function and sequence spaces. We use them in the study of the decay of entropy numbers of compact embeddings between two function spaces of dominating mixed smoothness reducing this problem to the same question on the sequence space level.

The considered scales cover many important specific spaces (Sobolev, Zygmund, Besov) and we get generalisations of respective assertions of Belinsky, Dinh Dung and Temlyakov.

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Preface

We study the function spaces with dominating mixed smoothness. First spaces of this type were defined by S. M. Nikol'skij in [21] and [22]. He introduced the spaces of Sobolev type

$$S_p^{\overline{r}}W(\mathbb{R}^2) = \Big\{ f | f \in L_p(\mathbb{R}^2), ||f|S_p^{\overline{r}}W(\mathbb{R}^2)|| = ||f|L_p|| + \Big|\Big| \frac{\partial^{r_1} f}{\partial x_1^{r_1}} \mid L_p \Big|\Big| + \\ + \Big|\Big| \frac{\partial^{r_2} f}{\partial x_2^{r_2}} \mid L_p \Big|\Big| + \Big|\Big| \frac{\partial^{r_1+r_2} f}{\partial x_1^{r_1} \partial x_2^{r_2}} \mid L_p \Big|\Big| < \infty \Big\},$$

where $1 < p < \infty, r_i = 0, 1, 2, \ldots; (i = 1, 2)$. The mixed derivative $\frac{\partial^{r_1+r_2} f}{\partial x_1^{r_1} \partial x_2^{r_2}}$ plays the dominant part here and gave the name to this class of spaces. The detailed study of spaces of such type was performed by many authors, for example T. I. Amanov, O. V. Besov, K. K. Golovkin, P. I. Lizorkin, S. M. Nikol'skij, M. K. Potapov and H.–J. Schmeisser. We refer to [1] for a systematic treatment of this topic. As in the theory of classical Sobolev spaces an alternative definition in terms of Fourier transform may be given (see (1.8) and (1.9)). This definition is based on a decomposition

$$f = \sum_{\overline{k} \in \mathbb{N}_0^d} (\varphi_{k_1} \otimes \cdots \otimes \varphi_{k_d} \hat{f})^\vee, \quad \text{convergence in } S'(\mathbb{R}^d),$$

where $\{\varphi_k\}_{k\in\mathbb{N}_0}$ is a decomposition of unity on $\mathbb{R}$ known from the theory of classical Besov spaces and $\varphi_{\overline{k}} = \varphi_{k_1} \otimes \cdots \otimes \varphi_{k_d}, \overline{k} = (k_1, \ldots, k_d)$, is a tensor product.

We refer mainly to [26], as far as the Fourier-analytic approach to these spaces is considered. In Chapter 2 of this book the classical theory of spaces with dominating mixed smoothness properties is developed. Several types of equivalent quasinorms, embedding and trace theorems and characterisation of these spaces by differences are proved there. One studies also basic properties of crucial operators on these spaces, namely of lifting and maximal operators and Fourier multipliers. We recall some facts from this book, which shall be useful later on, in Chapter 1. In contrary to [26], we do not restrict the dimension of the underlying Euclidean space to $d = 2$, hence we state these results formulated for general dimension $d \geq 2$. As mentioned in [26], this generalisation is obvious.

The second Chapter is devoted to local means, atomic, subatomic and wavelet decompositions of spaces with dominating mixed smoothness. We state the result for both Besov and Triebel-Lizorkin spaces but in some cases we give the proofs only for the Triebel-Lizorkin scale. The proofs for Besov-type spaces are omitted as they are very similar to the proofs presented here. First of all, we characterise this class of spaces by so-called local means. See Theorem 1.25 for details. This fundamental characterisation serves us as a basis for all three decomposition techniques.

By atomic decomposition of a function f one usually means a decomposition of a type

$$f(x) = \sum_\nu \sum_m \lambda_{\nu m} a_{\nu m}(x), \qquad \text{convergence in} \quad S'(\mathbb{R}^d),$$

where $a_{\nu m}$ are some simple building blocks, called *atoms*, and $\lambda_{\nu m}$ are complex numbers. A function f then belongs to some function space if and only if the sequence of coefficients $\{\lambda_{\nu m}\}_{\nu,m}$ belongs to some sequence space. For the exact formulation see Theorem 2.4. Let us mention that the atoms are specified only implicitly - a function a is an atom if and only if it satisfies some qualitative properties (see Definition 2.3).

By a subatomic decomposition we mean a decomposition of a type

$$f(x) = \sum_{\beta}\sum_{\nu}\sum_{m} \lambda^{\beta}_{\nu m} (\beta q\!u)_{\nu m}(x), \qquad \text{convergence in} \quad S'(\mathbb{R}^d),$$

where $(\beta q\!u)_{\nu m}(x)$ are so-called *quarks* and $\lambda^{\beta}_{\nu m}$ are complex numbers. A quark is a special type of atom defined explicitly by (2.36). Hence the basic building blocks, quarks, are much more specific in this kind of decomposition. The price one has to pay for that is a more complicated connection between f and $\{\lambda^{\beta}_{\nu m}\}$. It is described in detail in Theorem 2.6. In this sense each of these decompositions has its advantages and disadvantages. But all of them have something in common : they establish a connection between function spaces and sequence spaces. As the sequence spaces are simpler to deal with, it turns out that this connection is very useful in many situations (embeddings, traces, entropy numbers, ...). On this place we have to mention another important way how to switch from function spaces to sequence spaces — namely the so-called φ-transform of M. Frazier and B. Jawerth. We refer to [15] and references given there for details.

The classical theory of atomic decompositions of Besov and Triebel-Lizorkin spaces was developed mainly in the works M. Frazier and B. Jawerth ([12], [13]) and H. Triebel ([33], [34]). The subatomic decomposition of these spaces is due to H. Triebel ([35], [37]). We follow their ideas and prove similar decomposition theorems for spaces with dominating mixed derivatives. This is done in Chapter 2 and is one of the main results of this work.

The last decomposition technique developed here is the wavelet decomposition. In that case a class of compactly supported wavelets is used as the building blocks, see Theorems 2.10 and 2.11 for precise formulation. The main advantage of the wavelet decomposition is the uniqueness of the series obtained. The price paid for that is the limited smoothness of the compactly supported wavelets.

In the third chapter we study the entropy numbers of embeddings of sequence spaces associated with the function spaces with dominating mixed smoothness. The notion of entropy numbers has its roots in the study of metric entropy done in 1930's by Kolmogorov. Given a bounded linear operator T between two quasi-Banach spaces A and B ($T \in L(A,B)$), the quantity $e_k(T), k \in \mathbb{N}$, denotes, roughly speaking, the smallest radius $\epsilon > 0$ such that the image of the unit ball of A under the operator T may be covered by 2^{k-1} balls in B of radius ϵ. The sequence $\{e_k(T)\}_{k=1}^{\infty}$ tends to zero if, and only if, the operator T is compact. The decay of this sequence is then understood as a measure of compactness of T. The crucial property of entropy numbers was observed by Carl [6], who proved that the entropy numbers of a compact operator $T \in L(A,A)$ dominate in some sense its eigenvalues. In general, we use the method of [10] in this part.

We use the decomposition techniques to reduce this question to the sequence space level. Namely, it turns out that

$$e_k(id : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \hookrightarrow S^{\overline{r}_2}_{p_2,q_2}A(\Omega)) \approx e_k(id : s^{\overline{r}_1}_{p_1,q_1}a(\Omega) \hookrightarrow s^{\overline{r}_2}_{p_2,q_2}a(\Omega)), \tag{1}$$

where the constants of equivalence do not depend on $k \in \mathbb{N}$. So, in the third chapter we study mainly the entropy numbers of embeddings of sequence spaces. We restrict ourselves to the case $\overline{r}_1 = (r_1, \ldots, r_1) \in \mathbb{R}^d$ and $\overline{r}_2 = (r_2, \ldots, r_2) \in \mathbb{R}^d$. Unlike in the case of the classical Besov and Triebel-Lizorkin spaces, it turns out that the estimates of entropy numbers depend on the second, fine, summability parameter q. Unfortunately, the method used here gives the optimal answer only under some restriction on the parameters involved. We prove that the embeddings appearing in (1) is compact if, and only if,

$$\alpha = r_1 - r_2 - \max\Big(\frac{1}{p_1} - \frac{1}{p_2}, 0\Big) > 0. \tag{2}$$

But the direct method gives the estimates for (1) only for

$$\alpha > \frac{1}{\min(p_1, p_2, q_1)} - \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{\max(p_2, q_2)}.$$

We overcome this obstacle in Chapter 4 by the use of a complex interpolation method as developed by O. Mendez and M. Mitrea in [20]. Our final result may be summarised in the following way.

Under condition (2),

$$e_k(id : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A(\Omega)) \geq c\, k^{r_2-r_1}(\log k)^{(d-1)(r_1-r_2+\frac{1}{q_2}-\frac{1}{q_1})_+}.$$

If $r_1 - r_2 - \frac{1}{q_1} + \frac{1}{q_2} > 0$ then

$$e_k(id : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A(\Omega)) \leq c\, k^{r_2-r_1}(\log k)^{(d-1)(r_1-r_2+\frac{1}{q_2}-\frac{1}{q_1})}.$$

If $r_1 - r_2 - \frac{1}{q_1} + \frac{1}{q_2} \leq 0$ then for every $\varepsilon > 0$ there is a constant $c_\varepsilon > 0$ such that

$$e_k(id : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A(\Omega)) \leq c_\varepsilon\, k^{r_2-r_1}(\log k)^{\varepsilon}.$$

(See Theorem 4.11 for exact formulation). Finally, we compare results obtained by this method with estimates on entropy numbers of embeddings of function spaces with dominating mixed smoothness obtained by Belinsky [4], Dinh Dung [8] and Temlyakov [30].

I would like to thank to Prof. Schmeisser and Prof. Sickel for supervising my research and for many valuable discussions.

1 Function spaces on $\mathbb{R}^d$

Our aim in this Chapter is to recall the known aspects of the theory of function spaces with dominating mixed smoothness $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ and $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$. First of all, we introduce some basic notation which we shall need later on. Then we quote some definitions and theorems stated in [26] which are crucial in the sequel. In the last part we develop the so-called *local mean* characterisation of the spaces $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ and $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$.

1.1 Notation

As usual, $\mathbb{R}^d$ denotes the $d-$dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers and $\mathbb{C}$ denotes the plain of complex numbers.

We denote the points of the underlying Euclidean space by $x, y, z, \ldots$. Their components are numbered from 1 to d, hence $x = (x_1, \ldots, x_d)$. If $x, y \in \mathbb{R}^d$, we write $x > y$ if, and only if,

$x_i > y_i$ for every $i = 1, \dots, d$. Similarly, we define the relations $x \geq y, x < y, x \leq y$. Finally, in slight abuse of notation, we write $x > \lambda$ for $x \in \mathbb{R}^d, \lambda \in \mathbb{R}$ if $x_i > \lambda, i = 1, \dots, d$.

The $d-$dimensional vector indices will be denoted by $\overline{k}, \overline{l}, \overline{m}, \dots$ and their components are also numbered, hence $\overline{k} = (k_1, \dots, k_d)$. When $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ is a multi-index, we denote its length by $|\alpha| = \sum_{j=1}^d \alpha_j$. The derivatives $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}$ have the usual distributive meaning as well as the symbol $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$.

Let $S(\mathbb{R}^d)$ be the Schwartz space of all complex–valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$. We denote the $d-$dimensional Fourier transform of a function $\varphi \in S(\mathbb{R}^d)$ by $\mathcal{F}\varphi$, $\mathcal{F}(\varphi)$ or by $\hat{\varphi}$. Its inverse is denoted by $\mathcal{F}^{-1}\varphi$, $\mathcal{F}^{-1}(\varphi)$ or $\varphi^\vee$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to the dual Schwartz space $S'(\mathbb{R}^d)$ in the usual way. Sometimes, we need to distinguish between the $d-$dimensional and one-dimensional Fourier transform. In that case we denote the later by $\mathcal{F}_1$ or $^{\wedge_1}$ and its inverse by $\mathcal{F}_1^{-1}$ or $^{\vee_1}$. We point out that for functions $\varphi(x) = \varphi_1(x_1) \cdots \varphi_d(x_d) = (\varphi_1 \otimes \cdots \otimes \varphi_d)(x)$ the following formula connects $\mathcal{F}$ with $\mathcal{F}_1$

$$(\mathcal{F}\varphi)(\xi) = (\mathcal{F}_1\varphi_1)(\xi_1) \cdots (\mathcal{F}_1\varphi_d)(\xi_d) = ((\mathcal{F}_1\varphi_1) \otimes \cdots \otimes (\mathcal{F}_1\varphi_d))(\xi), \quad \xi \in \mathbb{R}^d. \tag{1.1}$$

Let $0 < p, q \leq \infty$. Having a sequence of complex-valued functions $\{f_{\overline{k}}\}_{\overline{k} \in \mathbb{N}_0^d}$ on $\mathbb{R}^d$, we put

$$||f_{\overline{k}}|\ell_q(L_p)|| = \Big(\sum_{\overline{k} \in \mathbb{N}_0^d} ||f_{\overline{k}}|L_p(\mathbb{R}^d)||^q\Big)^{1/q} = \Big(\sum_{\overline{k} \in \mathbb{N}_0^d} \Big(\int_{\mathbb{R}^d} |f_{\overline{k}}(x)\mathrm{d}x|^p\Big)^{q/p}\Big)^{1/q} \tag{1.2}$$

and

$$||f_{\overline{k}}|L_p(\ell_q)|| = \left|\left|\Big(\sum_{\overline{k} \in \mathbb{N}_0^d} |f_{\overline{k}}(x)|^q\Big)^{1/q} |L_p(\mathbb{R}^d)\right|\right| = \Big(\int_{\mathbb{R}^d} \Big(\sum_{\overline{k} \in \mathbb{N}_0^d} |f_{\overline{k}}(x)|^q\Big)^{p/q} \mathrm{d}x\Big)^{1/p}, \tag{1.3}$$

appropriately modified when p and/or $q = \infty$.

We denote $a_+ = \max(a, 0)$ for a real number $a \in \mathbb{R}$. Furthermore, let

$$\sigma_{pq} = \Big(\frac{1}{\min(p,q)} - 1\Big)_+ \quad \text{and} \quad \sigma_p = \Big(\frac{1}{p} - 1\Big)_+ \tag{1.4}$$

for every $0 < p \leq \infty$ and $0 < q \leq \infty$.

All unimportant constants are denoted by c. So, the meaning of the letter c may change from one occurrence to another. By $a_k \approx b_k$ we mean that there are two constants $c_1, c_2 > 0$ such that $c_1 a_k \leq b_k \leq c_2$ for every admissible k.

1.2 Definitions and basic properties

In this section we define the function spaces with dominating mixed smoothness on $\mathbb{R}^d$ and recall their basic properties as they are described in [26]. We quote the results for general dimension d of the underlying space $\mathbb{R}^d$, although they were stated and proved only for $d = 2$ in [26]. But, as mentioned there, this generalisation is rather obvious.

1.2.1 Definitions

Definition 1.1. Let $\Phi(\mathbb{R})$ be the collection of all systems $\{\varphi_j(t)\}_{j=0}^{\infty} \subset S(\mathbb{R})$ such that

$$\begin{cases} \operatorname{supp}\varphi_0 \subset \{t \in \mathbb{R} : \ |t| \leq 2\} \\ \operatorname{supp}\varphi_j \subset \{t \in \mathbb{R} : \ 2^{j-1} \leq |t| \leq 2^{j+1}\} \quad \text{if} \quad j = 1, 2, \dots; \end{cases} \tag{1.5}$$

for every $\alpha \in \mathbb{N}_0$ there exists a positive constant c_α such that

$$2^{j\alpha}|D^\alpha \varphi_j(t)| \leq c_\alpha \quad \text{for all} \quad j = 0, 1, 2, \dots \text{ and all } t \in \mathbb{R}, \tag{1.6}$$

and

$$\sum_{j=0}^{\infty} \varphi_j(t) = 1 \quad \text{for every} \quad t \in \mathbb{R}. \tag{1.7}$$

For $\overline{k} = (k_1, \dots, k_d) \in \mathbb{N}_0^d$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we define $\varphi_{\overline{k}}(x) = \varphi_{k_1}(x_1) \cdots \varphi_{k_d}(x_d)$. Using this kind of notation, we can give a definition of spaces $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ and $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$.

Definition 1.2. Let $\overline{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$, $0 < q \leq \infty$ and $\varphi = \{\varphi_j\}_{j=0}^{\infty} \in \Phi(\mathbb{R})$.
(i) Let $0 < p \leq \infty$. Then $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|S^{\overline{r}}_{p,q}B(\mathbb{R}^d)||_\varphi = \Big(\sum_{\overline{k}\in\mathbb{N}_0^d} 2^{q\overline{k}\cdot\overline{r}}||(\varphi_{\overline{k}}\hat{f})^\vee|L_p(\mathbb{R}^d)||^q\Big)^{1/q} = ||2^{\overline{k}\cdot\overline{r}}(\varphi_{\overline{k}}\hat{f})^\vee|\ell_q(L_p)|| \tag{1.8}$$

is finite.
(ii) Let $0 < p < \infty$. Then $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)||_\varphi = \Big|\Big|\Big(\sum_{\overline{k}\in\mathbb{N}_0^d} |2^{\overline{k}\cdot\overline{r}}(\varphi_{\overline{k}}\hat{f})^\vee(\cdot)|^q\Big)^{1/q}|L_p(\mathbb{R}^d)\Big|\Big| = ||2^{\overline{k}\cdot\overline{r}}(\varphi_{\overline{k}}\hat{f})^\vee|L_p(\ell_q)|| \tag{1.9}$$

is finite.

Remark 1.3. According to (1.7), we have

$$\sum_{k\in\mathbb{N}_0^d} \varphi_{\overline{k}}(x) = \Big(\sum_{k_1=0}^{\infty} \varphi_{k_1}(x_1)\Big) \cdots \Big(\sum_{k_d=0}^{\infty} \varphi_{k_d}(x_d)\Big) = 1 \quad \text{for all } x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

In this sense, $\{\varphi_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d}$ is also a decomposition of unity, in this case on $\mathbb{R}^d$.

Remark 1.4. The symbol $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ stays, as usual, for $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ and $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ respectively.

1.2.2 Basic inequalities

One of the most important questions in the theory of spaces $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ is the independence of Definition 1.2 on the system $\varphi = \{\varphi_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d}$. The answer is given by

Theorem 1.5. *Let* $\{\varphi_j\}_{j=0}^\infty, \{\psi_j\}_{j=0}^\infty \in \Phi(\mathbb{R})$. *Let* $\overline{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ *and* $0 < q \le \infty$.
(i) *Let* $0 < p \le \infty$. *Then* $||f|S^{\overline{r}}_{p,q}B(\mathbb{R}^d)||_\varphi$ *and* $||f|S^{\overline{r}}_{p,q}B(\mathbb{R}^d)||_\psi$ *are equivalent quasinorms. Furthermore,* $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ *is a quasi-Banach space (Banach space if* $\min(p,q) \ge 1$) *and*

$$S(\mathbb{R}^d) \subset S^{\overline{r}}_{p,q}B(\mathbb{R}^d) \subset S'(\mathbb{R}^d).$$

(ii) *Let* $0 < p < \infty$. *Then* $||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)||_\varphi$ *and* $||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)||_\psi$ *are equivalent quasinorms. Furthermore,* $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ *is a quasi-Banach space (Banach space if* $\min(p,q) \ge 1$) *and*

$$S(\mathbb{R}^d) \subset S^{\overline{r}}_{p,q}F(\mathbb{R}^d) \subset S'(\mathbb{R}^d).$$

For the proof in the case $d = 2$, see [26, pages 87, 93]. So, we may write $||f|S^{\overline{r}}_{p,q}B(\mathbb{R}^d)||$ and $||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)||$ without any index φ or ψ meaning one of these equivalent quasinorms.

Remark 1.6. The reader noticed that we did *not* define the spaces $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ for $p = \infty$. The reason is very similar to the case of classical Triebel–Lizorkin spaces. If one extends Definition 1.2 to the case $p = \infty$, which is actually possible, than there is no counterpart of Theorem 1.5. In particular, these spaces *do* depend on the choice of the system $\{\varphi_j\} \in \Phi(\mathbb{R})$.

We recall also the following version of the famous Nikol'skij inequality which is due to B. Stöckert [29] and A. P. Uninskij [39].

Theorem 1.7. (Nikol'skij inequality) *Let* $0 < p \le u \le \infty$ *and* $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$. *Let* $\overline{b} = (b_1, \ldots, b_d) > 0$ *and* $Q_{\overline{b}} = [-b_1, b_1] \times \cdots \times [-b_d, b_d] \subset \mathbb{R}^d$. *Then there exists a positive constant* c, *which is independent of* $\overline{b}$, *such that*

$$||D^\alpha f|L_u(\mathbb{R}^d)|| \le c\, b_1^{\alpha_1 + \frac{1}{p} - \frac{1}{u}} \cdots b_d^{\alpha_d + \frac{1}{p} - \frac{1}{u}} ||f|L_p(\mathbb{R}^d)||$$

holds for every $f \in S'(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)$ *with* $\operatorname{supp} \hat{f} \subset Q_{\overline{b}}$.

1.2.3 Lifting property

As in the case of classical Besov and Triebel-Lizorkin spaces, we can define a lifting operator.

Definition 1.8. Let $\overline{\rho} = (\rho_1, \ldots, \rho_d) \in \mathbb{R}^d$. Then we define the so-called lifting operator $I_{\overline{\rho}}$ by

$$I_{\overline{\rho}} f = \mathcal{F}^{-1}(1+\xi_1^2)^{\rho_1/2} \cdots (1+\xi_d^2)^{\rho_d/2} \mathcal{F} f, \quad f \in S'(\mathbb{R}^d). \tag{1.10}$$

Theorem 1.9. *Let* $0 < q \le \infty$, $\overline{\rho}, \overline{r} \in \mathbb{R}^d$.
(i) *Let* $0 < p \le \infty$. *Then* $I_{\overline{\rho}}$ *maps* $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ *isomorphically onto* $S^{\overline{r}-\overline{\rho}}_{p,q}B(\mathbb{R}^d)$ *and* $||I_{\overline{\rho}} f|S^{\overline{r}-\overline{\rho}}_{p,q}B(\mathbb{R}^d)||$ *is an equivalent quasinorm in* $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$.
(ii) *Let* $0 < p < \infty$. *Then* $I_{\overline{\rho}}$ *maps* $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ *isomorphically onto* $S^{\overline{r}-\overline{\rho}}_{p,q}F(\mathbb{R}^d)$ *and* $||I_{\overline{\rho}} f|S^{\overline{r}-\overline{\rho}}_{p,q}F(\mathbb{R}^d)||$ *is an equivalent quasinorm in* $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$.

The proof may be again found in [26, page 98].

1.2.4 Maximal operators

It has been observed throughout many decades that maximal operators (and their boundedness on appropriate function spaces) play a crucial role in harmonic analysis and function spaces theory. Our constructions given later are based on the Hardy–Littlewood maximal operator and the maximal operator of Peetre. Now we give the definition of the first one. For the definition of the latter one, see Section 1.3.1.

For every locally integrable function $f(x) \in L_1^{loc}(\mathbb{R}^d)$ we define the classical Hardy-Littlewood maximal operator

$$(Mf)(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(y)| \mathrm{d}y, \qquad x \in \mathbb{R}^d, \tag{1.11}$$

where the supremum is taken over all cubes Q centred at x with sides parallel with coordinate axes. The symbol $|Q|$ denotes the Lebesgue mass of the cube Q. The famous Hardy-Littlewood inequality tells that for every p with $1 < p \le \infty$ there is a c such that

$$||Mf|L_p(\mathbb{R}^d)|| \le c\,||f|L_p(\mathbb{R}^d)||, \qquad f \in L_p(\mathbb{R}^d). \tag{1.12}$$

The following theorem is a vector-valued generalisation of (1.12) and is due to C. Fefferman and E. M. Stein [11].

Theorem 1.10. *Let $1 < p < \infty$ and $1 < q \le \infty$. There exists a constant c such that*

$$||Mf_{\overline{k}}|L_p(\ell_q)|| \le c\,||f_{\overline{k}}|L_p(\ell_q)|| \tag{1.13}$$

holds for all sequences $\{f_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d}$ of locally Lebesgue-integrable functions on $\mathbb{R}^d$.

To reflect the tensor structure of the decomposition of unity $\varphi = \{\varphi_{\overline{k}}\}$ used in Definition 1.2, we consider following "directional" maximal operators. We define

$$(M_1 f)(x) = \sup_{s>0} \frac{1}{2s} \int_{x_1-s}^{x_1+s} |f(t, x_2, \dots, x_d)| \mathrm{d}t \tag{1.14}$$

and in a similar way for other variables. We denote the composition of these operators by $\overline{M} = M_d \circ \cdots \circ M_1$. The following maximal theorem is due to R. J. Bagby [2] (actually, it is a special case of more general theorem given there).

Theorem 1.11. *Let $1 < p < \infty$ and $1 < q \le \infty$. There exists a constant c such that*

$$||M_i f_{\overline{k}}|L_p(\ell_q)|| \le c\,||f_{\overline{k}}|L_p(\ell_q)||, \quad i = 1, \dots, d \tag{1.15}$$

holds for all sequences $\{f_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d} \subset L_p(\ell_q)$ of functions on $\mathbb{R}^d$.

Iteration of this theorem shows that the estimate (1.15) holds also for the operator $\overline{M}$.

1.2.5 Fourier multipliers

Let $\Omega = \{\Omega_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d}$ be the sequence of compact subsets of $\mathbb{R}^d$ with following properties

$$\Omega_{\overline{k}} = \{x \in \mathbb{R}^d : |x_1| \le a_{1,k_1}, \dots, |x_d| \le a_{d,k_d}\} \quad \text{with} \quad a_{1,k_1}, \dots, a_{d,k_d} > 0.$$

Theorem 1.12. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $\overline{r} = (r_1, \ldots, r_d) > \frac{1}{\min(p,q)} + \frac{1}{2}$. Let $\Omega = \{\Omega_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d}, a_{1,k_1}, \ldots, a_{d,k_d} > 0$ be the same sequences as above. Then there is a positive constant c such that*

$$||(\varrho_{\overline{k}}\hat{f}_{\overline{k}})^\vee|L_p(\ell_q)|| \leq c\Big(\sup_{\overline{k}\in\mathbb{N}_0^d} ||\varrho_{\overline{k}}(a_{1,k_1}\cdot, \ldots, a_{d,k_d}\cdot)|S^{\overline{r}}_{2,2}F(\mathbb{R}^d)||\Big) \cdot ||f_{\overline{k}}|L_p(\ell_q)||$$

holds for all systems $\{f_{\overline{k}}\} \in L_p(\ell_q)$ with $\operatorname{supp} \hat{f}_{\overline{k}} \subset \Omega_{\overline{k}}$ and all systems $\{\varrho_{\overline{k}}\} \subset S^{\overline{r}}_{2,2}F(\mathbb{R}^d)$.

Remark 1.13. The proof may be found in [26, page 77].

1.2.6 Littlewood-Paley Theory

We state also a theorem of Littlewood-Paley type for spaces with dominating mixed smoothness. But first we define the Sobolev spaces with dominating mixed smoothness. This is the very direct generalisation of the definition of Nikol'skij given in the Preface.

Definition 1.14. Let $1 < p < \infty$ and $\overline{r} = (r_1, \ldots, r_d) \in \mathbb{N}_0^d$. We put

$$S^{\overline{r}}_p W(\mathbb{R}^d) = \{f | f \in L_p(\mathbb{R}^d), ||f|S^{\overline{r}}_p W(\mathbb{R}^d)|| = \sum_{0\leq\alpha\leq\overline{r}} ||D^\alpha f|L_p(\mathbb{R}^d)|| < \infty\}.$$

Clearly, we have $S^{\overline{0}}_p W(\mathbb{R}^d) = L_p(\mathbb{R}^d)$. The connection between $S^{\overline{r}}_p W(\mathbb{R}^d)$ and $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ is then given by

Theorem 1.15. *Let $1 < p < \infty$ and $\overline{r} = (r_1, \ldots, r_d) \in \mathbb{N}_0^d$. Then*

$$S^{\overline{r}}_p W(\mathbb{R}^d) = S^{\overline{r}}_{p,2}F(\mathbb{R}^d)$$

where the corresponding norms are equivalent to each other.

Remark 1.16. See [26, page 104] for details.

1.3 Local means

In this part we present the main technical tool, namely, we characterise the spaces $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ by the so–called *local means.* In general, we follow the method presented by Rychkov [25]. Recall, that the spaces $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ were introduced by Definition 1.2 and, according to Theorem 1.5, we know that this definition does *not* depend on the choice of the decomposition of unity $\{\varphi_j\}_{j=0}^\infty \subset \Phi(\mathbb{R})$. Hence we may fix some specific system $\{\varphi_j\}_{j=0}^\infty$ for the rest of our work.

We fix $\varphi(x) \in S(\mathbb{R})$ with

$$\varphi(x) = 1 \quad \text{if} \quad |x| \leq \frac{4}{3} \quad \text{and} \quad \varphi(x) = 0 \quad \text{if} \quad |x| \geq \frac{3}{2}.$$

We put $\varphi_0 = \varphi, \varphi_1(x) = \varphi(\frac{x}{2}) - \varphi(x)$ and

$$\varphi_j(x) = \varphi_1(2^{-j+1}x), \quad x \in \mathbb{R}, j \in \mathbb{N}.$$

One verifies easily that (1.5)–(1.7) holds.

1.3.1 The Peetre maximal operator

Next we discuss the analogy of the Peetre maximal operator introduced in [23]. The construction of Peetre adapted to the case of function spaces with dominating mixed smoothness assigns to every system $\{\psi_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d} \subset S(\mathbb{R}^d)$, to every distribution $f \in S'(\mathbb{R}^d)$ and to every vector $\overline{a} > 0$ the following quantities

$$\sup_{y\in\mathbb{R}^d} \frac{|(\psi_{\overline{k}}\hat{f})^\vee(y)|}{\prod_{i=1}^d (1+|2^{k_i}(y_i-x_i)|^{a_i})}, \quad x\in\mathbb{R}^d, \quad \overline{k}\in\mathbb{N}_0^d. \tag{1.16}$$

As $\psi_{\overline{k}} \in S(\mathbb{R}^d)$ for every $\overline{k}\in\mathbb{N}_0^d$ then $\psi_{\overline{k}}\hat{f}$ is well defined for every $f\in S'(\mathbb{R}^d)$ and, according to the Theorem of Paley–Wiener–Schwartz (see [32] and references given there for details), $(\psi_{\overline{k}}\hat{f})^\vee$ is an analytic function. In particular, $(\psi_{\overline{k}}\hat{f})^\vee(y)$ makes sense pointwise.

Unfortunately, as we are interested also in non–smooth kernels (for details, see Section 2.4), we need to consider also kernels $\psi_{\overline{k}} \notin S(\mathbb{R}^d)$. We weaken in a natural way the definition of the Schwartz space $S(\mathbb{R}^d)$ and obtain the class of spaces $X^{\overline{S}}(\mathbb{R}^d)$ defined for every $\overline{S}\in\mathbb{N}_0^d$ by

$$X^{\overline{S}}(\mathbb{R}^d) = \{\varphi \in S_2^{\overline{S}}W(\mathbb{R}^d) : ||\varphi|X^{\overline{S}}(\mathbb{R}^d)|| < \infty\},$$
$$||\varphi|X^{\overline{S}}(\mathbb{R}^d)|| = \Big(\sum_{0\le\alpha,\beta\le\overline{S}} ||x^\beta D^\alpha\varphi(x)|L_2(\mathbb{R}^d)||^2\Big)^{1/2}.$$

We denote $\omega(x) = \prod_{i=1}^d (1+x_i^2)^{\frac{S_i}{2}}$ and observe that $\varphi\in X^{\overline{S}}(\mathbb{R}^d)$ if, and only if, $\omega\cdot D^\alpha\varphi \in L_2(\mathbb{R}^d)$ for every $0\le\alpha\le\overline{S}$. This is obviously equivalent to $D^\alpha(\omega\cdot\varphi)\in L_2(\mathbb{R}^d)$ for every $0\le\alpha\le\overline{S}$, which may be written as $\omega\cdot\varphi \in S_2^{\overline{S}}W(\mathbb{R}^d)$. Hence

$$\varphi\in X^{\overline{S}}(\mathbb{R}^d) \quad \text{if, and only if,} \quad \omega\cdot\varphi\in S_2^{\overline{S}}W(\mathbb{R}^d).$$

This allows us to characterise the dual of $X^{\overline{S}}(\mathbb{R}^d)$. We get

$$\psi\in(X^{\overline{S}}(\mathbb{R}^d))' \quad \text{if, and only if,} \quad \omega^{-1}\cdot\psi \in (S_2^{\overline{S}}W(\mathbb{R}^d))' = S_{2,2}^{-\overline{S}}F(\mathbb{R}^d).$$

As a trivial consequence of the embedding ($\overline{S}\in\mathbb{N}_0^d$)

$$X^{\overline{S}}(\mathbb{R}^d) \hookrightarrow S_2^{\overline{S}}W(\mathbb{R}^d) \hookrightarrow S_{\infty,\infty}^{\overline{S}-\frac{1}{2}}B(\mathbb{R}^d)$$

we get for every $\overline{K}\in\mathbb{N}_0^d$ and every $\overline{S}\ge\overline{K}+1$

$$X^{\overline{S}}(\mathbb{R}^d) \hookrightarrow C^{\overline{K}}(\mathbb{R}^d).$$

Having now a function $\Psi_{\overline{k}}\in X^{\overline{S}}(\mathbb{R}^d)$ and some distribution $f\in(X^{\overline{S}}(\mathbb{R}^d))'$, we write

$$(f*\Psi_{\overline{k}})(y) = \int_{\mathbb{R}^d} f(x)\Psi_{\overline{k}}(y-x)\mathrm{d}x = f(\Psi_{\overline{k}}(y-\cdot)), \quad y\in\mathbb{R}^d.$$

So, given a system $\{\psi_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d}\subset X^{\overline{S}}(\mathbb{R}^d)$ for some $\overline{S}\in\mathbb{N}_0^d$, we denote $\Psi_{\overline{k}} = \hat{\psi}_{\overline{k}} \in X^{\overline{S}}(\mathbb{R}^d)$ and define in analogy with (1.16) for every $f\in(X^{\overline{S}}(\mathbb{R}^d))'$

$$(\Psi_{\overline{k}}^*f)_{\overline{a}}(x) = \sup_{y\in\mathbb{R}^d}\frac{|(\Psi_{\overline{k}}*f)(y)|}{\prod_{i=1}^d(1+|2^{k_i}(y_i-x_i)|^{a_i})}, \quad x\in\mathbb{R}^d, \quad \overline{k}\in\mathbb{N}_0^d. \tag{1.17}$$

Furthermore, for $\overline{S}=\infty$, we put $X^{\overline{S}}(\mathbb{R}^d) = S(\mathbb{R}^d)$.

1.3.2 Helpful lemmas

We split the proof of the local–mean characteristics of Besov and Triebel–Lizorkin spaces and give in this subsection the technical lemmas. This will allow us a straightforward proof later on. The lemmas originate in [25] and we quote them only with some minor modifications, mainly forced by the tensor structure of function spaces with dominated mixed smoothness.

We start with lemma describing the use of the so–called moment conditions.

Lemma 1.17. *Let $K \in \mathbb{N}_0$ and $g, h \in X^{K+2}(\mathbb{R})$. Furthermore, let $M \geq -1, M \leq K$ be an integer and*

$$(D^\alpha \hat{g})(0) = 0, \quad 0 \leq \alpha \leq M.$$

Then for every $N \in \mathbb{N}_0$ with $0 \leq N \leq K$ there is a constant C_N such that

$$\sup_{z\in\mathbb{R}} |(g_b * h)(z)|(1 + |z|^N) \leq C_N b^{M+1}, \quad 0 < b < 1, \tag{1.18}$$

where $g_b(t) = b^{-1} g(t/b)$.

Proof. Using the elementary properties of the Fourier transform we get

$$\text{LHS}(1.18) \leq c \max_{0\leq\alpha\leq N} ||D^\alpha[(g_b * h)^\wedge]|L_1(\mathbb{R})||.$$

By Leibnitz formula,

$$|D^\alpha[\hat{g}(b\cdot)\hat{h}(\cdot)](\xi)| \leq c \sum_{0\leq\beta\leq\alpha} b^\beta |(D^\beta \hat{g})(b\xi)(D^{\alpha-\beta}\hat{h})(\xi)|, \quad \xi \in \mathbb{R}. \tag{1.19}$$

As $\hat{g} \in C^{M+1}(\mathbb{R})$, we may use the Taylor formula and get

$$|(D^\beta \hat{g})(b\xi)| \leq c|b\xi|^{M-\beta+1}, \quad 0 \leq \beta \leq M \tag{1.20}$$

for $|b\xi| \leq 1$. But, as $D^\beta \hat{g} \in C(\mathbb{R})$, (1.20) holds for all $b, \xi \in \mathbb{R}$. Hence, for $0 \leq \beta \leq M$, we get

$$b^\beta |(D^\beta \hat{g})(b\xi)(D^{\alpha-\beta}\hat{h})(\xi)| \leq c\, b^{M+1} |(D^{\alpha-\beta}\hat{h})(\xi)| \cdot |\xi|^{(M-\beta+1)_+}, \quad \xi \in \mathbb{R}. \tag{1.21}$$

If $M < \beta \leq K$ and $0 < b < 1$, we have $b^\beta \leq b^{M+1}$ which, together with $D^\beta \hat{g} \in C(\mathbb{R})$, gives (1.21) for all $0 \leq \beta \leq K$.

We put (1.21) into (1.19) and obtain (1.18). □

Furthermore, we shall need the following convolution inequality.

Lemma 1.18. *Let $0 < p, q \leq \infty$, $\delta > 0$. Let $\{g_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d}$ be a sequence of nonnegative measurable functions on $\mathbb{R}^d$ and let*

$$G_{\overline{\nu}}(x) = \sum_{\overline{k}\in\mathbb{N}_0^d} 2^{-|\overline{\nu}-\overline{k}|\delta} g_{\overline{k}}(x), \quad x \in \mathbb{R}^d, \quad \overline{\nu} \in \mathbb{N}_0^d. \tag{1.22}$$

Then there is some constant $C = C(p, q, \delta)$ such that

$$||G_{\overline{k}}|\ell_q(L_p)|| \leq C||g_{\overline{k}}|\ell_q(L_p)|| \tag{1.23}$$

$$||G_{\overline{k}}|L_p(\ell_q)|| \leq C||g_{\overline{k}}|L_p(\ell_q)||. \tag{1.24}$$

Proof. Step 1.

We start with the proof of (1.23). If $p \geq 1$, we get by triangle inequality

$$||G_{\overline{\nu}}|L_p(\mathbb{R}^d)|| \leq \sum_{\overline{k}\in\mathbb{N}_0^d} 2^{-|\overline{\nu}-\overline{k}|\delta}||g_{\overline{k}}|L_p(\mathbb{R}^d)||, \quad \overline{\nu}\in\mathbb{N}_0^d.$$

When $q \leq 1$, we use the embedding $\ell_q \hookrightarrow \ell_1$ and get

$$||G_{\overline{\nu}}|\ell_q(L_p)|| \leq \Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{k}\in\mathbb{N}_0^d} 2^{-|\overline{\nu}-\overline{k}|\delta q}||g_{\overline{k}}|L_p(\mathbb{R}^d)||^q\Big)^{1/q}.$$

Interchanging the order of summation, we get (1.23) with $C = C_1 = (\sum_{\overline{k}\in\mathbb{Z}^d} 2^{-|\overline{k}|\delta q})^{1/q}$.

If $q > 1$, we apply Young's inequality. We denote

$$\begin{aligned} \lambda_{\overline{k}} &= 2^{-|\overline{k}|\delta}, \quad \overline{k}\in\mathbb{Z}^d, \\ \gamma_{\overline{k}} &= ||g_{\overline{k}}|L_p(\mathbb{R}^d)||, \quad \overline{k}\in\mathbb{N}_0^d \quad \text{and} \quad \gamma_{\overline{k}} = 0 \quad \text{for} \quad \overline{k}\in\mathbb{Z}^d\setminus\mathbb{N}_0^d. \end{aligned} \tag{1.25}$$

Then we get

$$||G_{\overline{\nu}}|L_p(\mathbb{R}^d)|| \leq (\lambda * \gamma)(\overline{\nu}), \quad \overline{\nu}\in\mathbb{N}_0^d$$

and Young's convolution inequality gives

$$||\lambda * \gamma|\ell_q|| \leq ||\lambda|\ell_1||\cdot||\gamma|\ell_q||.$$

This proves (1.23) with $C = C_2 = ||\lambda|\ell_1||$.

If $p < 1$, we use the $\ell_p \hookrightarrow \ell_1$ embedding and get

$$\int_{\mathbb{R}^d} G_{\overline{\nu}}^p(x)\mathrm{d}x \leq \sum_{\overline{k}\in\mathbb{N}_0^d} 2^{-|\overline{\nu}-\overline{k}|\delta p}\int_{\mathbb{R}^d} g_{\overline{k}}^p(x)\mathrm{d}x$$

For $q/p \leq 1$ this implies

$$\sum_{\overline{\nu}\in\mathbb{N}_0^d} ||G_{\overline{\nu}}|L_p(\mathbb{R}^d)||^q \leq \sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{k}\in\mathbb{N}_0^d} 2^{-|\overline{\nu}-\overline{k}|\delta q}||g_{\overline{k}}|L_p(\mathbb{R}^d)||^q.$$

Now we interchange again the order of summation and take the $1/q$ power. This proves (1.23) with $C = C_1$.

Finally, if $q/p > 1$, we use again Young's inequality, with λ^p and γ^p instead of λ and γ. This gives

$$||G_{\overline{\nu}}|\ell_q(L_p)||^p \leq ||\lambda^p|\ell_1||\cdot||\gamma^p|\ell_{q/p}||,$$

which proves (1.23) with $C = ||\lambda|\ell_p||$.

Step 2.

Next we turn to (1.24). This is a trivial consequence of the pointwise inequality

$$||G_{\overline{\nu}}(x)|\ell_q|| \leq C||g_{\overline{\nu}}(x)|\ell_q||, \quad x\in\mathbb{R}^d, \tag{1.26}$$

with C independent of $x \in \mathbb{R}^d$.

To prove (1.26), just use the $\ell_q \hookrightarrow \ell_1$ embedding for $q \leq 1$ and Young's inequality for $q > 1$. We do not give details, which are very similar to the calculation in Step 1. $\square$

As we do not want to exclude the case of arbitrary smooth functions, we use the following notation. We say that the vector $\overline{N} = \infty$ if and only if $N_i = \infty$ for all $i = 1, \dots, d$. The symbol $\overline{N} \in \mathbb{N}_0^d \cup \{\infty\}$ then admits $\overline{N} = \infty$ or $\overline{N}$ to be a vector of nonnegative integers.

Lemma 1.19. *Let $0 < r \leq 1$, and let $\{\gamma_{\overline{\nu}}\}_{\overline{\nu} \in \mathbb{N}_0^d}$, $\{\beta_{\overline{\nu}}\}_{\overline{\nu} \in \mathbb{N}_0^d}$ be two sequences taking values in $(0, \infty)$. Assume that, for some $\overline{N}^0 \in \mathbb{N}_0^d$,*

$$\gamma_{\overline{\nu}} = O(2^{\overline{\nu} \cdot \overline{N}^0}), \quad |\overline{\nu}| \to \infty. \tag{1.27}$$

Furthermore, we assume that there is $\overline{N}^1 \in \mathbb{N}_0^d \cup \{\infty\}$ with $\overline{N}^1 \geq \overline{N}^0$ such that

$$\gamma_{\overline{\nu}} \leq C_{\overline{N}} \sum_{\overline{k} \in \mathbb{N}_0^d} 2^{-\overline{k} \cdot \overline{N}} \beta_{\overline{k}+\overline{\nu}} \gamma_{\overline{k}+\overline{\nu}}^{1-r}, \quad \overline{\nu} \in \mathbb{N}_0^d, \quad C_{\overline{N}} < \infty, \tag{1.28}$$

holds for every $0 \leq \overline{N} \leq \overline{N}^1$ if $\overline{N}^1$ is finite or for every $\overline{N} \in \mathbb{N}_0^d$ if $\overline{N}^1 = \infty$.
Then, for the same set of $\overline{N}$,

$$\gamma_{\overline{\nu}}^r \leq C_{\overline{N}} \sum_{\overline{k} \in \mathbb{N}_0^d} 2^{-\overline{k} \cdot \overline{N} r} \beta_{\overline{k}+\overline{\nu}}, \quad \overline{\nu} \in \mathbb{N}_0^d, \tag{1.29}$$

with the same constants $C_{\overline{N}}$.

Proof. Put

$$\Gamma_{\overline{\nu}, \overline{N}} = \sup_{\overline{k} \in \mathbb{N}_0^d} 2^{-\overline{k} \cdot \overline{N}} \gamma_{\overline{k}+\overline{\nu}}, \quad \overline{\nu}, \overline{N} \in \mathbb{N}_0^d.$$

By (1.28),

$$\begin{aligned} \Gamma_{\overline{\nu}, \overline{N}} &\leq C_{\overline{N}} \sup_{\overline{k} \in \mathbb{N}_0^d} \sum_{\overline{l} \in \mathbb{N}_0^d} 2^{-(\overline{k}+\overline{l}) \cdot \overline{N}} \beta_{\overline{l}+\overline{k}+\overline{\nu}} \gamma_{\overline{l}+\overline{k}+\overline{\nu}}^{1-r} \\ &= C_{\overline{N}} \sup_{\overline{k} \in \mathbb{N}_0^d} \sum_{\overline{l} \in \mathbb{N}_0^d + \overline{k}} 2^{-\overline{l} \cdot \overline{N}} \beta_{\overline{l}+\overline{\nu}} \gamma_{\overline{l}+\overline{\nu}}^{1-r} \\ &= C_{\overline{N}} \sum_{\overline{l} \in \mathbb{N}_0^d} 2^{-\overline{l} \cdot \overline{N}} \beta_{\overline{l}+\overline{\nu}} \gamma_{\overline{l}+\overline{\nu}}^{1-r} \\ &\leq C_{\overline{N}} \Gamma_{\overline{\nu}, \overline{N}}^{1-r} \sum_{\overline{l} \in \mathbb{N}_0^d} 2^{-\overline{l} \cdot \overline{N} r} \beta_{\overline{l}+\overline{\nu}} \end{aligned} \tag{1.30}$$

When $\Gamma_{\overline{\nu}, \overline{N}} < \infty$, we finish the proof by

$$\gamma_{\overline{\nu}}^r \leq \Gamma_{\overline{\nu}, \overline{N}}^r \leq C_{\overline{N}} \sum_{\overline{l} \in \mathbb{N}_0^d} 2^{-\overline{l} \cdot \overline{N} r} \beta_{\overline{l}+\overline{\nu}}. \tag{1.31}$$

From (1.27), $\Gamma_{\overline{\nu}, \overline{N}}$ is finite for all $\overline{N}^0 \leq \overline{N} \leq \overline{N}^1$ (or for all $\overline{N}^0 \leq \overline{N}$ if $\overline{N}^1 = \infty$). As the right–hand side of (1.29) decreases when $\overline{N}$ increases in any coordinate, this proves (1.29) also for all $\overline{N} \not\geq \overline{N}^0$ with the constant $C_{\overline{N}^*}$, where $\overline{N}_i^* = \max(\overline{N}_i^0, \overline{N}_i)$. Take now any $\overline{N} \not\geq \overline{N}^0$ and apply (1.29) with $C_{\overline{N}^*}$ instead of $C_{\overline{N}}$ to get

$$\begin{aligned}\Gamma_{\overline{\nu},\overline{N}} &= \sup_{\overline{k}\in\mathbb{N}_0^d} 2^{-\overline{k}\cdot\overline{N}} \gamma_{\overline{k}+\overline{\nu}} \\ &\le \sup_{\overline{k}\in\mathbb{N}_0^d} \big(C_{\overline{N}^*} \sum_{\overline{l}\in\mathbb{N}_0^d} 2^{-(\overline{k}+\overline{l})\cdot\overline{N}r} \beta_{\overline{l}+\overline{k}+\overline{\nu}}\big)^{1/r} \\ &= C_{\overline{N}^*}^{1/r} \big(\sum_{\overline{l}\in\mathbb{N}_0^d} 2^{-\overline{l}\cdot\overline{N}r} \beta_{\overline{l}+\overline{\nu}}\big)^{1/r},\end{aligned}$$

which is finite whenever the right–hand side of (1.29) is finite (otherwise there is nothing to prove). So, even in this case, we may apply (1.30) and (1.31) and finish the proof of the lemma. □

1.3.3 Comparison of different Peetre maximal operators

In this subsection we present one inequality between different Peetre maximal operators. This inequality (together with the boundedness of Peetre maximal operator) forms the basis for our characterisation of $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ through local means.

Because of the limited smoothness of our kernel functions (discussed in detail in section 2.4), we cannot expect to get such an inequality for all $f \in S'(\mathbb{R}^d)$.

We start with (given) functions ψ_0^i, ψ_1^i, $i = 1, \ldots, d$ defined on $\mathbb{R}$ and denote

$$\begin{aligned}\psi_j^i(t) &= \psi_1^i(2^{-j+1}t), \quad t \in \mathbb{R}, \quad j = 2, 3, \ldots, \\ \psi_{\overline{k}}(x) &= \prod_{i=1}^d \psi_{k_i}^i(x_i), \quad x \in \mathbb{R}^d, \overline{k} \in \mathbb{N}_0^d, \\ \Psi_{\overline{k}} &= \hat{\psi}_{\overline{k}}, \qquad \overline{k} \in \mathbb{N}_0^d.\end{aligned} \tag{1.32}$$

To (also given) functions $\phi_0^i, \phi_1^i, i = 1, \ldots, d$ we associate $\phi_{\overline{k}}$ and $\Phi_{\overline{k}}$ in the same way. Furthermore, we suppose that $\psi_{\overline{k}}, \phi_{\overline{k}} \in X^{\overline{S}}(\mathbb{R}^d)$ for some $\overline{S} \in \mathbb{N}_0^d$.

Using this notation we may state the main result of this section.

Theorem 1.20. *Let $\overline{a}, \overline{r} \in \mathbb{R}^d, \overline{R} \in \mathbb{N}_0^d, 0 < p, q \le \infty$ with $\overline{a} > 0$ and $\overline{r} < \overline{R} + 1$. If $\overline{S} > \overline{R}$ is large enough,*

$$D^l \psi^i(0) = 0, \quad i = 1, \ldots, d, \quad l = 0, 1, \ldots, R_i, \tag{1.33}$$

and, for every $i = 1, \ldots, d$ and some $\varepsilon > 0$,

$$|\phi_0^i(t)| > 0 \quad on \quad \{t \in \mathbb{R} : |t| < \varepsilon\} \tag{1.34}$$

$$|\phi_1^i(t)| > 0 \quad on \quad \{t \in \mathbb{R} : \varepsilon/2 < |t| < 2\varepsilon\} \tag{1.35}$$

then

$$||2^{\overline{k}\cdot\overline{r}}(\Psi_{\overline{k}}^* f)_{\overline{a}}|\ell_q(L_p)|| \le c||2^{\overline{k}\cdot\overline{r}}(\Phi_{\overline{k}}^* f)_{\overline{a}}|\ell_q(L_p)|| \tag{1.36}$$

$$||2^{\overline{k}\cdot\overline{r}}(\Psi_{\overline{k}}^* f)_{\overline{a}}|L_p(\ell_q)|| \le c||2^{\overline{k}\cdot\overline{r}}(\Phi_{\overline{k}}^* f)_{\overline{a}}|L_p(\ell_q)|| \tag{1.37}$$

for all $f \in (X^{\overline{S}}(\mathbb{R}^d))'$.

Proof. Step 1. — formal calculations.
It follows from (1.34) and (1.35) that there exist functions $\{\lambda_j^i\}_{j=0}^\infty$, $i=1,\ldots,d$ with

$$\sum_{j=0}^\infty \lambda_j^i(t)\phi_j^i(t)=1,\quad t\in\mathbb{R}, \tag{1.38}$$

$$\lambda_j^i(t)=\lambda_1^i(2^{-j+1}t),\quad t\in\mathbb{R},\quad j\in\mathbb{N}, \tag{1.39}$$

$$\operatorname{supp}\lambda_0^i\subset\{t\in\mathbb{R}:|t|\le\varepsilon\}\quad\text{and}\quad \operatorname{supp}\lambda_j^i\subset\{t\in\mathbb{R}:2^{j-2}\varepsilon\le|t|\le 2^j\varepsilon\},\quad j\in\mathbb{N}. \tag{1.40}$$

Now we define, as usually, $\lambda_{\overline{k}}(x)=\lambda_{k_1}^1(x_1)\cdots\lambda_{k_d}^d(x_d)$ for every $\overline{k}\in\mathbb{N}_0^d$. From (1.38) we obtain

$$\sum_{\overline{k}\in\mathbb{N}_0^d}\lambda_{\overline{k}}(x)\phi_{\overline{k}}(x)=1,\quad x\in\mathbb{R}^d.$$

Finally, we denote $\Lambda_{\overline{k}}=\hat{\lambda}_{\overline{k}}$, $\overline{k}\in\mathbb{N}_0^d$. This gives us the following identities

$$f=\sum_{\overline{k}\in\mathbb{N}_0^d}\Lambda_{\overline{k}}*\Phi_{\overline{k}}*f,\quad \Psi_{\overline{\nu}}*f=\sum_{\overline{k}\in\mathbb{N}_0^d}\Psi_{\overline{\nu}}*\Lambda_{\overline{k}}*\Phi_{\overline{k}}*f,\quad \overline{\nu}\in\mathbb{N}_0^d. \tag{1.41}$$

We have

$$\begin{aligned}|(\Psi_{\overline{\nu}}*\Lambda_{\overline{k}}*\Phi_{\overline{k}}*f)(y)|&\le\int_{\mathbb{R}^d}|(\Psi_{\overline{\nu}}*\Lambda_{\overline{k}})(z)|\cdot|(\Phi_{\overline{k}}*f)(y-z)|\mathrm{d}z\\&\le(\Phi_{\overline{k}}^*f)_{\overline{a}}(y)\int_{\mathbb{R}^d}|(\Psi_{\overline{\nu}}*\Lambda_{\overline{k}})(z)|\prod_{i=1}^d(1+|2^{k_i}z_i|^{a_i})\mathrm{d}z\\&\equiv(\Phi_{\overline{k}}^*f)_{\overline{a}}(y)I_{\overline{\nu}\overline{k}}=(\Phi_{\overline{k}}^*f)_{\overline{a}}(y)\prod_{i=1}^d I_{\nu_ik_i},\end{aligned} \tag{1.42}$$

where

$$I_{\nu_ik_i}=\int_{\mathbb{R}}|(\Psi_{\nu_i}^i*\Lambda_{k_i}^i)(z_i)|(1+|2^{k_i}z_i|^{a_i})\mathrm{d}z_i.$$

We claim that by Lemma 1.17,

$$I_{\nu_ik_i}\le C\begin{cases}2^{(k_i-\nu_i)(R_i+1)}, & \text{if } k_i\le\nu_i\\ 2^{(\nu_i-k_i)(a_i+|r_i|+1)}, & \text{if } k_i\ge\nu_i.\end{cases} \tag{1.43}$$

We namely have (for $1\le k_i<\nu_i$) with the change of variables $2^{k_i}z_i\to z_i$

$$\begin{aligned}I_{\nu_ik_i}&=\frac{1}{2}\int_{\mathbb{R}}|(\Psi_{\nu_i-k_i}^i*\Lambda_1^i(\cdot/2))(z_i)|(1+|z_i|^{a_i})\mathrm{d}z_i\\&\le c\sup_{z\in\mathbb{R}}|(\Psi_{\nu_i-k_i}^i*\Lambda_1^i(\cdot/2))(z_i)|(1+|z_i|^{a_i+2})\le c\,2^{(k_i-\nu_i)(R_i+1)},\end{aligned}$$

when S_i are chosen sufficiently large.
Analogously, for $1\le\nu_i<k_i$ with the change of variables $2^{\nu_i}z_i\to z_i$

$$\begin{aligned}I_{\nu_ik_i}&\le 2^{(k_i-\nu_i)a_i}\int_{\mathbb{R}}|(\Psi_1^i*\Lambda_{k_i-\nu_i}^i)(z_i)|(1+|z_i|^{a_i})\mathrm{d}z_i\\&\le c\,2^{(\nu_i-k_i)(-a_i+M+1)},\end{aligned}$$

where M may be taken as large as S_i allows. Taking $M > 2a_i + |r_i|$ (which is possible for S_i large enough), we get (1.43). This covers the cases where $\nu_i, k_i \geq 1, \nu_i \neq k_i$. The cases $k_i = \nu_i \geq 1$, $k_i > \nu_i = 0$ and $\nu_i > k_i = 0$ can be treated separately in the similar way. The needed moment conditions are always satisfied by (1.33) or (1.40), respectively. The case $k_i = \nu_i = 0$ is covered by the constant C in (1.43).

Next, we point out that

$$(\Phi^*_{\overline{k}} f)_{\overline{a}}(y) \leq (\Phi^*_{\overline{k}} f)_{\overline{a}}(x) \prod_{i=1}^{d} (1 + |2^{k_i}(x_i - y_i)|^{a_i})$$
$$\leq c\, (\Phi^*_{\overline{k}} f)_{\overline{a}}(x) \prod_{i=1}^{d} (1 + |2^{\nu_i}(x_i - y_i)|^{a_i}) \max(1, 2^{(k_i - \nu_i)a_i}).$$

We put this into (1.42) and use (1.43)

$$\sup_{y \in \mathbb{R}^d} \frac{|(\Psi_{\overline{\nu}} * \Lambda_{\overline{k}} * \Phi_{\overline{k}} * f)(y)|}{\prod_{i=1}^{d} (1 + |2^{\nu_i}(x_i - y_i)|^{a_i})} \leq (\Phi^*_{\overline{k}} f)_{\overline{a}}(x) \prod_{i=1}^{d} I_{\nu_i k_i} \max(1, 2^{(k_i - \nu_i)a_i})$$
$$\leq c\, (\Phi^*_{\overline{k}} f)_{\overline{a}}(x) \prod_{i=1}^{d} \begin{cases} 2^{(k_i - \nu_i)(R_i + 1)}, & \text{if } k_i \leq \nu_i \\ 2^{(\nu_i - k_i)(|r_i| + 1)}, & \text{if } k_i \geq \nu_i. \end{cases}$$

This inequality, together with (1.41) and (1.42), gives for

$$\delta = \min\{1, R_i + 1 - r_i; i = 1, \ldots, d\} > 0$$

the estimate

$$2^{\overline{\nu} \cdot \overline{r}} (\Psi^*_{\overline{\nu}} f)_{\overline{a}}(x) \leq c \sum_{\overline{k} \in \mathbb{N}_0^d} 2^{-|\overline{k} - \overline{\nu}|\delta} 2^{\overline{k} \cdot \overline{r}} (\Phi^*_{\overline{k}} f)_{\overline{a}}(x), \quad \overline{\nu} \in \mathbb{N}_0^d, \quad x \in \mathbb{R}^d.$$

Lemma 1.18 now gives immediately the desired result.

Step 2. — theoretical background.

In the Step 1 we did not took care about problems caused by limited smoothness of functions ψ^i_j, ϕ^i_j not to disturb the elegant calculation done there. Nevertheless, to complete the proof, we have to fill some gaps. We go through the proof of the Step 1 once more and discuss the theoretical aspects of the calculation.

- Functions λ^i_j

  By the choice $\lambda^i_j(t) = \varphi_j(\frac{3t}{2\varepsilon})/\phi^i_j(t)$ we ensure (1.38)–(1.40). The functions $\varphi_j, j \in \mathbb{N}_0$, were fixed in the beginning of Section 1.3. And by conditions (1.34) and (1.35) we get $\lambda_{\overline{k}} \in X^{\overline{S}}(\mathbb{R}^d)$.

- Identities (1.41)

  First, we point out that the expression $\Lambda_{\overline{k}} * \Phi_{\overline{k}} * f$ is well defined for every $\overline{k} \in \mathbb{N}_0^d$. As the function $\lambda_{\overline{k}} = \Lambda^\vee_{\overline{k}}$ has compact support, we have $\Lambda_{\overline{k}} * \Phi_{\overline{k}} = (\lambda_{\overline{k}} \phi_{\overline{k}})^\wedge \in X^{\overline{S}}(\mathbb{R}^d)$. The same holds for $\Psi_{\overline{\nu}} * \Lambda_{\overline{k}} * \Phi_{\overline{k}}$.

Next we prove the convergence of both sums in (1.41) for every $f \in (X^{\overline{S}}(\mathbb{R}^d))'$ and every $\overline{\nu} \in \mathbb{N}_0^d$ in $(X^{\overline{S}}(\mathbb{R}^d))'$. By the duality arguments, it is enough to prove that

$$\sum_{\overline{k}\in\mathbb{N}_0^d} \psi_{\overline{\nu}}\lambda_{\overline{k}}\phi_{\overline{k}}\mu \to \psi_{\overline{\nu}}\mu, \quad \overline{\nu} \in \mathbb{N}_0^d,$$

converges in $X^{\overline{S}}(\mathbb{R}^d)$ for every $\mu \in X^{\overline{S}}(\mathbb{R}^d)$. This follows from (1.38) and (1.40).

Finally, to come over from (1.41) to (1.42), we have to ensure that (1.41) converges also pointwise. Better said, we need to prove

$$|(\Psi_{\overline{\nu}} * f)(y)| \leq \sum_{\overline{k}\in\mathbb{N}_0^d} |(\Psi_{\overline{\nu}} * \Lambda_{\overline{k}} * \Phi_{\overline{k}} * f)(y)| \tag{1.44}$$

for all $\overline{\nu} \in \mathbb{N}_0^d$ and almost all $y \in \mathbb{R}^d$.

Fix $\overline{\nu} \in \mathbb{N}_0^d$ and let $f_{\overline{k}}(y) = (\Psi_{\overline{\nu}} * \Lambda_{\overline{k}} * \Phi_{\overline{k}} * f)(y)$. Then we know from (1.42) that

$$|f_{\overline{k}}(y)| \leq (\Phi^*_{\overline{k}}f)_{\overline{a}}(y) I_{\overline{\nu}\overline{k}}, \quad y \in \mathbb{R}^d.$$

By (1.43) (and by Hölder's inequality for $q > 1$)

$$\sum_{\overline{k}\in\mathbb{N}_0^d} ||f_{\overline{k}}|L_p(\mathbb{R}^d)|| \leq c||2^{\overline{k}\cdot\overline{r}}(\Phi^*_{\overline{k}}f)_{\overline{a}}|\ell_q(L_p)||.$$

So, whenever the right–hand side of (1.36) is finite, we obtain the L_p–convergence of the series $\sum_{\overline{k}\in\mathbb{N}_0^d} |f_{\overline{k}}|$. Hence, this series converges in the Lebesgue measure as well and therefore also pointwise almost everywhere. We recommend [19] as far as several types of convergence of sequences of functions are concerned. So, whenever the right hand side of (1.36) is finite, we get (1.44).

When the right–hand side in (1.37) is finite, we use

$$||2^{\overline{k}\cdot\overline{r}}(\Phi^*_{\overline{k}}f)_{\overline{a}}|\ell_{\max(p,q)}(L_p)|| \leq c||2^{\overline{k}\cdot\overline{r}}(\Phi^*_{\overline{k}}f)_{\overline{a}}|L_p(\ell_q)||$$

and apply the same arguments as above.

$\square$

Remark 1.21. The conditions (1.33) are usually called *moment conditions* while (1.34) and (1.35) are the so-called *Tauberian conditions.*

1.3.4 Boundedness of the Peetre maximal operator

In this subsection we present a theorem describing the boundedness of Peetre maximal operator in the framework of weighted $L_p(\ell_q)$ and $\ell_q(L_p)$ spaces. We use the notation explained in the beginning of section 1.3.3. Especially, we still suppose that the functions $\psi_{\overline{k}}, \overline{k} \in \mathbb{N}_0^d$, belong to the space $X^{\overline{S}}(\mathbb{R}^d)$, where the vector $\overline{S}$ will be specified later on. Our main result now reads as

Theorem 1.22. *Let $\overline{a}, \overline{r} \in \mathbb{R}^d, 0 < p, q \leq \infty$. Let for every $i = 1, \ldots, d$*

$$|\psi_0^i(t)| > 0 \quad on \quad \{t \in \mathbb{R} : |t| < \varepsilon\} \tag{1.45}$$

$$|\psi_1^i(t)| > 0 \quad on \quad \{t \in \mathbb{R} : \varepsilon/2 < |t| < 2\varepsilon\}. \tag{1.46}$$

(i) *If $\overline{a} > \frac{1}{p}$ and $\overline{S} > 0$ is large enough then*

$$||2^{\overline{k}\cdot\overline{r}}(\Psi_{\overline{k}}^* f)_{\overline{a}}|\ell_q(L_p)|| \leq c||2^{\overline{k}\cdot\overline{r}}(\Psi_{\overline{k}} * f)|\ell_q(L_p)|| \tag{1.47}$$

holds for all $f \in (X^{\overline{S}-\overline{a}-1}(\mathbb{R}^d))'$.
(ii) *If $\overline{a} > \frac{1}{\min(p,q)}$ and $\overline{S} > 0$ is large enough then*

$$||2^{\overline{k}\cdot\overline{r}}(\Psi_{\overline{k}}^* f)_{\overline{a}}|L_p(\ell_q)|| \leq c||2^{\overline{k}\cdot\overline{r}}(\Psi_{\overline{k}} * f)|L_p(\ell_q)|| \tag{1.48}$$

holds for all $f \in (X^{\overline{S}-\overline{a}-1}(\mathbb{R}^d))'$.

Proof. In analogy to (1.38)–(1.40) we find functions $\{\lambda_j^i\}_{j=0}^{\infty}, i = 1, \ldots, d$ with (1.39), (1.40) and

$$\sum_{j=0}^{\infty} \lambda_j^i(t)\psi_j^i(t) = 1, \quad t \in \mathbb{R}. \tag{1.49}$$

Instead of (1.41) we now get the identity

$$f = \sum_{\overline{k}\in\mathbb{N}_0^d} \Lambda_{\overline{k}} * \Psi_{\overline{k}} * f.$$

A dilation $t \to 2^{-\nu_i}t$ in (1.49) leads to

$$\Psi_{\overline{\nu}} * f = \sum_{\overline{k}\in\mathbb{N}_0^d} \Lambda_{\overline{k},\overline{\nu}} * \Psi_{\overline{k},\overline{\nu}} * \Psi_{\overline{\nu}} * f, \quad \overline{\nu} \in \mathbb{N}_0^d, \tag{1.50}$$

where

$$\Lambda_{\overline{k},\overline{\nu}}(\xi) = [\lambda_{\overline{k}}(2^{-\overline{\nu}}\cdot)]^\wedge(\xi) = 2^{|\overline{\nu}|}\Lambda_{\overline{k}}(2^{\overline{\nu}}\xi), \quad \overline{k}, \overline{\nu} \in \mathbb{N}_0^d.$$

$\Psi_{\overline{k},\overline{\nu}}$ is defined similarly. We recall that $2^{\overline{\nu}}\xi = (2^{\nu_1}\xi_1, \ldots, 2^{\nu_d}\xi_d)$. Hence, for $\overline{k} \geq 1, \overline{\nu} \in \mathbb{N}_0^d$, we obtain $\Psi_{\overline{k},\overline{\nu}} = \Psi_{\overline{k}+\overline{\nu}}$. To simplify the notation, we point out that

$$\psi_{\overline{k}}(2^{-\overline{\nu}}x)\psi_{\overline{\nu}}(x) = \sigma_{\overline{k},\overline{\nu}}(x)\psi_{\overline{k}+\overline{\nu}}(x), \quad \overline{k}, \overline{\nu} \in \mathbb{N}_0^d,$$

where

$$\sigma_{\overline{k},\overline{\nu}}(x) = \prod_{i=1}^{d} \sigma_{k_i,\nu_i}^i(x_i),$$

$$\sigma_{k_i,\nu_i}^i(x_i) = \begin{cases} \psi_{\nu_i}^i(x_i) & \text{if} \quad k_i > 0 \\ \psi_0^i(2^{-\nu_i}x_i) & \text{if} \quad k_i = 0. \end{cases}$$

Hence we may rewrite (1.50) as

$$\Psi_{\overline{\nu}} * f = \sum_{\overline{k}\in\mathbb{N}_0^d} \Lambda_{\overline{k},\overline{\nu}} * \hat{\sigma}_{\overline{k},\overline{\nu}} * \Psi_{\overline{k}+\overline{\nu}} * f, \quad \overline{\nu} \in \mathbb{N}_0^d. \tag{1.51}$$

By Lemma 1.17, the estimate

$$|(\Lambda_{\overline{k},\overline{\nu}} * \hat{\sigma}_{\overline{k},\overline{\nu}})(z)| \leq C_{\overline{N}} 2^{|\overline{\nu}|} \frac{2^{-\overline{k}\cdot\overline{N}}}{\prod_{i=1}^{d}(1+|2^{\nu_i} z_i|^{a_i})}$$

holds for $\overline{k}, \overline{\nu} \in \mathbb{N}_0^d$ with any $\overline{N} \leq \overline{S} - 2$. The last estimate, together with (1.51), gives

$$|(\Psi_{\overline{\nu}} * f)(y)| \leq C_{\overline{N}} 2^{|\overline{\nu}|} \sum_{\overline{k}\in\mathbb{N}_0^d} \int_{\mathbb{R}^d} \frac{2^{-\overline{k}\cdot\overline{N}}}{\prod_{i=1}^{d}(1+|2^{\nu_i}(y_i - z_i)|^{a_i})} |(\Psi_{\overline{k}+\overline{\nu}} * f)(z)| \mathrm{d}z \tag{1.52}$$

Fix now any $s \in (0,1]$. Divide both sides of (1.52) by $\prod_{i=1}^{d}(1+|2^{\nu_i}(x_i - y_i)|^{a_i})$, take the supremum over $y \in \mathbb{R}^d$ and apply following inequalities

$$(1+|2^{\nu_i}(y_i - z_i)|^{a_i})(1+|2^{\nu_i}(x_i - y_i)|^{a_i}) \geq c(1+|2^{\nu_i}(x_i - z_i)|^{a_i}),$$

$$|(\Psi_{\overline{k}+\overline{\nu}} * f)(z)| \leq |(\Psi_{\overline{k}+\overline{\nu}} * f)(z)|^s (\Psi^*_{\overline{k}+\overline{\nu}} f)_{\overline{a}}(x)^{1-s} \prod_{i=1}^{d}(1+|2^{k_i+\nu_i}(x_i - z_i)|^{a_i})^{1-s},$$

$$\frac{(1+|2^{k_i+\nu_i}(x_i - z_i)|^{a_i})^{1-s}}{(1+|2^{\nu_i}(x_i - z_i)|^{a_i})} \leq \frac{2^{k_i a_i}}{(1+|2^{k_i+\nu_i}(x_i - z_i)|^{a_i})^s}.$$

Finally, we get

$$(\Psi^*_{\overline{\nu}} f)_{\overline{a}}(x) \leq c_{\overline{N}} \sum_{\overline{k}\in\mathbb{N}_0^d} 2^{\overline{k}\cdot(\overline{a}-\overline{N}-1)} (\Psi^*_{\overline{k}+\overline{\nu}} f)_{\overline{a}}(x)^{1-s} \int_{\mathbb{R}^d} \frac{2^{|\overline{k}+\overline{\nu}|} |(\Psi_{\overline{k}+\overline{\nu}} * f)(z)|^s}{\prod_{i=1}^{d}(1+|2^{k_i+\nu_i}(x_i - z_i)|^{a_i})^s} \mathrm{d}z,$$

and apply Lemma 1.19 with

$$\gamma_{\overline{\nu}} = (\Psi^*_{\overline{\nu}} f)_{\overline{a}}(x), \quad \beta_{\overline{\nu}} = \int_{\mathbb{R}^d} \frac{2^{|\overline{\nu}|} |(\Psi_{\overline{\nu}} * f)(z)|^s}{\prod_{i=1}^{d}(1+|2^{\nu_i}(x_i - z_i)|^{a_i})^s} \mathrm{d}z, \quad \overline{\nu} \in \mathbb{N}_0^d,$$

$\overline{N}^1 = \overline{S} - \overline{a} - 1$ and $\overline{N}^0$ giving the order of the distribution f, which is finite for $\overline{S} = \infty$ and smaller than $\overline{S}$ if $\overline{S}$ is finite.

By Lemma 1.19, we obtain for every $\overline{N} \leq \overline{S} - \overline{a} - 1$, $x \in \mathbb{R}^d$ and $\overline{\nu} \in \mathbb{N}_0^d$

$$(\Psi^*_{\overline{\nu}} f)_{\overline{a}}(x)^s \leq C_{\overline{N}} \sum_{\overline{k}\in\mathbb{N}_0^d} 2^{-\overline{k}\cdot\overline{N}s} \int_{\mathbb{R}^d} \frac{2^{|\overline{k}+\overline{\nu}|} |(\Psi_{\overline{k}+\overline{\nu}} * f)(z)|^s}{\prod_{i=1}^{d}(1+|2^{k_i+\nu_i}(x_i - z_i)|^{a_i})^s} \mathrm{d}z. \tag{1.53}$$

We point out that (1.53) holds for $s > 1$ as well with much simpler proof. In that case, we take (1.52) with $\overline{a}+1$ instead of $\overline{a}$, divide by $\prod_{i=1}^{d}(1+|2^{\nu_i}(x_i - y_i)|^{a_i})$ and apply Hölder's inequality for series and integrals.

We now choose $s > 0$ with $\frac{1}{a_i} < s < p$ (or $\frac{1}{a_i} < s < \min(p,q)$, respectively) for every $i = 1, \ldots, d$. Then the function

$$\frac{1}{\prod_{i=1}^{d}(1+|z_i|)^{a_i s}} \in L_1(\mathbb{R}^d),$$

and by the majorant property of the Hardy–Littlewood maximal operator $\overline{M}$ (see [28, Chapter 2]) it follows

$$(\Psi^*_{\overline{\nu}} f)_{\overline{a}}(x)^s \leq C'_{\overline{N}} \sum_{\overline{k}\in\mathbb{N}_0^d} 2^{-\overline{k}\cdot\overline{N}s} \overline{M}(|\Psi_{\overline{k}+\overline{\nu}} * f|^s)(x). \tag{1.54}$$

We choose $\overline{N} > 0$ such that $\overline{N} > -\overline{r}$ and denote

$$g_{\overline{k}}(x) = 2^{\overline{k}\cdot\overline{r}s}\overline{M}(|\Psi_{\overline{k}} * f|^s)(x).$$

Then we get from (1.54)

$$G_{\overline{\nu}}(x) = 2^{\overline{\nu}\cdot\overline{r}s}(\Psi^*_{\overline{\nu}}f)_{\overline{a}}(x)^s \le C'_{\overline{N}} \sum_{\overline{k}\ge\overline{\nu}} 2^{s(\overline{k}-\overline{\nu})(-\overline{N}-\overline{r})} g_{\overline{k}}(x)$$

Hence, for $0 < \delta < \min\{N_i + r_i, i = 1, \ldots, d\}$, we may apply Lemma (1.18) with $L_{p/s}(\ell_{q/s})$ and $\ell_{q/s}(L_{p/s})$ norm respectively. This results into

$$||2^{\overline{k}\cdot\overline{r}s}(\Psi^*_{\overline{k}}f)_{\overline{a}}(x)|\ell_{q/s}(L_{p/s})|| \le c||2^{\overline{k}\cdot\overline{r}s}\overline{M}(|\Psi_{\overline{k}} * f|^s)(x)|\ell_{q/s}(L_{p/s})|| \tag{1.55}$$

and

$$||2^{\overline{k}\cdot\overline{r}s}(\Psi^*_{\overline{k}}f)_{\overline{a}}(x)|L_{p/s}(\ell_{q/s})|| \le c||2^{\overline{k}\cdot\overline{r}s}\overline{M}(|\Psi_{\overline{k}} * f|^s)(x)|L_{p/s}(\ell_{q/s})||. \tag{1.56}$$

In the first case, we rewrite the left–hand side of (1.55) and use the classical Hardy–Littlewood Theorem (see (1.12) for details, we recall that $s < p$),

$$||2^{\overline{k}\cdot\overline{r}}(\Psi^*_{\overline{k}}f)_{\overline{a}}(x)|\ell_q(L_p)|| \le c||2^{\overline{k}\cdot\overline{r}}(\Psi_{\overline{k}} * f)(x)|\ell_q(L_p)||.$$

In the second case, we rewrite the left–hand side of (1.56) and use Theorem 1.11 (now we recall that $s < \min(p, q)$),

$$||2^{\overline{k}\cdot\overline{r}}(\Psi^*_{\overline{k}}f)_{\overline{a}}(x)|L_p(\ell_q)|| \le c||2^{\overline{k}\cdot\overline{r}}(\Psi_{\overline{k}} * f)(x)|L_p(\ell_q)||,$$

which concludes the proof. □

1.3.5 Local means characterisation

We summarise sections 1.3.3 and 1.3.4 and give the usual formulation of the local means characterisation. We still use the tensor construction of functions $\psi_{\overline{k}}$ described in the beginning of section 1.3.3. The spaces $X^{\overline{S}}(\mathbb{R}^d)$ and the Peetre maximal function $(\Psi^*_{\overline{k}}f)_{\overline{a}}$ were defined in section 1.3.1. We still suppose that $\psi^i_0, \psi^i_1 \in X^{\overline{S}}(\mathbb{R}^d)$, where the vector $\overline{S}$ will be specified later on.

Theorem 1.23. (i) *Let $0 < p, q \le \infty$, $\overline{r}, \overline{a} \in \mathbb{R}^d$, $\overline{R}, \overline{S} \in \mathbb{Z}^d$ with $\overline{r} \le \overline{R} + 1$ and $\overline{a} > \frac{1}{p}$. If $\overline{S} > \overline{R}$ is large enough,*

$$D^\alpha \psi^i_1(0) = 0, \quad i = 1, \ldots, d, \quad \alpha = 0, 1, \ldots, R_i, \tag{1.57}$$

and

$$|\psi^i_0(t)| > 0 \quad on \quad \{t \in \mathbb{R} : |t| < \varepsilon\} \tag{1.58}$$

$$|\psi^i_1(t)| > 0 \quad on \quad \{t \in \mathbb{R} : \varepsilon/2 < |t| < 2\varepsilon\} \tag{1.59}$$

for some $\varepsilon > 0$, then

$$||f|S^{\overline{r}}_{p,q}B(\mathbb{R}^d)|| \approx ||2^{\overline{k}\cdot\overline{r}}(\Psi_{\overline{k}} * f)|\ell_q(L_p)|| \approx ||2^{\overline{k}\cdot\overline{r}}(\Psi^*_{\overline{k}}f)_{\overline{a}}|\ell_q(L_p)||$$

for all $f \in (X^{\overline{S}-\overline{a}-1}(\mathbb{R}^d))'$.

(ii) *Let* $0 < p < \infty, 0 < q \leq \infty$, $\overline{r}, \overline{a} \in \mathbb{R}^d, \overline{R}, \overline{S} \in \mathbb{Z}^d$ *with* $\overline{r} \leq \overline{R}+1$ *and* $\overline{a} > \frac{1}{\min(p,q)}$. *If* $\overline{S} > \overline{R}$ *is large enough, and* (1.57) – (1.59) *are satisfied then*

$$||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)|| \approx ||2^{\overline{k}\cdot\overline{r}}(\Psi_{\overline{k}} * f)|L_p(\ell_q)|| \approx ||2^{\overline{k}\cdot\overline{r}}(\Psi^*_{\overline{k}}f)_{\overline{a}}|L_p(\ell_q)||$$

for all $f \in (X^{\overline{S}-\overline{a}-1}(\mathbb{R}^d))'$.

Remark 1.24. 1. Theorem 1.23 is just reformulation of Theorem 1.20 and Theorem 1.22.

2. In the proof of Theorems 1.20 and 1.22 we followed essentially the approach described in [25]. We point out that recently very similar results were obtained in [3].

3. We may set $\overline{S} = \infty$ in Theorem 1.23. Then one obtains equivalent quasinorms on $S'(\mathbb{R}^d)$. By choosing $\overline{S}$ large, but finite, we may always ensure, that the new quasinorms are equivalent at least on $S^{\overline{r}}_{p,q}A(\mathbb{R}^d) \subset (X^{\overline{S}-\overline{a}-1}(\mathbb{R}^d))'$.

Next we reformulate Theorem 1.23 using the local means in the sense of [33].

Theorem 1.25. *Let* $0 < p,q \leq \infty$ *(with* $p < \infty$ *in the* F*–case),* $\overline{r} \in \mathbb{R}^d, \overline{S}^1, \overline{S}^2 \in \mathbb{N}_0^d$ *with* $\overline{S}^1 - \overline{S}^2 > \frac{1}{p}+1$ *in the* B*–case and* $\overline{S}^1 - \overline{S}^2 > \frac{1}{\min(p,q)}+1$ *in the* F*–case. Let* $\overline{R} \in \mathbb{N}_0^d$ *be a vector of* d *nonnegative integers with* $\overline{R} > \overline{r}$. *Further let* $k_0, k^1, \ldots, k^d$ *be* $d+1$ *complex-valued functions from* $X^{\overline{S}^1}(\mathbb{R})$ *whose supports lie in the set* $\{t \in \mathbb{R} : |t| < 1\}$ *and*

$$F_1(k_0)(0) \neq 0, \qquad F_1(k^i)(0) \neq 0,\ i = 1, \ldots, d. \tag{1.60}$$

Let us denote

$$k^i_0(t) = k_0(t) \qquad \text{and} \qquad k^i_n(t) = 2^n\left(\frac{\mathrm{d}^{R_i}}{\mathrm{d}t^{R_i}}k^i\right)(2^n t),\ i = 1, \ldots, d,\ n \in \mathbb{N},\ t \in \mathbb{R}.$$

As usually, we denote by $k_{\overline{\nu}}(x) = k^1_{\nu_1}(x_1)\cdots k^d_{\nu_d}(x_d), \overline{\nu} = (\nu_1, \ldots, \nu_d) \in \mathbb{N}_0^d$, *the tensor product of these functions.*

The corresponding local means are defined by

$$k_{\overline{\nu}}(f)(x) = \int_{\mathbb{R}^d} k_{\overline{\nu}}(y)f(x+y)\mathrm{d}y, \quad \overline{\nu} \in \mathbb{N}_0^d, \quad x \in \mathbb{R}^d, \tag{1.61}$$

appropriately interpreted for any $f \in (X^{\overline{S}^1}(\mathbb{R}^d))'$. *Then, if* $\overline{S}^2$ *is large enough,*

$$||2^{\overline{\nu}\cdot\overline{r}}k_{\overline{\nu}}(f)|L_p(\ell_q)|| \approx ||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)||, \quad f \in (X^{\overline{S}^2}(\mathbb{R}^d))', \tag{1.62}$$

and

$$||2^{\overline{\nu}\cdot\overline{r}}k_{\overline{\nu}}(f)|\ell_q(L_p)|| \approx ||f|S^{\overline{r}}_{p,q}B(\mathbb{R}^d)||, \quad f \in (X^{\overline{S}^2}(\mathbb{R}^d))'. \tag{1.63}$$

Proof. Put $\psi^i_0 = F_1^{-1}k_0$ and $\psi^i_1 = F_1^{-1}(\frac{\mathrm{d}^{R_i}}{\mathrm{d}t^{R_i}}k^i)$. Then the Tauberian conditions (1.58) and (1.59) are satisfied and (1.57) is also true. If we define $\psi_{\overline{\nu}}, \overline{\nu} \in \mathbb{N}_0^d$, as in (1.32), we get

$$\begin{aligned}(\psi_{\overline{\nu}}\hat{f})^\vee(x) &= c\int_{\mathbb{R}^d}(\psi_{\overline{\nu}})^\vee(y)f(x-y)\mathrm{d}y = c\int_{\mathbb{R}^d}(F\psi_{\overline{\nu}})(y)f(x+y)\mathrm{d}y \\ &= c\int_{\mathbb{R}^d}\left(\prod_{i=1}^d(F_1\psi^i_{\nu_i})(y_i)\right)f(x+y)\mathrm{d}y.\end{aligned} \tag{1.64}$$

Finally, if $\nu_i = 0$, we get $(F_1\psi_0^i)(y_i) = k_0^i(y_i)$ and if $\nu_i \geq 1$ we obtain in a similar way

$$(F_1\psi_{\nu_i}^i)(y_i) = (F_1(\psi^i(2^{-\nu_i}\cdot)))(y_i) = 2^{\nu_i}(F_1\psi^i)(2^{\nu_i}y_i) = 2^{\nu_i}\big(\frac{\mathrm{d}^{R_i}}{\mathrm{d}t^{R_i}}k^i\big)(2^{\nu_i}y_i) = k_{\nu_i}^i(y_i).$$

Using this calculation and (1.64) we get

$$(\psi_{\overline{\nu}}\hat{f})^\vee(x) = \int_{\mathbb{R}^d} k_{\overline{\nu}}(y)f(x+y)\mathrm{d}y, \quad \overline{\nu}\in\mathbb{N}_0^d, \quad x\in\mathbb{R}^d$$

and the theorem follows. □

Remark 1.26. We point out that $\overline{S}^1 = \overline{S}^2 = \infty$ is allowed in Theorem 1.25.

We shall need some other modifications of Theorem 1.23. But first we give some necessary notation. For $\overline{\nu}\in\mathbb{N}_0^d, \overline{m}\in\mathbb{Z}^d$ we denote by $Q_{\overline{\nu}\,\overline{m}}$ the cube with the centre at the point $2^{-\overline{\nu}}\overline{m} = (2^{-\nu_1}m_1,\dots,2^{-\nu_d}m_d)$ with sides parallel to coordinate axes and of lengths $2^{-\nu_1},\dots,2^{-\nu_d}$. Hence

$$Q_{\overline{\nu}\,\overline{m}} = \{x\in\mathbb{R}^d : |x_i - 2^{-\nu_i}m_i| \leq 2^{-\nu_i-1}, i=1,\dots,d\}, \qquad \overline{\nu}\in\mathbb{N}_0^d,\ \overline{m}\in\mathbb{Z}^d. \tag{1.65}$$

If $\gamma>0$ then $\gamma Q_{\overline{\nu}\,\overline{m}}$ denotes a cube concentric with $Q_{\overline{\nu}\,\overline{m}}$ with sides also parallel to coordinate axes and of lengths $\gamma 2^{-\nu_1},\dots,\gamma 2^{-\nu_d}$.

Defining the Peetre maximal function by (1.17), we get

$$(\Psi_{\overline{\nu}}^* f)_{\overline{a}}(x) \geq c \sup_{x-y\in\gamma Q_{\overline{\nu},0}} |(\Psi_{\overline{\nu}} * f)(y)|, \quad \overline{\nu}\in\mathbb{N}_0^d, \quad x\in\mathbb{R}^d,$$

where the constant c depends on $\overline{a}>0, \gamma>0$ but does not depend neither on x nor on $\overline{\nu}$. This very simple observation gives together with Theorem 1.23 following

Theorem 1.27. *Let $\overline{r}\in\mathbb{R}^d$, $0<p,q\leq\infty$ ($p<\infty$ in the F–case). Let $\overline{R}\in\mathbb{N}_0^d$ with $\overline{R}>\overline{r}$, $\overline{S}^1,\overline{S}^2\in\mathbb{N}_0^d$ and $k_{\overline{\nu}}$ be as in Theorem 1.25. Then, for any $\gamma>0$,*

$$\Bigg|\Bigg|\Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d} 2^{q\overline{\nu}\cdot\overline{r}} \sup_{x-y\in\gamma Q_{\overline{\nu},0}} |k_{\overline{\nu}}(f)(y)|^q\Big)^{1/q} |L_p(\mathbb{R}^d)\Bigg|\Bigg| \approx ||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)||, \quad f\in(X^{\overline{S}^2}(\mathbb{R}^d))' \tag{1.66}$$

and

$$\Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d} 2^{q\overline{\nu}\cdot\overline{r}}||\sup_{x-y\in\gamma Q_{\overline{\nu},0}} |k_{\overline{\nu}}(f)(y)||L_p(\mathbb{R}^d)||^q\Big)^{1/q} \approx ||f|S^{\overline{r}}_{p,q}B(\mathbb{R}^d)||, \quad f\in(X^{\overline{S}^2}(\mathbb{R}^d))'. \tag{1.67}$$

Another modification of Theorem 1.23 is rather technical and deals with 'directional' local means, namely with local means of the form ($d=2$):

$$\int_{\mathbb{R}} k_{\nu_1}^1(y_1)f(x_1+y_1,x_2)\mathrm{d}y_1.$$

To introduce these local means in the general dimension, we define for every $A\subset\{1,\dots,d\}$

$$k_{\overline{\nu},A}(f)(x) = \int_{\mathbb{R}^{|A|}} \big(\prod_{i\in A} k_{\nu_i}^i(y_i)\big) f(x_1+y_1\chi_A(1),\dots,x_d+y_d\chi_A(d))\big(\prod_{i\in A}\mathrm{d}y_i\big). \tag{1.68}$$

It means, we restrict the integration in (1.61) to those variables y_i for which $i\in A$. The others are left untouched.

Using this notation, we may state our next Lemma.

Lemma 1.28. *Let $0 < p < \infty, 0 < q \leq \infty, A \subset \{1, \ldots, d\}$ and $\gamma > 0$. Let $\overline{r} \in \mathbb{R}^d$ be such that $r_i > \frac{1}{\min(p,q)}$ for $i \notin A$. Let $R_i \in \mathbb{N}_0$ and $k^i_{\overline{\nu}}$ be as in Theorem 1.25 for every $i \in A$. Further let $k_{\overline{\nu},A}(f)$ be defined by (1.68). Then*

$$\left|\left|\left(\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^d\\ \nu_i=0, i\notin A}} 2^{q\overline{\nu}\cdot\overline{r}} \sup_{x-y\in\gamma Q_{\overline{\nu},0}} |k_{\overline{\nu},A}(f)(y)|^q\right)^{1/q} |L_p(\mathbb{R}^d)\right|\right| \leq c||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)|| \tag{1.69}$$

holds for every $f \in S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$. The sum is taken over all $\overline{\nu} = (\nu_1, \ldots, \nu_d) \in \mathbb{N}_0^d$ with $\nu_i = 0$ whenever $i \notin A$. The L_p-quasinorm is then taken with respect to x.

Remark 1.29. There is again a direct analogy of this Lemma for the B-scale and for non-smooth kernels. The proof of this Lemma follows the proof of Theorem 1.23.

2 Decomposition theorems

In this chapter we present three decomposition theorems. We give atomic, subatomic and wavelet decomposition characteristics of spaces with dominating mixed smoothness. But first of all we explain some notation used in connection with sequence spaces.

2.1 Sequence spaces

We recall that for $\overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d$ we denote by $Q_{\overline{\nu}\overline{m}}$ the cube with the centre at the point $2^{-\overline{\nu}}\overline{m} = (2^{-\nu_1}m_1, \ldots, 2^{-\nu_d}m_d)$ with sides parallel to coordinate axes and of lengths $2^{-\nu_1}, \ldots, 2^{-\nu_d}$. By $\chi^{(p)}_{\overline{\nu}\overline{m}}$ we denote a p-normalised characteristic function of $Q_{\overline{\nu}\overline{m}}$, it means that $\chi^{(p)}_{\overline{\nu}\overline{m}}(x) = 2^{\frac{|\overline{\nu}|}{p}}\chi_{Q_{\overline{\nu}\overline{m}}}(x)$. Furthermore, we write $\chi_{\overline{\nu}\overline{m}}(x) = \chi_{Q_{\overline{\nu}\overline{m}}}(x)$.

Definition 2.1. If $0 < p, q \leq \infty$, $\overline{r} \in \mathbb{R}^d$ and

$$\lambda = \{\lambda_{\overline{\nu}\overline{m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d\} \tag{2.1}$$

then we define

$$s^{\overline{r}}_{p,q}b = \left\{\lambda : ||\lambda|s^{\overline{r}}_{p,q}b|| = \left(\sum_{\overline{\nu}\in\mathbb{N}_0^d} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\left(\sum_{\overline{m}\in\mathbb{Z}^d} |\lambda_{\overline{\nu}\overline{m}}|^p\right)^{q/p}\right)^{1/q} < \infty\right\} \tag{2.2}$$

and

$$s^{\overline{r}}_{p,q}f = \left\{\lambda : ||\lambda|s^{\overline{r}}_{p,q}f|| = \left|\left|\left(\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} |2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu}\overline{m}}\chi_{\overline{\nu}\overline{m}}(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R}^d)\right|\right| < \infty\right\} \tag{2.3}$$

with the usual modification for p and/or q equal to ∞.

Remark 2.2. We point out that with λ given by (2.1) and $g_{\overline{\nu}}(x) = \sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu}\overline{m}}\chi_{\overline{\nu}\overline{m}}(x)$, we obtain that

$$||\lambda|s^{\overline{r}}_{p,q}b|| = ||2^{\overline{\nu}\cdot\overline{r}}g_{\overline{\nu}}|\ell_q(L_p)||, \qquad ||\lambda|s^{\overline{r}}_{p,q}f|| = ||2^{\overline{\nu}\cdot\overline{r}}g_{\overline{\nu}}|L_p(\ell_q)||.$$

Sequence spaces of this kind were denoted by E_{dis} in [14] and may be understood as a discrete version of $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ and $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$.

2.2 Atomic decomposition

Definition 2.3. Let $\overline{K} \in \mathbb{N}_0^d, \overline{L}+1 \in \mathbb{N}_0^d$, and $\gamma > 1$. A $\overline{K}$-times differentiable complex-valued function $a(x)$ is called $[\overline{K}, \overline{L}]$-atom centred at $Q_{\overline{\nu}\,\overline{m}}$ if

$$\operatorname{supp} a \subset \gamma Q_{\overline{\nu}\,\overline{m}}, \tag{2.4}$$

$$|D^\alpha a(x)| \le 2^{\alpha\cdot\overline{\nu}} \qquad \text{for} \quad 0 \le \alpha \le \overline{K} \tag{2.5}$$

and

$$\int_{\mathbb{R}} x_i^j a(x) \mathrm{d}x_i = 0 \qquad \text{if} \quad i = 1, \ldots, d;\ j = 0, \ldots, L_i \quad \text{and} \quad \nu_i \ge 1. \tag{2.6}$$

Using this notation we may state the atomic decomposition theorem.

Theorem 2.4. *Let $0 < p, q \le \infty$, ($p < \infty$ in the $F-$case) and $\overline{r} \in \mathbb{R}^d$. Fix $\overline{K} \in \mathbb{N}_0^d$ and $\overline{L}+1 \in \mathbb{N}_0^d$ with*

$$K_i \ge (1 + [r_i])_+ \quad \textit{and} \quad L_i \ge \max(-1, [\sigma_{pq} - r_i]), \qquad i = 1, \ldots, d. \tag{2.7}$$

($L_i \ge \max(-1, [\sigma_p - r_i])$ in the B-case).
(i) *If $\lambda \in s^{\overline{r}}_{p,q}a$ and $\{a_{\overline{\nu}\,\overline{m}}(x)\}_{\overline{\nu}\in\mathbb{N}_0^d, \overline{m}\in\mathbb{Z}^d}$ are $[\overline{K}, \overline{L}]$-atoms centred at $Q_{\overline{\nu}\,\overline{m}}$, then the sum*

$$\sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}} a_{\overline{\nu}\,\overline{m}}(x) \tag{2.8}$$

converges in $S'(\mathbb{R}^d)$, its limit f belongs to the space $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ and

$$||f|S^{\overline{r}}_{p,q}A(\mathbb{R}^d)|| \le c\,||\lambda|s^{\overline{r}}_{p,q}a||, \tag{2.9}$$

where the constant c is universal for all admissible λ and $a_{\overline{\nu}\,\overline{m}}$.
(ii) *For every $f \in S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ there is a $\lambda \in s^{\overline{r}}_{p,q}a$ and $[\overline{K}, \overline{L}]$-atoms centred at $Q_{\overline{\nu}\,\overline{m}}$ (denoted again by $\{a_{\overline{\nu}\,\overline{m}}(x)\}_{\overline{\nu}\in\mathbb{N}_0^d, \overline{m}\in\mathbb{Z}^d}$) such that the sum (2.8) converges in $S'(\mathbb{R}^d)$ to f and*

$$||\lambda|s^{\overline{r}}_{p,q}a|| \le c\,||f|S^{\overline{r}}_{p,q}A(\mathbb{R}^d)||. \tag{2.10}$$

The constant c is again universal for every $f \in S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$.

Proof. We give the proof only for the F-case. The proof for the B-scale is very similar.
Step 1.
First of all we prove the convergence of (2.8) in $S'(\mathbb{R}^d)$. Let $\varphi \in S(\mathbb{R}^d)$. We use the Taylor expansion of φ with respect to the first variable

$$\begin{aligned}\varphi(y) = \sum_{\alpha_1 \le L_1} & \frac{D^{(\alpha_1,0,\ldots,0)}\varphi(2^{-\nu_1}m_1, y_2, \ldots, y_d)}{\alpha_1!}(y_1 - 2^{\nu_1}m_1)^{\alpha_1} \\ &+ \frac{1}{L_1!}\int_{2^{-\nu_1}m_1}^{y_1} (t_1 - 2^{-\nu_1}m_1)^{L_1} D^{(L_1+1,0,\ldots,0)}\varphi(t_1, y_2, \ldots, y_d)\mathrm{d}t_1\end{aligned} \tag{2.11}$$

and (2.6) to obtain

$$\int_{\mathbb{R}^d} a_{\overline{\nu}\,\overline{m}}(y)\varphi(y)\mathrm{d}y = \int_{\mathbb{R}^d} \frac{a_{\overline{\nu}\,\overline{m}}(y)}{L_1!} \int_{2^{-\nu_1}m_1}^{y_1} (t_1 - 2^{-\nu_1}m_1)^{L_1} D^{(L_1+1,0,\ldots,0)}\varphi(t_1, y_2, \ldots, y_d)\mathrm{d}t_1\mathrm{d}y. \tag{2.12}$$

Using an analogy of (2.11) iteratively for the remaining $d-1$ variables we see that the left hand side of (2.12) is equal to

$$\int_{\mathbb{R}^d} \frac{a_{\overline{\nu}\,\overline{m}}(y)}{\overline{L}!} \int_{2^{-\nu_1}m_1}^{y_1} \cdots \int_{2^{-\nu_d}m_d}^{y_d} \prod_{i=1}^{d}(t_i - 2^{-\nu_i}m_i)^{L_i} D^{\overline{L}+1}\varphi(t_1, \ldots, t_d)\mathrm{d}t\mathrm{d}y.$$

Using the support property (2.4) of $a_{\overline{\nu}\,\overline{m}}$ we may estimate the absolute value of the inner $d-$dimensional integration from above by $(y \in \gamma Q_{\overline{\nu}\,\overline{m}})$

$$c\, 2^{-\overline{\nu}\cdot(\overline{L}+1)} \sup_{x\in\gamma Q_{\overline{\nu}\,\overline{m}}} |(D^{\overline{L}+1}\varphi)(x)| \le c_M 2^{-\overline{\nu}\cdot(\overline{L}+1)} \langle y\rangle^{-M} \sup_{x\in\gamma Q_{\overline{\nu}\,\overline{m}}} \langle x\rangle^M |(D^{\overline{L}+1}\varphi)(x)|,$$

where M is at our disposal. Here we denote $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$ for $x \in \mathbb{R}^d$.

Let us now suppose that $p \ge 1$ and use (2.5) and Hölder's inequality to get for M large enough

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}} a_{\overline{\nu}\,\overline{m}}(y)\varphi(y)\mathrm{d}y \right| \\
&\le c\, 2^{-\overline{\nu}\cdot(\overline{L}+1)} 2^{-\overline{\nu}\cdot\frac{1}{p}} \sup_{x\in\mathbb{R}^d} \langle x\rangle^M |(D^{\overline{L}+1}\varphi)(x)| \int_{\mathbb{R}^d} \left(\sum_{\overline{m}\in\mathbb{Z}^d} 2^{\overline{\nu}\cdot\frac{1}{p}} |\lambda_{\overline{\nu}\,\overline{m}}| \chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(y) \right) \langle y\rangle^{-M}\mathrm{d}y \\
&\le c\, 2^{-\overline{\nu}\cdot(\overline{r}+\overline{L}+1)} \cdot 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})} \left(\sum_{\overline{m}\in\mathbb{Z}^d} |\lambda_{\overline{\nu}\,\overline{m}}|^p \right)^{1/p} \cdot \sup_{x\in\mathbb{R}^d} \langle x\rangle^M |(D^{\overline{L}+1}\varphi)(x)|.
\end{aligned}$$

As $\lambda \in s^{\overline{r}}_{pq}f \subset s^{\overline{r}}_{p,\infty}b$ and $\overline{r} + \overline{L} + 1 > 0$, the convergence of (2.8) in $S'(\mathbb{R}^d)$ now follows.

If $p < 1$, we get a similar estimate

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}} a_{\overline{\nu}\,\overline{m}}(y)\varphi(y)\mathrm{d}y \right|^p \\
&\le c\, 2^{-\overline{\nu}\cdot(\overline{L}+1)p} \sup_{x\in\mathbb{R}^d} |(D^{\overline{L}+1}\varphi)(x)|^p \sum_{\overline{m}\in\mathbb{Z}^d} |\lambda_{\overline{\nu}\,\overline{m}}|^p \left| \int_{\mathbb{R}^d} \chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(y)\mathrm{d}y \right|^p \\
&\le c\, 2^{-\overline{\nu}\cdot(\overline{r}+\overline{L}+1-1/p+1)p} \sup_{x\in\mathbb{R}^d} |(D^{\overline{L}+1}\varphi)(x)|^p \sum_{\overline{m}\in\mathbb{Z}^d} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})p} |\lambda_{\overline{\nu}\,\overline{m}}|^p.
\end{aligned}$$

In this case we use the fact that $\overline{r} + \overline{L} + 1 - 1/p + 1 > 0$ and the embedding $s^{\overline{r}}_{p,q}f \subset s^{\overline{r}}_{p,\infty}b$.

Step 2.

Next we prove (2.9). We use the equivalent quasinorms in $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ given by (1.62). Let us choose $\overline{R} > \overline{K}$ and define the functions $k_{\overline{l}}$ for $\overline{l} \in \mathbb{N}_0^d$ as in Theorem 1.25. Then we have for all $\overline{l}, \overline{\nu} \in \mathbb{N}_0^d$ and all $\overline{m} \in \mathbb{Z}^d$

$$2^{\overline{l}\cdot\overline{r}} k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x) = 2^{\overline{l}\cdot\overline{r}} \int_{\mathbb{R}^d} k^1_{l_1}(y_1) \cdots k^d_{l_d}(y_d) a_{\overline{\nu}\,\overline{m}}(x+y)\mathrm{d}y. \tag{2.13}$$

Further calculation depends on the size of the supports of $k_{\overline{l}}$ and $a_{\overline{\nu}\,\overline{m}}$. Hence we have to distinguish between $l_i \geq \nu_i$ and $l_i < \nu_i$. This leads to 2^d cases. We describe the first one ($\overline{l} \geq \overline{\nu}$) and the last one ($\overline{l} < \overline{\nu}$) in the full detail and then we discuss the 'mixed' cases.

I. $\overline{l} \geq \overline{\nu}$.

We suppose that $\overline{l} > 0$. This only simplifies the notation, the terms with $l_i = \nu_i = 0$ may be incorporated afterwards. We use the definition of $k^i_{l_i}$ and make partial integration (K_i-times in the i^{th} variable) to obtain

$$2^{\overline{l}\cdot\overline{r}} k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x) = 2^{\overline{l}\cdot(\overline{r}+1)} \int_{\mathbb{R}^d} \prod_{i=1}^d \left(\frac{\mathrm{d}^{R_i}}{\mathrm{d}t^{R_i}} k^i \right) (2^{l_i} y_i) a_{\overline{\nu}\,\overline{m}}(x+y)\mathrm{d}y$$

$$= 2^{\overline{l}\cdot\overline{r}} \int_{\mathbb{R}^d} \prod_{i=1}^d \left(\frac{\mathrm{d}^{R_i}}{\mathrm{d}t^{R_i}} k^i \right) (y_i) a_{\overline{\nu}\,\overline{m}}(x_1 + 2^{-l_1} y_1, \ldots, x_d + 2^{-l_d} y_d)\mathrm{d}y$$

$$= 2^{\overline{l}\cdot(\overline{r}-\overline{K})} \int_{\mathbb{R}^d} \prod_{i=1}^d \left(\frac{\mathrm{d}^{R_i-K_i}}{\mathrm{d}t^{R_i-K_i}} k^i \right) (y_i) (D^{\overline{K}} a_{\overline{\nu}\,\overline{m}})(x_1 + 2^{-l_1} y_1, \ldots, x_d + 2^{-l_d} y_d)\mathrm{d}y.$$

Next we use the smoothness of k^i, the boundedness of their supports and the properties (2.4) and (2.5) to estimate the absolute value of this expression.

$$2^{\overline{l}\cdot\overline{r}} |k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x)| \leq c\, 2^{\overline{l}\cdot(\overline{r}-\overline{K})} 2^{\overline{\nu}\cdot\overline{K}} \cdot$$
$$\cdot \int_{\mathbb{R}^d} \left(\prod_{i=1}^d \chi_{\operatorname{supp} k^i}(y_i) \right) \chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x_1 + 2^{-l_1} y_1, \ldots, x_d + 2^{-l_d} y_d)\mathrm{d}y.$$

As $\operatorname{supp} k^i \subset \{t \in \mathbb{R} : |t| \leq 1\}, i = 1, \ldots, d$, it follows that

$$2^{\overline{l}\cdot\overline{r}} |k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x)| \leq c\, 2^{-(\overline{K}-\overline{r})(\overline{l}-\overline{\nu})} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})} \chi^{(p)}_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x). \tag{2.14}$$

II. $\overline{l} < \overline{\nu}$.

The integration in (2.13) may be restricted to $\{y : |y_i| \leq 2^{-l_i}\}$. We use the Taylor expansion of functions $k^i_{l_i}(y_i)$ with respect to the off-points $2^{-\nu_i} m_i - x_i$ up to order L_i

$$2^{-l_i} k^i_{l_i}(y_i) = \sum_{0 \leq \beta_i \leq L_i} c^i_{\beta_i}(x_i)(y_i - 2^{-\nu_i} m_i + x_i)^{\beta_i} + 2^{l_i(L_i+1)} O(|x_i + y_i - 2^{-\nu_i} m_i|^{L_i+1}) \tag{2.15}$$

and (2.6) to get

$$2^{\overline{l}\cdot\overline{r}} k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x) = 2^{\overline{l}\cdot(\overline{r}+1)} \int_{\{y:|y_i| \leq 2^{-l_i}\}} a_{\overline{\nu}\,\overline{m}}(x+y) \prod_{i=1}^d 2^{l_i(L_i+1)} O(|x_i + y_i - 2^{-\nu_i} m_i|^{L_i+1})\mathrm{d}y.$$

Since

$$|a_{\overline{\nu}\,\overline{m}}(x+y)| \leq \chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x+y)$$

we obtain

$$2^{\overline{l}\cdot\overline{r}} |k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x)| \leq c\, 2^{\overline{l}\cdot(\overline{r}+1)} 2^{(\overline{l}-\overline{\nu})\cdot(\overline{L}+1)} \int_{\{y:|y_i| \leq 2^{-l_i}\}} \chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x+y)\mathrm{d}y. \tag{2.16}$$

The last integral is always smaller then $c\,2^{-|\overline{\nu}|}$ and is zero if $\{y : x+y \in \gamma Q_{\overline{\nu}\,\overline{m}}\} \cap \{y : |y_i| \leq 2^{-l_i}\} = \emptyset$. Hence

$$\int_{\{y:|y_i|\leq 2^{-l_i}\}} \chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x+y)\mathrm{d}y \leq c\,2^{-|\overline{\nu}|}\chi_{c2^{\overline{\nu}-\overline{l}}Q_{\overline{\nu}\,\overline{m}}}(x). \tag{2.17}$$

But the last expression may be estimated from above with the use of maximal operators M_i defined by (1.14).

$$2^{|\overline{l}-\overline{\nu}|}\chi_{c2^{\overline{\nu}-\overline{l}}Q_{\overline{\nu}\,\overline{m}}}(x) \leq c\,(\overline{M}\chi_{\overline{\nu}\,\overline{m}})(x). \tag{2.18}$$

Let $0<\omega<\min(1,p,q)$. Taking the $1/\omega$-power of (2.18) and inserting it in (2.17) we obtain

$$\int_{\{y:|y_i|\leq 2^{-l_i}\}} \chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x+y)\mathrm{d}y \leq c\,2^{-|\overline{\nu}|}2^{|\overline{\nu}-\overline{l}|\frac{1}{\omega}}(\overline{M}\chi_{\overline{\nu}\,\overline{m}})^{\frac{1}{\omega}}(x). \tag{2.19}$$

Next we replace $\chi_{\overline{\nu}\,\overline{m}}$ by $\chi^{(p)}_{\overline{\nu}\,\overline{m}}$ in (2.19) and insert it in (2.16).

$$2^{\overline{l}\cdot\overline{r}}|k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x)| \leq c\,2^{(\overline{l}-\overline{\nu})\cdot(\overline{r}+1+\overline{L}+1-\frac{1}{\omega})}2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})}(\overline{M}\chi^{(p)\omega}_{\overline{\nu}\,\overline{m}})^{\frac{1}{\omega}}(x).$$

By (2.7) and (1.4) we may choose the number ω such that $\overline{\varkappa} = (\overline{r}+1+\overline{L}+1-\frac{1}{\omega}) > 0$.

III. Mixed terms.

We estimate for example the term with $l_1 \geq \nu_1$, $l_i < \nu_i, i=2,\ldots,d$.

First we apply (2.15) for $i=2,\ldots,d$ and use (2.6) to leave out the terms with $\beta \leq \overline{L}$. Then we use K_1 partial integration in the first variable. In the expression we get we use again the support properties of the functions involved and (2.5) to obtain

$$2^{\overline{l}\cdot\overline{r}}|k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x)| \leq 2^{\overline{\nu}\cdot\overline{r}}2^{(l_1-\nu_1)(r_1-K_1)}2^{\sum_{i=2}^d l_i(r_i+1)+(l_i-\nu_i)(L_i+1)-\nu_i r_i}$$
$$\int_{A_{\overline{l}}} \chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x_1+2^{-l_1}y_1, x_2+y_2,\ldots,x_d+y_d)\mathrm{d}y,$$

where $A_{\overline{l}} = \{y\in\mathbb{R}^d : |y_1|\leq 1, |y_i|\leq 2^{-l_i}, i=2,\ldots,d\}$. Due to the product structure of the integrated function we may split the last integral into a one-dimensional integral with respect to $\mathrm{d}y_1$ and $d-1$ dimensional integral with respect to the remaining variables. The first integral then may be estimated from above by $c\chi_{\{t:|t-2^{-\nu_1}m_1|\leq 2^{-\nu_i}\}}(x_1)$. Finally we use the maximal operators $M_i, i=2,\ldots,d$ to estimate the second integral. And, exactly as in the second step, it turns out, that there is some vector $\overline{\varrho}>0$ such that

$$2^{\overline{l}\cdot\overline{r}}|k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x)| \leq c\,2^{-\sum_{i=1}^d|l_i-\nu_i|\varrho_i}2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})}(\overline{M}\chi^{(p)\omega}_{\overline{\nu}\,\overline{m}})^{\frac{1}{\omega}}(x). \tag{2.20}$$

Let us observe that also (2.14) may be estimated from above by the right-hand side of(2.20). Hence the estimate (2.20) is valid for all $\overline{l},\overline{\nu}\in\mathbb{N}_0^d$.

Using this estimate, we get for $q\leq 1$,

$$\left|2^{\overline{l}\cdot\overline{r}}k_{\overline{l}}\Big(\sum_{\overline{\nu},\overline{m}}\lambda_{\overline{\nu}\,\overline{m}}a_{\overline{\nu}\,\overline{m}}\Big)(x)\right|^q \leq c\sum_{\overline{\nu},\overline{m}}|\lambda_{\overline{\nu}\,\overline{m}}|^q 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}2^{-q\sum_{i=1}^d|l_i-\nu_i|\varrho_i}(\overline{M}\chi^{(p)\omega}_{\overline{\nu}\,\overline{m}})^{\frac{q}{\omega}}(x).$$

For $q>1$, the same estimate is justified by Hölder's inequality.

We sum over $\overline{l}$, take the $\frac{1}{q}-$power and then we apply the L_p-quasinorm with respect to x. Denoting $g_{\overline{\nu}\,\overline{m}} = 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})}\lambda_{\overline{\nu}\,\overline{m}}\chi^{(p)}_{\overline{\nu}\,\overline{m}}$ we arrive at

$$\begin{aligned}\Bigg|\Bigg|\Bigg(\sum_{\overline{l}\in\mathbb{N}_0^d}\Bigg|2^{\overline{l}\cdot\overline{r}}k_{\overline{l}}\Bigg(\sum_{\overline{\nu},\overline{m}}\lambda_{\overline{\nu}\,\overline{m}}a_{\overline{\nu}\,\overline{m}}\Bigg)(x)\Bigg|^q\Bigg)^{\frac{1}{q}}|L_p(\mathbb{R}^d)\Bigg|\Bigg|&\\ \le c\,\Bigg|\Bigg|\Bigg(\sum_{\overline{\nu},\overline{m}}2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}|\lambda_{\overline{\nu}\,\overline{m}}|^q(\overline{M}\chi^{(p)\omega}_{\overline{\nu}\,\overline{m}})^{\frac{q}{\omega}}(x)\Bigg)^{\frac{1}{q}}|L_p(\mathbb{R}^d)\Bigg|\Bigg|&\\ = c\,\Bigg|\Bigg|\Bigg(\sum_{\overline{\nu},\overline{m}}(\overline{M}g^{\omega}_{\overline{\nu}\,\overline{m}})^{\frac{q}{\omega}}(x)\Bigg)^{\frac{\omega}{q}}|L_{\frac{p}{\omega}}(\mathbb{R}^d)\Bigg|\Bigg|^{\frac{1}{\omega}}.&\end{aligned}$$

Using Theorem 1.11 and the definition of ω, we see that this expression may be estimated from above by $c\,||\lambda|s^{\overline{r}}_{p,q}f||$. On the other hand, from Theorem 1.23, we see that this already ensures that f belongs to $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ and proves (2.9).

Step 3.

It remains to prove (ii). Let us assume first that

$$\overline{L}=-1,\quad \overline{K}>\overline{r},\quad \overline{r}>\sigma_{pq},\quad 0<p<\infty,\quad 0<q\le\infty. \tag{2.21}$$

Furthermore, let $\overline{N}\in\mathbb{N}_0^d$ be vector of integers with $\overline{N}>\overline{r}$. According to the construction given at [34, page 68], we may find functions $k_0, k^1,\dots,k^d$ such that

$$k_0, k^1,\dots,k^d\in S(\mathbb{R}); \tag{2.22}$$

$$\operatorname{supp}k_0, \operatorname{supp}k^i\subset\{t\in\mathbb{R}:|t|\le 1\}, i=1,\dots,d; \tag{2.23}$$

$$1=F_1(k_0)(\xi)+\sum_{\nu_i=1}^{\infty}F_1(\mathrm{d}^{N_i}k^i)(2^{-\nu_i}\xi),\quad \xi\in\mathbb{R}, i=1,\dots,d; \tag{2.24}$$

$$F_1k_0(0)=1; \tag{2.25}$$

$$F_1(\mathrm{d}^{N_i}k^i)(\xi)=(F_1k_0)(\xi)-(F_1k_0)(2\xi),\quad \xi\in\mathbb{R}, i=1,\dots,d. \tag{2.26}$$

We define $k_{\overline{l}}(x)$ and $k_{\overline{l}}(f)(x)$ as in Theorem 1.25.

We claim that then

$$f=\sum_{\overline{l}\in\mathbb{N}_0^d}k_{\overline{l}}(f)(x)=\lim_{P\to\infty}\sum_{\overline{l}\le P}k_{\overline{l}}(f),\quad \text{convergence in } S'(\mathbb{R}^d). \tag{2.27}$$

To prove this, fix $\varphi\in S(\mathbb{R}^d)$. Since the Fourier transform is isomorphic mapping from $S'(\mathbb{R}^d)$ onto itself and

$$(k_{\overline{l}}(f))^\wedge(\xi)=\Big(\prod_{i=1}^d F_1(k^i_{l_i})(-\xi_i)\Big)\hat{f}(\xi),$$

it is enough to show that

$$\varphi(\xi)\sum_{\overline{l}\le P}\Big(\prod_{i=1}^d F_1(k^i_{l_i})(-\xi_i)\Big)\to\varphi(\xi)\quad \text{in } S(\mathbb{R}^d). \tag{2.28}$$

The last sum may be rewritten using (2.26) as

$$\sum_{\overline{l}\le P}\Big(\prod_{i=1}^d F_1(k^i_{l_i})(-\xi_i)\Big)=\prod_{i=1}^d\Big((F_1k_0)(-\xi_i)+\sum_{l_i=1}^P(F_1(\mathrm{d}^{N_i}k^i))(-2^{-l_i}\xi_i)\Big)=\prod_{i=1}^d(F_1k_0)(-2^{-P}\xi_i).$$

We denote the last expression by $1-\Phi(2^{-P}\xi)$ and fix $M\in\mathbb{N}$. Using the fact that $\varphi\in S(\mathbb{R}^d)$ we obtain

$$\begin{aligned} p_M(\varphi(\xi)\Phi(2^{-P}\xi)) &\le c\sup_{\substack{0\le\overline{\alpha},\overline{\beta}\le M\\ \xi\in\mathbb{R}^d}} 2^{-P\cdot\overline{\beta}}(D^{\overline{\alpha}}\varphi)(\xi)(D^{\overline{\beta}}\Phi)(2^{-P}\xi)\prod_{i=1}^d\langle\xi_i\rangle^M\\ &\le c\sup_{\substack{0\le\overline{\beta}\le M\\ \xi\in\mathbb{R}^d}} 2^{-P\cdot\overline{\beta}}(D^{\overline{\beta}}\Phi)(2^{-P}\xi)\prod_{i=1}^d\langle\xi_i\rangle^{-1}\end{aligned}$$

where the constant c doesn't depend on P (but depends on M). p_M are the functionals defining the topology on $S(\mathbb{R}^d)$, namely $p_M(\varphi)=\sup_{0\le\overline{\alpha}\le M,x\in\mathbb{R}^d}|D^\alpha\varphi(x)|\langle x\rangle^M$.

If at least one of $\beta_i>0$, then this expression tends to zero if $P\to\infty$. If $\overline{\beta}=0$, then we split the supremum into $\sup_{|\xi|\ge 2^P}$ and $\sup_{|\xi|<2^P}$. The first supremum may be estimated from above by $c2^{-P}$. To estimate the second one, we notice that $|\Phi(\xi)|\le c|\xi|$ in $\{\xi:|\xi|\le 1\}$. Hence

$$c\sup_{|\xi|\le 2^P}\Phi(2^{-P}\xi)\prod_{i=1}^d\langle\xi_i\rangle^{-1}\le c\sup_{\xi\in\mathbb{R}^d}\frac{2^{-P}|\xi|}{\langle\xi\rangle}$$

and $p_M(\varphi(\xi)\Phi(2^{-P}\xi))\to 0$ as $P\to\infty$. This proves (2.28) and, consequently, also (2.27).

Next we find nonnegative function ψ which satisfies

$$\psi\in S(\mathbb{R}),\quad \operatorname{supp}\psi \text{ is compact and } \sum_{\overline{m}\in\mathbb{Z}^d}\psi(x-\overline{m})=1 \text{ for } x\in\mathbb{R}^d, \tag{2.29}$$

and we define for $\overline{\nu}\in\mathbb{N}_0^d$ and $\overline{m}\in\mathbb{Z}^d$ the function $\psi_{\overline{\nu}\,\overline{m}}(x)=\psi(2^{\overline{\nu}}x-\overline{m})$. Then there is a γ such that

$$\operatorname{supp}\psi_{\overline{\nu}\,\overline{m}}\subset\gamma Q_{\overline{\nu}\,\overline{m}},\quad \overline{\nu}\in\mathbb{N}_0^d,\overline{m}\in\mathbb{Z}^d. \tag{2.30}$$

We multiply (2.27) by these decompositions of unity and obtain

$$f=\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d}\psi_{\overline{\nu}\,\overline{m}}(x)k_{\overline{\nu}}(f)(x)=\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d}\lambda_{\overline{\nu}\,\overline{m}}a_{\overline{\nu}\,\overline{m}}(x), \tag{2.31}$$

where

$$\lambda_{\overline{\nu}\,\overline{m}}=\sum_{0\le\alpha\le\overline{K}}\sup_{y\in\gamma Q_{\overline{\nu}\,\overline{m}}}|D^\alpha[k_{\overline{\nu}}(f)](y)|$$

and

$$a_{\overline{\nu}\,\overline{m}}(x)=\lambda^{-1}_{\overline{\nu}\,\overline{m}}\psi_{\overline{\nu}\,\overline{m}}(x)k_{\overline{\nu}}(f)(x).$$

(If some $\lambda_{\overline{\nu}\,\overline{m}}=0$, then we take $a_{\overline{\nu}\,\overline{m}}(x)=0$ as well). It follows that $a_{\overline{\nu}\,\overline{m}}$ are $[\overline{K},\overline{L}]-$atoms centred at $Q_{\overline{\nu}\,\overline{m}}$. The properties (2.4) and (2.6) are satisfied trivially (recall that $\overline{L}=-1$), and the property (2.5) is fulfilled up to some constant c independent of $\overline{\nu}, \overline{m}$ and x. To prove that this decomposition satisfies (2.10), write

$$||\lambda|s^{\overline{r}}_{p,q}f|| \le c \sum_{0\le\alpha\le\overline{K}} \left|\left| \left(\sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in\mathbb{Z}^d} 2^{\overline{\nu}\cdot\overline{r}q} 2^{\overline{\nu}\cdot\frac{q}{p}} \sup_{x-y\in\gamma Q_{\overline{\nu}\,\overline{m}}} |D^\alpha[k_{\overline{\nu}}(f)(y)]|^q \right)^{1/q} |L_p \right|\right| \qquad (2.32)$$

and use Theorem 1.27 with $D^{\alpha_i}k_0$ and $D^{\alpha_i}k^i$ in the place of k_0 and k^i. We lose the Tauberian conditions (1.60) for these new kernels but according to Theorem 1.20, they are not necessary in the proof of (2.32).

Step 4.

Now we prove the existence of the optimal decomposition for all $\overline{r}\in\mathbb{R}^d$ and $\overline{L}$ restricted by (2.7). To simplify the notation, we restrict ourselves in this step to $d=2$. So, let us take $f\in S^{\overline{r}}_{p,q}F(\mathbb{R}^2)$. In Definition 1.8 we may substitute $(1+x^2)^{\overline{\rho}}$ by $(1+x_1^{2\rho_1})(1+x_2^{2\rho_2})$ for $\overline{\rho}\in\mathbb{N}_0^2$ and (using twice Theorem 1.12) we obtain the respective counterpart of Theorem 1.9. Hence f can be decomposed as

$$f = g + \frac{\partial^{2M_1}g}{\partial x_1^{2M_1}} + \frac{\partial^{2M_2}g}{\partial x_2^{2M_2}} + \frac{\partial^{2M_1+2M_2}g}{\partial x_1^{2M_1}x_2^{2M_2}}, \qquad (2.33)$$

where $\overline{M}=(M_1,M_2)\in 2\mathbb{N}_0^2$ is at our disposal and may be chosen arbitrary large, $g\in S^{\overline{r}+2\overline{M}}_{p,q}F(\mathbb{R}^2)$ and $||g|S^{\overline{r}+2\overline{M}}_{p,q}F(\mathbb{R}^2)||\approx ||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^2)||$.

The optimal decomposition of f will be obtained as a sum of decompositions of these four terms.

To decompose the first term, choose $\overline{M}$ such that

$$||g|S^{\overline{K}}\mathcal{C}(\mathbb{R}^2)|| \le c\,||g|S^{\overline{r}+2\overline{M}}_{p,q}F(\mathbb{R}^2)||.$$

This is possible according to [26, Theorem 2.4.1.]. Then we decompose

$$g(x) = \sum_{\overline{m}\in\mathbb{Z}^d} \psi(x-\overline{m})g(x) = \sum_{\overline{m}\in\mathbb{Z}^d} \lambda^1_{0\,\overline{m}} a^1_{0\,\overline{m}},$$

where

$$\lambda^1_{0\,\overline{m}} = c_1 \sum_{0\le\alpha\le\overline{K}} \sup_{|y-\overline{m}|\le c_2} |(D^\alpha g)(y)|$$

and

$$a^1_{0\,\overline{m}} = \frac{1}{\lambda^1_{0\,\overline{m}}}\psi(x-\overline{m})g(x)$$

for c_1, c_2 sufficiently large and for ψ with (2.29) and (2.30). Then $a^1_{0\,\overline{m}}$ are $[\overline{K},\overline{L}]$-atoms centred at $Q_{0\,\overline{m}}$. Furthermore, according to Lemma 1.28, we have

$$||\lambda^1|s^{\overline{r}}_{p,q}f|| = \left(\sum_{\overline{m}\in\mathbb{Z}^d} |\lambda^1_{0\,\overline{m}}|^p\right)^{1/p} \le c_1 \sum_{\alpha\le\overline{K}} \left|\left| \sup_{\cdot-y\in\gamma Q_{0\,0}} |(D^\alpha g)(y)|\;|L_p(\mathbb{R}^d)\right|\right|$$
$$\le c||g|S^{\overline{r}+2\overline{M}}_{p,q}F(\mathbb{R}^2)|| \le c||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^2)||.$$

We have used Lemma 1.28 with $d=2$ and $A=\emptyset$.

As for the last term in the decomposition (2.33), we may assume that $\overline{M}$ is large enough to apply Step 3. So we may assume that we have a decomposition (2.31) for g with, let's say, $\lambda^4_{\overline{\nu}\,\overline{m}}$ and $a^4_{\overline{\nu}\,\overline{m}}(x)$ instead of $\lambda_{\overline{\nu}\,\overline{m}}$ and $a_{\overline{\nu}\,\overline{m}}(x)$ and $||\lambda^4_{\overline{\nu}\,\overline{m}}|s^{\overline{r}+2\overline{M}}_{p,q}f|| \le c\,||g|S^{\overline{r}+2\overline{M}}_{p,q}F(\mathbb{R}^2)||$. As $a^4_{\overline{\nu}\,\overline{m}}(x)$ are $[\overline{K}+2\overline{M},-1]$-atoms, the functions $2^{2\overline{\nu}\cdot\overline{M}}D^{2(M_1,M_2)}a^4_{\overline{\nu}\,\overline{m}}(x)$ are $[\overline{K},2\overline{M}-1]$-atoms.

In the case of the second term we use the decomposition

$$g(x)=\sum_{\substack{\overline{\nu}\in\mathbb{N}^2_0\\ \nu_2=0}}\sum_{\overline{m}\in\mathbb{Z}^d}\psi_{\overline{\nu}\,\overline{m}}(x)k_{\overline{\nu},A}(g)(x)=\sum_{\substack{\overline{\nu}\in\mathbb{N}^2_0\\ \nu_2=0}}\sum_{\overline{m}\in\mathbb{Z}^d}\lambda^2_{\overline{\nu}\,\overline{m}}a^2_{\overline{\nu}\,\overline{m}}(x),$$

where $A=\{1\}$, $k_{\overline{\nu},A}(g)(x)$ are defined by (1.68),

$$\lambda^2_{\overline{\nu}\,\overline{m}}=c_1 2^{2\nu_1 M_1}\sum_{\beta\le\overline{K}+(2M_1,0)}\sup_{y\in c_2Q_{\overline{\nu}\,\overline{m}}}|D^\beta(k_{\overline{\nu},A}(g))(y)|$$

and

$$a^2_{\overline{\nu}\,\overline{m}}(x)=\frac{1}{\lambda^2_{\overline{\nu}\,\overline{m}}}\psi_{\overline{\nu}\,\overline{m}}(x)k_{\overline{\nu},A}(g)(x).$$

If c_1 and c_2 are large enough, then $D^{(2M_1,0)}a^2_{\overline{\nu}\,\overline{m}}(x)$ are $[\overline{K},\overline{L}]$-atoms for $L_1\le 2M_1-1$. Finally, we use Lemma 1.28 to estimate $||\lambda^2|s^{\overline{r}}_{p,q}f||$.

$$\begin{aligned}||\lambda^2|s^{\overline{r}}_{p,q}f||&\le c_1\sum_{\beta\le\overline{K}+(2M_1,0)}\left|\left|\left(\sum_{\substack{\overline{\nu}\in\mathbb{N}^2_0\\ \nu_2=0}}2^{q\nu_1(2M_1+r_1)}\sup_{\cdot-y\in c_2Q_{\overline{\nu}\,0}}|D^\beta(k_{\overline{\nu},A}(g))(y)|^q\right)^{1/q}|L_p\right|\right|\\ &\le c\,||g|S^{\overline{r}+2\overline{M}}_{p,q}F(\mathbb{R}^d)||\le c\,||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)||,\end{aligned}$$

if $\overline{M}$ is chosen sufficiently large. We have used Lemma 1.28 with $D^{\beta_1}k_1$ and $D^{\beta_2}g$ instead of k_1 and f. The third term can be estimated in a similar way. The sum of these four decompositions then gives the decomposition for f.

In general dimension d one has to use the full generality of Lemma 1.28 but the idea of the proof is still the same. □

2.3 Subatomic decomposition

In this section we describe the subatomic decomposition for spaces $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$. We follow closely [35] and [37].

First of all, we shall introduce some special building blocks called quarks.

Definition 2.5. Let $\psi\in S(\mathbb{R})$ be a non-negative function with

$$\operatorname{supp}\psi\subset\{t\in\mathbb{R}:|t|<2^\phi\}\qquad(2.34)$$

for some $\phi\ge 0$ and

$$\sum_{n\in\mathbb{Z}}\psi(t-n)=1,\quad t\in\mathbb{R}.\qquad(2.35)$$

We define $\Psi(x)=\psi(x_1)\cdots\psi(x_d)$ and $\Psi^\beta(x)=x^\beta\Psi(x)$ for $x=(x_1,\dots,x_d)$ and $\beta\in\mathbb{N}^d_0$. Further let $\overline{r}\in\mathbb{R}^d$ and $0<p\le\infty$. Then

$$(\beta qu)_{\overline{\nu}\,\overline{m}}(x)=\Psi^\beta(2^{\overline{\nu}}x-\overline{m}),\quad \overline{\nu}\in\mathbb{N}^d_0,\ \overline{m}\in\mathbb{Z}^d\qquad(2.36)$$

is called an β-quark related to $Q_{\overline{\nu}\,\overline{m}}$.

Recall that the spaces $s^{\overline{r}}_{p,q}a$ were defined by (2.2) and (2.3).

Theorem 2.6. *Let $0 < p,q \le \infty$ (with $p < \infty$ in the F-case) and $\overline{r} \in \mathbb{R}^d$ be such that*

$$\overline{r} > \sigma_p \text{ in the B-case and } \overline{r} > \sigma_{pq} \text{ in the F-case.}$$

(i) *Let*

$$\lambda = \{\lambda^\beta : \beta \in \mathbb{N}_0^d\} \quad \text{with} \quad \lambda^\beta = \{\lambda^\beta_{\overline{\nu}\,\overline{m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d\}$$

and let $\varrho > \phi$, where ϕ is the number from (2.34). *If*

$$\sup_{\beta \in \mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^\beta|s^{\overline{r}}_{p,q}a|| < \infty$$

then the series

$$\sum_{\beta\in\mathbb{N}_0^d}\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} \lambda^\beta_{\overline{\nu}\,\overline{m}}(\beta q u)_{\overline{\nu}\,\overline{m}}(x) \tag{2.37}$$

converges in $S'(\mathbb{R}^d)$, its limit f belongs to $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ and

$$||f|S^{\overline{r}}_{p,q}A(\mathbb{R}^d)|| \le c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^\beta|s^{\overline{r}}_{p,q}a||. \tag{2.38}$$

$(\beta q u)_{\overline{\nu}\,\overline{m}}$ has the same meaning as in (2.36).
(ii) *Every $f \in S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ can be represented by* (2.37) *with convergence in $S'(\mathbb{R}^d)$ and*

$$\sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^\beta|s^{\overline{r}}_{p,q}a|| \le c||f|S^{\overline{r}}_{p,q}A(\mathbb{R}^d)||. \tag{2.39}$$

Proof. We give the proof again only for the F-scale. The proof for the B-scale is very similar.
Step 1.
First of all, we shall discuss convergence of (2.37). It turns out that this series converges not only in $S'(\mathbb{R}^d)$ but also in some $L_u(\mathbb{R}^d), u \ge 1$.
Let $1 \le p < \infty$. Then $\overline{r} > 0$ and we get

$$|f(x)| \le c \sum_{\beta\in\mathbb{N}_0^d}\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} 2^{\phi|\beta|}|\lambda^\beta_{\overline{\nu}\,\overline{m}}|\tilde{\chi}_{\overline{\nu}\,\overline{m}}(x), \tag{2.40}$$

where $\tilde{\chi}_{\overline{\nu}\,\overline{m}}$ is a characteristic function of $2^{\phi+1}Q_{\overline{\nu}\,\overline{m}}$. Using two times the Hölder's inequality we get for every $\epsilon > 0$

$$|f(x)| \le c \sup_{\beta\in\mathbb{N}_0^d} 2^{(\phi+\epsilon)|\beta|} \sup_{\overline{\nu}\in\mathbb{N}_0^d} 2^{|\overline{\nu}|\epsilon} \sup_{\overline{m}\in\mathbb{Z}^d} |\lambda^\beta_{\overline{\nu}\,\overline{m}}|\tilde{\chi}_{\overline{\nu}\,\overline{m}}(x).$$

Taking the p-power and replacing the suprema with sums we get

$$|f(x)|^p \le c \sum_{\beta\in\mathbb{N}_0^d}\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} 2^{(\phi+\epsilon)|\beta|p}2^{|\overline{\nu}|\epsilon p}|\lambda^\beta_{\overline{\nu}\,\overline{m}}|^p\tilde{\chi}_{\overline{\nu}\,\overline{m}}(x). \tag{2.41}$$

Let us denote $\tilde{q} = \max(p,q)$ and choose ϵ such that $0 < 2\epsilon < \varrho - \phi$ and $\epsilon < \overline{r}$. Integration of (2.41) and the Hölder's inequality result in

$$\begin{aligned} ||f|L_p(\mathbb{R}^d)|| &\leq c \sup_{\beta\in\mathbb{N}_0^d} 2^{(\phi+2\epsilon)|\beta|} \left(\sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in\mathbb{Z}^d} 2^{-\overline{\nu}\cdot(\frac{1}{p}-\epsilon)p} |\lambda^\beta_{\overline{\nu}\overline{m}}|^p \right)^{1/p} \\ &\leq c \sup_{\beta\in\mathbb{N}_0^d} 2^{(\phi+2\epsilon)|\beta|} \left(\sum_{\overline{\nu}\in\mathbb{N}_0^d} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})\tilde{q}} \left(\sum_{\overline{m}\in\mathbb{Z}^d} |\lambda^\beta_{\overline{\nu}\overline{m}}|^p \right)^{\tilde{q}/p} \right)^{1/\tilde{q}} \qquad (2.42) \\ &\leq c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|} ||\lambda^\beta|s^{\overline{r}}_{p,\tilde{q}}b|| \leq c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^\beta|s^{\overline{r}}_{p,q}f||. \end{aligned}$$

Therefore, for $1 \leq p < \infty$, (2.37) converges in $L_p(\mathbb{R}^d)$.

If $p = \infty$, we get the uniform pointwise convergence of (2.37) by similar arguments.

Let $0 < p < 1$. Then $\overline{r} > \frac{1}{p} - 1$ and we get again (2.40). Integrating this estimate and using Hölder's inequality, we get for every $\epsilon > 0$

$$\begin{aligned} ||f|L_1(\mathbb{R}^d)|| &\leq c \sum_{\beta\in\mathbb{N}_0^d} \sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in\mathbb{Z}^d} 2^{\phi|\beta|} 2^{-|\overline{\nu}|} |\lambda^\beta_{\overline{\nu}\overline{m}}| \\ &\leq c \sup_{\beta\in\mathbb{N}_0^d} 2^{(\phi+\epsilon)|\beta|} \sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in\mathbb{Z}^d} 2^{-|\overline{\nu}|} |\lambda^\beta_{\overline{\nu}\overline{m}}|. \end{aligned}$$

By similar arguments as in (2.42) we get

$$||f|L_1(\mathbb{R}^d)|| \leq c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^\beta|s^{\overline{r}}_{p,q}f||$$

and (2.37) converges in $L_1(\mathbb{R}^d)$.

Step 2.

We now prove that the function f defined as a limit of (2.37) belongs to $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ and the estimate (2.38).

We decompose (2.37) into

$$f = \sum_{\beta\in\mathbb{N}_0^d} f^\beta \qquad (2.43)$$

with

$$f^\beta = \sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in\mathbb{Z}^d} \lambda^\beta_{\overline{\nu}\overline{m}} (\beta q u)_{\overline{\nu}\overline{m}}(x). \qquad (2.44)$$

We show that $(\beta q u)_{\overline{\nu}\overline{m}}$ are (up to some normalising constants) $[\overline{K}, -1]$-atoms centred at $Q_{\overline{\nu}\overline{m}}$ for every $\overline{K} \in \mathbb{N}_0^d$. The conditions (2.4) and (2.6) are satisfied trivially. To prove (2.5) we chose $0 \leq \alpha \leq \overline{K}$ and estimate

$$D^\alpha(\beta q u)_{\overline{\nu}\overline{m}}(x) = \prod_{i=1}^d 2^{\nu_i\alpha_i} D^{\alpha_i}(\psi^{\beta_i})(2^{\nu_i}x_i - m_i)$$

where $\psi^{\beta_i}(t) = t^{\beta_i}\psi(t)$. But for $0 \leq \alpha_i \leq K_i$ and any $t \in \operatorname{supp}\psi$ we get by Leibnitz rule

$$|D^{\alpha_i}(\psi^{\beta_i})(t)| \leq c_{K_i} \sup_{\gamma_1\leq K_i} \sup_{\gamma_2\leq K_i} |D^{\gamma_1}t^{\beta_i}| \cdot |(D^{\gamma_2}\psi)(t)| \leq c_{K_i,\psi} \sup_{\gamma_1\leq K_i} |D^{\gamma_1}t^{\beta_i}|.$$

The last absolute value may be estimated from above by $(1+\beta_i)^{K_i}2^{\phi\beta_i}$. Hence we obtain

$$|D^{\alpha_i}(\psi^{\beta_i})(t)| \le c_{K_i,\psi}(1+\beta_i)^{K_i}2^{\phi\beta_i}$$

and

$$|D^{\alpha}(\beta qu)_{\overline{\nu}\,\overline{m}}(x)| \le c_1\, 2^{\alpha\cdot\overline{\nu}}(1+\beta)^{\overline{K}}2^{\phi|\beta|} \le c_2\, 2^{\alpha\cdot\overline{\nu}}2^{(\phi+\epsilon)|\beta|}$$

for every $\epsilon>0$. The constant c_2 is independent of β but may depend on $\overline{K}$, ψ and ϵ.
It follows that the functions $c_2^{-1}2^{-(\phi+\epsilon)|\beta|}(\beta qu)_{\overline{\nu}\,\overline{m}}(x)$ are $[\overline{K},-1]$-atoms and (2.44) may be understood as an atomic decomposition of f^β. By Theorem 2.4 it follows that

$$||f^\beta|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)|| \le c2^{(\phi+\epsilon)|\beta|}||\lambda^\beta|s^{\overline{r}}_{p,q}f||$$

and for $\eta=\min(1,p,q)$ get by the triangle inequality for $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$-quasinorms

$$\begin{aligned}||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)||^\eta &\le \sum_{\beta\in\mathbb{N}_0^d}||f^\beta|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)||^\eta\\ &\le c\sum_{\beta\in\mathbb{N}_0^d}2^{(\phi+\epsilon)\eta|\beta|}||\lambda^\beta|s^{\overline{r}}_{p,q}f||^\eta\\ &\le c\sup_{\beta\in\mathbb{N}_0^d}2^{(\phi+2\epsilon)\eta|\beta|}||\lambda^\beta|s^{\overline{r}}_{p,q}f||^\eta.\end{aligned}$$

If we choose $\epsilon>0$ so small that $\phi+2\epsilon<\varrho$ we obtain (2.38). This finishes the proof of part (i).

Step 3.

By Remark 1.3 we have

$$\hat{f}(\xi)=\sum_{\overline{\nu}\in\mathbb{N}_0^d}\varphi_{\overline{\nu}}(\xi)\hat{f}(\xi)$$

with convergence in $S'(\mathbb{R}^d)$. Let $Q_{\overline{\nu}}$ be a cube in $\mathbb{R}^d$ centred at the origin with side lengths $2\pi2^{\nu_1},\ldots,2\pi2^{\nu_d}$. Hence $\operatorname{supp}\varphi_{\overline{\nu}}\subset Q_{\overline{\nu}}$ and we may interpret $\varphi_{\overline{\nu}}\hat{f}$ as a periodic distribution. Using its expansion into a Fourier series we get

$$(\varphi_{\overline{\nu}}\hat{f})(\xi)=\sum_{\overline{m}\in\mathbb{Z}^d}b_{\overline{\nu}\,\overline{m}}e^{-i(2^{-\overline{\nu}}\overline{m})\cdot\xi},\quad \xi\in Q_{\overline{\nu}}, \tag{2.45}$$

with

$$b_{\overline{\nu}\,\overline{m}}=c2^{-|\overline{\nu}|}\int_{Q_{\overline{\nu}}}e^{-i(2^{-\overline{\nu}}\overline{m})\cdot\xi}(\varphi_{\overline{\nu}}\hat{f})(\xi)\mathrm{d}\xi=c'2^{-|\overline{\nu}|}(\varphi_{\overline{\nu}}\hat{f})^\vee(2^{-\overline{\nu}}\overline{m}).$$

Here we used again the notation $2^{-\overline{\nu}}\overline{m}=(2^{-\nu_1}m_1,\ldots,2^{-\nu_d}m_d)$ for $\overline{\nu}\in\mathbb{N}_0^d$ and $\overline{m}\in\mathbb{Z}^d$.
Let now $\omega\in S(\mathbb{R}^d)$ with $\operatorname{supp}\omega\subset Q_0$ and $\omega(\xi)=1$ if $|\xi_i|\le 2$ for all $i=1,\ldots,d$. Then the functions $\omega_{\overline{\nu}}(\xi)=\omega(2^{-\overline{\nu}}\xi)$ satisfy

$$\operatorname{supp}\omega_{\overline{\nu}}\subset Q_{\overline{\nu}},\quad \omega_{\overline{\nu}}(\xi)=1\quad\text{if}\quad \xi\in\operatorname{supp}\varphi_{\overline{\nu}}$$

for all $\overline{\nu}\in\mathbb{N}_0^d$. We multiply (2.45) with $\omega_{\overline{\nu}}$, extend it by zero outside $Q_{\overline{\nu}}$, and take the inverse Fourier transform

$$(\varphi_{\overline{\nu}}\hat{f})^\vee(x)=\sum_{\overline{m}\in\mathbb{Z}^d}b_{\overline{\nu}\,\overline{m}}\omega^\vee_{\overline{\nu}}(x-2^{-\overline{\nu}}\overline{m})=\sum_{\overline{m}\in\mathbb{Z}^d}2^{|\overline{\nu}|}b_{\overline{\nu}\,\overline{m}}\omega^\vee(2^{\overline{\nu}}x-\overline{m}),\quad x\in\mathbb{R}^d.$$

Using (2.35) and the definition of Ψ, we get

$$(\varphi_{\overline{\nu}}\hat{f})^\vee(x) = \sum_{\overline{m}\in\mathbb{Z}^d} 2^{|\overline{\nu}|} b_{\overline{\nu}\,\overline{m}} \sum_{\overline{l}\in\mathbb{Z}^d} \Psi(2^{\overline{\nu}}x-\overline{l})\omega^\vee(2^{\overline{\nu}}x-\overline{m}).$$

Expanding the entire analytic function $\omega^\vee(2^{\overline{\nu}}\cdot-\overline{m})$ with respect to the off-point $2^{-\overline{\nu}}\overline{l}$ we arrive at

$$\begin{aligned}(\varphi_{\overline{\nu}}\hat{f})^\vee(x) &= \sum_{\overline{m}\in\mathbb{Z}^d} 2^{|\overline{\nu}|} b_{\overline{\nu}\,\overline{m}} \sum_{\overline{l}\in\mathbb{Z}^d} \Psi(2^{\overline{\nu}}x-\overline{l}) \sum_{\beta\in\mathbb{N}_0^d} 2^{\overline{\nu}\cdot\beta}\frac{(D^\beta\omega^\vee)(\overline{l}-\overline{m})}{\beta!}(x-2^{-\overline{\nu}}\overline{l})^\beta\\ &= \sum_{\overline{m}\in\mathbb{Z}^d} 2^{|\overline{\nu}|} b_{\overline{\nu}\,\overline{m}} \sum_{\overline{l}\in\mathbb{Z}^d}\sum_{\beta\in\mathbb{N}_0^d} \Psi^\beta(2^{\overline{\nu}}x-\overline{l})\frac{(D^\beta\omega^\vee)(\overline{l}-\overline{m})}{\beta!}.\end{aligned}$$

Hence

$$f = \sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\beta\in\mathbb{N}_0^d}\sum_{\overline{l}\in\mathbb{Z}^d} \lambda^\beta_{\overline{\nu}\,\overline{l}}\Psi^\beta(2^{\overline{\nu}}x-\overline{l}) = \sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\beta\in\mathbb{N}_0^d}\sum_{\overline{l}\in\mathbb{Z}^d} \lambda^\beta_{\overline{\nu}\,\overline{l}}(\beta qu)_{\overline{\nu}\,\overline{l}}(x),$$

where

$$\lambda^\beta_{\overline{\nu}\,\overline{l}} = 2^{|\overline{\nu}|}\sum_{\overline{m}\in\mathbb{Z}^d} b_{\overline{\nu}\,\overline{m}}\frac{(D^\beta\omega^\vee)(\overline{l}-\overline{m})}{\beta!} = c\sum_{\overline{m}\in\mathbb{Z}^d}(\varphi_{\overline{\nu}}\hat{f})^\vee(2^{-\overline{\nu}}\overline{m})\frac{(D^\beta\omega^\vee)(\overline{l}-\overline{m})}{\beta!}.$$

It remains to prove (2.39). For this reason we define

$$\Lambda_{\overline{\nu}\,\overline{m}} = (\varphi_{\overline{\nu}}\hat{f})^\vee(2^{-\overline{\nu}}\overline{m})$$

and prove that

$$\sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^\beta|s^{\overline{r}}_{p,q}f|| \le c||\Lambda|s^{\overline{r}}_{p,q}f|| \le c'||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)||. \tag{2.46}$$

We start with the second inequality in (2.46).

Let $x\in Q_{\overline{\nu}\,\overline{m}}$ be fixed. Then

$$|(\varphi_{\overline{\nu}}\hat{f})^\vee(2^{-\overline{\nu}}\overline{m})| \le \sup_{x-y\in Q_{\overline{\nu},0}}|(\varphi_{\overline{\nu}}\hat{f})^\vee(y)| \le c(\varphi^*_{\overline{\nu}}f)_{\overline{a}}(x) \tag{2.47}$$

for every $\overline{a}\in\mathbb{R}^d_+$. We multiply (2.47) by $2^{\overline{\nu}\cdot\overline{r}}$, take the q-power and sum over $\overline{m}\in\mathbb{Z}^d$ to get

$$2^{\overline{\nu}\cdot\overline{r}q}\sum_{\overline{m}\in\mathbb{Z}^d}|\Lambda_{\overline{\nu}\,\overline{m}}|^q|\chi_{\overline{\nu}\,\overline{m}}(x)|^q \le c\,2^{\overline{\nu}\cdot\overline{r}q}(\varphi^*_{\overline{\nu}}f)^q_{\overline{a}}(x), \quad x\in\mathbb{R}^d, \overline{\nu}\in\mathbb{N}_0^d.$$

Taking $\overline{a}>\frac{n}{\min(p,q)}$, we get finally with the help of Theorem 1.22

$$\begin{aligned}||\Lambda|s^{\overline{r}}_{p,q}f|| &= \left|\left|\left(\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d}2^{\overline{\nu}\cdot\overline{r}q}|\Lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}}(x)|^q\right)^{1/q}|L_p(\mathbb{R}^d)\right|\right|\\ &\le c\left|\left|\left(\sum_{\overline{\nu}\in\mathbb{N}_0^d}2^{\overline{\nu}\cdot\overline{r}q}(\varphi^*_{\overline{\nu}}f)^q_{\overline{a}}(x)\right)^{1/q}|L_p(\mathbb{R}^d)\right|\right|\\ &\le c||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)||.\end{aligned}$$

To prove the first inequality in (2.46), we mention that

$$\lambda^\beta_{\overline{\nu}\,\overline{l}} = \frac{1}{\beta!}\sum_{\overline{m}\in\mathbb{Z}^d}\Lambda_{\overline{\nu}\,\overline{m}}(D^\beta\omega^\vee)(\overline{l}-\overline{m}) \tag{2.48}$$

and recall a result proven in [36], namely that for any given $a>0$ there are constants $c_a>0$ and $C>0$ such that

$$|D^\beta\omega^\vee(x)| \le c_a 2^{C|\beta|}(1+|x|^2)^{-a}, \quad x\in\mathbb{R}^d, \beta\in\mathbb{N}^d_0. \tag{2.49}$$

Furthermore, we define

$$h^\beta_{\overline{\nu}}(x) = 2^{\overline{\nu}\cdot\overline{r}}\sum_{\overline{l}\in\mathbb{Z}^d}\lambda^\beta_{\overline{\nu}\,\overline{l}}\chi_{\overline{\nu}\,\overline{l}}(x), \tag{2.50}$$

$$H_{\overline{\nu}}(x) = 2^{\overline{\nu}\cdot\overline{r}}\sum_{\overline{l}\in\mathbb{Z}^d}\Lambda_{\overline{\nu}\,\overline{l}}\chi_{\overline{\nu}\,\overline{l}}(x) \tag{2.51}$$

and let $0<\kappa<\min(1,p,q)$. We prove (2.46) by the following chain of inequalities

$$\begin{aligned} 2^{\varrho|\beta|}||\lambda^\beta|s^{\overline{r}}_{p,q}f|| &= 2^{\varrho|\beta|}||h^\beta_{\overline{\nu}}|L_p(\ell_q)|| = 2^{\varrho|\beta|}||\ |h^\beta_{\overline{\nu}}|^\kappa\ |L_{\frac{p}{\kappa}}(\ell_{\frac{q}{\kappa}})||^{\frac{1}{\kappa}} \\ &\le c2^{\varrho|\beta|}\left(\frac{2^{C|\beta|}}{\beta!}\right)^\kappa ||\overline{M}(|H_{\overline{\nu}}|^\kappa)|L_{\frac{p}{\kappa}}(\ell_{\frac{q}{\kappa}})||^{\frac{1}{\kappa}} \\ &\le c'||\ |H_{\overline{\nu}}|^\kappa\ |L_{\frac{p}{\kappa}}(\ell_{\frac{q}{\kappa}})||^{\frac{1}{\kappa}} = ||\Lambda|s^{\overline{r}}_{p,q}f||. \end{aligned} \tag{2.52}$$

The equalities in (2.52) involve only definitions of corresponding spaces. The second inequality follows from Theorem 1.10, choice of κ and the growth of $\beta!$ for $|\beta|\to\infty$. Hence only the first inequality in (2.52) needs to be proven.

To prove it, put (2.49) into (2.48) to obtain for every $a>0$

$$|\lambda^\beta_{\overline{\nu}\,\overline{l}}| \le \frac{c_a 2^{C|\beta|}}{\beta!}\sum_{\overline{m}\in\mathbb{Z}^d}\frac{|\Lambda_{\overline{\nu}\,\overline{m}}|}{(1+|\overline{l}-\overline{m}|^2)^a}. \tag{2.53}$$

Let us take $x\in Q_{\overline{\nu}\,\overline{l}}$. Using the definition of $h^\beta_{\overline{\nu}}$ from (2.50), (2.53) and the property $\kappa<1$ we get

$$|h^\beta_{\overline{\nu}}(x)|^\kappa = 2^{\overline{\nu}\cdot\overline{r}\kappa}|\lambda^\beta_{\overline{\nu}\,\overline{l}}|^\kappa \le \frac{c_a^\kappa 2^{C|\beta|\kappa}}{(\beta!)^\kappa}2^{\overline{\nu}\cdot\overline{r}\kappa}\sum_{\overline{m}\in\mathbb{Z}^d}\frac{|\Lambda_{\overline{\nu}\,\overline{m}}|^\kappa}{(1+|\overline{l}-\overline{m}|^2)^{a\kappa}}. \tag{2.54}$$

We split the summation over $\overline{m}\in\mathbb{Z}^d$ into two sums according to the size of $|\overline{l}-\overline{m}|$

$$\sum_{\overline{m}\in\mathbb{Z}^d}\frac{|\Lambda_{\overline{\nu}\,\overline{m}}|^\kappa}{(1+|\overline{l}-\overline{m}|^2)^{a\kappa}} = \sum_{k=0}^\infty\frac{1}{(1+k^2)^{a\kappa}}\sum_{\overline{m}:|\overline{l}-\overline{m}|=k}|\Lambda_{\overline{\nu}\,\overline{m}}|^\kappa. \tag{2.55}$$

Finally, we estimate the last sum using the iterated maximal operator $\overline{M}$

$$\begin{aligned} \sum_{\overline{m}:|\overline{l}-\overline{m}|=k}|\Lambda_{\overline{\nu}\,\overline{m}}|^\kappa &\le 2^{-\overline{\nu}\cdot\overline{r}\kappa}2^{|\overline{\nu}|}\int_{y:y-x\in(k+2)Q_{\overline{\nu},0}}|H_{\overline{\nu}}(y)|^\kappa\mathrm{d}y \\ &\le 2^{-\overline{\nu}\cdot\overline{r}\kappa}(k+2)^d\overline{M}(|H_{\overline{\nu}}|^\kappa)(x). \end{aligned} \tag{2.56}$$

We combine (2.54), (2.55) and (2.56) and arrive at

$$|h^{\beta}_{\overline{\nu}}(x)|^{\kappa} \leq c'_a \frac{2^{C|\beta|\kappa}}{(\beta!)^{\kappa}} \overline{M}(|H_{\overline{\nu}}|^{\kappa})(x)$$

for every $a > \frac{d+1}{2\kappa}$. This finishes the proof of (2.52) and, consequently, also the proof of (2.46) and hence also of the part (ii) of Theorem 2.6. □

Next we shall deal with subatomic decompositions in the general case. Namely, we would like to prove an analogy of Theorem 2.6 without the restriction $\overline{r} > \sigma_{pq}$.

Remark 2.7. For the need of this section we introduce temporarily following notation. Let $A \subset \{1, \dots, d\}$ and $\overline{N} = (N_1, \dots, N_d) \in \mathbb{R}^d$. Then we define the vector $\overline{N}^A = (N_1^A, \dots, N_d^A)$ by

$$N_i^A = \begin{cases} N_i & \text{if} \quad i \in A, \\ 0 & \text{if} \quad i \notin A. \end{cases}$$

Furthermore, we denote by D_i^{γ} the operator

$$D_i^{\gamma} = \frac{\partial^{\gamma}}{\partial x_i}, \quad i = 1, \dots, n, \quad \gamma \in \mathbb{N}_0$$

and by $D_A^{\overline{L}}$ the operator

$$D_A^{\overline{L}} = \prod_{i \in A} D_i^{L_i} = D^{\overline{L}_A}, \quad A \subset \{1, \dots, n\}, \quad \overline{L} \in \mathbb{N}_0^d.$$

Theorem 2.8. *Let* $0 < p, q \leq \infty$ *(*$p < \infty$ *in the F-case) and* $\overline{r} \in \mathbb{R}^d$*. Further let* $\overline{L} + 1 \in \mathbb{N}_0^d$ *and* $\overline{\sigma} \in \mathbb{R}^d$ *satisfy*

$$L_i \geq \max(-1, [\sigma_p - r_i]), \quad \sigma_i > \max(\sigma_p, r_i), \quad i = 1, \dots, d,$$

in the B-case and

$$L_i \geq \max(-1, [\sigma_{pq} - r_i]), \quad \sigma_i > \max(\sigma_{pq}, r_i), \quad i = 1, \dots, d,$$

in the F-case.

(i) *Let for every set* $A \subset \{1, \dots, d\}$

$$\lambda^A = \{\lambda^{A,\beta} : \beta \in \mathbb{N}_0^d\} \quad \text{with} \quad \lambda^{A,\beta} = \{\lambda^{A,\beta}_{\overline{\nu}\,\overline{m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d\}$$

and let $\varrho > \phi$*, where* ϕ *is the number from* (2.34)*. If*

$$\sup_{A \subset \{1,\dots,d\}} \sup_{\beta \in \mathbb{N}_0^d} 2^{\varrho|\beta|} ||\lambda^{A,\beta}| s^{\overline{r}}_{p,q} a|| < \infty$$

then the series

$$\sum_{A \subset \{1,\dots,d\}} \sum_{\beta \in \mathbb{N}_0^d} \sum_{\overline{\nu} \in \mathbb{N}_0^d} \sum_{\overline{m} \in \mathbb{Z}^d} \left(\prod_{i \notin A} 2^{\nu_i (r_i - \sigma_i)} \right) \lambda^{A,\beta}_{\overline{\nu}\,\overline{m}} \left[D_A^{\overline{L}+1} \Psi^{\beta} \right] (2^{\overline{\nu}} x - \overline{m}) \qquad (2.57)$$

converges in $S'(\mathbb{R}^d)$, its limit f belongs to $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ and

$$||f|S^{\overline{r}}_{p,q}A(\mathbb{R}^d)|| \le c \sup_{A\subset\{1,\dots,d\}} \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^{A,\beta}|s^{\overline{r}}_{p,q}a||. \tag{2.58}$$

(ii) *Every $f \in S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ can be represented by (2.57) with convergence in $S'(\mathbb{R}^d)$ and*

$$\sup_{A\subset\{1,\dots,d\}} \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^{A,\beta}|s^{\overline{r}}_{p,q}a|| \le c||f|S^{\overline{r}}_{p,q}A(\mathbb{R}^d)||. \tag{2.59}$$

Remark 2.9. Because of the difficulties with notation we shall give the proof only for $d = 2$. Furthermore, we deal only with the F-scale. The proof for the B-scale is again similar and technically simpler.

Proof of Theorem 2.8 for $d = 2$. Step 1.

First we discuss the convergence of (2.57). As the first sum is only finite, we may discuss the convergence of the triple sum over $\beta, \overline{\nu}$ and $\overline{m}$ separately for each $A \subset \{1, 2\}$. Let us do this for example for $A = \{1\}$. Then we may rewrite the terms in (2.57) as

$$2^{\nu_2(r_2-\sigma_2)}[D^{(L_1+1,0)}\Psi^\beta](2^{\overline{\nu}}x - \overline{m}) = 2^{\nu_2(r_2-\sigma_2)}2^{-\nu_1(L_1+1)}[D^{(L_1+1,0)}(\beta qu)_{\overline{\nu}\,\overline{m}}](x) \tag{2.60}$$

where $(\beta qu)_{\overline{\nu}\,\overline{m}}(x)$ are β-quarks according to Definition 2.5. As $L_1 + 1 > 0$ and $\sigma_2 - r_2 > 0$, we may use the same arguments as in the proof of Theorem 2.6 and obtain the same kind of convergence. Especially, the convergence of (2.57) in $S'(\mathbb{R}^d)$ is ensured.

Step 2.

Let us assume that the function f is given by (2.57). Then we may understand this decomposition as

$$f = \sum_{A\subset\{1,2\}} f^A. \tag{2.61}$$

We shall prove that, for every admissible set A,

$$||f^A|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)|| \le c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^{A,\beta}|s^{\overline{r}}_{p,q}f||. \tag{2.62}$$

If $A = \emptyset$ then the decomposition of $f^\emptyset$ in the triple sum according to (2.57) can be understood as a subatomic decomposition of $f^\emptyset$ in the space $S^{\overline{\sigma}}_{p,q}F(\mathbb{R}^d)$ and from Theorem 2.6 it follows that

$$f \in S^{\overline{\sigma}}_{p,q}F(\mathbb{R}^d) \subset S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$$

and

$$||f^\emptyset|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)|| \le c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||2^{\overline{\nu}\cdot(\overline{r}-\overline{\sigma})}\lambda^{\emptyset,\beta}|s^{\overline{\sigma}}_{p,q}f|| = c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^{\emptyset,\beta}|s^{\overline{r}}_{p,q}f||.$$

If $A = \{1\}$ then we use (2.60) and obtain that $f^{\{1\}} = D^{(L_1+1,0)}g$, where

$$g \in S^{(r_1+L_1+1,\sigma_2)}_{p,q}F(\mathbb{R}^d) \quad \text{and} \quad ||g|S^{(r_1+L_1+1,\sigma_2)}_{p,q}F(\mathbb{R}^d)|| \le c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^{\{1\},\beta}|s^{\overline{r}}_{p,q}f||.$$

Hence

$$\begin{aligned} ||f^{\{1\}}|S^{(r_1,r_2)}_{p,q}F(\mathbb{R}^d)|| &\le ||f^{\{1\}}|S^{(r_1,\sigma_2)}_{p,q}F(\mathbb{R}^d)|| = ||D^{(L_1+1,0)}g|S^{(r_1,\sigma_2)}_{p,q}F(\mathbb{R}^d)|| \\ &\le ||g|S^{(r_1+L_1+1,\sigma_2)}_{p,q}F(\mathbb{R}^d)|| \le c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^{\{1\},\beta}|s^{\overline{r}}_{p,q}f||. \end{aligned} \tag{2.63}$$

Using similar technique we prove (2.62) also for $A = \{2\}$ and $A = \{1,2\}$. Now (2.61) together with (2.62) shows that (2.58) holds.

Step 3.

We prove the part (ii) of the theorem. By similar arguments as in the Step 4. of the proof of Theorem 2.4 we prove in analogy with (2.33) that for every $\overline{M} \in \mathbb{N}_0^d$ such that

$$\overline{r} + \overline{M} + 1 \geq \overline{\sigma}, \quad \overline{M} \geq \overline{L}, \quad \text{and} \quad \overline{M} + 1 \in 4\mathbb{N}^2$$

there is a function $g \in S^{\overline{r}+\overline{M}+1}_{p,q}F(\mathbb{R}^d)$ with

$$f = g + \frac{\partial^{M_1+1} g}{\partial x_1^{M_1+1}} + \frac{\partial^{M_2+1} g}{\partial x_2^{M_2+1}} + \frac{\partial^{M_1+1+M_2+1} g}{\partial x_1^{M_1+1} x_2^{M_2+1}}. \tag{2.64}$$

Furthermore

$$||g|S^{\overline{r}+\overline{M}+1}_{p,q}F(\mathbb{R}^d)|| \approx ||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)||. \tag{2.65}$$

Let us define

$$g_1 = g, \quad g_2 = D^{(M_1-L_1,0)}g, \quad g_3 = D^{(0,M_2-L_2)}g \quad \text{and} \quad g_4 = D^{(M_1-L_1,M_2-L_2)}g.$$

Then we can rewrite (2.64) and (2.65) as

$$f = g_1 + \frac{\partial^{L_1+1} g_2}{\partial x_1^{L_1+1}} + \frac{\partial^{L_2+1} g_3}{\partial x_2^{L_2+1}} + \frac{\partial^{L_1+1+L_2+1} g_4}{\partial x_1^{L_1+1} x_2^{L_2+1}} \tag{2.66}$$

with

$$\begin{cases} g_1 \in S^{\overline{r}+\overline{M}+1}_{p,q}F(\mathbb{R}^d) \subset S^{\overline{\sigma}}_{p,q}F(\mathbb{R}^d), \\ g_2 \in S^{(r_1+L_1+1,r_2+M_2+1)}_{p,q}F(\mathbb{R}^d) \subset S^{(r_1+L_1+1,\sigma_2)}_{p,q}F(\mathbb{R}^d), \\ g_3 \in S^{(r_1+M_1+1,r_2+L_2+1)}_{p,q}F(\mathbb{R}^d) \subset S^{(\sigma_1,r_2+L_2+1)}_{p,q}F(\mathbb{R}^d), \\ g_4 \in S^{\overline{r}+\overline{L}+1}_{p,q}F(\mathbb{R}^d). \end{cases} \tag{2.67}$$

Furthermore, the norm of g_i in the corresponding space may be estimated from above by $||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)||$ for all $i = 1, \ldots, 4$. We may use Theorem 2.6 for each function g_i to get four optimal decompositions and corresponding analogy of (2.39). Putting these estimates into (2.67) and using (2.60) we get (2.59). □

2.4 Wavelet decomposition

In this subsection we describe the wavelet decomposition for spaces $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$. In general, we follow the ideas expressed in [38]. First of all, we recall following crucial theorem from the wavelet theory.

Theorem 2.10. *For any $s \in \mathbb{N}$ there are real–valued compactly supported functions*

$$\psi_0(t) \in C^s(\mathbb{R}) \quad \textit{and} \quad \psi_1(t) \in C^s(\mathbb{R}) \tag{2.68}$$

with

$$\int_{\mathbb{R}} t^\alpha \psi_1(t)\mathrm{d}t = 0, \quad \alpha = 0, 1, \ldots, s \tag{2.69}$$

such that

$$\{2^{j/2}\psi_{jm}(t) : j \in \mathbb{N}_0, m \in \mathbb{Z}\} \tag{2.70}$$

with

$$\psi_{jm}(t) = \begin{cases} \psi_0(t-m) & \text{if} \quad j=0, m \in \mathbb{Z} \\ \sqrt{2^{-1}}\psi_1(2^{j-1}t-m) & \text{if} \quad j \in \mathbb{N}, m \in \mathbb{Z} \end{cases} \tag{2.71}$$

is an orthonormal basis in $L_2(\mathbb{R})$.

We have already observed in previous sections the importance of tensor product constructions in the theory of function spaces with dominating mixed derivative. Following this idea, we consider a tensor product version of Theorem 2.10. Let ψ_0 and ψ_1 be the functions from Theorem 2.10 satisfying (2.68) and (2.69). Let ψ_{jm} be defined by (2.71). Then we define their tensor product counterparts by

$$\Psi_{\overline{k}\,\overline{m}}(x) = \psi_{k_1 m_1}(x_1) \cdot \ldots \cdot \psi_{k_d m_d}(x_d), \tag{2.72}$$

where

$$x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \quad \overline{k} = (k_1, \ldots, k_d) \in \mathbb{N}_0^d \quad \text{and} \quad \overline{m} = (m_1, \ldots, m_d) \in \mathbb{Z}^d. \tag{2.73}$$

The tensor version of Theorem 2.10 then reads

Theorem 2.11. *For any $s \in \mathbb{N}$ there are real compactly supported functions*

$$\psi_0(t) \in C^s(\mathbb{R}) \quad \textit{and} \quad \psi_1(t) \in C^s(\mathbb{R})$$

with (2.69) *such that*

$$\{2^{|\overline{k}|/2}\Psi_{\overline{k}\overline{m}}(x) : \overline{k} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d\}, \tag{2.74}$$

with $\Psi_{\overline{k}\overline{m}}$ defined by (2.72) *and* (2.71), *is an orthonormal basis in* $L_2(\mathbb{R}^d)$.

Now we have all the necessary definitions at hand and we may state our wavelet decomposition theorem. As usual $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ stands for $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ or $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$, and $s^{\overline{r}}_{p,q}a$ for $s^{\overline{r}}_{p,q}b$ or $s^{\overline{r}}_{p,q}f$ respectively.

Theorem 2.12. *Let*

$$\overline{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d, \quad 0 < p \leq \infty, \quad 0 < q \leq \infty$$

with $p < \infty$ in the F-case. Then there is a natural number $s(\overline{r}, p, q)$ such that the following statements hold.

(i) *Let $\lambda \in s^{\overline{r}}_{p,q}a$. Then*

1. The sum

$$\sum_{\overline{k}\in\mathbb{N}_0^d, \overline{m}\in\mathbb{Z}^d} \lambda_{\overline{k}\overline{m}}\Psi_{\overline{k}\overline{m}} \tag{2.75}$$

converges in $S'(\mathbb{R}^d)$ to some distribution f.

2. $f \in S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ and

$$||f|S^{\overline{r}}_{p,q}A(\mathbb{R}^d)|| \leq c||\lambda|s^{\overline{r}}_{p,q}a||, \tag{2.76}$$

where the constant c does not depend on λ.

3. The sum (2.75) converges unconditionally in $S^{\overline{r}-\epsilon}_{p,q}A(\mathbb{R}^d)$ *for any* $\epsilon > 0$.

4. If $\max(p,q) < \infty$ *then the sum (2.75) converges unconditionally in* $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$.

(ii) *Let* $f \in S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$. *Then we may define the sequence* λ *by*

$$\lambda_{\overline{k}\overline{m}} = 2^{|\overline{k}|}(f, \Psi_{\overline{k}\overline{m}}), \quad \overline{k} \in \mathbb{N}^d_0, \quad \overline{m} \in \mathbb{Z}^d, \tag{2.77}$$

and it holds

1. $\lambda \in s^{\overline{r}}_{p,q}a$ *and*

$$||\lambda|s^{\overline{r}}_{p,q}a|| \le c||f|S^{\overline{r}}_{p,q}A(\mathbb{R}^d)||, \tag{2.78}$$

where the constant c *does not depend on* f.

2. The sum (2.75) converges in $S'(\mathbb{R}^d)$ *to* f.

3. If $\gamma \in s^{\overline{r}}_{p,q}a$ *and the sum* $\sum_{\overline{k}\in\mathbb{N}^d_0, \overline{m}\in\mathbb{Z}^d} \gamma_{\overline{k}\overline{m}}\Psi_{\overline{k}\overline{m}}$ *converges in* $S'(\mathbb{R}^d)$ *to* f *then* $\gamma = \lambda$.

Before we come to the proof of Theorem 2.12 we clarify the technical problems caused by the limited smoothness of the functions $\Psi_{\overline{k}\overline{m}}$.

2.4.1 Duality

As the functions $\Psi_{\overline{k}\overline{m}}$ are of bounded smoothness, they do *not* belong to $S(\mathbb{R}^d)$. According to (2.68), (2.71) and (2.72), we have only $\Psi_{\overline{k}\overline{m}} \in C^{(s,\dots,s)}(\mathbb{R}^d)$. Hence it is impossible to understand the expression $(f, \Psi_{\overline{k}\overline{m}})$ in the distributional sense for every $f \in S'(\mathbb{R}^d)$.

To give a meaning to the symbol $(f, \Psi_{\overline{k}\overline{m}})$, one has to study the dual spaces of $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ first. As far as the Fourier–analytic version of classical Besov and Triebel–Lizorkin spaces is considered, the corresponding theory was presented in [32], Chapter 2.11. It is not difficult to see that one may adopt these results to the spaces with dominating mixed smoothness. We do not intend to give some exhaustive theory. The only fact we need is

$$[S^{\overline{r}}_{p,p}B(\mathbb{R}^d)]' = S^{-\overline{r}+\sigma_p}_{p'p'}B(\mathbb{R}^d), \quad \overline{r} \in \mathbb{R}^d, \quad 0 < p < \infty,$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1 \quad \text{for} \quad 1 < p < \infty$$

and

$$p' = \infty \quad \text{for} \quad p \le 1.$$

The functions

$$D^\alpha \Psi_{\overline{k}\overline{m}}, \quad 0 \le \alpha \le (s,\dots,s),$$

are bounded functions with compact support. Using Hölder's inequality, we see that

$$||D^\alpha \Psi_{\overline{k}\overline{m}}|L_{\tilde{p}}(\mathbb{R}^d)|| < \infty$$

for every

$$0 \le \alpha \le (s,\dots,s), \quad 0 < \tilde{p} \le \infty.$$

Using the Littlewood–Paley theory, we get

$$\Psi_{\overline{k}\overline{m}} \in S^{\overline{s}}_{\tilde{p},2}F(\mathbb{R}^d), \quad 1 < \tilde{p} < \infty$$

for $\overline{s} = (s, \ldots, s)$. And, by the Sobolev embedding,

$$S^{\overline{s}}_{\tilde{p},2}F(\mathbb{R}^d) \hookrightarrow [S^{\overline{r}-\epsilon}_{p,p}B(\mathbb{R}^d)]' = S^{-\overline{r}+\epsilon+\sigma_p}_{p',p'}B(\mathbb{R}^d)$$

for s large enough and every $\epsilon > 0$.

So, for

$$f \in S^{\overline{r}}_{p,q}A(\mathbb{R}^d) \hookrightarrow S^{\overline{r}-\epsilon}_{p,p}B(\mathbb{R}^d)$$

we may interpret $\Psi_{\overline{k}\overline{m}}$ as a bounded linear functional on a space f belongs to. And $(f, \Psi_{\overline{k}\overline{m}})$ is then the value of this functional at f.

We may also reverse these arguments. The functions $\Psi_{\overline{k}\overline{m}}$ belong to

$$S^{\overline{s}}_{\tilde{p},2}F(\mathbb{R}^d), \quad 1 < \tilde{p} < \infty$$

and

$$S^{\overline{s}}_{\tilde{p},2}F(\mathbb{R}^d) \hookrightarrow S^{\overline{s}-\epsilon}_{\tilde{p},\tilde{p}}B(\mathbb{R}^d).$$

Hence, for s large, we get

$$f \in [S^{\overline{s}-\epsilon}_{\tilde{p},\tilde{p}}B(\mathbb{R}^d)]'.$$

In this case we may interpret f as a linear bounded functional on a space $\Psi_{\overline{k}\overline{m}}$ belongs to. $(f, \Psi_{\overline{k}\overline{m}})$ is then the value of this functional at $\Psi_{\overline{k}\overline{m}}$.

Proof of Theorem 2.12, Part (i). Let $\lambda \in s^{\overline{r}}_{p,q}f$. If

$$s > \max\{(1 + [r_i])_+, [\sigma_{pq} - r_i], i = 1, \ldots, d\}$$

and $\overline{s} = (s, \ldots, s) \in \mathbb{R}^d$ then $\Psi_{\overline{k}\overline{m}}$ are $[\overline{s}, \overline{s}]-$atoms cantered at $Q_{\overline{k}\overline{m}}$. So, for s large, all the assumptions of Theorem 2.4 are satisfied and, according to this theorem, (2.75) converges in $S'(\mathbb{R}^d)$. We denote its limit by f. The same theorem tells us that $f \in S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ and implies even the estimate (2.76). Hence the points 1. and 2. are proven. Very similar arguments apply also to the B-case.

For $\lambda \in s^{\overline{r}}_{p,q}a$ and natural number μ we define

$$\lambda^\mu = \{\lambda^\mu_{\overline{k}\overline{m}} : \overline{k} \in \mathbb{N}^d_0, \overline{m} \in \mathbb{Z}^d\}$$

by

$$\lambda^\mu_{\overline{k}\overline{m}} = \begin{cases} \lambda_{\overline{k}\overline{m}} & \text{if } |\overline{k}| > \mu \\ 0 & \text{otherwise.} \end{cases}$$

If $\max(p, q) < \infty$ then

$$\lim_{\mu\to\infty} ||\lambda^\mu|s^{\overline{r}}_{p,q}a|| = 0. \tag{2.79}$$

This is clear in the $b-$case and one has to use Lebesgue's dominated convergence theorem in the $f-$case. Using (2.76), already proven, we finish the proof of 4.

In the proof of the third point, we replace (2.79) by

$$\lim_{\mu\to\infty} ||\lambda^\mu|s^{\overline{r}-\epsilon}_{p,q}a|| = 0. \tag{2.80}$$

To see that (2.80) holds, one uses the same reasoning as in (2.79), and Hölder's inequality. This finishes the proof of part (i). □

Proof of Theorem 2.12, part (ii).
The meaning of the expression $(f, \Psi_{\overline{km}})$ was already discussed in section 2.4.1. For the rest of the proof we consider only the $F-$case. The proof for $B-$spaces is very similar.

Before we prove the first statement of the second part we do some calculation. We may rewrite the norm in $s^{\overline{r}}_{p,q}f$ as

$$||\lambda|s^{\overline{r}}_{p,q}f|| = ||2^{\overline{k}\cdot\overline{r}}g_{\overline{k}}|L_p(\ell_q)||, \tag{2.81}$$

where

$$g_{\overline{k}}(x) = \sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{km}}\chi_{\overline{km}}(x). \tag{2.82}$$

If $x \in Q_{\overline{km}}$ and λ is defined by (2.77) we use (2.82)

$$g_{\overline{k}}(x) = \lambda_{\overline{km}} = 2^{|\overline{k}|}\int_{\mathbb{R}^d} \Psi_{\overline{km}}(y)f(y)\mathrm{d}y = 2^{|\overline{k}|}\int_{\mathbb{R}^d} \psi_{k_1m_1}(y_1)\cdot\ldots\cdot\psi_{k_dm_d}(y_d)f(y)\mathrm{d}y.$$

We assume that $\overline{k} \geq 1$, insert the Definition (2.71) and substitute $z_i = y_i - 2^{-k_i}m_i$

$$\begin{aligned} g_{\overline{k}}(x) &= 2^{|\overline{k}|}\int_{\mathbb{R}^d} \psi_1(2^{k_1}z_1)\cdot\ldots\cdot\psi_1(2^{k_d}z_d)f(2^{-k_1}m_1+z_1,\ldots,2^{-k_d}m_d+z_d)\mathrm{d}z \\ &= \mathcal{K}_{\overline{k}}(f)(2^{-\overline{k}}\overline{m}). \end{aligned}$$

Here $\mathcal{K}_{\overline{k}}(f)(2^{-\overline{k}}\overline{m})$ denotes the local means

$$\mathcal{K}_{\overline{k}}(f)(y) = \int_{\mathbb{R}^d} \mathcal{K}_{\overline{k}}(z)f(y+z)\mathrm{d}z, \qquad y \in \mathbb{R}^d. \tag{2.83}$$

for the kernel

$$\mathcal{K}_{\overline{k}}(z) = 2^{|\overline{k}|}\psi_1(2^{k_1}z_1)\cdot\ldots\cdot\psi_1(2^{k_d}z_d)$$

We point out that all integrals have to be interpreted in the distributional sense. If one (or more) $k_i = 0$, only notational changes are necessary. Hence, for every $x \in Q_{\overline{km}}$,

$$|g_{\overline{k}}(x)| \leq \sup_{y-x\in Q_{\overline{k},0}} |\mathcal{K}_{\overline{k}}(f)(y)|.$$

Applying Theorem 1.27 we see that

$$||\lambda|s^{\overline{r}}_{p,q}f|| = ||2^{\overline{k}\cdot\overline{r}}g_{\overline{k}}|L_p(\ell_q)|| \leq c\,||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)||.$$

This finishes the proof of 1.

To prove the second statement, we define a new function g by

$$g = \sum_{\overline{k}\in\mathbb{N}_0^d,\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{km}}\Psi_{\overline{km}}, \tag{2.84}$$

where $\lambda_{\overline{km}}$ are given by (2.77). The convergence of this sum is guaranteed by $\lambda \in s^{\overline{r}}_{p,q}f$ (which we have just proved) and by part (i). It shows even that $g \in S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$. We need to prove that $g = f$ or, equivalently, that

$$(g,\varphi) = (f,\varphi) \quad \text{for every} \quad \varphi \in S(\mathbb{R}^d).$$

First we consider the expressions $(g, \Psi_{\overline{k}'\overline{m}'})$. As $\lambda \in s^{\overline{r}}_{p,q}f$, (2.84) converges in any $S^{\overline{r}-\epsilon}_{p,2}F(\mathbb{R}^d)$, where $\epsilon > 0$ may be chosen arbitrarily. If the number s is chosen sufficiently large then, according to Section 2.4.1, $\Psi_{\overline{k}'\overline{m}'} \in [S^{\overline{r}-\epsilon}_{p,2}F(\mathbb{R}^d)]'$. Hence

$$(g, \Psi_{\overline{k}'\overline{m}'}) = \lim_{\mu\to\infty}\left(\sum_{|\overline{k}|\le\mu,\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{k}\overline{m}}\Psi_{\overline{k}\overline{m}}, \Psi_{\overline{k}'\overline{m}'}\right) = \lim_{\mu\to\infty}\sum_{|\overline{k}|\le\mu,\overline{m}\in\mathbb{Z}^d} 2^{|\overline{k}|}(f, \Psi_{\overline{k}\overline{m}})(\Psi_{\overline{k}\overline{m}}, \Psi_{\overline{k}'\overline{m}'}).$$

Using orthogonality of system (2.74) we arrive at

$$(g, \Psi_{\overline{k}'\overline{m}'}) = (f, \Psi_{\overline{k}'\overline{m}'}), \quad \overline{k}' \in \mathbb{N}^d_0, \quad \overline{m}' \in \mathbb{Z}^d.$$

One may extend this argument to any finite linear combination of $\Psi_{\overline{k}'\overline{m}'}$. For a general function $\varphi \in S(\mathbb{R}^d)$ we consider its Fourier series decomposition with respect to system (2.74):

$$\varphi = \sum_{\overline{k},\overline{m}} 2^{|\overline{k}|}(\varphi, \Psi_{\overline{k}\overline{m}})\Psi_{\overline{k}\overline{m}}. \tag{2.85}$$

As $S(\mathbb{R}^d)$ is a subset of all Fourier-analytic Besov and Triebel-Lizorkin spaces, we see that (for s large enough) (2.85) converges also in the space $[S^{\overline{r}-\epsilon}_{p,2}F(\mathbb{R}^d)]'$. Hence we get

$$(g, \varphi) = \lim_{\mu\to\infty}\sum_{|\overline{k}|\le\mu,\overline{m}\in\mathbb{Z}^d} 2^{|\overline{k}|}(\varphi, \Psi_{\overline{k}\overline{m}})(g, \Psi_{\overline{k}\overline{m}}) = \lim_{\mu\to\infty}\sum_{|\overline{k}|\le\mu,\overline{m}\in\mathbb{Z}^d} 2^{|\overline{k}|}(\varphi, \Psi_{\overline{k}\overline{m}})(f, \Psi_{\overline{k}\overline{m}}) = (f, \varphi).$$

Hence the sum (2.75) converges to f.

The final step, namely the proof of the third statement, follows now very easily. Suppose that the assumptions are satisfied. We define the coefficients $\lambda_{\overline{k}\overline{m}}$ by (2.77) and g by (2.84). Then we get $f = g$ according to point 2. And by the same duality arguments as there we obtain

$$\gamma_{\overline{k}\overline{m}} = 2^{|\overline{k}|/2}(f, \Psi_{\overline{k}\overline{m}}) = 2^{|\overline{k}|/2}(g, \Psi_{\overline{k}\overline{m}}) = \lambda_{\overline{k}\overline{m}}, \quad \overline{k} \in \mathbb{N}^d_0, \quad \overline{m} \in \mathbb{Z}^d.$$

□

3 Entropy numbers - direct results

3.1 Notation and definitions

We have seen in the previous section the sharp connection between function spaces $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ and corresponding sequence spaces $s^{\overline{r}}_{p,q}a$ given by several decomposition techniques. We would like to use these results to study the entropy numbers of embeddings of function spaces with dominating mixed smoothness on domains.

First, we define function spaces on domains by restrictions of function spaces defined on $\mathbb{R}^d$.

Definition 3.1. Let Ω be an arbitrary bounded domain in $\mathbb{R}^d$. Then $S^{\overline{r}}_{p,q}A(\Omega)$ is the restriction of $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ to Ω:

$$S^{\overline{r}}_{p,q}A(\Omega) = \{f \in D'(\Omega) : \exists g \in S^{\overline{r}}_{p,q}A(\mathbb{R}^d) \quad \text{with} \quad g|\Omega = f\} \tag{3.1}$$

$$||f|S^{\overline{r}}_{p,q}A(\Omega)|| = \inf ||g|S^{\overline{r}}_{p,q}A(\mathbb{R}^d)||, \tag{3.2}$$

where the infimum is taken over all $g \in S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ such that its restriction to Ω, denoted by $g|\Omega$, coincides in $D'(\Omega)$ with f.

Next, we define the sequence spaces corresponding to $S^{\overline{r}}_{p,q}A(\Omega)$. The change with respect to $s^{\overline{r}}_{p,q}a$ is rather simple. In Definition 2.2 the sum over $\overline{m} \in \mathbb{Z}^d$ represents a discrete analogy of $L_p(\mathbb{R}^d)$-norm and the sum over $\overline{\nu} \in \mathbb{N}_0^d$ the sum over all coverings of plane with dyadic cubes. So, to adapt Definition 2.2 to suit well to function spaces on domains, we have to restrict the sum to those $\overline{m}$ which are in some relation with Ω.

For that reason we define for every bounded domain $\Omega \subset \mathbb{R}^d$

$$A^{\Omega}_{\overline{\nu}} = \{\overline{m} \in \mathbb{Z}^d : Q_{\overline{\nu}\,\overline{m}} \cap \Omega \neq \emptyset\}, \quad \overline{\nu} \in \mathbb{N}_0^d.$$

The sequence spaces associated with a bounded domain Ω are then defined by

Definition 3.2. If $0 < p \leq \infty$, $0 < q \leq \infty, \overline{r} \in \mathbb{R}^d$ and

$$\lambda = \{\lambda_{\overline{\nu}\,\overline{m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in A^{\Omega}_{\overline{\nu}}\}$$

then we define

$$s^{\overline{r},\Omega}_{p,q}b = \left\{\lambda : ||\lambda|s^{\overline{r},\Omega}_{p,q}b|| = \left(\sum_{\overline{\nu}\in\mathbb{N}_0^d} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\Big(\sum_{\overline{m}\in A^{\Omega}_{\overline{\nu}}} |\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\right)^{1/q} < \infty\right\} \tag{3.3}$$

and

$$s^{\overline{r},\Omega}_{p,q}f = \left\{\lambda : ||\lambda|s^{\overline{r},\Omega}_{p,q}f|| = \left\|\left(\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in A^{\Omega}_{\overline{\nu}}} |2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}(\cdot)}|^q\right)^{1/q}|L_p(\mathbb{R}^d)\right\| < \infty\right\}. \tag{3.4}$$

Furthermore, we define corresponding *building blocks*.

Definition 3.3. Let $0 < p \leq \infty$, $0 < q \leq \infty, \overline{r} \in \mathbb{R}^d$ and let $\mu \in \mathbb{N}_0$ be fixed. If

$$\lambda = \{\lambda_{\overline{\nu}\,\overline{m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^d, |\overline{\nu}| = \mu, \overline{m} \in A^{\Omega}_{\overline{\nu}}\}$$

then we define

$$(s^{\overline{r},\Omega}_{p,q}b)_\mu = \left\{\lambda : ||\lambda|(s^{\overline{r},\Omega}_{p,q}b)_\mu|| = \left(\sum_{|\overline{\nu}|=\mu} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\Big(\sum_{\overline{m}\in A^{\Omega}_{\overline{\nu}}} |\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\right)^{1/q} < \infty\right\} \tag{3.5}$$

and

$$(s^{\overline{r},\Omega}_{p,q}f)_\mu = \left\{\lambda : ||\lambda|(s^{\overline{r},\Omega}_{p,q}f)_\mu|| = \left\|\left(\sum_{|\overline{\nu}|=\mu}\sum_{\overline{m}\in A^{\Omega}_{\overline{\nu}}} |2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}}(\cdot)|^q\right)^{1/q}|L_p(\mathbb{R}^d)\right\| < \infty\right\}. \tag{3.6}$$

Remark 3.4. 1. We point out that, that for the number of elements of $A^{\Omega}_{\overline{\nu}}$ we have trivially

$$\#(A^{\Omega}_{\overline{\nu}}) \approx 2^{|\overline{\nu}|}, \quad \overline{\nu} \in \mathbb{N}_0^d \tag{3.7}$$

where the constants in this equivalence depend only on Ω. The dimension of $(s^{\overline{r},\Omega}_{p,q}a)_\mu$ will be denoted by

$$D_\mu := \sum_{|\overline{\nu}|=\mu} \#(A^{\Omega}_{\overline{\nu}}), \quad \mu \in \mathbb{N}_0. \tag{3.8}$$

2. As usual, we write $s^{\overline{r},\Omega}_{p,q}a$ for $s^{\overline{r},\Omega}_{p,q}b$ or $s^{\overline{r},\Omega}_{p,q}f$ respectively. The same holds for $(s^{\overline{r},\Omega}_{p,q}a)_\mu$.

Next we define the notion of entropy numbers and recall its basic properties. We refer to [10] and references given there for details.

Definition 3.5. Let A, B be quasi-Banach spaces and let T be a bounded linear operator $T \in L(A, B)$. Let U_A and U_B denote the unit ball in the spaces A and B, respectively. Then for every $k \in \mathbb{N}$ we define the k-th dyadic entropy number by

$$e_k(T) := \inf\{\epsilon > 0 : T(U_A) \subset \bigcup_{j=1}^{2^{k-1}} (b_j + \epsilon U_B)\}$$

for some $b_1, \ldots, b_{2^{k-1}} \in B$.

Definition 3.6. Given any $p \in (0, 1]$ and a quasi-Banach space B, we say that B is a $p-$Banach space, if

$$||x + y|B||^p \leq ||x|B||^p + ||y|B||^p \quad \text{for all} \quad x, y \in B. \tag{3.9}$$

It can be shown that if $||\cdot|B||_1$ is a quasinorm on B, then there is $p \in (0, 1]$ and a quasinorm $||\cdot|B||_2$ with (3.9) on B which is equivalent to $||\cdot|B||_1$. We refer again to [10] and references given there for details.

Theorem 3.7. *Let A, B, C be quasi-Banach spaces, $S, T \in L(A, B)$, $R \in L(B, C)$. Then*

- $||T|| \geq e_1(T) \geq e_2(T) \geq \cdots \geq 0.$
- $e_{k+l-1}(R \circ S) \leq e_k(R) e_l(S), \quad k, l \in \mathbb{N}.$
- *If B is p-Banach space, then* $e^p_{k+l-1}(S + T) \leq e^p_k(S) + e^p_l(T)$

Remark 3.8. We refer to the first property of entropy numbers from Theorem 3.7 as *monotonicity*, the second is called *submultiplicativity* and the last one is quoted by *subaditivity.*

Although we shall not need it in sequel, we quote the fundamental result of Carl (see [6], [7] and [10] for details). It illustrates the importance of estimates of entropy numbers in the study of spectral properties of compact operators.

Theorem 3.9. *Let A be a quasi-Banach space and let $T \in L(A, A) = L(A)$ be a compact operator on A. We denote its non-zero eigenvalues with respect to multiplicity by*

$$|\lambda_1(T)| \geq |\lambda_2(T)| \geq |\lambda_3(T)| \geq \cdots > 0.$$

Then

$$|\lambda_k(T)| \leq \sqrt{2} e_k(T).$$

In what follows we restrict ourselves to $\overline{r} = (r_1, r_2, \ldots, r_d) \in \mathbb{R}^d$ with $r_1 = r_2 = \cdots = r_d$.

3.2 Basic lemmas

Now we collect some basic properties of the building blocks defined by (3.5) and (3.6). We start with the following

Lemma 3.10. *1. Let $0 < p_1, p_2 \leq \infty$ and $N \in \mathbb{N}$. Then*

$$||id : \ell_{p_1}^N \to \ell_{p_2}^N|| = \begin{cases} 1, & p_1 \leq p_2, \\ N^{\frac{1}{p_2} - \frac{1}{p_1}}, & p_1 \geq p_2. \end{cases} \tag{3.10}$$

2. Let $0 < p \leq \infty$ and $\overline{r} = (r, \dots, r) \in \mathbb{R}^d$. Then

$$(s_{p,p}^{\overline{r},\Omega} b)_\mu = (s_{p,p}^{\overline{r},\Omega} f)_\mu = 2^{\mu(r-\frac{1}{p})} \ell_p^{D_\mu}, \quad \mu \in \mathbb{N}_0 \tag{3.11}$$

and

$$s_{p,p}^{\overline{r},\Omega} b = s_{p,p}^{\overline{r},\Omega} f. \tag{3.12}$$

The number D_μ is given by (3.8).
3. Let $0 < p_2 \leq p_1 \leq \infty$, $0 < q \leq \infty$ and $\overline{r} = (r, \dots, r) \in \mathbb{R}^d$. Then

$$||id : (s_{p_1,q}^{\overline{r},\Omega} a)_\mu \to (s_{p_2,q}^{\overline{r},\Omega} a)_\mu|| \approx 1, \quad \mu \in \mathbb{N}_0. \tag{3.13}$$

4. Let $0 < q_2 \leq q_1 \leq \infty$, $0 < p \leq \infty$ and $\overline{r} = (r, \dots, r) \in \mathbb{R}^d$. Then

$$||id : (s_{p,q_1}^{\overline{r},\Omega} a)_\mu \to (s_{p,q_2}^{\overline{r},\Omega} a)_\mu|| \approx \mu^{(d-1)(\frac{1}{q_2} - \frac{1}{q_1})}, \quad \mu \in \mathbb{N}. \tag{3.14}$$

All constants of equivalence involved in (3.13) and (3.14) do not depend $\mu \in \mathbb{N}_0$.

Proof. The proof of 1. and 2. involves only (3.5) and (3.6). For the proof of 3. in the case $a = b$ we write

$$\begin{aligned}
||\lambda | s_{p_2,q}^{\overline{r},\Omega} b|| &= \Bigg(\sum_{|\overline{\nu}|=\mu} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p_2})q} \Big(\sum_{\overline{m} \in A_{\overline{\nu}}^{\Omega}} |\lambda_{\overline{\nu}\,\overline{m}}|^{p_2} \Big)^{q/p_2} \Bigg)^{1/q} \\
&= 2^{\mu(r-\frac{1}{p_2})} \Bigg(\sum_{|\overline{\nu}|=\mu} \Big(\sum_{\overline{m} \in A_{\overline{\nu}}^{\Omega}} |\lambda_{\overline{\nu}\,\overline{m}}|^{p_2} \Big)^{q/p_2} \Bigg)^{1/q} \\
&\leq c\, 2^{\mu(r-\frac{1}{p_2})} 2^{\mu(\frac{1}{p_2}-\frac{1}{p_1})} \Bigg(\sum_{|\overline{\nu}|=\mu} \Big(\sum_{\overline{m} \in A_{\overline{\nu}}^{\Omega}} |\lambda_{\overline{\nu}\,\overline{m}}|^{p_1} \Big)^{q/p_1} \Bigg)^{1/q} \\
&= c\, ||\lambda | s_{p_1,q}^{\overline{r},\Omega} b||,
\end{aligned}$$

where we have used (3.10).
In the case $a = f$, we get by Hölder's inequality and boundedness of Ω

$$\begin{aligned}
||\lambda | s_{p_2,q}^{\overline{r},\Omega} f|| &= \Bigg|\Bigg| \Bigg(\sum_{\overline{\nu} \in \mathbb{N}_0^d} \sum_{\overline{m} \in A_{\overline{\nu}}^{\Omega}} |2^{\overline{\nu}\cdot\overline{r}} \lambda_{\overline{\nu}\,\overline{m}} \chi_{\overline{\nu}\,\overline{m}(\cdot)}|^q \Bigg)^{1/q} |L_{p_2}(\mathbb{R}^d) \Bigg|\Bigg| \\
&\leq c \Bigg|\Bigg| \Bigg(\sum_{\overline{\nu} \in \mathbb{N}_0^d} \sum_{\overline{m} \in A_{\overline{\nu}}^{\Omega}} |2^{\overline{\nu}\cdot\overline{r}} \lambda_{\overline{\nu}\,\overline{m}} \chi_{\overline{\nu}\,\overline{m}(\cdot)}|^q \Bigg)^{1/q} |L_{p_1}(\mathbb{R}^d) \Bigg|\Bigg| \\
&= c\, ||\lambda | s_{p_1,q}^{\overline{r},\Omega} f||.
\end{aligned}$$

The proof of 4. involves only 1. and

$$\#\{\overline{\nu} \in \mathbb{N}_0^d : |\overline{\nu}| = \mu\} \approx \mu^{d-1}, \quad \mu \in \mathbb{N}.$$

□

Next, we recall a fundamental result which is essentially due to Schütt [27] and Kühn [17].

Lemma 3.11. (i) *If* $0 < p_1 \le p_2 \le \infty$ *and* k *and* N *are natural numbers, then*

$$e_k(id : \ell_{p_1}^N \to \ell_{p_2}^N) \approx \begin{cases} 1 & \text{if} \quad 1 \le k \le \log 2N, \\ \left(k^{-1}\log(1+\frac{N}{k})\right)^{\frac{1}{p_2}-\frac{1}{p_1}} & \text{if} \quad \log 2N \le k \le 2N, \\ 2^{-\frac{k}{2N}} N^{\frac{1}{p_2}-\frac{1}{p_1}} & \text{if} \quad 2N \le k, \end{cases} \tag{3.15}$$

where the constants of equivalence do not depend on k *and* N*.*
(ii) *If* $0 < p_2 < p_1 \le \infty$ *and* k *and* N *are natural numbers, then*

$$e_k(id : \ell_{p_1}^N \to \ell_{p_2}^N) \approx 2^{-\frac{k}{2N}} N^{\frac{1}{p_2}-\frac{1}{p_1}} \tag{3.16}$$

where the corresponding constants again do not depend on k *and* N*.*

Remark 3.12. We refer to [27], [17], [10] and references given there for the proofs of this fundamental result.

Lemma 3.13. *Let*

$$\overline{r}_1 = (r_1, \ldots, r_1) \in \mathbb{R}^d, \overline{r}_2 = (r_2, \ldots, r_2) \in \mathbb{R}^d, \quad 0 < p_1, p_2, q_1, q_2 \le \infty.$$

Let $k \ge 2D_\mu$*. Then*

$$e_k(id : (s_{p_1,q_1}^{\overline{r}_1,\Omega} a)_\mu \to (s_{p_2,q_2}^{\overline{r}_2,\Omega} a^\dagger)_\mu) \approx 2^{-\frac{k}{2D_\mu}} \mu^{(d-1)\left(\frac{1}{q_2}-\frac{1}{q_1}\right)} 2^{\mu(r_2-r_1)} \tag{3.17}$$

with constants of equivalence independent of k *and* μ*.*

Remark 3.14. The symbols a and $a^\dagger$ stand for b or f, not necessary for the same letter. Hence the formula (3.17) represents actually *four* different equivalences and, consequently, eight inequalities are to be proven.

Proof. Let us denote

$$\gamma_1 = \min(p_1, q_1), \quad \gamma_2 = \min(p_2, q_2) \tag{3.18}$$
$$\delta_1 = \max(p_1, q_1), \quad \delta_2 = \max(p_2, q_2). \tag{3.19}$$

Step 1.
In the first Step we use the following diagram to estimate $e_k(id)$ from above.

$$\begin{array}{ccc} (s_{p_1,q_1}^{\overline{r}_1,\Omega} a)_\mu & \xrightarrow{id} & (s_{p_2,q_2}^{\overline{r}_2,\Omega} a^\dagger)_\mu \\ {\scriptstyle id_1}\big\downarrow & & \big\uparrow{\scriptstyle id_3} \\ (s_{\gamma_1,\gamma_1}^{\overline{r}_1,\Omega} a)_\mu & \xrightarrow{id_2} & (s_{\delta_2,\delta_2}^{\overline{r}_2,\Omega} a^\dagger)_\mu. \end{array} \tag{3.20}$$

Using the submultiplicativity of entropy numbers (see Theorem 3.7) we get

$$e_k(id) \le ||id_1|| \cdot ||id_3|| \cdot e_k(id_2) \tag{3.21}$$

To estimate $||id_1||$ and $||id_3||$ we use (3.13), resp. (3.14) and get

$$||id_1|| \le c\,\mu^{(d-1)\left(\frac{1}{\gamma_1}-\frac{1}{q_1}\right)_+}, \quad ||id_3|| \le c\,\mu^{(d-1)\left(\frac{1}{q_2}-\frac{1}{\delta_2}\right)_+}. \tag{3.22}$$

To estimate $e_k(id_2)$ we use Lemma 3.11 and (3.11)

$$(s^{\overline{r}_1,\Omega}_{\gamma_1,\gamma_1}a)_\mu \approx 2^{\mu(r_1-\frac{1}{\gamma_1})}\ell^{D_\mu}_{\gamma_1}$$

and its counterpart for $(s^{\overline{r}_2,\Omega}_{\delta_2,\delta_2}a^\dagger)_\mu$. This gives

$$e_k(id_2) \le c\,2^{\mu(-r_1+\frac{1}{\gamma_1}+r_2-\frac{1}{\delta_2})}2^{-\frac{k}{2D_\mu}}D_\mu^{\frac{1}{\delta_2}-\frac{1}{\gamma_1}}. \tag{3.23}$$

Putting (3.22) and (3.23) into (3.21) and using $D_\mu \approx \mu^{d-1}2^\mu$ we get the desired result and finish the Step 1.

Step 2.

We prove now the estimates from below. Let $\gamma_1, \gamma_2, \delta_1, \delta_2$ be still defined by (3.18) and (3.19), respectively. We use following diagram.

$$\begin{array}{ccc} (s^{\overline{r}_1,\Omega}_{p_1,q_1}a)_\mu & \xrightarrow{id} & (s^{\overline{r}_2,\Omega}_{p_2,q_2}a^\dagger)_\mu \\ id_1 \uparrow & & \downarrow id_3 \\ (s^{\overline{r}_1,\Omega}_{\delta_1,\delta_1}a)_\mu & \xrightarrow{id_2} & (s^{\overline{r}_2,\Omega}_{\gamma_2,\gamma_2}a^\dagger)_\mu. \end{array} \tag{3.24}$$

As $id_2 = id_1 \circ id \circ id_3$ we may use again the submultiplicativity of entropy numbers. The estimate for the entropy numbers of id_2 is given by Lemma 3.11

$$e_k(id_2) \ge c\;2^{\mu(-r_1+r_2+\frac{1}{\delta_1}-\frac{1}{\gamma_2})}2^{-\frac{k}{2D_\mu}}D_\mu^{\frac{1}{\gamma_2}-\frac{1}{\delta_1}}$$

and for $||id_1||$ and $||id_3||$ we use similar estimates as in the Step 1.

$$||id_1|| \le c\,\mu^{(d-1)\left(\frac{1}{q_1}-\frac{1}{\delta_1}\right)_+}, \quad ||id_3|| \le c\,\mu^{(d-1)\left(\frac{1}{\gamma_2}-\frac{1}{q_2}\right)_+}. \tag{3.25}$$

From this the result immediately follows. □

Lemma 3.13 is a generalisation of Lemma 3.11 as far as the third line of (3.15) and (3.16) is concerned. So, for $k \ge 2D_\mu$, the estimate (3.17) provides four equivalences with constants independent of k and μ. In the case $k \le 2D_\mu$ the situation is not so simple any more; we give two different estimates from above.

Lemma 3.15. *Let*

$$\overline{r}_1 = (r_1,\dots,r_1) \in \mathbb{R}^d, \overline{r}_2 = (r_2,\dots,r_2) \in \mathbb{R}^d, \quad 0 < p_1, p_2, q_1, q_2 \le \infty$$

with $p_1 \le p_2$. Let $k \le 2D_\mu$. Then

$$e_k(id : (s^{\overline{r}_1,\Omega}_{p_1,q_1}a)_\mu \to (s^{\overline{r}_2,\Omega}_{p_2,q_2}a^\dagger)_\mu) \le c\,\mu^{(d-1)\left(\frac{1}{\gamma_1}-\frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{\delta_2}\right)}2^{\mu(-r_1+\frac{1}{\gamma_1}+r_2-\frac{1}{\delta_2})}\cdot \tag{3.26}$$
$$\cdot\left[k^{-1}\log\left(\frac{\mu^{d-1}2^\mu}{k}+1\right)\right]^{\frac{1}{\gamma_1}-\frac{1}{\delta_2}},$$

where $\gamma_1, \gamma_2, \delta_1, \delta_2$ are given by (3.18) and (3.19). The constant c is independent of k and μ.

The proof of Lemma 3.15 copies exactly the first step of the proof of Lemma 3.13.

The second estimate from above follows closely the idea of Kühn, Leopold, Sickel and Skrzypczak expressed in [18].

Lemma 3.16. *Let*

$$\overline{r}_1 = (r_1, \dots, r_1) \in \mathbb{R}^d, \overline{r}_2 = (r_2, \dots, r_2) \in \mathbb{R}^d, \quad 0 < p_1, p_2, q_1, q_2 \leq \infty$$

with

$$p_1 \leq p_2, \qquad \frac{1}{p_1} - \frac{1}{p_2} > \frac{1}{q_1} - \frac{1}{q_2}.$$

Let $(d-1)\mu^{d-1}\log\mu \leq k \leq 2D_\mu = 2\sum_{|\overline{\nu}|=\mu} \# A^\Omega_{\overline{\nu}}$. *Then*

$$e_k(id : (s^{\overline{r}_1,\Omega}_{p_1,q_1}b)_\mu \to (s^{\overline{r}_2,\Omega}_{p_2,q_2}b)_\mu) \leq c\, 2^{\mu(-r_1+r_2+\frac{1}{p_1}-\frac{1}{p_2})}\mu^{\frac{1}{p_1}-\frac{1}{p_2}+\frac{1}{q_2}-\frac{1}{q_1}} \cdot \tag{3.27}$$

$$\cdot k^{\frac{1}{p_2}-\frac{1}{p_1}}\left[\log\left(\frac{\mu^{d-1}2^\mu}{k}+1\right)\right]^{\frac{1}{p_1}-\frac{1}{p_2}}$$

Proof. We denote $X_i = (s^{\overline{r}_i,\Omega}_{p_i,q_i}b)_\mu, i = 1, 2$. We shall construct an $\epsilon-$net of X_2-balls covering a unit ball B_{X_1} of X_1. For that reason we fix some ordering of the set $\{\overline{\nu} \in \mathbb{N}^d_0 : |\overline{\nu}| = \mu\} = \{\overline{\nu}^1, \dots, \overline{\nu}^{S(\mu,d)}\}$, where

$$S_{\mu,d} = \#\{\overline{\nu} \in \mathbb{N}^d_0 : |\overline{\nu}| = \mu\} = \binom{\mu+d-1}{\mu}, \quad \mu \in \mathbb{N}_0. \tag{3.28}$$

First we consider the subset of B_{X_1}

$$B = \{\lambda \in B_{X_1} : ||\lambda_{\overline{\nu}^1}|X_1|| \geq ||\lambda_{\overline{\nu}^2}|X_1|| \geq \dots \geq ||\lambda_{\overline{\nu}^{S(\mu,d)}}|X_1||\}$$

and construct an $\epsilon-$net $\mathcal{N}$ in X_2 for B. Then, if Π is any permutation of the index set $\{1, \dots, S_{\mu,d}\}$ and

$$B_\Pi = \{\lambda \in B_{X_1} : ||\lambda_{\overline{\nu}^{\Pi(1)}}|X_1|| \geq ||\lambda_{\overline{\nu}^{\Pi(2)}}|X_1|| \geq \dots \geq ||\lambda_{\overline{\nu}^{\Pi(S(\mu,d))}}|X_1||\}$$

we get, by permutation of the coordinates, $\epsilon-$nets $\mathcal{N}_\Pi$ for B_Π, all having the same cardinality as $\mathcal{N}$, say 2^k.

Clearly, $B_{X_1} = \cup_\Pi B_\Pi$, where the union is taken over all permutations Π of the set $\{1, \dots, S(\mu, d)\}$. Hence $\cup_\Pi \mathcal{N}_\Pi$ is an $\epsilon-$ net in X_2 for B_{X_1} of cardinality

$$S(\mu, d)!2^k \leq \mu^{(d-1)\mu^{d-1}}2^k = 2^{(d-1)\mu^{d-1}\log\mu+k}.$$

It remains to construct an $\epsilon-$net for B_X in X_2. For $\lambda \in B$ we have $||\lambda_{\overline{\nu}^j}|X_1|| \leq j^{-1/q_1}$. If $k_1, \dots, k_{S(\mu,d)}$ are arbitrary natural numbers, we set

$$\epsilon_j := cj^{-1/q_1}2^{\mu\left(-r_1+\frac{1}{p_1}+r_2-\frac{1}{p_2}\right)}\left[k_j^{-1}\log\left(\frac{2^\mu}{k_j}+1\right)\right]^{\frac{1}{p_1}-\frac{1}{p_2}}$$

and, according to Lemma 3.11, we find ϵ_j- net $\mathcal{N}_j$ in $2^{\mu\left(r_2-\frac{1}{p_2}\right)}\ell^{A_j}_{p_2}$ for $j^{1/q_1}B_Y$, where $Y = 2^{\mu\left(r_1-\frac{1}{p_1}\right)}\ell^{A_j}_{p_1}$ and $A_j = \#(A^\Omega_{\overline{\nu}^j})$.

Thus $\mathcal{N}_1 \times \ldots \times \mathcal{N}_{S(\mu,d)}$ is an ϵ−net in X_2 for B of cardinality $2^{k_1+\ldots+k_{S(\mu,d)}}$, where

$$\epsilon = \Big(\sum_{j=1}^{S(\mu,d)} \epsilon_j^{q_2} \Big)^{\frac{1}{q_2}}.$$

Finally, we choose $k_j, j = 1, \ldots, S(\mu, d)$. Fix $m \in \mathbb{N}$ and set

$$k_j = 2^m j^{-\alpha},$$

where $0 < \alpha < 1$ is chosen such that

$$\alpha\big(\frac{1}{p_1} - \frac{1}{p_2}\big) > \frac{1}{q_1} - \frac{1}{q_2}$$

Then

$$k = \sum_{j=1}^{S(\mu,d)} k_j \approx 2^m \mu^{(d-1)(-\alpha+1)} \tag{3.29}$$

and

$$\Big(\sum_{j=1}^{S(\mu,d)} \epsilon_j^{q_2} \Big)^{\frac{1}{q_2}} \approx 2^{\mu\left(-r_1+\frac{1}{p_1}+r_2-\frac{1}{p_2}\right)} 2^{m\left(\frac{1}{p_2}-\frac{1}{p_1}\right)} S(\mu,d)^{\alpha\left(\frac{1}{p_1}-\frac{1}{p_2}\right)-\frac{1}{q_1}+\frac{1}{q_2}} \big[\log(2^{\mu-m}\mu^{\alpha}+1)\big]^{\frac{1}{p_1}-\frac{1}{p_2}}.$$

Substituting for 2^m from (3.29) we get

$$\Big(\sum_{j=1}^{S(\mu,d)} \epsilon_j^{q_2} \Big)^{\frac{1}{q_2}} \approx 2^{\mu\left(-r_1+\frac{1}{p_1}+r_2-\frac{1}{p_2}\right)} k^{\frac{1}{p_2}-\frac{1}{p_1}} \mu^{(d-1)\left(\frac{1}{p_1}-\frac{1}{p_2}+\frac{1}{q_2}-\frac{1}{q_1}\right)} \big[\log\big(\frac{\mu^{d-1}2^{\mu}}{k}+1\big)\big]^{\frac{1}{p_1}-\frac{1}{p_2}},$$

which finishes the proof. □

3.3 Main result

In this subsection we present our main results concerning sequence spaces. Our aim is to estimate the entropy numbers of

$$id : s^{\overline{r}_1,\Omega}_{p_1,q_1} a \to s^{\overline{r}_2,\Omega}_{p_2,q_2} a^{\dagger}. \tag{3.30}$$

First we split the identity (3.30) into a sum of identities between building blocks

$$id = \sum_{\mu=0}^{\infty} id_{\mu}, \quad id_{\mu} : s^{\overline{r}_1,\Omega}_{p_1,q_1} a \to s^{\overline{r}_2,\Omega}_{p_2,q_2} a^{\dagger}, \tag{3.31}$$

where

$$(id_{\mu}\lambda)_{\overline{\nu}\,\overline{m}} = \begin{cases} \lambda_{\overline{\nu}\,\overline{m}}, & \text{if } |\overline{\nu}| = \mu \\ 0, & \text{otherwise} \end{cases} \tag{3.32}$$

for all $\overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in A^{\Omega}_{\overline{\nu}}$.

Next we observe that

$$e_k(id_\mu) = e_k(id'_\mu), \quad k \in \mathbb{N}, \quad \mu \in \mathbb{N}_0, \tag{3.33}$$

where

$$id'_\mu : (s^{\overline{r}_1,\Omega}_{p_1,q_1}a)_\mu \to (s^{\overline{r}_2,\Omega}_{p_2,q_2}a^\dagger)_\mu, \quad \mu \in \mathbb{N}_0 \tag{3.34}$$

are the natural identities between our building blocks.

First, we characterise when the embedding (3.30) is compact.

Theorem 3.17. *Let*

$$\overline{r}_1 = (r_1, \ldots, r_1) \in \mathbb{R}^d, \overline{r}_2 = (r_2, \ldots, r_2) \in \mathbb{R}^d, \quad 0 < p_1, p_2, q_1, q_2 \leq \infty. \tag{3.35}$$

Then the embedding (3.30) *is compact if and only if*

$$\alpha = r_1 - r_2 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0. \tag{3.36}$$

Proof. Part 1.

In the first part we prove that (3.36) is sufficient for compactness of (3.30). First we restrict to the case

• $0 < p_1 \leq p_2 \leq \infty$ and $a = a^\dagger = b$.

It is an easy exercise to show that

$$||id_\mu|s^{\overline{r}_1,\Omega}_{p_1,q_1}b \to s^{\overline{r}_2,\Omega}_{p_2,q_2}b|| = ||id'_\mu|(s^{\overline{r}_1,\Omega}_{p_1,q_1}b)_\mu \to (s^{\overline{r}_2,\Omega}_{p_2,q_2}b)_\mu|| \leq 2^{-\mu(r_1-r_2+\frac{1}{p_2}-\frac{1}{p_1})}S(\mu,d)^{\left(\frac{1}{q_2}-\frac{1}{q_1}\right)_+},$$

where the number $S(\mu, d)$ was defined by (3.28). So, if (3.36) is satisfied, then we may approximate the operator id by finite ranks operators $P_j = \sum_{\mu=0}^{j} id_\mu$.

• $0 < p_1 \leq p_2 \leq \infty$

In this case we choose $\epsilon > 0$ such that

$$r_1 - r_2 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right) > 2\epsilon$$

and use following trivial embeddings

$$s^{\overline{r}_1,\Omega}_{p_1,q_1}a \to s^{\overline{r}_1-\epsilon,\Omega}_{p_1,q_1}b \to s^{\overline{r}_2+\epsilon,\Omega}_{p_2,q_2}b \to s^{\overline{r}_2,\Omega}_{p_2,q_2}a^\dagger.$$

All these embeddings are continuous, the middle one is even compact.

• $0 < p_2 \leq p_1 \leq \infty$.

Now we use the following line of embeddings

$$s^{\overline{r}_1,\Omega}_{p_1,q_1}a \to s^{\overline{r}_1,\Omega}_{p_2,q_1}a \to s^{\overline{r}_2,\Omega}_{p_2,q_2}a^\dagger.$$

We have already proven, that the second embedding is compact. As the first embedding is continuous, it finishes the proof of part 1.

Part 2. If (3.36) is *not* satisfied, we construct a sequence $\{e_\mu\}_{\mu=0}^\infty$ from the unit ball of $s^{\overline{r}_1,\Omega}_{p_1,q_1}a$ such that $||e_\mu - e_{\mu'}|s^{\overline{r}_2,\Omega}_{p_2,q_2}a^\dagger|| \geq c > 0$ for $\mu \neq \mu'$.

Let us start with the case $p_1 \le p_2$. For $\mu \in \mathbb{N}_0$ fixed, we choose one $\overline{\nu}_\mu \in \mathbb{N}_0^d$ with $|\overline{\nu}_\mu| = \mu$ and one $\overline{m}_\mu \in A^\Omega_{\overline{\nu}_\mu}$. Then we set

$$(e_\mu)_{\overline{\nu}\,\overline{m}} = \begin{cases} 2^{-\mu(r_1-1/p_1)} & \text{for} \quad \overline{\nu} = \overline{\nu}_\mu, \overline{m} = \overline{m}_\mu, \\ 0 & \text{otherwise.} \end{cases}$$

When $p_1 > p_2$ we fix again one $\overline{\nu}_\mu \in \mathbb{N}_0^d$ with $|\overline{\nu}_\mu| = \mu$ and define $(e_\mu)_{\overline{\nu}\,\overline{m}} = 2^{-\mu r_1}$ for $\overline{\nu} = \overline{\nu}_\mu$ and $\overline{m} \in A^\Omega_{\overline{\nu}_\mu}$ and $(e_\mu)_{\overline{\nu}\,\overline{m}} = 0$ otherwise. $\square$

It is our main task to estimate the decay of $e_k(id)$ for id given by (3.30) when this sequence tends to zero, it means when (3.36) is satisfied. First we get the estimates from below.

Theorem 3.18. *Let $\overline{r}_1, \overline{r}_2, p_1, p_2, q_1, q_2$ be given by (3.35) with (3.36). Then*

$$e_k(id : s^{\overline{r}_1,\Omega}_{p_1,q_1}a \to s^{\overline{r}_2,\Omega}_{p_2,q_2}a^\dagger) \ge c\, k^{r_2-r_1}(\log k)^{(d-1)\left(r_1-r_2+\frac{1}{q_2}-\frac{1}{q_1}\right)_+}, \quad k \ge 2, \tag{3.37}$$

where the constant c does not depend on k.

Proof. Step 1.

For every $\mu \in \mathbb{N}$ we consider the following diagram:

$$\begin{array}{ccc} (s^{\overline{r}_1,\Omega}_{p_1,q_1}a)_\mu & \xrightarrow{id'_\mu} & (s^{\overline{r}_2,\Omega}_{p_2,q_2}a^\dagger)_\mu \\ {\scriptstyle id_1}\downarrow & & \uparrow{\scriptstyle id_2} \\ s^{\overline{r}_1,\Omega}_{p_1,q_1}a & \xrightarrow{id} & s^{\overline{r}_2,\Omega}_{p_2,q_2}a^\dagger \end{array} \tag{3.38}$$

The meaning of id and id'_μ was explained by (3.30) – (3.34). id_1 extends a given finite sequence by zeros while id_2 is the identity restricted to the $\mu-$th building block. Hence

$$id_1(\{\lambda_{\overline{\nu}\,\overline{m}}\} : |\overline{\nu}| = \mu, \overline{m} \in A^\Omega_{\overline{\nu}}) = \\ (\{\gamma_{\overline{\nu}\,\overline{m}}\} : \gamma_{\overline{\nu}\,\overline{m}} = \lambda_{\overline{\nu}\,\overline{m}} \quad \text{for } |\overline{\nu}| = \mu \text{ and } \quad \gamma_{\overline{\nu}\,\overline{m}} = 0 \text{ otherwise})$$

and

$$id_2(\{\lambda_{\overline{\nu}\,\overline{m}}\} : \overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in A^\Omega_{\overline{\nu}}) = (\{\lambda_{\overline{\nu}\,\overline{m}}\} : |\overline{\nu}| = \mu).$$

For

$$k = 2D_\mu \tag{3.39}$$

we get by Lemma 3.13

$$c\,\mu^{\left(\frac{1}{q_2}-\frac{1}{q_1}\right)} 2^{\mu(r_2-r_1)} \le e_k(id'_\mu) \le ||id_1|| \cdot ||id_2|| \cdot e_k(id) = e_k(id).$$

If k is given by (3.39) we get $\mu \approx \log k$ and $2^\mu \approx \frac{k}{\log^{d-1} k}$. Hence

$$e_k(id) \ge c\, k^{r_2-r_1}(\log k)^{(d-1)\left(r_1-r_2+\frac{1}{q_2}-\frac{1}{q_1}\right)}.$$

By monotonicity, we extend this results to all $k \ge 2$.

Step 2. We repeat the same arguments with different building blocks. The diagram (3.38) is replaced by

$$\begin{array}{ccc} 2^{\mu\left(r_1-\frac{1}{p_1}\right)}\ell_{p_1}^{A_\mu} & \xrightarrow{id'_\mu} & 2^{\mu\left(r_2-\frac{1}{p_2}\right)}\ell_{p_2}^{A_\mu} \\ id_1 \big\downarrow & & \big\uparrow id_2 \\ s_{p_1,q_1}^{\overline{r}_1,\Omega}a & \xrightarrow{id} & s_{p_2,q_2}^{\overline{r}_2,\Omega}a^\dagger \end{array}, \tag{3.40}$$

where $A_\mu = \#(A_{\overline{\nu}}^{\Omega})$ for some $\overline{\nu}$ with $|\overline{\nu}| = \mu$. Instead of Lemma 3.13 we use Lemma 3.11 to get for $k = 2A_\mu$

$$c\,2^{\mu(r_2-r_1)} \le e_k(id'_\mu) \le ||id_1|| \cdot ||id_2|| \cdot e_k(id) = e_k(id).$$

Finally, we substitute $2^\mu \approx k$, get

$$e_k(id) \ge c\,k^{r_2-r_1}$$

and use monotonicity arguments to extend the result to all $k \ge 2$. $\square$

Theorem 3.19. *Let $\overline{r}_1, \overline{r}_2, p_1, p_2, q_1, q_2$ be given by* (3.35) *with* (3.36)*. If*

$$\alpha > V_1(p_1,q_1,p_2,q_2) := \frac{1}{\min(p_1,q_1)} - \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{\max(p_2,q_2)} \tag{3.41}$$

for $p_1 \le p_2$ and

$$\alpha > V_1(p_2,q_1,p_2,q_2) := \frac{1}{\min(p_2,q_1)} - \frac{1}{\max(p_2,q_2)} \tag{3.42}$$

for $p_1 > p_2$ then

$$e_k(id : s_{p_1,q_1}^{\overline{r}_1,\Omega}a \to s_{p_2,q_2}^{\overline{r}_2,\Omega}a^\dagger) \le c\,k^{r_2-r_1}(\log k)^{(d-1)\left(r_1-r_2+\frac{1}{q_2}-\frac{1}{q_1}\right)}, \quad k \ge 2, \tag{3.43}$$

where the constant c does not depend on k.

Proof. Step 1.

We restrict ourselves first to the case $p_1 \le p_2$.

We split id as indicated in (3.31)

$$id = \sum_{\mu=0}^{J} id_\mu + \sum_{\mu=J+1}^{L} id_\mu + \sum_{\mu=L+1}^{\infty} id_\mu,$$

where the numbers $J \le L$ shall be specified later on. Furthermore, we shall later define natural numbers k_μ, $\mu = 0,\ldots,L$ and $k = \sum_{\mu=0}^{L} k_\mu$. This will supply the fundamental estimate

$$e_k^\varrho(id) \le \sum_{\mu=0}^{J} e_{k_\mu}^\varrho(id_\mu) + \sum_{\mu=J+1}^{L} e_{k_\mu}^\varrho(id_\mu) + \sum_{\mu=L+1}^{\infty} ||id_\mu||^\varrho, \qquad \varrho = \min(1,p_2,q_2). \tag{3.44}$$

We recall that by (3.33) one may substitute $e_{k_\mu}(id_\mu)$ by $e_{k_\mu}(id'_\mu)$.

Step 2. Fix now $J \in \mathbb{N}$. We show how to choose the numbers L and k_μ (in dependence of J) and we estimate the three sums in (3.44).

We start with the last one. First we remark that

$$||id_\mu|| \le c\,2^{-\mu\alpha}\mu^{(d-1)\left(\frac{1}{q_2}-\frac{1}{q_1}\right)_+}, \qquad \mu \in \mathbb{N}$$

and

$$\sum_{\mu=L+1}^{\infty} ||id_\mu||^\varrho \le c \sum_{\mu=L+1}^{\infty} 2^{-\varrho\mu\alpha}\mu^{\varrho(d-1)\left(\frac{1}{q_2}-\frac{1}{q_1}\right)_+} \le c\,2^{-\varrho\alpha L}L^{\varrho(d-1)\left(\frac{1}{q_2}-\frac{1}{q_1}\right)_+},$$

Finally, we choose $L \ge J$ large such that the last expression may be estimated from above by

$$\sum_{\mu=L+1}^{\infty} ||id_\mu||^\varrho \le c\,2^{J(r_2-r_1)}J^{(d-1)\left(\frac{1}{q_2}-\frac{1}{q_1}\right)}.$$

Step 3. We estimate the first sum in (3.44). We define

$$k_\mu = 2D_\mu 2^{(J-\mu)\epsilon} \ge 2D_\mu, \quad \mu = 0, \ldots, J,$$

where ϵ is an arbitrary fixed number with $0 < \epsilon < 1$. Then we get

$$\sum_{\mu=0}^{J} k_\mu \approx J^{d-1}2^J. \tag{3.45}$$

By Lemma 3.13

$$e_{k_\mu}(id_\mu) \approx 2^{-2^{(J-\mu)\epsilon}}\mu^{(d-1)\left(\frac{1}{q_2}-\frac{1}{q_1}\right)}2^{\mu(r_2-r_1)}, \quad \sum_{\mu=0}^{J} e^\varrho_{k_\mu}(id_\mu) \approx J^{\varrho(d-1)\left(\frac{1}{q_2}-\frac{1}{q_1}\right)}2^{\varrho J(r_2-r_1)}. \tag{3.46}$$

Step 4. We estimate the second sum in (3.44). We set

$$k_\mu = 2D_\mu 2^{(J-\mu)\varkappa} \le 2D_\mu, \quad J+1 \le \mu \le L$$

where $\varkappa$ is chosen such that

$$\varkappa > 1, \quad r_1 - r_2 > \varkappa\left(\frac{1}{\gamma_1} - \frac{1}{\delta_2}\right). \tag{3.47}$$

γ_1 and δ_2 was defined by (3.18) and (3.19), respectively. Then we get

$$\sum_{\mu=J+1}^{L} k_\mu \approx J^{d-1}2^J. \tag{3.48}$$

By Lemma 3.15 we get for $e_{k_\mu}(id_\mu)$

$$e_{k_\mu}(id_\mu) \le c\,\mu^{(d-1)\left(\frac{1}{q_2}-\frac{1}{q_1}\right)}2^{\mu(r_2-r_1)}2^{(J-\mu)\varkappa\left(\frac{1}{\delta_2}-\frac{1}{\gamma_1}\right)}\left[\log\left(c2^{-(J-\mu)\varkappa}+1\right)\right]^{\frac{1}{\gamma_1}-\frac{1}{\delta_2}}.$$

By (3.47) we get

$$\sum_{\mu=J+1}^{L} e^\varrho_{k_\mu}(id_\mu) \approx J^{\varrho(d-1)\left(\frac{1}{q_2}-\frac{1}{q_1}\right)}2^{\varrho J(r_2-r_1)}. \tag{3.49}$$

Finally, we put (3.45), (3.48) together with (3.46) and (3.49) into (3.44) and obtain

$$e_{c_1 J^{d-1} 2^J}(id) \leq c_2\, J^{(d-1)\left(\frac{1}{q_2}-\frac{1}{q_1}\right)} 2^{J(r_2-r_1)}.$$

Substituting $k = c_1 J^{d-1} 2^J$ and using monotonicity arguments, we finish the proof of the theorem for $p_1 \leq p_2$.

Step 5. In the case $p_1 > p_2$ we use the chain of embeddings

$$s^{\overline{r}_1,\Omega}_{p_1,q_1} a \hookrightarrow s^{\overline{r}_1,\Omega}_{p_2,q_1} a \hookrightarrow s^{\overline{r}_2,\Omega}_{p_2,q_2} a^\dagger.$$

The first embedding is then continuous (as $p_1 > p_2$ and Ω is bounded), the second is covered by previous steps. Altogether, it finishes the proof. □

Remark 3.20. 1. One notices immediately a gap between (3.36) and (3.41). To eliminate this gap we use a complex interpolation method in the next chapter.

2. Lemma 3.16 allows us to reduce the gap a bit in a special case, where $a = a^\dagger = b$. If we use Lemma 3.16 instead of Lemma 3.15 in the Step 4. in the previous proof, we get the same result, namely (3.43), for

$$p_1 \leq p_2, \quad r_1 - r_2 + \frac{1}{p_2} - \frac{1}{p_1} > 0, \quad \frac{1}{p_1} - \frac{1}{p_2} > \frac{1}{q_1} - \frac{1}{q_2}.$$

4 Complex interpolation

In Theorem 3.18 we obtained an estimate from below for entropy numbers of the embedding

$$id : s^{\overline{r}_1,\Omega}_{p_1,q_1} a \rightarrow s^{\overline{r}_2,\Omega}_{p_2,q_2} a^\dagger. \tag{4.1}$$

The corresponding estimate from above was obtained in Theorem 3.19 for

$$\alpha = r_1 - r_2 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > \frac{1}{\min(p_1,p_2,q_1)} - \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{\max(p_2,q_2)}. \tag{4.2}$$

So for any p_1, p_2, q_1, q_2 we have one natural boundary for $r_1 - r_2$ which ensures compactness of (4.1), see Theorem 3.17, and a second one, in general larger and given by (4.2), where the estimates from above and from below for entropy numbers of (4.1) coincide. The main purpose of this chapter is to eliminate this gap using a complex interpolation method. We follow closely [20].

4.1 Abstract background

In this subsection we briefly describe the complex interpolation method of [20]. We quote only the minimum needed for our purpose.

We say that two quasi-Banach spaces X_0, X_1 form an *interpolation couple* (X_0, X_1) if there is a Hausdorff topological vector space X such that X_0 and X_1 are continuously embedded in X. Given an interpolation couple (X_0, X_1), we define the space $X_0 \cap X_1$ by

$$X_0 \cap X_1 = \{x \in X : ||x|X_0 \cap X_1|| < \infty\},$$

where

$$||x|X_0 \cap X_1|| = \max\{||x|X_0||, ||x|X_1||\}.$$

Similarly, we define the space $X_0 + X_1$ by

$$X_0 + X_1 = \{x \in X : ||x|X_0 + X_1|| < \infty\},$$

where

$$||x|X_0 + X_1|| = \inf\{||x_0|X_0|| + ||x_1|X_1|| : x = x_0 + x_1, x_j \in X_j, j = 0, 1\}.$$

It is easy to verify that $X_0 \cap X_1$ and $X_0 + X_1$ are quasi-Banach spaces, see for example [5] for details.

If X is a quasi-Banach space and $\Omega \subset \mathbb{C}$ is an open subset then $f : \Omega \to X$ is called *analytic* if for each $z_0 \in \Omega$ there exists $r > 0$ such that there is a power expansion $f(z) = \sum_{n=0}^{\infty} x_n z^n$, $x_n \in X$, converging uniformly for $|z - z_0| < r$.

Given an interpolation couple (X_0, X_1) of quasi-Banach spaces, we consider the class $\mathcal{F}$ of all functions f with values in $X_0 + X_1$, which are bounded and continuous on the strip

$$S = \{z \in \mathbb{C} : 0 \leq \text{Re } z \leq 1\},$$

and analytic in the open strip

$$S_0 = \{z \in \mathbb{C} : 0 < \text{Re } z < 1\},$$

and moreover, the functions $t \to f(j + it)(j = 0, 1)$ are bounded continuous functions into X_j.

We endow $\mathcal{F}$ with the quasinorm

$$||f|\mathcal{F}|| = \max\Big\{\sup_{t\in\mathbb{R}} ||f(it)|X_0||, \sup_{t\in\mathbb{R}} ||f(1 + it)|X_1||\Big\}. \tag{4.3}$$

Finally, we set

$$[X_0, X_1]_\theta := \Big\{x \in X_0 + X_1 : x = f(\theta) \text{ for some } f \in \mathcal{F}\Big\}, \quad 0 < \theta < 1.$$

This space is equipped with the quasinorm

$$||x|[X_0, X_1]_\theta|| := \inf\{||f|\mathcal{F}|| : f \in \mathcal{F}, f(\theta) = x\}, \quad x \in [X_0, X_1]_\theta.$$

As far as the classical complex interpolation theory of Peetre is considered, we refer again to [5] and references given there. However, it is well known, that the extension of this complex interpolation method to the quasi-Banach spaces is not possible due to the possible failure of the Maximum Modulus Principle in the quasi-Banach context. However, there is a significant class of quasi-Banach spaces, called *A–convex*, in which the Maximum Modulus Principle is valid. As far as the study of this class is concerned, see [20] and references given there for details. We quote only the minimum from this theory needed in the sequel.

Definition 4.1. A quasi-Banach space $(X, ||\cdot|X||)$ is called *A–convex* if there is a constant C such that for every polynomial $P : \mathbb{C} \to X$ we have

$$||P(0)|X|| \leq C \max_{|z|=1} ||P(z)|X||.$$

Next theorem shows that in the frame of A-convex quasi-Banach spaces the Maximum Modulus Principle holds.

Theorem 4.2. *For a quasi-Banach space $(X, ||\cdot|X||)$ the following conditions are equivalent:*

(i) *X is A–convex*

(ii) *there exists C such that*

$$\max\{||f(z)|X|| : z \in S_0\} \leq C \max\{||f(z)|X|| : z \in S \setminus S_0\}$$

for any function $f : S \to X$ analytic on S_0 and continuous and bounded on S.

In the special case when X_0 and X_1 are quasi-Banach lattices, it was observed by Calderón that the interpolation space $[X_0, X_1]_\theta$ coincides with the so-called *Calderón product* of spaces X_0 and X_1, usually denoted by $X_0^{1-\theta} X_1^\theta$. We quote again necessary definitions and corresponding theorems from [20].

First, let $(\mathfrak{X}, \mathcal{S}, \mu)$ be a $\sigma-$finite measure space and let $\mathfrak{M}$ be the class of all complex–valued, $\mu-$measurable functions on $\mathfrak{X}$. Then a quasi-Banach space $X \subset \mathfrak{M}$ is called *a quasi-Banach lattice of functions* if for every $f \in X$ and $g \in \mathfrak{M}$ with $|g(x)| \leq |f(x)|$ for $\mu-$a.e. $x \in \mathfrak{X}$ one has $g \in X$ with $||g|X|| \leq ||f|X||$.

Furthermore, a quasi-Banach lattice of functions $(X, ||\cdot|X||)$ is called *lattice r-convex* if

$$\Big|\Big|\Big(\sum_{j=1}^{m} |f_j|^r\Big)^{1/r} |X\Big|\Big| \leq \Big(\sum_{j=1}^{m} ||f_j|X||^r\Big)^{1/r}$$

for any finite family $\{f_j\}_{1 \leq j \leq m}$ of functions from X.

The following theorem gives a very simple condition for lattice of functions to be $A-$convex.

Theorem 4.3. *Let X be a complex quasi-Banach lattice of functions. Then the following assertions are equivalent*

(i) *X is A–convex*

(ii) *X is lattice r-convex for some $r > 0$.*

Finally, if $(X_j, ||\cdot|X_j||), j = 0, 1$ are quasi-Banach lattices of functions and $0 < \theta < 1$ then the *Calderón product* $X_0^{1-\theta} X_1^\theta$ is the function spaces defined by the quasinorm

$$||f|X_0^{1-\theta} X_1^\theta|| := \inf\Big\{||f_0|X_0||^{1-\theta} ||f_1|X_1||^\theta : |f| \leq |f_0|^{1-\theta} |f_1|^\theta, f_j \in X_j, j = 0, 1\Big\}.$$

The connection between complex interpolation and Calderón products is given by

Theorem 4.4. *Let $(\mathfrak{X}, \mathcal{S}, \mu)$ be a complete separable metric space, let μ be a $\sigma-$finite Borel measure on $\mathfrak{X}$, and let X_0, X_1 be a pair of quasi-Banach lattices of functions on $(\mathfrak{X}, \mu)$.*

Then if both X_0 and X_1 are A–convex and separable, it follows that $X_0 + X_1$ is A-convex and $[X_0, X_1]_\theta = X_0^{1-\theta} X_1^\theta, 0 < \theta < 1$.

As pointed out in [20] in the case of quasi-Banach sequence lattices, only one of the spaces in 4.4 must be separable.

4.2 Interpolation of $s^{\overline{r},\Omega}_{p,q}a$

Now we apply Theorem 4.4 to interpolate the sequence spaces $s^{\overline{r},\Omega}_{p,q}a$. First, we have to prove, that these spaces are A–convex. According to Theorem 4.3 it is enough to prove that they are *lattice s–convex* for some $s > 0$. Trivially, $s = \min(1, p, q)$ works fine in both b and f case.

Hence, it is enough to compute the Calderón products

$$(s^{\overline{r}_1,\Omega}_{p_1,q_1}a)^{1-\theta}(s^{\overline{r}_2,\Omega}_{p_2,q_2}a)^{\theta}, \quad 0 < \theta < 1.$$

The answer is given by

Theorem 4.5. *Let*

$$\overline{r}_1, \overline{r}_2 \in \mathbb{R}^d, \quad 0 < p_1, p_2, q_1, q_2 \leq \infty, \quad 0 < \theta < 1. \tag{4.4}$$

If $\overline{r}, p$ and q are given by

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad \overline{r} = (1-\theta)\overline{r}_1 + \theta\overline{r}_2, \tag{4.5}$$

we get

$$(s^{\overline{r}_1,\Omega}_{p_1,q_1}a)^{1-\theta}(s^{\overline{r}_2,\Omega}_{p_2,q_2}a)^{\theta} = s^{\overline{r},\Omega}_{p,q}a.$$

Proof. Step 1. First, let $\lambda \in s^{\overline{r},\Omega}_{p,q}a$ and $\lambda^j \in s^{\overline{r}_j,\Omega}_{p_j,q_j}a, j = 1, 2$ with

$$|\lambda_{\overline{\nu}\,\overline{m}}| \leq |\lambda^1_{\overline{\nu}\,\overline{m}}|^{1-\theta} \cdot |\lambda^2_{\overline{\nu}\,\overline{m}}|^{\theta}, \quad \overline{\nu} \in \mathbb{N}^d_0, \quad \overline{m} \in A^{\Omega}_{\overline{\nu}}. \tag{4.6}$$

We have to show that

$$||\lambda|s^{\overline{r},\Omega}_{p,q}a|| \leq ||\lambda^1|s^{\overline{r}_1,\Omega}_{p_1,q_1}a||^{1-\theta} \cdot ||\lambda^2|s^{\overline{r}_2,\Omega}_{p_2,q_2}a||^{\theta}.$$

But this is a simple exercise on Hölder's inequality in both b and f case.

Step 2. Now we prove the reverse inequality for $a = b$.

To $\lambda \in s^{\overline{r},\Omega}_{p,q}b$ given, we will find $\lambda^j \in s^{\overline{r}_j,\Omega}_{p_j,q_j}b, j = 1, 2$ with (4.6) such that

$$||\lambda|s^{\overline{r},\Omega}_{p,q}b|| = ||\lambda^1|s^{\overline{r}_1,\Omega}_{p_1,q_1}b||^{1-\theta} \cdot ||\lambda^2|s^{\overline{r}_2,\Omega}_{p_2,q_2}b||^{\theta}. \tag{4.7}$$

First we deal with the case $p_j, q_j < \infty, j = 1, 2$.

We choose

$$\lambda^j_{\overline{\nu}\,\overline{m}} = c^j_{\overline{\nu}}|\lambda_{\overline{\nu}\,\overline{m}}|^{p/p_j}, \quad j = 1, 2, \quad \overline{\nu} \in \mathbb{N}^d_0, \quad \overline{m} \in A^{\Omega}_{\overline{\nu}}, \tag{4.8}$$

where

$$c^j_{\overline{\nu}} = 2^{(\overline{\nu}\cdot\overline{r})\frac{q}{q_j}}2^{-\overline{\nu}\cdot\overline{r}_j}\Lambda_{\overline{\nu}}^{\frac{q}{q_j}-\frac{p}{p_j}}, \quad j = 1, 2, \quad \overline{\nu} \in \mathbb{N}^d_0, \tag{4.9}$$

and

$$\Lambda_{\overline{\nu}} = \Big(\sum_{\overline{m}\in A^{\Omega}_{\overline{\nu}}} |\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{1/p}, \quad \overline{\nu} \in \mathbb{N}^d_0. \tag{4.10}$$

(If $\Lambda_{\overline{\nu}} = 0$ for some $\overline{\nu} \in \mathbb{N}^d_0$ we set $c_{\overline{\nu}} = 0$.)

By this choice we see that

$$|\lambda^1_{\overline{\nu}\,\overline{m}}|^{1-\theta}\cdot|\lambda^2_{\overline{\nu}\,\overline{m}}|^{\theta}=2^{\overline{\nu}\cdot\overline{r}q[\frac{1-\theta}{q_1}+\frac{\theta}{q_2}]}2^{-\overline{\nu}\cdot\overline{r}_1(1-\theta)-\overline{\nu}\cdot\overline{r}_2\theta}\Lambda_{\overline{\nu}}^{q[\frac{1-\theta}{q_1}+\frac{\theta}{q_2}]-p[\frac{1-\theta}{p_1}+\frac{\theta}{p_2}]}|\lambda_{\overline{\nu}\,\overline{m}}|^{p[\frac{1-\theta}{p_1}+\frac{\theta}{p_2}]}=|\lambda_{\overline{\nu}\,\overline{m}}|.$$

This proves (4.6).

To prove (4.7) we use (4.8), (4.9) and (4.10) to get

$$||\lambda^j|s^{\overline{r}_j,\Omega}_{p_j,q_j}b||=\Big[\sum_{\overline{\nu}\in\mathbb{N}_0^d}2^{\overline{\nu}\cdot\overline{r}_jq_j}(c^j_{\overline{\nu}})^{q_j}\Big(\sum_{\overline{m}\in A^\Omega_{\overline{\nu}}}|\lambda_{\overline{\nu}\,\overline{m}}|^{\frac{p}{p_j}p_j}\Big)^{q_j/p_j}\Big]^{1/q_j}=\Big[\sum_{\overline{\nu}\in\mathbb{N}_0^d}2^{\overline{\nu}\cdot\overline{r}q}\Lambda^{\frac{q}{q_j}}_{\overline{\nu}}\Big]^{1/q_j}.$$

From this (4.7) follows immediately.

If $\max(p_1,q_1,p_2,q_2)=\infty$ only notational changes are necessary.

Step 3. As far as the f–case is considered, one may modify slightly the proof for sequence spaces $f^s_{p,q}$ given in [13], Theorem 8.2.

We start again with given $\lambda\in s^{\overline{r},\Omega}_{p,q}f$ and we need to find $\lambda^j\in s^{\overline{r}_j,\Omega}_{p_j,q_j}f$, $j=1,2$ with (4.6) such that

$$||\lambda^1|s^{\overline{r}_1,\Omega}_{p_1,q_1}f||^{1-\theta}||\lambda^2|s^{\overline{r}_2,\Omega}_{p_2,q_2}f||^{\theta}\le c||\lambda|s^{\overline{r},\Omega}_{p,q}f||. \tag{4.11}$$

First we deal with the case $q_j<\infty$, $j=1,2$.

For every $k\in\mathbb{Z}$, let

$$A_k=\Big\{x\in\mathbb{R}^d:\Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d,\overline{m}\in A^\Omega_{\overline{\nu}}}2^{\overline{\nu}\cdot\overline{r}q}|\lambda_{\overline{\nu}\,\overline{m}}|^q\chi_{\overline{\nu}\,\overline{m}}(x)\Big)^{1/q}>2^k\Big\}$$

and

$$C_k=\{(\overline{\nu},\overline{m}):|Q_{\overline{\nu},\overline{m}}\cap A_k|\ge\frac{|Q_{\overline{\nu}\,\overline{m}}|}{2}\text{ and }|Q_{\overline{\nu},\overline{m}}\cap A_{k+1}|<\frac{|Q_{\overline{\nu}\,\overline{m}}|}{2}\}.$$

We note that if $(\overline{\nu},\overline{m})\notin\cup_{k\in\mathbb{Z}}C_k$, then $\lambda_{\overline{\nu}\,\overline{m}}=0$.

We define the sequences λ^j, $j=1,2$ by

$$\lambda^1_{\overline{\nu}\,\overline{m}}=2^{k\gamma}2^{\overline{\nu}\cdot\overline{u}}|\lambda_{\overline{\nu}\,\overline{m}}|^{q/q_1}\quad\text{and}\quad\lambda^1_{\overline{\nu}\,\overline{m}}=\quad 2^{k\delta}2^{\overline{\nu}\cdot\overline{v}}|\lambda_{\overline{\nu}\,\overline{m}}|^{q/q_2},$$

where

$$\gamma=\frac{p}{p_1}-\frac{q}{q_1},\qquad\delta=\frac{p}{p_2}-\frac{q}{q_2}$$
$$\overline{u}=q\theta[\frac{\overline{r}_2}{q_1}-\frac{\overline{r}_1}{q_2}],\quad\overline{v}=q(1-\theta)[\frac{\overline{r}_1}{q_2}-\frac{\overline{r}_2}{q_1}]$$

if $(\overline{\nu},\overline{m})\in C_k$, and $\lambda^1_{\overline{\nu}\,\overline{m}}=\lambda^2_{\overline{\nu}\,\overline{m}}=0$ if $(\overline{\nu},\overline{m})\notin\cup_{k\in\mathbb{Z}}C_k$.

We point out that

$$(1-\theta)\gamma+\delta\theta=(1-\theta)\overline{u}+\theta\overline{v}=0.$$

An easy calculation shows that

$$|\lambda^1_{\overline{\nu}\,\overline{m}}|^{1-\theta}\cdot|\lambda^2_{\overline{\nu}\,\overline{m}}|^{\theta}=2^{k[(1-\theta)\gamma+\theta\delta]+\overline{\nu}\cdot[(1-\theta)\overline{u}+\theta\overline{v}]}|\lambda_{\overline{\nu}\,\overline{m}}|^{q\left(\frac{1-\theta}{q_1}+\frac{\theta}{q_2}\right)}=|\lambda_{\overline{\nu}\,\overline{m}}|.$$

In the sequel we assume that $\gamma\ge 0$, since the contrary case follows from interchanging $s^{\overline{r}_1,\Omega}_{p_1q_1}f$ with $s^{\overline{r}_2,\Omega}_{p_2q_2}f$ and θ with $1-\theta$.

We prove that

$$||\lambda^j|s^{\overline{r}_j,\Omega}_{p_jq_j}f|| \le c||\lambda|s^{\overline{r},\Omega}_{pq}f||^{p/p_j}, \quad j=1,2. \tag{4.12}$$

From this, (4.11) clearly follows.

To prove (4.12) for $j=1$ we write

$$\begin{aligned}||\lambda^1|s^{\overline{r}_1,\Omega}_{p_1q_1}f|| &= \left|\left|\left(\sum_{k=-\infty}^{\infty}\sum_{(\overline{\nu},\overline{m})\in C_k}|2^{\overline{\nu}\cdot\overline{r}_1}\lambda^1_{\overline{\nu}\,\overline{m}}|^{q_1}\chi_{\overline{\nu}\,\overline{m}}(x)\right)^{1/q_1}|L_{p_1}\right|\right| \\ &\le c\left|\left|\left(\sum_{k=-\infty}^{\infty}\sum_{(\overline{\nu},\overline{m})\in C_k}|2^{\overline{\nu}\cdot\overline{r}_1}\lambda^1_{\overline{\nu}\,\overline{m}}|^{q_1}\chi_{Q_{\overline{\nu}\,\overline{m}}\cap A_k}(x)\right)^{1/q_1}|L_{p_1}\right|\right|,\end{aligned}$$

where on the second line we use the definition of the set C_k and the boundedness of the maximal operator $\overline{M}$ as described by Theorem 1.11.

We denote

$$D_k = \bigcup_{l=-\infty}^{k} C_l$$

and continue

$$\begin{aligned}||\lambda^1|s^{\overline{r}_1,\Omega}_{p_1q_1}f|| &\le c\left|\left|\sum_{k=-\infty}^{\infty}\chi_{A_k\setminus A_{k+1}}(x)\left(\sum_{(\overline{\nu},\overline{m})\in D_k}|2^{\overline{\nu}\cdot\overline{r}_1}\lambda^1_{\overline{\nu}\,\overline{m}}|^{q_1}\chi_{\overline{\nu}\,\overline{m}}(x)\right)^{1/q_1}|L_{p_1}\right|\right| \\ &\le c\left|\left|\sum_{k=-\infty}^{\infty}\chi_{A_k\setminus A_{k+1}}(x)2^{k\gamma}\left(\sum_{(\overline{\nu},\overline{m})\in D_k}2^{\overline{\nu}\cdot\overline{r}_1q_1}2^{\overline{\nu}\cdot\overline{u}q_1}|\lambda_{\overline{\nu}\,\overline{m}}|^{q}\chi_{\overline{\nu}\,\overline{m}}(x)\right)^{1/q_1}|L_{p_1}\right|\right| \\ &\le c\left|\left|\sum_{k=-\infty}^{\infty}\chi_{A_k\setminus A_{k+1}}(x)2^{k\gamma}\left(\sum_{\overline{\nu}\in\mathbb{N}_0^d,\overline{m}\in A^{\Omega}_{\overline{\nu}}}2^{\overline{\nu}\cdot\overline{r}q}|\lambda_{\overline{\nu}\,\overline{m}}|^{q}\chi_{\overline{\nu}\,\overline{m}}(x)\right)^{1/q_1}|L_{p_1}\right|\right| \\ &\le c\left|\left|\left(\sum_{\overline{\nu}\in\mathbb{N}_0^d,\overline{m}\in A^{\Omega}_{\overline{\nu}}}2^{\overline{\nu}\cdot\overline{r}q}|\lambda_{\overline{\nu}\,\overline{m}}|^{q}\chi_{\overline{\nu}\,\overline{m}}(x)\right)^{\frac{p}{qp_1}}|L_{p_1}\right|\right| \\ &= c||\lambda|s^{\overline{r},\Omega}_{pq}f||^{p/p_1}.\end{aligned}$$

The second estimate in (4.12) is similar. □

After these preparations we are ready to present the main result of this section. Recall, that the spaces $S^{\overline{r}}_{p,q}A(\Omega)$ were defined by (3.1) and (3.2).

Theorem 4.6. *Let* $\overline{r}_j, p_j, q_j$ *for* $j=1,2$ *be given by* (4.4). *Let* $0<\theta<1$ *and define* $\overline{r}, p$ *and* q *by* (4.5). *Also suppose that* $\min(q_1,q_2)<\infty$.

(i) *Then*

$$[s^{\overline{r}_1,\Omega}_{p_1,q_1}b, s^{\overline{r}_2,\Omega}_{p_2,q_2}b]_\theta = s^{\overline{r},\Omega}_{p,q}b. \tag{4.13}$$

(ii) *Furthermore, if* $p_j<\infty, j=1,2$,

$$[s^{\overline{r}_1,\Omega}_{p_1,q_1}f, s^{\overline{r}_2,\Omega}_{p_2,q_2}f]_\theta = s^{\overline{r},\Omega}_{p,q}f. \tag{4.14}$$

Proof. The proof of (4.13) and (4.14) follows immediately from Theorem 4.4 and 4.5. □

4.3 Interpolation properties of entropy numbers.

Now we shall discuss the connection between the complex interpolation method developed above with entropy numbers. We use Theorem 1.3.2 from [10]. We recall that for $t > 0$, an interpolation couple (B_0, B_1) and $b \in B_0 + B_1$, the Peetre's $K-$functional is given by

$$K(t, b, B_0, B_1) = \inf\{||b_0|B_0|| + t||b_1|B_1|| : b = b_0 + b_1, b_0 \in B_0, b_1 \in B_1\}.$$

Theorem 4.7. (i) *Let A be a quasi-Banach space and let (B_0, B_1) be an interpolation couple of $p-$Banach spaces. Let $0 < \theta < 1$ and let B_θ be a quasi-Banach space such that $B_0 \cap B_1 \subset B_\theta \subset B_0 + B_1$ and*

$$||b|B_\theta|| \leq ||b|B_0||^{1-\theta} \cdot ||b|B_1||^\theta \quad \text{for all} \quad b \in B_0 \cap B_1.$$

Let $T \in L(A, B_0 \cap B_1)$. Then for all $k_0, k_1 \in \mathbb{N}$,

$$e_{k_0+k_1-1}(T : A \to B_\theta) \leq 2^{1/p} e_{k_0}^{1-\theta}(T : A \to B_0) e_{k_1}^\theta(T : A \to B_1).$$

(ii) *Let (A_0, A_1) be an interpolation couple of quasi-Banach spaces and let B be a $p-$Banach space. Let $0 < \theta < 1$ and let A be a quasi-Banach space such that $A \subset A_0 + A_1$ and*

$$t^{-\theta} K(t, a, A_0, A_1) \leq ||a|A|| \quad \text{for all} \quad a \in A \quad \text{and all} \quad t > 0.$$

Let $T : A_0 + A_1 \to B$ be linear and such that its restriction to A_0 and A_1 are continuous. Then its restriction to A is also continuous and for all $k_0, k_1 \in \mathbb{N}$,

$$e_{k_0+k_1-1}(T : A \to B) \leq 2^{1/p} e_{k_0}^{1-\theta}(T : A_0 \to B) e_{k_1}^\theta(T : A_1 \to B).$$

So, we only have to verify that the complex interpolation satisfies the assumptions of this theorem.

Theorem 4.8. *Let B_0, B_1 be an interpolation couple of A-convex quasi-Banach spaces and let $0 < \theta < 1$. Then*

(i)

$$||b|[B_0, B_1]_\theta|| \leq ||b|B_0||^{1-\theta} \cdot ||b|B_1||^\theta \quad \text{for all} \quad b \in B_0 \cap B_1.$$

(ii) *Let the functionals in B_i' separate the points of B_i, $i = 0, 1$. Then*

$$t^{-\theta} K(t, b, B_0, B_1) \leq ||b|[B_0, B_1]_\theta|| \text{ for all } b \in [B_0, B_1]_\theta \quad \text{and all} \quad t > 0.$$

Proof. Step 1. Fix $b \in B_0 \cap B_1$, set $M_j = ||b|B_j||$, $j = 0, 1$ and define $g(z) = M_0^{z-1} M_1^{-z} b$. Then $||g|\mathcal{F}|| = 1$ and

$$||M_0^{\theta-1} M_1^{-\theta} b|[B_0, B_1]_\theta|| \leq ||g(\theta)|[B_0, B_1]_\theta|| \leq 1.$$

This proves (i).

Step 2. One may follow [31], 1.10.3. There one may find a proof dealing with classical complex–interpolation method and Banach spaces. Nevertheless, the proof works also for the generalised method, as described above, and quasi-Banach sequence spaces. Especially, the Hahn–Banach Theorem needed there still holds for all sequence spaces which come to play. □

4.4 Filling the gaps

Now we use the complex interpolation and its relation to entropy numbers to close the gap mentioned in the beginning of Section 4. Namely, we are interested in those combination of "input" parameters which satisfy

$$V_1(p_1,q_1,p_2,q_2) := \frac{1}{\min(p_1,p_2,q_1)} - \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{\max(p_2,q_2)} \geq r_1 - r_2 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0. \quad (4.15)$$

Our main result on the sequence space level states

Theorem 4.9. *Let* $\overline{r}_j = (r_j,\dots,r_j) \in \mathbb{R}^d, 0 < p_j, q_j \leq \infty, j = 1,2$ *with*

$$r_1 - r_2 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0. \quad (4.16)$$

Furthermore, let $p_j < \infty$ *in the* $f-$*case.*

(i) *If* $r_1 - r_2 - \frac{1}{q_1} + \frac{1}{q_2} > 0$ *then*

$$e_k(id : s^{\overline{r}_1,\Omega}_{p_1,q_1}a \to s^{\overline{r}_2,\Omega}_{p_2,q_2}a) \approx k^{r_2-r_1}(\log k)^{(d-1)(r_1-r_2-\frac{1}{q_1}+\frac{1}{q_2})}, \quad k \geq 2.$$

(ii) *If* $r_1 - r_2 - \frac{1}{q_1} + \frac{1}{q_2} \leq 0$ *and* $\varepsilon > 0$ *then there are constants* c *and* C_ε *such that*

$$ck^{r_2-r_1} \leq e_k(id : s^{\overline{r}_1,\Omega}_{p_1,q_1}a \to s^{\overline{r}_2,\Omega}_{p_2,q_2}a) \leq C_\varepsilon k^{r_2-r_1}(\log k)^\varepsilon, \quad k \geq 2.$$

Remark 4.10. Unlike Theorems 3.18 and 3.19, this theorem deals only with embeddings which stay either in the b–scale or in the f–scale. We see also that this theorem closes the gap mentioned above up to the $(\log k)^\varepsilon$ term. Furthermore, the estimate from below is covered by Theorem 3.18. Hence we will concentrate on the estimates from above in the proof.

Proof. In the proof we shall distinguish several cases. First of all, we suppose that $p_1 \leq p_2$.

I. $p_1 \leq q_1, q_2 \leq p_2$. In this case the condition (4.15) is empty and the result is covered by Theorem 3.19.

II. $q_1 \leq p_1 \leq p_2 \leq q_2$. We start with the *sub*-case

IIa. $r_1 - r_2 - \frac{1}{q_1} + \frac{1}{q_2} > 0$.

In this case we have

$$r_1 - r_2 - \frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{q_1} - \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{q_2} = V_1(p_1,q_1,p_2,q_2)$$

and the result is again provided by Theorem 3.19.

IIb. $r_1 - r_2 - \frac{1}{q_1} + \frac{1}{q_2} \leq 0$

The second subcase *IIb.* introduces the $\log^\varepsilon$–gap. So we fix $\varepsilon > 0$ and use the following embedding

$$s^{\overline{r}_1,\Omega}_{p_1,q_1}a \hookrightarrow s^{\overline{r}_1,\Omega}_{p_1,q}a \hookrightarrow s^{\overline{r}_2,\Omega}_{p_2,q'}a \hookrightarrow s^{\overline{r}_2,\Omega}_{p_2,q_2}a. \quad (4.17)$$

The newly introduced indices q, q' are supposed to satisfy following conditions

$$\begin{aligned} &0 < q_1 \leq q \leq p_1 \leq p_2 \leq q' \leq q_2 \leq \infty, \\ &\frac{1}{p_1} - \frac{1}{p_2} < \frac{1}{q} - \frac{1}{q'} < r_1 - r_2 < \frac{1}{q} - \frac{1}{q'} + \varepsilon. \end{aligned} \quad (4.18)$$

The existence of these indices follows from (4.16) and the condition *IIb.* Hence we may apply the step *IIa.* to the middle embedding in (4.17). All the other embeddings are bounded which gives finally

$$e_k(id) \le ck^{r_2-r_1}(\log k)^{\varepsilon}$$

III. $q_1 < p_1, q_2 < p_2$.

We make the same splitting as in the case *II.* to the subcases *IIIa.* and *IIIb.*

IIIa. $r_1 - r_2 - \frac{1}{q_1} + \frac{1}{q_2} > 0$.

We choose $0 < \theta < 1$ such that

$$r_1 - r_2 - \frac{1}{q_1} + \frac{1}{q_2} > (1-\theta)\Big(\frac{1}{q_2} - \frac{1}{p_2}\Big) > 0. \tag{4.19}$$

and use the interpolation scheme

$$\begin{array}{l} s^{\overline{r},\Omega}_{p,q}a \ \searrow \\ s^{\overline{r}_1,\Omega}_{p_1,q_1}a \to s^{\overline{r}_2,\Omega}_{p_2,q_2}a \\ s^{\overline{r}_2,\Omega}_{p_2,q_2}a \ \nearrow \end{array} \tag{4.20}$$

with corresponding equations for r, p and q.

$$r_1 = (1-\theta)r + \theta r_2, \tag{4.21}$$

$$\frac{1}{p_1} = \frac{1-\theta}{p} + \frac{\theta}{p_2}, \tag{4.22}$$

$$\frac{1}{q_1} = \frac{1-\theta}{q} + \frac{\theta}{q_2}. \tag{4.23}$$

We have to verify that

$$r - r_2 - \frac{1}{p} + \frac{1}{p_2} > V_1(p, q, p_2, q_2). \tag{4.24}$$

If $q \ge p$, then $V_1(p, q, p_2, q_2) = 0$ and (4.24) is equivalent to (4.16). (One makes use of trivial calculation

$$(1-\theta)\Big(\frac{1}{p} - \frac{1}{p_2}\Big) = \frac{1}{p_1} - \frac{1}{p_2} \tag{4.25}$$

which follows directly from (4.22).)

If $q \le p$, then $V_1(p, q, p_2, q_2) = \frac{1}{q} - \frac{1}{p}$ and (4.24) is equivalent to (4.19).

In both cases $q \le p$, $q \ge p$ we may apply Theorem 3.19 to the upper embedding in (4.20). This leads to

$$e_k(id) \le c\Big(k^{r_2-r}(\log k)^{r-r_2-\frac{1}{q}+\frac{1}{q_2}}\Big)^{1-\theta} = ck^{r_2-r_1}(\log k)^{r_1-r_2-\frac{1}{q_1}+\frac{1}{q_2}}.$$

We have used the analogy of (4.25) for q's and r's.

Let us also mention that the condition $\min(q, q_2) < \infty$ needed to apply Theorem 4.6 is in the case *III.* always satisfied.

In the case *IIIb*, $r_1 - \frac{1}{q_1} + \frac{1}{q_2} \le 0$ ($\Longrightarrow$ $q_1 \le q_2$), we use the chain of embeddings (4.17) with (4.18) and

$$q_1 \le q \le p_1, \qquad q' = q_2.$$

Applying now the step *IIIa.* to the middle embedding we get the same result as in the case *IIb.*

IV. $p_1 < q_1, p_2 < q_2, p_1 \le p_2$.

As this case is dual to the the third case, we proceed in the same way.

IVa. $r_1 - \frac{1}{q_1} + \frac{1}{q_2} > 0$.

We choose $0 < \theta < 1$ such that

$$r_1 - r_2 - \frac{1}{p_1} + \frac{1}{p_2} > (1-\theta)\Big(\frac{1}{p_2} - \frac{1}{q_2}\Big) > 0. \tag{4.26}$$

Now we apply the interpolation scheme (4.20) with corresponding equations (4.21)–(4.23). We have to verify that

$$r - \frac{1}{p} + \frac{1}{p_2} > V_1(p, q, p_2, q_2). \tag{4.27}$$

If $q \ge p$, then $V_1(p, q, p_2, q_2) = \frac{1}{p_2} - \frac{1}{q_2}$ and (4.27) is equivalent to (4.26).

If $q \le p$, then $V_1(p, q, p_2, q_2) = \frac{1}{q} - \frac{1}{p} + \frac{1}{p_2} - \frac{1}{q_2}$ and (4.27) is equivalent to the condition IVa..

In both cases $q \le p$, $q \ge p$ we may apply Theorem 3.19 to the upper embedding in (4.20). This leads again to

$$e_k \le c\Big(k^{-r}(\log k)^{r - \frac{1}{q} + \frac{1}{q_2}}\Big)^{1-\theta} = ck^{-r_1}(\log k)^{r_1 - \frac{1}{q_1} + \frac{1}{q_2}}.$$

This finishes the discussion of the case *IVa.* as far as $\min(q, q_2) < \infty$ which is equivalent to $\min(q_1, q_2) < \infty$. If $q_1 = q_2 = \infty$ then we have to modify the arguments given above. In this case there is in general no hope to identify the interpolation space $[s^{\overline{r}_1,\Omega}_{p_1,\infty}a, s^{\overline{r}_2,\Omega}_{p_2,\infty}a]_\theta$ with the corresponding Calderón product $s^{\overline{r},\Omega}_{p,\infty}a$. But, according to [16], IV.1.11, one embedding still holds, namely

$$[s^{\overline{r}_1,\Omega}_{p_1,\infty}a, s^{\overline{r}_2,\Omega}_{p_2,\infty}a]_\theta \to s^{\overline{r},\Omega}_{p,\infty}a.$$

So we may use following interpolation schema

$$\begin{array}{ccc} & \nearrow s^{\overline{r},\Omega}_{p,\infty}a & \\ s^{\overline{r}_1,\Omega}_{p_1,\infty}a \to & [s^{\overline{r},\Omega}_{p,\infty}a, s^{\overline{r}_1,\Omega}_{p_1,\infty}a]_\theta & \to s^{\overline{r}_2,\Omega}_{p_2,\infty}a \\ & \searrow s^{\overline{r}_1,\Omega}_{p_1,\infty}a, & \end{array}$$

where p and r is given by (4.22) and (4.21). Then the choice of $0 < \theta < 1$ with

$$r_1 - r_2 > (1-\theta)\frac{1}{p_1}$$

ensures that we may proceed as in the Step *IIIa* and get the same result.

In the case *IVb*, $r_1 - \frac{1}{q_1} + \frac{1}{q_2} \le 0$ ($\Longrightarrow$ $q_1 \le q_2$), we use the chain of embeddings (4.17) with (4.18) and

$$q_1 = q, \qquad p_2 \le q' \le q_2.$$

Applying now the step *IVa.* to the middle embedding we get the same result as in the case *IIb.* □

4.5 Entropy numbers - conclusion

In the second section we have developed a strong tool connecting the function spaces $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ with sequence spaces $s^{\overline{r}}_{p,q}a$. In the third and fourth section we have studied the entropy numbers of embeddings of these sequence spaces. Finally, we combine these two concepts and obtain estimates for entropy numbers of embeddings of function spaces.

We recall that the function spaces on domains were defined by (3.1) and (3.2).

Our main result reads

Theorem 4.11. *Let Ω be a bounded domain in $\mathbb{R}^d$ with $d \geq 2$. Let $0 < p_1, q_1, p_2, q_2 \leq \infty$ with $p_1, p_2 < \infty$ in the $F-$case. Let $\overline{r}_i = (r_i, \ldots, r_i) \in \mathbb{R}^d, i = 1, 2$.*

(i) *The embedding*

$$id : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A^{\dagger}(\Omega) \tag{4.28}$$

is compact if and only if

$$r_1 - r_2 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0. \tag{4.29}$$

(ii) *In that case*

$$e_k(id : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A^{\dagger}(\Omega)) \geq c\, k^{r_2-r_1}(\log k)^{(d-1)(r_1-r_2+\frac{1}{q_2}-\frac{1}{q_1})_+}, \quad k \geq 2. \tag{4.30}$$

with c independent of k.

(iii) *If $A = A^{\dagger} = B$ or $A = A^{\dagger} = F$ and $r_1 - r_2 - \frac{1}{q_1} + \frac{1}{q_2} > 0$ then*

$$e_k(id : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A^{\dagger}(\Omega)) \leq c\, k^{r_2-r_1}(\log k)^{(d-1)(r_1-r_2+\frac{1}{q_2}-\frac{1}{q_1})}, \quad k \geq 2. \tag{4.31}$$

with c independent of k.

(iv) *If $A = A^{\dagger} = B$ or $A = A^{\dagger} = F$ and $r_1 - r_2 - \frac{1}{q_1} + \frac{1}{q_2} \leq 0$ then for every $\varepsilon > 0$ there is a constant $c_\varepsilon > 0$ such that*

$$e_k(id : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A^{\dagger}(\Omega)) \leq c_\varepsilon\, k^{r_2-r_1}(\log k)^{\varepsilon}, \quad k \geq 2. \tag{4.32}$$

(v) *For general $A, A^{\dagger}$ and*

$$r_1 - r_2 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > V_1(\min(p_1,p_2), q_1, p_2, q_2)$$

we get finally

$$e_k(id : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A^{\dagger}(\Omega)) \leq c\, k^{r_2-r_1}(\log k)^{(d-1)(r_1-r_2+\frac{1}{q_2}-\frac{1}{q_1})}, \quad k \geq 2.$$

Proof. Step 1. First we give some notation. If $f \in S^{\overline{r}_1}_{p_1,q_1}A(\Omega)$ then according to Definition 3.1 there is a function $g \in S^{\overline{r}_1}_{p_1,q_1}A(\mathbb{R}^d)$ such that

$$||g|S^{\overline{r}_1}_{p_1,q_1}A(\mathbb{R}^d)|| \leq 2||f|S^{\overline{r}_1}_{p_1,q_1}A(\Omega)||$$

with $g|\Omega = f$. We denote this function $g = extf$. Hence ext represents a (non-linear) bounded operator

$$ext : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_1}_{p_1,q_1}A(\mathbb{R}^d).$$

On the other hand, the natural restriction of $g \in S^{\overline{r}_1}_{p_1,q_1}A(\mathbb{R}^d)$ to $D'(\Omega)$ represents a linear bounded operator denoted by tr_Ω

$$tr_\Omega : S^{\overline{r}_1}_{p_1,q_1}A(\mathbb{R}^d) \to S^{\overline{r}_1}_{p_1,q_1}A(\Omega).$$

Step 2. To prove the first statement we introduce two diagrams which shall be of use even later on. In the first one, we start with $f \in S^{\overline{r}_1}_{p_1,q_1}A(\Omega)$ and extend it to $g = extf \in S^{\overline{r}_1}_{p_1,q_1}A(\mathbb{R}^d)$. Then we apply the wavelet decomposition to g as described in 2.12. This allows us to represent g in the form

$$g = \sum_{\overline{\nu}\in\mathbb{N}_0^d, \overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}} \Psi_{\overline{\nu}\,\overline{m}}. \tag{4.33}$$

In this way, we obtain a sequence $\lambda = \{\lambda_{\overline{\nu}\,\overline{m}} : \overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d\} \in s^{\overline{r}_1}_{p_1,q_1}a$. According to Theorem 2.12, the mapping which orders to a given function g its wavelet coefficients λ (and which shall be denoted by $\mathcal{W}$) is bounded

$$\mathcal{W} : S^{\overline{r}_1}_{p_1,q_1}A(\mathbb{R}^d) \to s^{\overline{r}_1}_{p_1,q_1}a.$$

As the distribution g doesn't need to have a bounded support, we restrict the sum in (4.33) to those $\overline{m} \in \mathbb{Z}^d$ such that $\operatorname{supp} \Psi_{\overline{\nu}\,\overline{m}} \cap \Omega \neq \emptyset$. Furthermore, we may always find a domain Ω' such that

$$\{\overline{m} \in \mathbb{Z}^d : \operatorname{supp} \Psi_{\overline{\nu}\,\overline{m}} \cap \Omega \neq \emptyset\} \subset A^{\Omega'}_{\overline{\nu}}, \quad \overline{\nu} \in \mathbb{N}_0^d.$$

This natural restriction will be formally realised by the the operator

$$id' : s^{\overline{r}_1}_{p_1,q_1}a \to s^{\overline{r}_1,\Omega'}_{p_1,q_1}a.$$

Finally, given a sequence $\lambda \in s^{\overline{r}_2,\Omega'}_{p_2,q_2}a^\dagger$, we denote by $S(\lambda)$ the distribution which arise as a wavelet sum with coefficients $\lambda_{\overline{\nu}\,\overline{m}}$.

$$S(\lambda) = \sum_{\overline{\nu}\in\mathbb{N}_0^d, \overline{m}\in A^{\Omega'}_{\overline{\nu}}} \lambda_{\overline{\nu}\,\overline{m}} \Psi_{\overline{\nu}\,\overline{m}}.$$

Using all this information we obtain the commutative diagram

$$\begin{array}{ccccccc} S^{\overline{r}_1}_{p_1,q_1}A(\Omega) & \xrightarrow{ext} & S^{\overline{r}_1}_{p_1,q_1}A(\mathbb{R}^d) & \xrightarrow{\mathcal{W}} & s^{\overline{r}_1}_{p_1,q_1}a & \xrightarrow{id'} & s^{\overline{r}_1,\Omega'}_{p_1,q_1}a \\ id_1 \Big\downarrow & & & & & & id_2 \Big\downarrow \\ S^{\overline{r}_2}_{p_2,q_2}A^\dagger(\Omega) & \xleftarrow{tr_\Omega} & S^{\overline{r}_2}_{p_2,q_2}A^\dagger(\mathbb{R}^d) & \xleftarrow{S} & & & s^{\overline{r}_2,\Omega'}_{p_2,q_2}a^\dagger \end{array} \tag{4.34}$$

All the operators involved are bounded, under hypothesis (4.29) the embedding id_2 is even compact. This proves that the condition (4.29) is sufficient for compactness of (4.28).

To prove that this condition is also necessary, we follow the reasoning given in the proof of Theorem 3.17. Suppose, that (4.29) is *not* satisfied. We shall construct a sequence $\{f_\mu\}$ bounded in $S^{\overline{r}_1}_{p_1,q_1}A(\Omega)$ such that each its two different members have mutual distance measured in $S^{\overline{r}_2}_{p_2,q_2}A^\dagger(\Omega)$ greater than some constant $c > 0$.

If $p_1 \leq p_2$, we proceed in this way: for every $\mu \geq \mu'$ there is $\overline{\nu}_\mu$ and $\overline{m}_\mu$ with $|\overline{\nu}_\mu| = \mu$ and $CQ_{\overline{\nu}_\mu,\overline{m}_\mu} \subset \Omega$. We set

$$f_\mu = \Psi_{\overline{\nu}_\mu \overline{m}_\mu}, \ \mu \geq \mu'.$$

If $p_1 > p_2$, we choose for every $\mu \geq \mu''$ some $\overline{\nu}_\mu$ with $|\overline{\nu}_\mu| = \mu$ and such that $\#\{\overline{m} \in \mathbb{Z}^d : CQ_{\overline{\nu}_\mu \overline{m}} \subset \Omega\} \approx 2^\mu$. Then we set

$$f_\mu = \sum_{\overline{m}: CQ_{\overline{\nu}_\mu \overline{m}} \subset \Omega} \Psi_{\overline{\nu}_\mu \overline{m}}, \ \mu \geq \mu''.$$

Step 3. Till now we have used (4.34) only to prove the compactness of (4.28). But one may use it also for the estimates of entropy numbers of (4.28). This gives

$$e_k(id_1) \leq c e_k(id_2), \quad k \in \mathbb{N},$$

where the constant c covers all the bounded operators $ext, \mathcal{W}, id', S$ and tr_Ω. This allows us to overtake the estimate from above obtained on the sequence space level to the function space level.

Step 4. Now we prove the estimate from below, namely (4.30). To this effect we consider sets

$$B_{\overline{\nu}}^\Omega = \{\overline{m} \in \mathbb{Z}^d : CQ_{\overline{\nu}\overline{m}} \subset \Omega\}, \quad \overline{\nu} \in \mathbb{N}_0^d.$$

These sets form a certain counterpart to $A_{\overline{\nu}}^\Omega$. There are, nevertheless, some important differences. One notices that we cannot hope for a straightforward equivalence of (3.7). Instead of that, we see that there are constants μ_0, c_1 and c_2 such that for every $\mu > \mu_0$ the cardinality of the set

$$\{\overline{\nu} : |\overline{\nu}| = \mu, c_1\, 2^\mu \leq \#(B_{\overline{\nu}}^\Omega) \leq c_2\, 2^\mu\}$$

is equivalent to μ^{d-1}. It means that (3.7) doesn't hold in general for all $\nu \in \mathbb{N}_0^d$ but only for almost all $\overline{\nu}$ with $|\overline{\nu}|$ large enough.

Following the proof of Theorem 3.18 we have to choose two kinds of building blocks. In the first case, we use the sequence spaces given by the quasinorm

$$||\lambda|(s_{p,q}^{\overline{r},\Omega} b)'_\mu|| = \left(\sum_{|\overline{\nu}|=\mu} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q} \Big(\sum_{\overline{m} \in B_{\overline{\nu}}^\Omega} |\lambda_{\overline{\nu}\overline{m}}|^p \Big)^{q/p} \right)^{1/q}$$

and

$$||\lambda|(s_{p,q}^{\overline{r},\Omega} f)'_\mu|| = \left|\left| \left(\sum_{|\overline{\nu}|=\mu} \sum_{\overline{m} \in B_{\overline{\nu}}^\Omega} |2^{\overline{\nu}\cdot\overline{r}} \lambda_{\overline{\nu}\overline{m}} \chi_{\overline{\nu}\overline{m}}(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^d) \right|\right|.$$

To estimate the entropy numbers of

$$e_k(id : (s_{p_1,q_1}^{\overline{r}_1,\Omega} a)'_\mu \to (s_{p_2,q_2}^{\overline{r}_2,\Omega} a^\dagger)'_\mu)$$

for $\mu \geq \mu_0$ large enough one may use the same arguments (and get the same results) as presented in Lemma 3.13.

Hence for $\mu \geq \mu_0$ we use the diagram (with $k = \mu^{d-1} 2^\mu$)

$$\begin{array}{ccc} (s_{p_1,q_1}^{\overline{r}_1,\Omega} a)'_\mu & \xrightarrow{S} & S_{p_1,q_1}^{\overline{r}_1} A(\Omega) \\ id_1 \Big\downarrow & & id_2 \Big\downarrow \\ (s_{p_2,q_2}^{\overline{r}_2,\Omega} a^\dagger)'_\mu & \xleftarrow{\mathcal{W}} & S_{p_2,q_2}^{\overline{r}_2} A^\dagger(\Omega) \end{array} \tag{4.35}$$

to get

$$e_k(id : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A^\dagger(\Omega)) \geq c\, k^{r_2-r_1}(\log k)^{(d-1)(r_1-r_2+\frac{1}{q_2}-\frac{1}{q_1})}, \quad k \geq 2.$$

On the other hand, the diagram (and the choice $k = 2^\mu$)

$$\begin{array}{ccc} 2^{\mu\left(r_1-\frac{1}{p_1}\right)}\ell^{B_\mu}_{p_1} & \xrightarrow{\ S\ } & S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \\ id_1 \Big\downarrow & & id_2 \Big\downarrow \\ 2^{\mu\left(r_2-\frac{1}{p_2}\right)}\ell^{B_\mu}_{p_2} & \xleftarrow{\ \mathcal{W}\ } & S^{\overline{r}_2}_{p_2,q_2}A^\dagger(\Omega) \end{array} \tag{4.36}$$

gives

$$e_k(id : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A^\dagger(\Omega)) \geq c\, k^{r_2-r_1}, \quad k \geq 2.$$

Here $B_\mu = \#(B^\Omega_{\overline{\nu}})$ for some $\overline{\nu}$ with $|\overline{\nu}| = \mu$ is chosen such that $B_\mu \approx \mu^{d-1}2^\mu$, $\mu \geq \mu_0$.

Step 5. The proof of (v) involves the same arguments as given in previous Steps and Theorem 3.19. □

Remark 4.12. Theorem 4.11 describes in detail the entropy numbers of

$$id : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A^\dagger(\Omega)$$

if $A = A^\dagger$. In this case it gives (up to the $(\log k)^\epsilon$-gap) the final answer. Let us look a bit closer on the situation where $A = B$ and $A^\dagger = F$. The estimate from below is covered by (4.30). If $q_1 \leq p_1$ we may use the embeddings

$$S^{\overline{r}_1}_{p_1,q_1}B(\Omega) \hookrightarrow S^{\overline{r}_1}_{p_1,q_1}F(\Omega) \hookrightarrow S^{\overline{r}_2}_{p_2,q_2}F(\Omega) \tag{4.37}$$

to overtake the results obtained for $F \hookrightarrow F$ embedding also to $B \hookrightarrow F$. If $q_2 \leq p_2$, we replace (4.37) by

$$S^{\overline{r}_1}_{p_1,q_1}B(\Omega) \hookrightarrow S^{\overline{r}_2}_{p_2,q_2}B(\Omega) \hookrightarrow S^{\overline{r}_2}_{p_2,q_2}F(\Omega). \tag{4.38}$$

But if $p_1 < q_1$ and $p_2 < q_2$ (and, for simplicity, $p_1 \leq p_2$), no trivial embedding would help. In that case we get (4.31) only for

$$r_1 - r_2 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right) > \frac{1}{p_2} - \frac{1}{q_2}.$$

In the case of $A = F$ and $A^\dagger = B$ the situation is similar. We may get (4.31) whenever (4.37) is compact and $p_1 \leq q_1$ or $p_2 \leq q_2$. If $q_1 < p_1, q_2 < p_2$ and $p_1 \leq p_2$, we get the same result only for

$$r_1 - r_2 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right) > \frac{1}{q_1} - \frac{1}{p_1}.$$

4.6 Comparison with known results

As the function spaces with dominating mixed smoothness have been studied systematically by many authors, there are also many important results on the estimates of the decay of entropy numbers available in the literature. Here, we compare our results supplied by decomposition techniques with those ones obtained by Belinsky [4], Temlyakov [30] and Dinh Dung [8].

Unfortunately, the classes of functions studied by them differ slightly from the scales $S^{\overline{r}}_{p,q}B(\Omega)$ and $S^{\overline{r}}_{p,q}F(\Omega)$. Let us sketch briefly their setting. They consider 1-periodic functions of d real variables. Hence, their domain Ω is fixed $\Omega = [0,1)^d$. Belinsky considered four main scales of spaces with dominating mixed smoothness, $W^{\overline{r}}_p, H^{\overline{r}}_p$ on the one hand and $L_p, B^0_{\infty,1}$ on the other hand.

As far as $1 < p < \infty$, the space L_p of periodic functions is a direct counterpart of $S^0_{p,2}F(\Omega)$. Similarly, $B^0_{\infty,1}$ is called $S^0_{\infty,1}B(\Omega)$ in our terminology. The spaces $W^{\overline{r}}_p$ defined by Belinsky by the means of Weyl derivatives represent for $1 < p < \infty$ the Sobolev spaces of dominating mixed smoothness $S^{\overline{r}}_{p,2}F(\Omega)$ and, finally, the spaces $H^{\overline{r}}_p$ are sometimes called Nikol'skij spaces and have their counterpart in $S^{\overline{r}}_{p,\infty}B(\Omega)$. To simplify the comparison of our results with those one of Belinsky, we denote the spaces $W^{\overline{r}}_p, H^{\overline{r}}_p, L_p$ and $B^0_{\infty,1}$ by $\widetilde{S}^{\overline{r}}_{p,2}F, \widetilde{S}^{\overline{r}}_{p,\infty}B, \widetilde{S}^0_{p,2}F$ and $\widetilde{S}^0_{\infty,1}B$.

We now quote four results of Belinsky and compare them with corresponding analogy obtained by our method earlier. We set the smoothness involved to be (as in our case) $\overline{r} = (r, \dots, r) \in \mathbb{R}^d$ although the results are presented in a bit greater generality in [4].

Theorem 4.13. (i) *Let $r > 1/p - 1/q$, and $1 < p \le q < \infty$. Then*

$$e_k(id : \widetilde{S}^{\overline{r}}_{p,2}F \to \widetilde{S}^0_{q,2}F) \approx \Big(\frac{\log^{d-1} k}{k}\Big)^r. \tag{4.39}$$

(ii) *Let $r > 1/p - 1/q$, and $1 < p \le q < \infty$. Then*

$$e_k(id : \widetilde{S}^{\overline{r}}_{p,\infty}B \to \widetilde{S}^0_{q,2}F) \approx \Big(\frac{\log^{d-1} k}{k}\Big)^r \log^{\frac{d-1}{2}} k. \tag{4.40}$$

(iii) *Let $r > 1/2$. Then*

$$e_k(id : \widetilde{S}^{\overline{r}}_{2,2}F \to \widetilde{S}^0_{\infty,1}B) \approx \Big(\frac{\log^{d-1} k}{k}\Big)^r \log^{\frac{d-1}{2}} k. \tag{4.41}$$

(iv) *Let $r > 1/2$. Then*

$$e_k(id : \widetilde{S}^{\overline{r}}_{2,\infty}B \to \widetilde{S}^0_{\infty,1}B) \approx \Big(\frac{\log^{d-1} k}{k}\Big)^r \log^{d-1} k. \tag{4.42}$$

Remark 4.14. We point out that according to Theorem 3.17, all the bounds for r in Theorem 4.13 are optimal. Due to Theorem 4.11, we achieved the same results as in (i), (iii) and (iv). The embedding appearing in (4.40) corresponds to

$$id : S^{\overline{r}}_{p,\infty}B(\Omega) \to S^0_{q,2}F(\Omega)$$

in our setting. In this case, we get for

$$r - \Big(\frac{1}{p} - \frac{1}{q}\Big) > V_1(p, \infty, q, 2) = \frac{1}{q} - \frac{1}{\max(q,2)}$$

by Theorem 4.11

$$e_k(id) \le c\, k^{-r} (\log k)^{(d-1)(r+\frac{1}{2})}, \quad k \ge 2.$$

So, for $q \ge 2$, our result is optimal for all possible r, but for $q < 2$ we get the optimal result only for $r > \frac{1}{p} - \frac{1}{2} > \frac{1}{p} - \frac{1}{q}$.

In [30], Temlyakov obtained other important results on entropy numbers of embeddings of spaces with dominating mixed smoothness. Using our notation, they maybe summarised as follows.

Theorem 4.15. (i) *Let $r > 1$. Then*

$$e_k(id : S^{\overline{r}}_{1,\infty}B \to S^0_{\infty,2}B) \le c\, k^{-r}(\log k)^{(d-1)(r+\frac{1}{2})}. \tag{4.43}$$

(ii) *Let $r > 0$. Then*

$$e_k(id : S^{\overline{r}}_{\infty,\infty}B \to L_1) \ge c\, k^{-r}(\log k)^{(d-1)(r+\frac{1}{2})}. \tag{4.44}$$

(iii) *Let $r > 1$ and $1 < p, q < \infty$. Then*

$$e_k(id : S^{\overline{r}}_{q,2}F \to S^0_{p,2}F) \le c\, k^{-r}(\log k)^{(d-1)r}. \tag{4.45}$$

(iv) *Let $r > 0$ and $1 < q < \infty$. Then*

$$e_k(id : S^{\overline{r}}_{q,2}F \to L_1) \ge c\, k^{-r}(\log k)^{(d-1)r}. \tag{4.46}$$

Remark 4.16. We discuss briefly these results. We point out, that the bound for r is always optimal up to the case (iii). Namely, the embedding in (4.45) is compact if and only if $r > (\frac{1}{q} - \frac{1}{p})_+$. The inequalities (4.43) and (4.45) are completely covered by Theorem 4.11. But as for (4.44) and (4.46), these results are of a different nature. Namely, they deal with the space $L_1(\Omega)$. This space does *not* fit into our scales $S^{\overline{r}}_{p,q}A(\Omega)$. All the known decomposition techniques fail to give some decomposition of this space and, therefore, no reduction to the sequence space level is possible. The same holds for embeddings to other spaces of this kind, especially $L_\infty(\Omega)$.

Finally, we discuss the results obtained by Dinh Dung in [8].

Theorem 4.17. *Let $1 < p_1, p_2 < \infty$, $0 < q \le \infty$ and $r > 0$. Then we have*
(i) *for either $r > \frac{1}{p_1}$ and $q \ge p_1$ or $r > (\frac{1}{p_1} - \frac{1}{p_2})_+$ and $q \ge \min(p_2, 2)$*

$$e_k(id : S^{\overline{r}}_{p_1,q}B \to S^0_{p_2,2}F) \approx k^{-r}(\log k)^{(d-1)(r+\frac{1}{2}-\frac{1}{q})}, \tag{4.47}$$

(ii) *and for $r > (\frac{1}{p_1} - \frac{1}{p_2})_+$*

$$e_k(id : S^{\overline{r}}_{p_1,2}F \to S^0_{p_2,2}F) \approx k^{-r}(\log k)^{(d-1)r}. \tag{4.48}$$

The embedding (4.48) is (for $p_1 \le p_2$) covered by (4.39) and for general p_1 and p_2 by (4.30) and (4.31). We therefore concentrate on (4.47). In [9], Dinh Dung comments that the conditions on r and q in Theorem 4.17 ensure the positivity of the power of logarithm in (4.47). In view of our general estimate (4.30), this should really be so. But unfortunately, the conditions given in Theorem 4.17 do *not* ensure that $r + \frac{1}{2} - \frac{1}{q} > 0$. To see that, set $p_1 = p_2 < q < 2$ and $0 < r < \frac{1}{q} - \frac{1}{2}$. A closer inspection of the proof of Theorem 2 in [8] shows that in the case $r > (\frac{1}{p_1} - \frac{1}{p_2})_+$ and $q \ge \min(p_2, 2)$ Dinh Dung proves actually a bit weaker result, namely

$$e_k(id : S^{\overline{r}}_{p_1,q}B \to S^0_{p_2,2}F) \le c\, k^{-r}(\log k)^{(d-1)(r+\frac{1}{\min(p_2,2)}-\frac{1}{q})}, \quad k \ge 2. \tag{4.49}$$

In this result, the power of logarithm is always positive and, therefore, no contradiction with (4.30) occurs. We point out, that our results covers and improves (4.49) as far as the set of parameters is concerned.

We start with $p_1 \le p_2$. By Remark 4.12, we get (4.47) for all $r > \frac{1}{p_1} - \frac{1}{p_2}$ with $r > \frac{1}{q} - \frac{1}{2}$ if $q \le p_1$ or $2 \le p_2$. Moreover, for $r \le \frac{1}{q} - \frac{1}{2}$ we get (4.30) and analogy of (4.32). Finally, if $r > \frac{1}{p_1} - \frac{1}{2}$ we get (4.47) even if $q > p_1$ and $2 > p_2$. Similar discussion may be done for $p_1 > p_2$.

Next we present some special cases of Theorem 4.11 which were not discussed separately yet, but which may be of some interest on its own.

Theorem 4.18. *Let* $\overline{r} = (r, \dots, r) \in \mathbb{R}^d$.

(i) *The embedding*

$$id : S^{\overline{r}}_{1,1}B(\Omega) \to S^0_{\infty,\infty}B(\Omega)$$

is compact if and only if $r > 1$ *and in that case*

$$e_k(id) \approx k^{-r}(\log k)^{(d-1)(r-1)}, \quad k \ge 2.$$

(ii) *The embedding*

$$id : S^{\overline{r}}_{\infty,1}B(\Omega) \to S^0_{\infty,\infty}B(\Omega)$$

is compact if and only if $r > 0$. *If* $r > 1$

$$e_k(id) \approx k^{-r}(\log k)^{(d-1)(r-1)}, \quad k \ge 2,$$

and for $0 < r \le 1$ *and every* $\epsilon > 0$ *there are constants* c *and* c_ϵ *such that*

$$c\, k^{-r} \le e_k(id) \le c_\epsilon k^{-r}(\log k)^{\epsilon}, \quad k \ge 2.$$

(iii) *Let* $0 < p \le q < \infty$. *The embedding*

$$id : S^{\overline{r}}_{p,2}F(\Omega) \to S^0_{q,\infty}B(\Omega)$$

is compact if and only if $r > \frac{1}{p} - \frac{1}{q}$. *If in this case* $r > \frac{1}{2}$ *then*

$$e_k(id) \approx k^{-r}(\log k)^{(d-1)(r-\frac{1}{2})}, \quad k \ge 2,$$

and for $\frac{1}{p} - \frac{1}{q} < r \le \frac{1}{2}$ *and every* $\epsilon > 0$ *there are constants* c *and* c_ϵ *such that*

$$c\, k^{-r} \le e_k(id) \le c_\epsilon k^{-r}(\log k)^{\epsilon}, \quad k \ge 2.$$

Proof. The proof follows from Theorem 4.11 and Remark 4.12. □

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