# Non-smooth atomic decompositions, traces on Lipschitz domains, and pointwise multipliers in function spaces

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January 10, 2012

**Abstract**

We provide non-smooth atomic decompositions for Besov spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$, $s > 0$, $0 < p, q \le \infty$, defined via differences. The results are used to compute the trace of Besov spaces on the boundary $\Gamma$ of bounded Lipschitz domains $\Omega$ with smoothness $s$ restricted to $0 < s < 1$ and no further restrictions on the parameters $p, q$. We conclude with some more applications in terms of pointwise multipliers.

**Math Subject Classifications (MSC2010):** 46E35, 42B35, 47B38.

**Keywords and Phrases:** Lipschitz domains, Besov spaces, differences, real interpolation, atoms, traces, pointwise multipliers.

## Introduction

Besov spaces – sometimes briefly denoted as B-spaces in the sequel – of positive smoothness, have been investigated for many decades already, resulting, for instance, from the study of partial differential equations, interpolation theory, approximation theory, harmonic analysis.
There are several definitions of Besov spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ to be found in the literature. Two of the most prominent approaches are the *Fourier-analytic approach* using Fourier transforms on the one hand and the *classical approach* via higher order differences involving the modulus of smoothness on the other. These two definitions are equivalent only with certain restrictions on the parameters, in particular, they differ for $0 < p < 1$ and $0 < s \le n(\frac{1}{p} - 1)$, but may otherwise share similar properties.
In the present paper we focus on the *classical approach*, which introduces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ as those subspaces of $L_p(\mathbb{R}^n)$ such that

$$\|f|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\|_r = \|f|L_p(\mathbb{R}^n)\| + \left(\int_0^1 t^{-sq}\omega_r(f,t)_p^q \,\frac{\mathrm{d}t}{t}\right)^{1/q}$$

is finite, where $0 < p, q \le \infty$, $s > 0$, $r \in \mathbb{N}$ with $r > s$, and $\omega_r(f,t)_p$ is the usual $r$-th modulus of smoothness of $f \in L_p(\mathbb{R}^n)$.
These spaces occur naturally in nonlinear approximation theory. Especially important is the case $p < 1$, which is needed for the description of approximation classes of classical methods such as rational approximation and approximation by splines with free knots. For more details we refer to the introduction of [7].
For our purposes it will be convenient to use an equivalent characterization for the classical Besov spaces,

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cf. [17], [45, Sect. 9.2], and also [34, Th. 2.11], relying on *smooth atomic decompositions*. They which allow us to characterize $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ as the space of those $f \in L_p(\mathbb{R}^n)$ which can be represented as

$$f(x) = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{j,m} a_{j,m}(x), \quad x \in \mathbb{R}^n, \tag{0.1}$$

with the sequence of coefficients $\lambda = \{\lambda_{j,m} \in \mathbb{C} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ belonging to some appropriate sequence space $b^s_{p,q}$, where $s > 0$, $0 < p, q \leq \infty$, and with smooth atoms $a_{j,m}(x)$.

It is one of the aims of the present paper to develop non-smooth atomic decompositions for Besov spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$, cf. Theorem 2.6 and Corollary 2.8. We will show that one can relax the assumptions on the smoothness of the atoms $a_{j,m}$ used in the representation (0.1) and, thus, replace these atoms with more general ones without loosing any crucial information compared smooth atomic decompositions for functions $f \in \mathbf{B}^s_{p,q}(\mathbb{R}^n)$.
There are only few forerunners dealing with non-smooth atomic decompositions in function spaces so far. We refer to the papers [44], [26], and [4], all mainly considering the different Fourier-analytic approach for Besov spaces and having in common that they restrict themselves to the technically simpler case when $p = q$. Our approach generalizes and extends these results and seems to be the first one covering the full range of indices $0 < p, q \leq \infty$. The reader may also consult [31] for another generalization of the classical atomic decomposition technique using building blocks of limited smoothness.

The additional freedom we gain in the choice of suitable non-smooth atoms $a_{j,m}$ for the atomic decompositions of $f \in \mathbf{B}^s_{p,q}(\mathbb{R}^n)$ makes this approach well suited to further investigate Besov spaces $\mathbf{B}^s_{p,q}(\Omega)$ on non-smooth domains $\Omega$ and their boundaries $\Gamma$. In particular, we shall focus on bounded Lipschitz domains and start by obtaining some interesting new properties concerning interpolation and equivalent quasi-norms for these spaces as well as an atomic decomposition for Besov spaces $\mathbf{B}^s_{p,q}(\Gamma)$, defined on the boundary $\Gamma = \partial\Omega$ of a Lipschitz domain.

But the main goal of this article is to demonstrate the strength of the newly developed non-smooth atomic decompositions in view of trace results. The trace is taken with respect to the boundary $\Gamma$ of bounded Lipschitz domains $\Omega$. Our main result reads as

$$\operatorname{Tr} \mathbf{B}^{s+\frac{1}{p}}_{p,q}(\Omega) = \mathbf{B}^s_{p,q}(\Gamma),$$

where $n \geq 2$, $0 < s < 1$, and $0 < p, q \leq \infty$, cf. Theorem 4.11. Its proof reveals how well suited non-smooth atoms are in order to tackle this problem. The limiting case $s = 0$ is also considered in Corollary 4.13.
In the range $0 < s < 1$, our results are optimal in the sense that there are no further restrictions on the parameters $p$, $q$. The fact that we now also cover traces in Besov spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ with $p < 1$ could be of particular interest in nonlinear approximation theory.
Moreover, as a by-product we obtain corresponding trace results on Lipschitz domains for Triebel-Lizorkin spaces, defined via atomic decompositions.
The papers [33] and [34], dealing with traces on hyperplanes and smooth domains, respectively, might be considered as forerunners of the trace results established in this paper. Nevertheless, the methods we use now are completely different.
The same question for $s \geq 1$ was studied in [20]. It turns out that in this case the function spaces on the boundary look very different and also the extension operator must be changed. Moreover, based on the seminal work [19], traces on Lipschitz domains were studied in [22, Th. 1.1.3] for the Fourier-analytic Besov spaces with the natural restrictions

$$(n-1)\max\left(\frac{1}{p}-1, 0\right) < s < 1 \quad \text{and} \quad \frac{n-1}{n} < p. \tag{0.2}$$

Our Theorem 4.11 actually covers and extends [22, Th. 1.1.3], as for the parameters restricted by (0.2) the Besov spaces defined by differences coincide with the Fourier-analytic Besov spaces.
In contrast to MAYBORODA we make use of the classical Whitney extension operator and the cone property of Lipschitz domains in order to establish our results instead of potential layers and interpolation. Moreover, the extension operator we construct is not linear – and in fact cannot be whenever $s < (n-1)\max(\frac{1}{p}-1,0)$ – compared to the extension operator in [22, Th. 1.1.3]. Let us recall that the importance of non-linear extension operators is known in the theory of differentiable spaces since the pioneering work of Gagliardo [14], cf. also [2, Chapter 5].
Finally, we shall use the non-smooth atomic decompositions again to deal with pointwise multipliers in the respective function spaces. Let $\mathbf{B}^s_{p,q,\mathrm{selfs}}(\mathbb{R}^n)$ denote the self-similar spaces introduced in Definition 5.1 and $M(\mathbf{B}^s_{p,q}(\mathbb{R}^n))$ the set of all pointwise multipliers of $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$. We prove for $s > 0$, $0 < p, q \leq \infty$ in Theorem 5.4 the relationship

$$\bigcup_{\sigma > s} \mathbf{B}^s_{p,q,\mathrm{selfs}}(\mathbb{R}^n) \subset M(\mathbf{B}^s_{p,q}(\mathbb{R}^n)) \hookrightarrow \mathbf{B}^s_{p,q,\mathrm{selfs}}(\mathbb{R}^n). \tag{0.3}$$

Additionally, if $0 < p \leq 1$, one even has a coincidence in terms of $M(\mathbf{B}^s_{p,p}(\mathbb{R}^n)) = \mathbf{B}^s_{p,p,\mathrm{selfs}}(\mathbb{R}^n)$. Our results generalize the multiplier assertions from [44] to the case when $p \neq q$. Moreover, they extend previous results to classical Besov spaces with small parameters $s$ and $p$. In this context we refer to [23], [24], and [25], where pointwise multipliers in Besov spaces with $p, q \geq 1$ and $p = q$ were studied in detail.
We conclude using (0.3) in order to discuss under which circumstances the characteristic function $\chi_\Omega$ of a bounded domain $\Omega$ in $\mathbb{R}^n$ is a pointwise multiplier in $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ – establishing a connection between pointwise multipliers and certain fundamental notion of fractal geometry, so-called $h$-sets, cf. Definition 5.6. In particular, if a boundary $\Gamma = \partial\Omega$ is an $h$-set satisfying

$$\sup_{j \in \mathbb{N}_0} \sum_{k=0}^{\infty} 2^{k\sigma q} \left( \frac{h(2^{-j})}{h(2^{-j-k})} 2^{-kn} \right)^{q/p} < \infty,$$

where $\sigma > 0$, $0 < p < \infty$, and $0 < q \leq \infty$, then Theorem 5.8 shows that

$$\chi_\Omega \in \mathbf{B}^\sigma_{p,q,\mathrm{selfs}}(\mathbb{R}^n).$$

The present paper is organized as follows: Section 1 contains notation, definitions, and preliminary assertions on smooth atomic decompositions. The main investigation starts in Section 2, where we construct non-smooth atomic decompositions for the spaces under focus. Afterwards Section 3 provides new insights (and helpful results) concerning function spaces on Lipschitz domains and their boundaries. These powerful techniques are then used in Section 4 in order to compute traces on Lipschitz domains – the heart of this article. Finally, we conclude with some further applications of non-smooth atomic decompositions in terms of pointwise multipliers in Section 5.

# 1 Preliminaries

We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be euclidean $n$-space, $n \in \mathbb{N}$, $\mathbb{C}$ the complex plane. The set of multi-indices $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_i \in \mathbb{N}_0$, $i = 1, \ldots, n$, is denoted by $\mathbb{N}_0^n$, with $|\beta| = \beta_1 + \cdots + \beta_n$, as usual. Moreover, if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ we put $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$.
We use the symbol '$\lesssim$' in

$$a_k \lesssim b_k \quad \text{or} \quad \varphi(x) \lesssim \psi(x)$$

always to mean that there is a positive number $c_1$ such that

$$a_k \leq c_1\, b_k \quad \text{or} \quad \varphi(x) \leq c_1\, \psi(x)$$

for all admitted values of the discrete variable $k$ or the continuous variable $x$, where $\{a_k\}_k$, $\{b_k\}_k$ are non-negative sequences and $\varphi$, $\psi$ are non-negative functions. We use the equivalence '$\sim$' in

$$a_k \sim b_k \quad \text{or} \quad \varphi(x) \sim \psi(x)$$

for

$$a_k \lesssim b_k \quad \text{and} \quad b_k \lesssim a_k \qquad \text{or} \qquad \varphi(x) \lesssim \psi(x) \quad \text{and} \quad \psi(x) \lesssim \varphi(x).$$

If $a \in \mathbb{R}$, then $a_+ := \max(a, 0)$ and $[a]$ denotes the integer part of $a$.
Given two (quasi-) Banach spaces $X$ and $Y$, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of $X$ into $Y$ is continuous. All unimportant positive constants will be denoted by $c$, occasionally with subscripts. For convenience, let both $\mathrm{d}x$ and $|\cdot|$ stand for the ($n$-dimensional) Lebesgue measure in the sequel. $L_p(\mathbb{R}^n)$, with $0 < p \leq \infty$, stands for the usual quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\|f|L_p(\mathbb{R}^n)\| := \left(\int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}$$

with the appropriate modification if $p = \infty$. Throughout the paper $\Omega$ will denote a domain in $\mathbb{R}^n$ and the Lebesgue space $L_p(\Omega)$ is defined in the usual way.

We denote by $C^K(\mathbb{R}^n)$ the space of all $K$-times continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ equipped with the norm

$$\|f|C^K(\mathbb{R}^n)\| = \max_{|\alpha|\leq K} \sup_{x\in\mathbb{R}^n} |D^\alpha f(x)|.$$

Additionally, $C^\infty(\mathbb{R}^n)$ contains the set of smooth and bounded functions on $\mathbb{R}^n$, i.e.,

$$C^\infty(\mathbb{R}^n) := \bigcap_{K\in\mathbb{N}} C^K(\mathbb{R}^n),$$

whereas $C_0^\infty(\mathbb{R}^n)$ denotes the space of smooth functions with compact support.
Furthermore, $B(x_0, R)$ stands for an open ball with radius $R > 0$ around $x_0 \in \mathbb{R}^n$,

$$B(x_0, R) = \{x \in \mathbb{R}^n : |x - x_0| < R\}. \tag{1.1}$$

Let $Q_{j,m}$ with $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ denote a cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centered at $2^{-j}m$, and with side length $2^{-j+1}$. For a cube $Q$ in $\mathbb{R}^n$ and $r > 0$, we denote by $rQ$ the cube in $\mathbb{R}^n$ concentric with $Q$ and with side length $r$ times the side length of $Q$. Furthermore, $\chi_{j,m}$ stands for the characteristic function of $Q_{j,m}$.

Let $G \subset \mathbb{R}^n$ and $j \in \mathbb{N}_0$. We use the abbreviation

$$\sum_{m\in\mathbb{Z}^n}^{G,j} = \sum_{m\in\mathbb{Z}^n, Q_{j,m}\cap G\neq\emptyset}, \tag{1.2}$$

where $G$ will usually denote either a domain $\Omega$ in $\mathbb{R}^n$ or its boundary $\Gamma$.

## 1.1 Smooth atomic decompositions in function spaces

We introduce the Besov spaces $\mathbf{B}^s_{p,q}(\Omega)$ through their decomposition properties. This provides a constructive definition expanding functions $f$ via smooth atoms (excluding any moment conditions) and suitable coefficients, where the latter belong to certain sequence spaces denoted by $b^s_{p,q}(\Omega)$ defined below.

**Definition 1.1** *Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$. Furthermore, let $\Omega \subset \mathbb{R}^n$ and $\lambda = \{\lambda_{j,m} \in \mathbb{C} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$. Then*

$$b^s_{p,q}(\Omega) = \left\{ \lambda : \|\lambda | b^s_{p,q}(\Omega)\| = \left( \sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \left( \sum_{m \in \mathbb{Z}^n}^{\Omega,j} |\lambda_{j,m}|^p \right)^{q/p} \right)^{1/q} < \infty \right\}$$

(*with the usual modification if $p = \infty$ and/or $q = \infty$*).

**Remark 1.2** If $\Omega = \mathbb{R}^n$, we simply write $b^s_{p,q}$ and $\sum_m$ instead of $b^s_{p,q}(\Omega)$ and $\sum_m^{\Omega,j}$, respectively.

Now we define the smooth atoms.

**Definition 1.3** *Let $K \in \mathbb{N}_0$ and $d > 1$. A $K$-times continuously differentiable complex-valued function $a$ on $\mathbb{R}^n$ (continuous if $K = 0$) is called a $K$-atom if for some $j \in \mathbb{N}_0$*

$$\operatorname{supp} a \subset dQ_{j,m} \quad \text{for some } m \in \mathbb{Z}^n, \tag{1.3}$$

*and*

$$|D^\alpha a(x)| \leq 2^{|\alpha|j} \quad \text{for } |\alpha| \leq K. \tag{1.4}$$

It is convenient to write $a_{j,m}(x)$ instead of $a(x)$ if this atom is located at $Q_{j,m}$ according to (1.3). Furthermore, $K$ denotes the smoothness of the atom, cf. (1.4).

We define Besov spaces $\mathbf{B}^s_{p,q}(\Omega)$ using the *atomic approach*.

**Definition 1.4** *Let $s > 0$ and $0 < p, q \leq \infty$. Let $d > 1$ and $K \in \mathbb{N}_0$ with*

$$K \geq (1 + [s])$$

*be fixed. Then $f \in L_p(\Omega)$ belongs to $\mathbf{B}^s_{p,q}(\Omega)$ if, and only if, it can be represented as*

$$f(x) = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n}^{\Omega,j} \lambda_{j,m} a_{j,m}(x), \tag{1.5}$$

*where the $a_{j,m}$ are $K$-atoms ($j \in \mathbb{N}_0$) with*

$$\operatorname{supp} a_{j,m} \subset dQ_{j,m}, \qquad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n,$$

*and $\lambda \in b^s_{p,q}(\Omega)$, convergence being in $L_p(\Omega)$. Furthermore,*

$$\|f | \mathbf{B}^s_{p,q}(\Omega)\| := \inf \|\lambda | b^s_{p,q}(\Omega)\|, \tag{1.6}$$

*where the infimum is taken over all admissible representations* (1.5).

**Remark 1.5** According to [45], based on [17], the above defined spaces are independent of $d$ and $K$. This may justify our omission of $K$ and $d$ in (1.6).
Since the atoms $a_{j,m}$ used in Definition 1.4 are defined also outside of $\Omega$, the spaces $\mathbf{B}^s_{p,q}(\Omega)$ can as well be regarded as restrictions of the corresponding spaces on $\mathbb{R}^n$ in the usual interpretation, i.e.,

$$\mathbf{B}^s_{p,q}(\Omega) = \{f \in L_p(\Omega) : \quad \text{there exists} \quad g \in \mathbf{B}^s_{p,q}(\mathbb{R}^n) \quad \text{with} \quad g\big|_\Omega = f\},$$

furnished with the norm

$$\|f | \mathbf{B}^s_{p,q}(\Omega)\| = \inf \left\{ \|g | \mathbf{B}^s_{p,q}(\mathbb{R}^n)\| \quad \text{with} \quad g\big|_\Omega = f \right\},$$

where $g\big|_\Omega = f$ denotes the restriction of $g$ to $\Omega$. Therefore, well-known embedding results for B-spaces defined on $\mathbb{R}^n$ carry over to those defined on domains $\Omega$. Let $s > 0$, $\varepsilon > 0$, $0 < q, u \leq \infty$, and $q \leq v \leq \infty$. Then we have

$$\mathbf{B}^{s+\varepsilon}_{p,u}(\Omega) \hookrightarrow \mathbf{B}^s_{p,q}(\Omega) \qquad \text{and} \qquad \mathbf{B}^s_{p,q}(\Omega) \hookrightarrow \mathbf{B}^s_{p,u}(\Omega),$$

cf. [18, Th. 1.15], where also further embeddings for Besov spaces may be found.

**Classical approach** Originally Besov spaces were defined merely using higher order differences instead of atomic decompositions. The question arises whether this *classical approach* coincides with our *atomic approach*. This might not always be the case but is true for spaces defined on $\mathbb{R}^n$ and on so-called $(\varepsilon,\delta)$-domains which we introduce next.

Recall that domain always stands for open set. The boundary of $\Omega$ is denoted by $\Gamma = \partial\Omega$.

**Definition 1.6** *Let $\Omega$ be a domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Then $\Omega$ is said to be an $(\varepsilon,\delta)$-domain, where $0<\varepsilon<\infty$ and $0<\delta<\infty$, if it is connected and if for any $x \in \Omega$, $y \in \Omega$ with $|x-y|<\delta$ there is a curve $L \subset \Omega$, connecting $x$ and $y$ such that $|L| \leq \varepsilon^{-1}|x-y|$ and*

$$\operatorname{dist}(z,\Gamma) \geq \varepsilon \min(|x-z|,|y-z|), \qquad z \in L. \tag{1.7}$$

**Remark 1.7** All domains we will be concerned with in the sequel are $(\varepsilon,\delta)$-domains. In particular, the definition includes *minimally smooth* domains in the sense of Stein, cf. [37, p. 189], and therefore bounded Lipschitz domains (as will be considered in Section 3).
Furthermore, the half space $\mathbb{R}^n_+ := \{x : x = (x', x_n) \in \mathbb{R}^n, x' \in \mathbb{R}^{n-1}, x_n > 0\}$ is another example.

It is well-known that $(\varepsilon,\delta)$-domains play a crucial role concerning questions of extendability. It is precisely this property which was used in [34, Th. 2.10] to show that for $(\varepsilon,\delta)$-domains the atomic approach for B-spaces is equivalent to the *classical approach* (in terms of equivalent quasi-norms), which introduces $\mathbf{B}^s_{p,q}(\Omega)$ as the subspace of $L_p(\Omega)$ such that

$$\|f|\mathbf{B}^s_{p,q}(\Omega)\|_r = \|f|L_p(\Omega)\| + \left(\int_0^1 t^{-sq}\omega_r(f,t,\Omega)_p^q \,\frac{\mathrm{d}t}{t}\right)^{1/q} \tag{1.8}$$

is finite, where $0<p,q\leq\infty$ (with the usual modification if $q=\infty$), $s>0$, $r\in\mathbb{N}$ with $r>s$. Here $\omega_r(f,t,\Omega)_p$ stands for the usual $r$-th modulus of smoothness of a function $f \in L_p(\Omega)$,

$$\omega_r(f,t,\Omega)_p = \sup_{|h|\leq t} \|\Delta^r_h f(\cdot,\Omega) \mid L_p(\Omega)\|, \quad t>0, \tag{1.9}$$

where

$$\Delta^r_h f(x,\Omega) := \begin{cases} \Delta^r_h f(x), & x, x+h, \ldots, x+rh \in \Omega, \\ 0, & \text{otherwise}, \end{cases} \tag{1.10}$$

This approach for the spaces $\mathbf{B}^s_{p,q}(\Omega)$ was used in [8]. The proof of the coincidence uses the fact that the classical and atomic approach can be identified for spaces defined on $\mathbb{R}^n$, which follows from results by Hedberg, Netrusov [17] on atomic decompositions and by Triebel [45, Section 9.2] on the reproducing formula.

The classical scale of Besov spaces contains many well-known function spaces. For example, if $p=q=\infty$, one recovers the Hölder-Zygmund spaces $\mathcal{C}^s(\mathbb{R}^n)$, i.e.,

$$\mathbf{B}^s_{\infty,\infty}(\mathbb{R}^n) = \mathcal{C}^s(\mathbb{R}^n), \quad s>0. \tag{1.11}$$

Later on we will need the following homogeneity estimate proved recently in [39, Th. 2] based on [3].

**Theorem 1.8** *Let $0<\lambda\leq 1$ and $f \in \mathbf{B}^s_{p,q}(\mathbb{R}^n)$ with $\operatorname{supp} f \subset B(0,\lambda)$. Then*

$$\|f(\lambda\cdot)|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| \sim \lambda^{s-n/p}\|f|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\|. \tag{1.12}$$

## 2 Non-smooth atomic decompositions

Our aim is to provide a non-smooth atomic characterization of Besov spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$, i.e., relaxing the assumptions about the smoothness of the atoms $a_{j,m}$ in Definition 1.3. Note that condition (1.4) is equivalent to

$$\|a(2^{-j}\cdot)|C^K(\mathbb{R}^n)\| \le 1. \tag{2.1}$$

We replace the $C^K$-norm with $K > s$ by a Besov quasi-norm $\mathbf{B}^\sigma_{p,p}(\mathbb{R}^n)$ with $\sigma > s$ or in case of $0 < s < 1$ by a norm in the space of Lipschitz functions $\mathrm{Lip}(\mathbb{R}^n)$.
The following non-smooth atoms were introduced in [43]. They will be very adequate when considering (non-smooth) atomic decompositions of spaces defined on Lipschitz domains (or on the boundary of a Lipschitz domain, respectively).

**Definition 2.1** *(i) The space of Lipschitz functions* $\mathrm{Lip}(\mathbb{R}^n)$ *is defined as the collection of all real-valued functions* $f : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$\|f|\mathrm{Lip}(\mathbb{R}^n)\| = \max\left\{\sup_x |f(x)|, \quad \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|}\right\} < \infty.$$

*(ii) We say that* $a \in \mathrm{Lip}(\mathbb{R}^n)$ *is a* Lip*-atom, if for some* $j \in \mathbb{N}_0$

$$\operatorname{supp} a \subset dQ_{j,m}, \quad m \in \mathbb{Z}^n,\ d > 1, \tag{2.2}$$

*and*

$$|a(x)| \le 1, \qquad |a(x) - a(y)| \le 2^j|x-y|. \tag{2.3}$$

**Remark 2.2** One might use alternatively in (2.3) that

$$\|a(2^{-j}\cdot)|\mathrm{Lip}(\mathbb{R}^n)\| \le 1. \tag{2.4}$$

We use the abbreviation

$$\mathbf{B}^s_p(\mathbb{R}^n) = \mathbf{B}^s_{p,p}(\mathbb{R}^n) \quad \text{with} \quad 0 < p \le \infty, \quad s > 0.$$

In particular, in view of (1.11),

$$\mathcal{C}^s(\mathbb{R}^n) = \mathbf{B}^s_\infty(\mathbb{R}^n), \quad s > 0,$$

are the Hölder-Zygmund spaces.

**Definition 2.3** *Let* $0 < p \le \infty$, $\sigma > 0$ *and* $d > 1$*. Then* $a \in \mathbf{B}^\sigma_p(\mathbb{R}^n)$ *is called a* $(\sigma, p)$*-atom if for some* $j \in \mathbb{N}_0$

$$\operatorname{supp} a \subset dQ_{j,m} \qquad \text{for some } m \in \mathbb{Z}^n, \tag{2.5}$$

*and*

$$\|a(2^{-j}\cdot)|\mathbf{B}^\sigma_p(\mathbb{R}^n)\| \le 1. \tag{2.6}$$

**Remark 2.4** Note that if $\sigma < \frac{n}{p}$ then $(\sigma, p)$-atoms might be unbounded. Roughly speaking, they arise by dilating $\mathbf{B}^\sigma_p$-normalized functions. Obviously, the condition (2.6) is a straightforward modification of (2.1) and (2.4).

In general, it is convenient to write $a_{j,m}(x)$ instead of $a(x)$ if the atoms are located at $Q_{j,m}$ according to (2.2) and (2.5), respectively. Furthermore, $\sigma$ denotes the 'non-smoothness' of the atom, cf. (1.4).

The non-smooth atoms we consider in Definition 2.3, are renormalized versions of the non-smooth $(s, p)^\sigma$-atoms considered in [44] and [48], where (2.6) is replaced by

$$a \in B^\sigma_p(\mathbb{R}^n) \qquad \text{with} \qquad \|a(2^{-j}\cdot)|B^\sigma_p(\mathbb{R}^n)\| \le 2^{j(\sigma-s)},$$

resulting in corresponding changes concerning the definition of the sequence spaces $b^s_{p,q}$ used for the atomic decomposition.
However, the function spaces we consider are different from the ones considered there. Furthermore, for our purposes (studying traces later on) it is convenient to shift the factors $2^{j(s-\frac{n}{p})}$ to the sequence spaces.

We wish to compare these atoms with the smooth atoms in Definition 1.3.

**Proposition 2.5** *Let $0 < p \leq \infty$ and $0 < \sigma < K$. Furthermore, let $d > 1$, $j \in \mathbb{N}_0$, and $m \in \mathbb{Z}^n$. Then any $K$-atom $a_{j,m}$ is a $(\sigma, p)$-atom.*

P r o o f : Since the functions $a_{j,m}(2^{-j}\cdot)$ have compact support, we obtain

$$\|a_{j,m}(2^{-j}\cdot)|\mathbf{B}^\sigma_p(\mathbb{R}^n)\| \lesssim \|a_{j,m}(2^{-j}\cdot)|C^K(\mathbb{R}^n)\| \leq 1,$$

with constants independent of $j$, giving the desired result for non-smooth atoms from Definition 2.3. □

The use of atoms with limited smoothness (i.e. finite element functions or splines) was studied already in [27], where the author deals with spline approximation (and traces) in Besov spaces.

The following theorem contains the main result of this section. It gives the counterpart of Definition 1.4 and provides a non-smooth atomic decomposition of the spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$.

**Theorem 2.6** *Let $0 < p, q \leq \infty$, $0 < s < \sigma$, and $d > 1$. Then $f \in L_p(\mathbb{R}^n)$ belongs to $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} a_{j,m}, \tag{2.7}$$

*where the $a_{j,m}$ are $(\sigma, p)$-atoms ($j \in \mathbb{N}_0$) with $\operatorname{supp} a_{j,m} \subset dQ_{j,m}$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, and $\lambda \in b^s_{p,q}$, convergence being in $L_p(\mathbb{R}^n)$. Furthermore,*

$$\|f|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| = \inf \|\lambda|b^s_{p,q}\|, \tag{2.8}$$

*where the infimum is taken over all admissible representations* (2.7)*.*

P r o o f : We have the atomic decomposition based on smooth $K$-atoms according to Definition 1.4. By Proposition 2.5 classical $K$-atoms are special $(\sigma, p)$-atoms. Hence, it is enough to prove that

$$\|f|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| \lesssim \left( \sum_{k=0}^{\infty} 2^{k(s-\frac{n}{p})q} \left( \sum_{l \in \mathbb{Z}^n} |\lambda_{k,l}|^p \right)^{q/p} \right)^{1/q} \tag{2.9}$$

for any atomic decomposition

$$f = \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \lambda_{k,l} a^{k,l}, \tag{2.10}$$

where $a^{k,l}$ are $(\sigma, p)$-atoms according to Definition 2.3.
For this purpose we expand each function $a^{k,l}(2^{-k}\cdot)$ optimally in $\mathbf{B}^\sigma_p(\mathbb{R}^n)$ with respect to classical $K$-atoms $b^{j,w}_{k,l}$ where $\sigma < K$,

$$a^{k,l}(2^{-k}x) = \sum_{j=0}^{\infty} \sum_{w \in \mathbb{Z}^n} \eta^{k,l}_{j,w} b^{j,w}_{k,l}(x), \quad x \in \mathbb{R}^n, \tag{2.11}$$

with

$$\operatorname{supp} b^{j,w}_{k,l} \subset Q_{j,w}, \qquad \left| D^\alpha b^{j,w}_{k,l}(x) \right| \leq 2^{|\alpha| j}, \quad |\alpha| \leq K, \tag{2.12}$$

and

$$\left(\sum_{j=0}^{\infty} 2^{j(\sigma-\frac{n}{p})p} \sum_{w\in\mathbb{Z}^n} |\eta^{k,l}_{j,w}|^p\right)^{\frac{1}{p}} = \|\eta^{k,l}|b^{\sigma}_{p,p}\| \sim \|a^{k,l}(2^{-k}\cdot)|\mathbf{B}^{\sigma}_p(\mathbb{R}^n)\| \lesssim 1. \tag{2.13}$$

Hence,

$$a^{k,l}(x) = \sum_{j=0}^{\infty}\sum_{w\in\mathbb{Z}^n} \eta^{k,l}_{j,w} b^{j,w}_{k,l}(2^k x),$$

where the functions $b^{j,w}_{k,l}(2^k\cdot)$ are supported by cubes with side lengths $\sim 2^{-k-j}$. By (2.12) we have

$$\left|D^\alpha b^{j,w}_{k,l}(2^k x)\right| = 2^{k|\alpha|}\left|(D^\alpha b^{j,w}_{k,l})(2^k x)\right| \le 2^{(j+k)|\alpha|}.$$

Replacing $j+k$ by $j$ and putting $d^{j,w}_{k,l}(x) := b^{j-k,w}_{k,l}(2^k x)$, we obtain that

$$a^{k,l}(x) = \sum_{j=k}^{\infty}\sum_{w\in\mathbb{Z}^n} \eta^{k,l}_{j-k,w} d^{j,w}_{k,l}(x), \tag{2.14}$$

where $d^{j,w}_{k,l}$ are classical $K$-atoms supported by cubes with side lengths $\sim 2^{-j}$. We insert (2.14) into the expansion (2.10). We fix $j\in\mathbb{N}_0$ and $w\in\mathbb{Z}^n$, and collect all non-vanishing terms $d^{j,w}_{k,l}$ in the expansions (2.14). We have $k\le j$. Furthermore, multiplying (2.11) if necessary with suitable cut-off functions it follows that there is a natural number $N$ such that for fixed $k$ only at most $N$ points $l\in\mathbb{Z}^n$ contribute to $d^{j,w}_{k,l}$. We denote this set by $(j,w,k)$. Hence its cardinality is at most $N$, where $N$ is independent of $j,w,k$. Then

$$d^{j,w}(x) = \frac{\sum_{k\le j}\sum_{l\in(j,w,k)} \eta^{k,l}_{j-k,w}\cdot\lambda_{k,l}\cdot d^{j,w}_{k,l}(x)}{\sum_{k\le j}\sum_{l\in(j,w,k)} |\eta^{k,l}_{j-k,w}|\cdot|\lambda_{k,l}|}$$

are correctly normalized smooth $K$-atoms located in cubes with side lengths $\sim 2^{-j}$ and centered at $2^{-j}w$. Let

$$\nu_{j,w} = \sum_{k\le j}\sum_{l\in(j,w,k)} |\eta^{k,l}_{j-k,w}|\cdot|\lambda_{k,l}|. \tag{2.15}$$

Then we obtain a classical atomic decomposition in the sense of Definition 1.4

$$f = \sum_j\sum_w \nu_{j,w} d^{j,w}(x),$$

where $d^{j,w}$ are $K$-atoms and

$$\|f|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| \lesssim \|\nu|b^s_{p,q}\|.$$

Therefore, in order to prove (2.9), it is enough to show, that

$$\|\nu|b^s_{p,q}\| \lesssim \|\lambda|b^s_{p,q}\| \tag{2.16}$$

if (2.13) holds.
Let $0<\varepsilon<\sigma-s$. Then we obtain by (2.15) that (assuming $p<\infty$)

$$|\nu_{j,w}|^p \lesssim \sum_{k\le j}\sum_{l\in(j,w,k)} 2^{(j-k)p\varepsilon}|\eta^{k,l}_{j-k,w}|^p|\lambda_{k,l}|^p, \tag{2.17}$$

where we used the bounded cardinality of the sets $(j,w,k)$.
This gives for $q/p\le 1$

$$\|\nu|b^s_{p,q}\|^q = \sum_{j=0}^{\infty} 2^{j(s-n/p)q}\left(\sum_{w\in\mathbb{Z}^n}|\nu_{j,w}|^p\right)^{q/p}$$

$$\lesssim \sum_{j=0}^{\infty} 2^{j(s-n/p)q} \left( \sum_{w\in\mathbb{Z}^n} \sum_{k=0}^{j} \sum_{l\in(j,w,k)} 2^{(j-k)p\varepsilon} |\eta_{j-k,w}^{k,l}|^p |\lambda_{k,l}|^p \right)^{q/p}$$

$$\le \sum_{j=0}^{\infty} 2^{j(s-n/p)q} \sum_{k=0}^{j} \left( \sum_{w\in\mathbb{Z}^n} \sum_{l\in(j,w,k)} 2^{(j-k)p\varepsilon} |\eta_{j-k,w}^{k,l}|^p |\lambda_{k,l}|^p \right)^{q/p}$$

$$= \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{j(s-n/p)q} \left( \sum_{w\in\mathbb{Z}^n} \sum_{l\in(j,w,k)} 2^{(j-k)p\varepsilon} |\eta_{j-k,w}^{k,l}|^p |\lambda_{k,l}|^p \right)^{q/p}$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} 2^{(j+k)(s-n/p)q} \left( \sum_{w\in\mathbb{Z}^n} \sum_{l\in(j+k,w,k)} 2^{jp\varepsilon} |\eta_{j,w}^{k,l}|^p |\lambda_{k,l}|^p \right)^{q/p}$$

$$= \sum_{k=0}^{\infty} 2^{k(s-n/p)q} \sum_{j=0}^{\infty} 2^{j(s-\sigma+\varepsilon)q} \left( \sum_{w\in\mathbb{Z}^n} \sum_{l\in(j+k,w,k)} 2^{j(\sigma-n/p)p} |\eta_{j,w}^{k,l}|^p |\lambda_{k,l}|^p \right)^{q/p}$$

$$\lesssim \sum_{k=0}^{\infty} 2^{k(s-n/p)q} \left( \sum_{j=0}^{\infty} \sum_{w\in\mathbb{Z}^n} \sum_{l\in(j+k,w,k)} 2^{j(\sigma-n/p)p} |\eta_{j,w}^{k,l}|^p |\lambda_{k,l}|^p \right)^{q/p}$$

$$\le \sum_{k=0}^{\infty} 2^{k(s-n/p)q} \left( \sum_{j=0}^{\infty} \sum_{w\in\mathbb{Z}^n} \sum_{l\in\mathbb{Z}^n} 2^{j(\sigma-n/p)p} |\eta_{j,w}^{k,l}|^p |\lambda_{k,l}|^p \right)^{q/p}$$

$$= \sum_{k=0}^{\infty} 2^{k(s-n/p)q} \left( \sum_{l\in\mathbb{Z}^n} |\lambda_{k,l}|^p \sum_{j=0}^{\infty} \sum_{w\in\mathbb{Z}^n} 2^{j(\sigma-n/p)p} |\eta_{j,w}^{k,l}|^p \right)^{q/p}$$

$$\lesssim \sum_{k=0}^{\infty} 2^{k(s-n/p)q} \left( \sum_{l\in\mathbb{Z}^n} |\lambda_{k,l}|^p \right)^{q/p} = \|\lambda|b^s_{p,q}\|^q.$$

We have used (2.13) in the last inequality.

If $q/p > 1$, we shall use the following inequality, which holds for every non-negative sequence $\{\gamma_{j,k}\}_{0\le k\le j<\infty}$, every $\alpha \ge 1$ and every $\varepsilon > 0$.

$$\sum_{j=0}^{\infty} \left( \sum_{k=0}^{j} 2^{-(j-k)\varepsilon} \gamma_{j,k} \right)^{\alpha} \le c_{\alpha,\varepsilon} \sum_{k=0}^{\infty} \left( \sum_{j=k}^{\infty} \gamma_{j,k} \right)^{\alpha}. \tag{2.18}$$

If $\alpha = \infty$, (2.18) has to be modified appropriately. To prove (2.18) for $\alpha < \infty$, we use Hölder's inequality and the embedding $\ell_1 \hookrightarrow \ell_\alpha$

$$\sum_{j=0}^{\infty} \left( \sum_{k=0}^{j} 2^{-(j-k)\varepsilon} \gamma_{j,k} \right)^{\alpha} \le \sum_{j=0}^{\infty} \left( \sum_{k=0}^{j} 2^{-(j-k)\varepsilon\alpha'} \right)^{\alpha/\alpha'} \left( \sum_{k=0}^{j} \gamma_{j,k}^{\alpha} \right)^{\alpha/\alpha}$$
$$\lesssim \sum_{j=0}^{\infty} \sum_{k=0}^{j} \gamma_{j,k}^{\alpha} = \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \gamma_{j,k}^{\alpha} \le \sum_{k=0}^{\infty} \left( \sum_{j=k}^{\infty} \gamma_{j,k} \right)^{\alpha}.$$

We use (2.17) and (2.18) with $p(\sigma - s - \varepsilon)$ instead of $\varepsilon$ and $\alpha = q/p > 1$,

$$\|\nu|b^s_{p,q}\|^q \lesssim \sum_{j=0}^{\infty} 2^{j(\sigma-\frac{n}{p})q} \left( \sum_{w\in\mathbb{Z}^n} \sum_{k=0}^{j} \sum_{l\in(j,w,k)} 2^{(j-k)p\varepsilon} |\eta_{j-k,w}^{k,l}|^p |\lambda_{k,l}|^p \right)^{q/p}$$

$$= \sum_{j=0}^{\infty} \left( \sum_{k=0}^{j} 2^{-(j-k)p(\sigma-s-\varepsilon)} \sum_{w\in\mathbb{Z}^n} \sum_{l\in(j,w,k)} 2^{k(s-n/p)p} 2^{(j-k)(\sigma-\frac{n}{p})p} |\eta^{k,l}_{j-k,w}|^p |\lambda_{k,l}|^p \right)^{q/p}$$

$$\lesssim \sum_{k=0}^{\infty} \left( \sum_{j=k}^{\infty} \sum_{w\in\mathbb{Z}^n} \sum_{l\in(j,w,k)} 2^{k(s-n/p)p} 2^{(j-k)(\sigma-\frac{n}{p})p} |\eta^{k,l}_{j-k,w}|^p |\lambda_{k,l}|^p \right)^{q/p}$$

$$= \sum_{k=0}^{\infty} 2^{k(s-n/p)q} \left( \sum_{j=0}^{\infty} \sum_{w\in\mathbb{Z}^n} \sum_{l\in(j+k,w,k)} 2^{j(\sigma-\frac{n}{p})p} |\eta^{k,l}_{j,w}|^p |\lambda_{k,l}|^p \right)^{q/p}$$

$$= \sum_{k=0}^{\infty} 2^{k(s-n/p)q} \left( \sum_{l\in\mathbb{Z}^n} \sum_{j=0}^{\infty} \sum_{w\in\mathbb{Z}^n : l\in(j+k,w,k)} 2^{j(\sigma-\frac{n}{p})p} |\eta^{k,l}_{j,w}|^p |\lambda_{k,l}|^p \right)^{q/p}$$

$$\lesssim \sum_{k=0}^{\infty} 2^{k(s-n/p)q} \left( \sum_{l\in\mathbb{Z}^n} |\lambda_{k,l}|^p \sum_{j=0}^{\infty} \sum_{w\in\mathbb{Z}^n} 2^{j(\sigma-\frac{n}{p})p} |\eta^{k,l}_{j,w}|^p \right)^{q/p}$$

$$\leq \sum_{k=0}^{\infty} 2^{k(s-n/p)q} \left( \sum_{l\in\mathbb{Z}^n} |\lambda_{k,l}|^p \right)^{q/p} = \|\lambda | b^s_{p,q}\|^q.$$

The proof of (2.16) is finished. We again used (2.13) in the last inequality. If $p$ and/or $q$ are equal to infinity, only notational changes are necessary. $\square$

**Remark 2.7** Our results generalize [44, Th. 2] and [48, Th. 2.3], where non-smooth atomic decompositions for spaces $\mathbf{B}^s_{p,p}(\mathbb{R}^n)$ with $s > \max(n(1/p-1), 0)$ can be found, to $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ with no restrictions on the parameters. In particular, the case when $p \neq q$ is completely new.

Using the Lip-atoms from Definition 2.1 and the embedding

$$\mathrm{Lip}(\mathbb{R}^n) \hookrightarrow B^1_\infty(\mathbb{R}^n),$$

cf. [41, p.89/90], as a Corollary we now obtain the following non-smooth atomic decomposition for Besov spaces with smoothness $0 < s < 1$.

**Corollary 2.8** *Let $0 < p, q \leq \infty$, $0 < s < 1$, and $d > 1$. Then $f \in L_p(\mathbb{R}^n)$ belongs to $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{j,m} a_{j,m}, \tag{2.19}$$

*where the $a_{j,m}$ are* Lip-*atoms* ($j \in \mathbb{N}_0$) *with* $\operatorname{supp} a_{j,m} \subset dQ_{j,m}$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, *and* $\lambda \in b^s_{p,q}$, *convergence being in $L_p(\mathbb{R}^n)$. Furthermore,*

$$\|f|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| = \inf \|\lambda | b^s_{p,q}\|, \tag{2.20}$$

*where the infimum is taken over all admissible representations* (2.19).

# 3 Spaces on Lipschitz domains and their boundaries

We call a one-to-one mapping $\Phi : \mathbb{R}^n \mapsto \mathbb{R}^n$, a *Lipschitz diffeomorphism*, if the components $\Phi_k(x)$ of $\Phi(x) = (\Phi_1(x), \ldots, \Phi_n(x))$ are Lipschitz functions on $\mathbb{R}^n$ and

$$|\Phi(x) - \Phi(y)| \sim |x-y|, \quad x, y \in \mathbb{R}^n, \ |x-y| \leq 1,$$

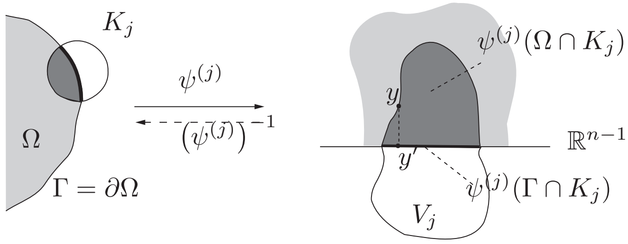
where the equivalence constants are independent of $x$ and $y$. Of course the inverse of $\Phi^{-1}$ is also a Lipschitz diffeomorphism on $\mathbb{R}^n$.

**Definition 3.1** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Then $\Omega$ is said to be a Lipschitz domain, if there exist $N$ open balls $K_1, \ldots, K_N$ such that $\bigcup_{j=1}^{N} K_j \supset \Gamma$ and $K_j \cap \Gamma \neq \emptyset$ if $j = 1, \ldots, N$, with the following property: for every ball $K_j$ there are Lipschitz diffeomorphisms $\psi^{(j)}$ such that*

$$\psi^{(j)} : K_j \longrightarrow V_j, \quad j = 1, \ldots, N,$$

*where $V_j := \psi^{(j)}(K_j)$ and*

$$\psi^{(j)}(K_j \cap \Omega) \subset \mathbb{R}^n_+, \qquad \psi^{(j)}(K_j \cap \Gamma) \subset \mathbb{R}^{n-1}.$$




**Remark 3.2** The maps $\psi^{(j)}$ can be extended outside $K_j$ in such a way that the extended vector functions (denoted by $\psi^{(j)}$ as well) yield diffeomorphic mappings from $\mathbb{R}^n$ onto itself (Lipschitz diffeomorphisms). There are several equivalent definitions of Lipschitz domains in the literature. Our approach follows [5]. Another version as can be found in [37], which defines first a *special* (*unbounded*) *Lipschitz domain* $\Omega$ in $\mathbb{R}^n$ as simply the domain above the graph of a Lipschitz function $h : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$, i.e.,

$$\Omega = \{(x', x_n) : h(x') < x_n\}.$$

Then a *bounded Lipschitz domain* $\Omega$ in $\mathbb{R}^n$ is defined as a bounded domain where the boundary $\Gamma = \partial\Omega$ can be covered by finitely many open balls $B_j$ in $\mathbb{R}^n$ with $j = 1, \ldots, J$, centered at $\Gamma$ such that

$$B_j \cap \Omega = B_j \cap \Omega_j \qquad \text{for } j = 1, \ldots, J,$$

where $\Omega_j$ are rotations of suitable special Lipschitz domains in $\mathbb{R}^n$.
We shall occasionally use this alternative definition, in particular, since it usually suffices to consider special Lipschitz domains in our proofs (the related covering involves only finitely many balls), simplifying the notation considerably.
Consider a covering $\Omega \subset K_0 \cup \left(\bigcup_{j=1}^{N} K_j\right)$, where $K_0$ is an inner domain with $\overline{K}_0 \subset \Omega$. Let $\{\varphi_j\}_{j=0}^{N}$ be a related *resolution of unity* of $\overline{\Omega}$, i.e., $\varphi_j$ are smooth nonnegative functions with support in $K_j$ additionally satisfying

$$\sum_{j=0}^{N} \varphi_j(x) = 1 \quad \text{if } x \in \overline{\Omega}. \tag{3.1}$$

Obviously, the restriction of $\varphi_j$ to $\Gamma$ is a resolution of unity with respect to $\Gamma$.

## 3.1 Atomic decompositions for Besov spaces on boundaries

The boundary $\partial\Omega = \Gamma$ of a bounded Lipschitz domain $\Omega$ will be furnished in the usual way with a surface measure $\mathrm{d}\sigma$. The corresponding complex-valued Lebesgue spaces $L_p(\Gamma)$, $0 < p \leq \infty$, are normed by

$$\|g|L_p(\Gamma)\| = \left(\int_\Gamma |g(\gamma)|^p \mathrm{d}\sigma(\gamma)\right)^{1/p}$$

(with obvious modifications if $p = \infty$). We require the introduction of Besov spaces on $\Gamma$. We rely on the resolution of unity according to (3.1) and the local Lipschitz diffeomorphisms $\psi^{(j)}$ mapping $\Gamma_j = \Gamma \cap K_j$ onto $W_j = \psi^{(j)}(\Gamma_j)$, recall Definition 3.1. We define

$$g_j(y) := (\varphi_j f) \circ (\psi^{(j)})^{-1}(y), \qquad j = 1, \ldots, N,$$

which restricted to $y = (y', 0) \in W_j$,

$$g_j(y') = (\varphi_j f) \circ (\psi^{(j)})^{-1}(y'), \qquad j = 1, \ldots, N, \quad f \in L_p(\Gamma),$$

makes sense. This results in functions $g_j \in L_p(W_j)$ with compact supports in the $(n-1)$-dimensional Lipschitz domain $W_j$. We do not distinguish notationally between $g_j$ and $(\psi^{(j)})^{-1}$ as functions of $(y', 0)$ and of $y'$.

Our constructions enable us to transport Besov spaces naturally from $\mathbb{R}^{n-1}$ to the boundary $\Gamma$ of a (bounded) Lipschitz domain via pull-back and a partition of unity.

**Definition 3.3** *Let $n \geq 2$, and let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$ with boundary $\Gamma$, and $\varphi_j$, $\psi^{(j)}$, $W_j$ be as above. Assume $0 < s < 1$ and $0 < p, q \leq \infty$. Then we introduce*

$$\mathbf{B}^s_{p,q}(\Gamma) = \{f \in L_p(\Gamma) : g_j \in \mathbf{B}^s_{p,q}(W_j),\ j = 1, \ldots, N\},$$

*equipped with the quasi-norm* $\|f|\mathbf{B}^s_{p,q}(\Gamma)\| := \sum_{j=1}^{N} \|g_j|\mathbf{B}^s_{p,q}(W_j)\|.$

**Remark 3.4** The spaces $\mathbf{B}^s_{p,q}(\Gamma)$ turn out to be independent of the particular choice of the resolution of unity $\{\varphi_j\}_{j=1}^N$ and the local diffeomorphisms $\psi^{(j)}$ (the proof is similar to the proof of [41, Prop. 3.2.3(ii)], making use of Propositions 3.11 and 3.12 below). We furnish $\mathbf{B}^s_{p,q}(W_j)$ with the intrinsic $(n-1)$-dimensional norms according to Definition 1.4. Note that we could furthermore replace $W_j$ in the definition of the norm above by $\mathbb{R}^{n-1}$ if we extend $g_j$ outside $W_j$ with zero, i.e.,

$$\|f|\mathbf{B}^s_{p,q}(\Gamma)\| \sim \sum_{j=1}^{N} \|g_j|\mathbf{B}^s_{p,q}(\mathbb{R}^{n-1})\|. \tag{3.2}$$

In particular, the equivalence (3.2) yields that characterizations for B-spaces defined on $\mathbb{R}^{n-1}$ can be generalized to B-spaces defined on $\Gamma$. This will be done in Theorem 3.8 for non-smooth atomic decompositions and is very likely to work as well for characterizations in terms of differences.

**Atomic decompositions for $\mathbf{B}^s_{p,q}(\Gamma)$** Similar to the non-smooth atomic decompositions constructed in Section 2 we now establish corresponding atomic decompositions for Besov spaces defined on Lipschitz boundaries. They will be very useful when investigating traces on Lipschitz domains in Section 3

The relevant sequence spaces and Lipschitz-atoms on the boundary $\Gamma$ we shall define next are closely related to the sequence spaces $b^s_{p,q}(\Omega)$ and Lip-atoms used for the non-smooth atomic decompositions as used in Corollary 2.8.

**Definition 3.5** *Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$. Furthermore, let $\Gamma$ be the boundary of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, and $\lambda = \{\lambda_{j,m} \in \mathbb{C} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$. Then*

$$b^s_{p,q}(\Gamma) = \left\{ \lambda : \|\lambda|b^s_{p,q}(\Gamma)\| = \left( \sum_{j=0}^{\infty} 2^{j(s-\frac{n-1}{p})q} \left( \sum_{m\in\mathbb{Z}^n}^{\Gamma,j} |\lambda_{j,m}|^p \right)^{q/p} \right)^{1/q} < \infty \right\}$$

(*with the usual modification if $p = \infty$ and/or $q = \infty$*).

**Definition 3.6** *Let $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $d > 1$, and let $\Gamma$ be the boundary of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. Put $Q^{\Gamma}_{j,m} := dQ_{j,m} \cap \Gamma \neq \emptyset$. A function $a \in \mathrm{Lip}(\Gamma)$ is a $\mathrm{Lip}^{\Gamma}$-atom, if*

$$\operatorname{supp} a \subset Q^{\Gamma}_{j,m}, \qquad d > 1,$$

$$\|a|L_\infty(\Gamma)\| \le 1 \quad \textit{and} \qquad \sup_{\substack{x,y\in\Gamma,\\ x\neq y}} \frac{|a(x)-a(y)|}{|x-y|} \le 2^j. \tag{3.3}$$

**Remark 3.7** Note that if we put $2^j\Gamma := \{2^j x : x \in \Gamma\}$, we can state (3.3) like $\|a(2^{-j}\cdot)|\mathrm{Lip}(2^j\Gamma)\| \le 1$.

The theorem below provides atomic decompositions for the spaces $\mathbf{B}^s_{p,q}(\Gamma)$.

**Theorem 3.8** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $0 < s < 1$, $0 < p, q \le \infty$. Then $f \in L_p(\Gamma)$ belongs to $\mathbf{B}^s_{p,q}(\Gamma)$ if, and only if,*

$$f = \sum_{j,m} \lambda_{j,m} a_{j,m},$$

*where $a_{j,m}$ are $\mathrm{Lip}^{\Gamma}$-atoms with $\operatorname{supp} a_{j,m} \subset Q^{\Gamma}_{j,m}$ and $\lambda \in b^s_{p,q}(\Gamma)$, convergence being in $L_p(\Gamma)$. Furthermore,*

$$\|f|\mathbf{B}^s_{p,q}(\Gamma)\| = \inf \|\lambda|b^s_{p,q}(\Gamma)\|,$$

*where the infimum is taken over all possible representations.*

Proof :
Step 1: Fix $f \in \mathbf{B}^s_{p,q}(\Gamma)$. For simplicity, we suppose that $\operatorname{supp} f \subset \{x \in \Gamma : \varphi_l(x) = 1\}$ for some $l \in \{1, 2, \dots, N\}$. If this is not the case the arguments have to be slightly modified to incorporate the decomposition of unity (3.1). To simplify the notation we write $\varphi$ instead of $\varphi_l$ and $\psi$ instead of $\psi^{(l)}$. Then we obtain

$$\|f|\mathbf{B}^s_{p,q}(\Gamma)\| = \|f \circ \psi^{-1}|\mathbf{B}^s_{p,q}(\mathbb{R}^{n-1})\|.$$

We use Corollary 2.8 with $n$ replaced by $n-1$ to obtain an optimal atomic decomposition

$$f \circ \psi^{-1} = \sum_{j,m} \lambda_{j,m} a_{j,m} \quad \text{where} \quad \|f \circ \psi^{-1}|\mathbf{B}^s_{p,q}(\mathbb{R}^{n-1})\| \sim \|\lambda|b^s_{p,q}(\mathbb{R}^{n-1})\|. \tag{3.4}$$

For $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^{n-1}$ fixed, we consider the function $a_{j,m}(\psi(x))$. Due to the Lipschitz properties of $\psi$, this function is supported in $Q^{\Gamma}_{j,l}$ for some $l \in \mathbb{Z}^n$ and we denote it by $a^{\Gamma}_{j,l}(x)$. Furthermore, we set $\lambda'_{j,l} = \lambda_{j,m}$. This leads to the decomposition

$$f = \sum_{j,l} \lambda'_{j,l} a^{\Gamma}_{j,l}. \tag{3.5}$$

It is straightforward to verify that $a^{\Gamma}_{j,l}$ are $\mathrm{Lip}^{\Gamma}$-atoms since $\|a^{\Gamma}_{j,l}|L_\infty(\Gamma)\| \lesssim \|a_{j,m}|L_\infty(W_l)\| \lesssim 1$ and

$$\frac{|a^{\Gamma}_{j,l}(x) - a^{\Gamma}_{j,l}(y)|}{|x-y|} = \frac{|a_{j,m}(x') - a_{j,m}(y')|}{|\psi^{-1}(x') - \psi^{-1}(y')|} \sim \frac{|a_{j,m}(x') - a_{j,m}(y')|}{|x'-y'|} \lesssim 2^j, \qquad x, y \in \Gamma.$$

Furthermore, we have the estimate

$$\|f|\mathbf{B}^s_{p,q}(\Gamma)\| = \|f \circ \psi^{-1}|\mathbf{B}^s_{p,q}(\mathbb{R}^{n-1})\| \sim \|\lambda|b^s_{p,q}(\mathbb{R}^{n-1})\| = \|\lambda'|b^s_{p,q}(\Gamma)\|.$$

Step 2:

The proof of the opposite direction follows along the same lines. If $f$ on $\Gamma$ is given by

$$f = \sum_{j,l} \lambda'_{j,l} a^{\Gamma}_{j,l},$$

then $f \circ \psi^{-1} = \sum_{j,m} \lambda_{j,m} a_{j,m}$, where $a_{j,m}(x) = a^{\Gamma}_{j,l}(\psi^{-1}(x))$ and $\lambda_{j,m} = \lambda'_{j,l}$ for suitable $m \in \mathbb{Z}^{n-1}$. Again it follows that $a_{j,m}$ are Lip-atoms on $\mathbb{R}^{n-1}$ and

$$\|f|\mathbf{B}^s_{p,q}(\Gamma)\| = \|f \circ \psi^{-1}|\mathbf{B}^s_{p,q}(\mathbb{R}^{n-1})\| \lesssim \|\lambda|b^s_{p,q}(\mathbb{R}^{n-1})\| = \|\lambda'|b^s_{p,q}(\Gamma)\|.$$

Step 3: The convergence in $L_p(\Gamma)$ of the representation $f = \sum^{j,\Gamma}_{j,m} \lambda_{j,m} a^{\Gamma}_{j,m}$, follows for $p \le 1$ by

$$\begin{aligned} \left\| \sum^{j,\Gamma}_{j,m} \lambda_{j,m} a^{\Gamma}_{j,m} | L_p(\Gamma) \right\|^p &\le \sum^{j,\Gamma}_{j,m} |\lambda_{j,m}|^p \|a^{\Gamma}_{j,m}|L_p(\Gamma)\|^p \\ &\lesssim \sum_j 2^{-j(n-1)} \sum^{j,\Gamma}_m |\lambda_{j,m}|^p = \|\lambda|b^0_{p,p}(\Gamma)\|^p \lesssim \|\lambda|b^s_{p,q}(\Gamma)\|^p \end{aligned} \tag{3.6}$$

and using

$$\begin{aligned} \left\| \sum^{j,\Gamma}_{j,m} \lambda_{j,m} a^{\Gamma}_{j,m} | L_p(\Gamma) \right\| &\le \sum_j \left\| \sum^{j,\Gamma}_{m} \lambda_{j,m} a^{\Gamma}_{j,m} | L_p(\Gamma) \right\| \lesssim \sum_j 2^{-j(n-1)/p} \left( \sum^{j,\Gamma}_m |\lambda_{j,m}|^p \right)^{1/p} \\ &= \|\lambda|b^0_{p,1}(\Gamma)\| \lesssim \|\lambda|b^s_{p,q}(\Gamma)\| \end{aligned} \tag{3.7}$$

for $p > 1$. □

## 3.2 Interpolation results

Interpolation results for $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ as obtained in [7, Cor. 6.2, 6.3] carry over to the spaces $\mathbf{B}^s_{p,q}(\Gamma)$, which follows immediately from their definition and properties of real interpolation.

**Theorem 3.9** *Let $\Omega$ be a bounded Lipschitz domain with boundary $\Gamma$.*

*(i) Let $0 < p, q, q_0, q_1 \le \infty$, $s_0 \neq s_1$, and $0 < s_i < 1$. Then*

$$\left(\mathbf{B}^{s_0}_{p,q_0}(\Gamma), \mathbf{B}^{s_1}_{p,q_1}(\Gamma)\right)_{\theta,q} = \mathbf{B}^s_{p,q}(\Gamma),$$

*where $0 < \theta < 1$ and $s = (1-\theta)s_0 + \theta s_1$.*

*(ii) Let $0 < p_i, q_i \le \infty$, $s_0 \neq s_1$ and $0 < s_i < 1$. Then for each $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and for $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ we have*

$$\left(\mathbf{B}^{s_0}_{p_0,q_0}(\Gamma), \mathbf{B}^{s_1}_{p_1,q_1}(\Gamma)\right)_{\theta,q} = \mathbf{B}^s_{p,q}(\Gamma),$$

*provided $p = q$.*

P r o o f : By definition of the spaces $\mathbf{B}^s_{p,q}(\Gamma)$ we can construct a well-defined and bounded linear operator

$$E : \mathbf{B}^s_{p,q}(\Gamma) \longrightarrow \oplus_{1 \le j \le N} \mathbf{B}^s_{p,q}(\mathbb{R}^{n-1}),$$

$$(Ef)_j := (\varphi_j f) \circ {\psi^{(j)}}^{-1} \quad \text{on } \mathbb{R}^{n-1}, \quad 1 \leq j \leq N,$$

which has a bounded and linear left inverse given by

$$R : \oplus_{1\leq j\leq N} \mathbf{B}^s_{p,q}(\mathbb{R}^{n-1}) \longrightarrow \mathbf{B}^s_{p,q}(\Gamma)$$

$$R\left((g_j)_{1\leq j\leq N}\right) := \sum_{j=1}^{N} \Psi_j\left(g_j \circ \psi_j\right) \qquad \text{on} \quad \Gamma,$$

where $\Psi_j \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}\Psi_j \subseteq K_j$, $\Psi \equiv 1$ in a neighborhood of $\operatorname{supp}\varphi_j$.

Straightforward calculation shows for $f \in \mathbf{B}^s_{p,q}(\Gamma)$

$$(R\circ E)f = R(Ef) = R\left(\left((\varphi_j f)\circ {\psi^{(j)}}^{-1}\right)_{1\leq j\leq N}\right) = \sum_{j=1}^{N} \Psi_j \varphi_j f = \sum_{j=1}^{N} \varphi_j f = f,$$

i.e.,

$$R\circ E = I, \quad \text{the identity operator on } \mathbf{B}^s_{p,q}(\Gamma).$$

One arrives at a standard situation in interpolation theory. Hence, by the method of retraction-coretraction, cf. [40, Sect. 1.2.4, 1.17.1], the results for $\mathbf{B}^s_{p,q}(\mathbb{R}^{n-1})$ carry over to the spaces $\mathbf{B}^s_{p,q}(\Gamma)$. Therefore, (i) and (ii) are a consequence of [7, Cor. 6.2, 6.3]. $\square$

Furthermore, we briefly show that the interpolation results for Besov spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ also hold for spaces on domains $\mathbf{B}^s_{p,q}(\Omega)$. This is not automatically clear in our context since the extension operator

$$\mathrm{Ex} : \mathbf{B}^s_{p,q}(\Omega) \longrightarrow \mathbf{B}^s_{p,q}(\mathbb{R}^n)$$

constructed in [8] is not linear. The situation is different for spaces $B^s_{p,q}(\Omega)$. Here Rychkov's (linear) extension operator, cf. [30], automatically yields interpolation results for B-spaces on domains.

**Theorem 3.10** *Let $\Omega$ be a bounded Lipschitz domain.*

*(i) Let $0 < p, q, q_0, q_1 \leq \infty$, $s_0 \neq s_1$, and $0 < s_i < 1$. Then*

$$\left(\mathbf{B}^{s_0}_{p,q_0}(\Omega), \mathbf{B}^{s_1}_{p,q_1}(\Omega)\right)_{\theta,q} = \mathbf{B}^s_{p,q}(\Omega),$$

*where $0 < \theta < 1$ and $s = (1-\theta)s_0 + \theta s_1$.*

*(ii) Let $0 < p_i, q_i \leq \infty$, $s_0 \neq s_1$ and $0 < s_i < 1$. Then for each $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and for $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ we have*

$$\left(\mathbf{B}^{s_0}_{p_0,q_0}(\Omega), \mathbf{B}^{s_1}_{p_1,q_1}(\Omega)\right)_{\theta,q} = \mathbf{B}^s_{p,q}(\Omega),$$

*provided $p = q$.*

Proof : In spite of our remarks before the theorem, we can nevertheless use the extension operator

$$\mathrm{Ex} : \mathbf{B}^s_{p,q}(\Omega) \longrightarrow \mathbf{B}^s_{p,q}(\mathbb{R}^n)$$

constructed in [8] to show that interpolation results for spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ carry over to spaces $\mathbf{B}^s_{p,q}(\Omega)$. Let $X_i(\Omega) := \mathbf{B}^{s_i}_{p_i,q_i}(\Omega)$. By the explanations given in [8, p. 859] we have the estimate

$$K(f, t, X_0(\Omega), X_1(\Omega)) \sim K(\mathrm{Ex}\, f, t, X_0(\mathbb{R}^n), X_1(\mathbb{R}^n)) \tag{3.8}$$

although the operator Ex is not linear. Let $\mathbf{B}^\theta(\Omega) := \left(\mathbf{B}^{s_0}_{p_0,q_0}(\Omega), \mathbf{B}^{s_1}_{p_1,q_1}(\Omega)\right)_{\theta,q}$ with the given restrictions on the parameters given in (i) and (ii), respectively. We have to prove that

$$\mathbf{B}^\theta(\Omega) = \mathbf{B}^s_{p,q}(\Omega),$$

but this follows immediately from [7, Cor. 6.2,6.3] using (3.8), since

$$\|f|\mathbf{B}^\theta(\Omega)\| \sim \|\mathrm{Ex}\, f|\mathbf{B}^\theta(\mathbb{R}^n)\| \sim \|\mathrm{Ex}\, f|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| \sim \|f|\mathbf{B}^s_{p,q}(\Omega)\|.$$

□

## 3.3 Properties of Besov spaces on Lipschitz domains

The non-smooth atomic decomposition enables us to generalize [33, Prop. 2.5] and obtain new results concerning diffeomorphisms and pointwise multipliers in $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ in the following way. For related matters we also refer to [22, Th. 3.3.3].

**Proposition 3.11** *Let $0 < p, q \le \infty$, $0 < s < 1$ and $\sigma > s$.*

(i) (*Diffeomorphisms*)
*Let $\psi$ be a Lipschitz diffeomorphism. Then $f \longrightarrow f \circ \psi$ is a linear and bounded operator from $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ onto itself.*

(ii) (*Pointwise multipliers*)
*Let $h \in \mathcal{C}^\sigma(\mathbb{R}^n)$. Then $f \longrightarrow hf$ is a linear and bounded operator from $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ into itself.*

Proof : Concerning (i), we make use of the atomic decomposition as in (2.19) with the Lip-atoms from Definition 2.1. Then we have

$$f \circ \psi = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} a_{j,m} \circ \psi$$

and $a \circ \psi$ is a Lip-atom based on a new cube, and multiplied with a constant depending on $\psi$, since

$$|(a_{j,m} \circ \psi)(x) - (a_{j,m} \circ \psi)(y)| \le 2^j |\psi(x) - \psi(y)| \lesssim 2^j |x - y|$$

To prove (ii) we argue as follows. First, we may suppose that $0 < s < \sigma < 1$. Furthermore, we choose a real parameter $\sigma'$ with $s < \sigma' < \sigma$. We take the smooth atomic decomposition (1.5) with $K$-atoms $a_{j,m}$, where $K = 1$. Multiplied with $h \in \mathcal{C}^\sigma$, it gives a new (non-smooth) atomic decomposition of $hf$. Its convergence in $L_p(\mathbb{R}^n)$ follows from the convergence of (1.5) in $L_p(\mathbb{R}^n)$ and the boundedness of $h$.
It remains to verify, that $ha_{j,m}$ are non-smooth $(\sigma', p)$-atoms. The support property follows immediately from the support property of $a_{j,m}$. We use the bounded support of $(ha_{j,m})(2^{-j}\cdot)$ and the multiplier assertion for $\mathbf{B}^\sigma_\infty(\mathbb{R}^n)$ as presented in [29, Section 4.6.1,Theorem 2] to get

$$\begin{aligned}
\|(ha_{j,m})(2^{-j}\cdot)|\mathbf{B}^{\sigma'}_p(\mathbb{R}^n)\| &\le \|(ha_{j,m})(2^{-j}\cdot)|\mathbf{B}^\sigma_\infty(\mathbb{R}^n)\| \\
&= \|h(2^{-j}\cdot) \cdot a_{j,m}(2^{-j}\cdot)|\mathbf{B}^\sigma_\infty(\mathbb{R}^n)\| \\
&\lesssim \|h(2^{-j}\cdot)|\mathbf{B}^\sigma_\infty(\mathbb{R}^n)\| \cdot \|a_{j,m}(2^{-j}\cdot)|\mathbf{B}^\sigma_\infty(\mathbb{R}^n)\|.
\end{aligned}$$

The last product is bounded by a constant due to the inequality

$$\|h(2^{-j}\cdot)|\mathbf{B}^\sigma_\infty(\mathbb{R}^n)\| \lesssim \|h|\mathbf{B}^\sigma_\infty(\mathbb{R}^n)\|, \quad j \in \mathbb{N}_0,$$

which may be verified directly (or found in [1, Section 1.7] or [10, Section 2.3.1]), combined with the fact that $a_{j,m}$ are $K$-atoms for $K = 1$. □

Furthermore, we establish an equivalent quasi-norm for $\mathbf{B}^s_{p,q}(\Omega)$.

**Proposition 3.12** *Let $0 < p, q \le \infty$, $0 < s < 1$, and $\Omega$ be a bounded Lipschitz domain. Then*

$$\|\varphi_0 f|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| + \sum_{j=1}^{N} \|(\varphi_j f)(\psi^{(j)}(\cdot))^{-1}|\mathbf{B}^s_{p,q}(\mathbb{R}^n_+)\| \tag{3.9}$$

*is an equivalent quasi-norm in $\mathbf{B}^s_{p,q}(\Omega)$.*

P r o o f : Let $\Omega_1$ be a bounded domain with

$$\overline{\Omega}_1 \subset \left\{ x \in \mathbb{R}^n : \sum_{j=0}^{N} \varphi_j(x) = 1 \right\}$$

and $\overline{\Omega} \subset \Omega_1$. Let $f \in \mathbf{B}^s_{p,q}(\Omega)$. If we restrict the infimum in (1.5) to $g \in \mathbf{B}^s_{p,q}(\mathbb{R}^n)$ with

$$g\big|_{\Omega} = f \qquad \text{and} \qquad \operatorname{supp} g \subset \Omega_1, \tag{3.10}$$

then we obtain a new equivalent quasi-norm in $\mathbf{B}^s_{p,q}(\Omega)$. This follows from Proposition 3.11(ii) if one multiplies an arbitrary element $g \in \mathbf{B}^s_{p,q}(\mathbb{R}^n)$ with a fixed infinitely differentiable function $\varkappa(x)$ with

$$\varkappa(x) = 1 \quad \text{if} \quad x \in \Omega \qquad \text{and} \qquad \operatorname{supp} \varkappa \subset \Omega_1.$$

For elements $g \in \mathbf{B}^s_{p,q}(\mathbb{R}^n)$ with (3.10),

$$\sum_{k=0}^{N} \|\varphi_k g|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\|$$

is an equivalent quasi-norm. This is also a consequence of Proposition 3.11(ii). Applying part (i) of that proposition to $g(x) \to g(\psi^{(j)}(x))$, we see that

$$\|\varphi_0 g|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| + \sum_{k=1}^{N} \|(\varphi_k g)(\psi^{(k)}(\cdot))^{-1}|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\|$$

is an equivalent quasi-norm for all $g \in \mathbf{B}^s_{p,q}(\mathbb{R}^n)$ with (3.10). But the infimum over all admissible $g$ with (3.10) yields (3.9). $\square$

# 4 Trace results on Lipschitz domains

Now we can look for traces of $f \in \mathbf{B}^s_{p,q}(\Omega)$ on the boundary $\Gamma$. We briefly explain our understanding of the trace operator since when dealing with $L_p(\mathbb{R}^n)$ functions the pointwise trace has no obvious meaning. Let $Y(\Gamma)$ denote one of the spaces $\mathbf{B}^\sigma_{u,v}(\Gamma)$ or $L_u(\Gamma)$. Since $\mathcal{S}(\Omega)$ is dense in $\mathbf{B}^s_{p,q}(\Omega)$ for $0 < p, q < \infty$ (both spaces can be interpreted as restrictions of their counterparts defined on $\mathbb{R}^n$), one asks first whether there is a constant $c > 0$ such that

$$\|\operatorname{Tr} \varphi|Y(\Gamma)\| \le c\|\varphi|\mathbf{B}^s_{p,q}(\Omega)\| \quad \text{for all } \varphi \in \mathcal{S}(\Omega), \tag{4.1}$$

where $\mathcal{S}(\Omega)$ stands for the restriction of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ to a domain $\Omega$. If this is the case, then one defines $\operatorname{Tr} f \in Y(\Gamma)$ for $f \in \mathbf{B}^s_{p,q}(\Omega)$ by completion and obtains

$$\|\operatorname{Tr} f|Y(\Gamma)\| \le c\|f|\mathbf{B}^s_{p,q}(\Omega)\|, \quad f \in \mathbf{B}^s_{p,q}(\Omega),$$

for the linear and bounded trace operator

$$\operatorname{Tr} : \mathbf{B}^s_{p,q}(\Omega) \hookrightarrow Y(\Gamma).$$

**Remark 4.1** We can extend (4.1) to spaces $\mathbf{B}^s_{p,q}(\Omega)$ with $p=\infty$ and/or $q=\infty$ by using embeddings for B- and F-spaces from [18, 32]. The results stated there can be generalized to domains $\Omega$, since the spaces $\mathbf{B}^s_{p,q}(\Omega)$ are defined by restriction of the corresponding spaces on $\mathbb{R}^n$, cf. Remark 1.5.
If $p=\infty$, we have that $\mathbf{B}^s_{\infty,q}(\Omega)$ with $s>0$ is embedded in the space of continuous functions and Tr makes sense pointwise. If $q=\infty$,

$$\mathbf{B}^s_{p,\infty}(\Omega)\hookrightarrow \mathbf{B}^{s-\varepsilon}_{p,1}(\Omega) \qquad \text{for any} \quad \varepsilon>0.$$

Let $s>\frac{1}{p}$ and $\varepsilon>0$ be small enough such that one has

$$s>s-\varepsilon>\frac{1}{p}.$$

Since by [46, Rem. 13] traces are independent of the source spaces and of the target spaces one can now define Tr for $\mathbf{B}^s_{p,\infty}(\Omega)$ by restriction of Tr for $\mathbf{B}^{s-\varepsilon}_{p,1}(\Omega)$ to $\mathbf{B}^s_{p,\infty}(\Omega)$. Hence (4.1) is always meaningful.

## 4.1 Boundedness of the trace operator

Now we are able to state and prove our first main theorem concerning traces of Besov spaces on Lipschitz domains.

**Theorem 4.2** *Let $n\geq 2$, $0<p,q\leq\infty$, $0<s<1$, and let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$ with boundary $\Gamma$. Then the operator*

$$\mathrm{Tr}\,:\mathbf{B}^{s+\frac{1}{p}}_{p,q}(\Omega)\longrightarrow \mathbf{B}^s_{p,q}(\Gamma) \tag{4.2}$$

*is linear and bounded.*

Proof : The linearity of the operator follows directly from its definition as discussed above. To prove the boundedness, we take an optimal representation of a smooth function $f\in\mathbf{B}^{s+\frac{1}{p}}_{p,q}(\Omega)$ as described in (1.5), i.e.,

$$f=\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}^{j,\Omega}\lambda_{j,m}a_{j,m} \quad \text{with} \quad \|f|\mathbf{B}^{s+\frac{1}{p}}_{p,q}(\Omega)\|\sim\|\lambda|b^{s+\frac{1}{p}}_{p,q}(\Omega)\|. \tag{4.3}$$

We put

$$\mathrm{Tr}\,f:=\left(\sum_{j,m}^{j,\Omega}\lambda_{j,m}a_{j,m}\right)\Bigg|_{\Gamma}=\sum_{j,m}^{j,\Gamma}\lambda_{j,m}a_{j,m}\Big|_{\Gamma}=\sum_{j,m}^{j,\Gamma}\lambda_{j,m}a^{\Gamma}_{j,m}. \tag{4.4}$$

The proof follows by Theorem 3.8 and the following four facts:

(i) $a^{\Gamma}_{j,m}$ are $\mathrm{Lip}^{\Gamma}$-atoms,

(ii) $\|\lambda|b^s_{p,q}(\Gamma)\|\lesssim\|\lambda|b^{s+\frac{1}{p}}_{p,q}(\Omega)\|$,

(iii) the decomposition (4.4) converges in $L_p(\Gamma)$,

(iv) the trace operator Tr coincides with the trace operator discussed above.

To prove the first point, we observe that

$$\operatorname{supp} a^{\Gamma}_{j,m}\subseteq\operatorname{supp} a_{j,m}\cap\Gamma\subseteq Q^{\Gamma}_{j,m}.$$

Furthermore, we have $\|a^{\Gamma}_{j,m}|L_\infty(\Gamma)\|\leq\|a_{j,m}|L_\infty(dQ_{j,m})\|\leq c$ and

$$\sup_{\substack{x,y\in Q^{\Gamma}_{j,m}\\ x\neq y}}\frac{a^{\Gamma}_{j,m}(x)-a^{\Gamma}_{j,m}(y)}{|x-y|}\leq\sup_{\substack{x,y\in dQ_{j,m}\\ x\neq y}}\frac{a_{j,m}(x)-a_{j,m}(y)}{|x-y|}\lesssim 2^j.$$

The proof of the second point follows directly by

$$\begin{aligned}\|\lambda|b^s_{p,q}(\Gamma)\| &= \left(\sum_j 2^{j(s-\frac{n-1}{p})q}\left(\sum_m^{j,\Gamma}|\lambda_{j,m}|^p\right)^{q/p}\right)^{1/p}\\ &\le \left(\sum_j 2^{j\left[(s+\frac{1}{p})-\frac{n}{p}\right]q}\left(\sum_m^{j,\Omega}|\lambda_{j,m}|^p\right)^{q/p}\right)^{1/p} = \|\lambda|b^{s+\frac{1}{p}}_{p,q}(\Omega)\|.\end{aligned}$$

The proof of the third point follows in the same way as the proof in Step 3 of Theorem 3.8.
The proof of (iv) is based on the fact that for $f\in\mathcal{S}(\Omega)$ there is an optimal atomic decomposition (4.3) which converges also pointwise. This may be observed by a detailed inspection of [17]. Therefore also the series (4.4) converges pointwise and the trace operator Tr may be understood in the pointwise sense for smooth $f$. □

## 4.2 Extension of atoms

In order to compute the exact trace space we still need to construct an extension operator

$$Ext:\mathbf{B}^s_{p,q}(\Gamma)\longrightarrow\mathbf{B}^{s+\frac{1}{p}}_{p,q}(\Omega)$$

and show its boundedness. The main problem will be to show that we can extend the $\mathrm{Lip}^\Gamma$-atoms from the source spaces in a nice way to obtain suitable atoms for the target spaces. We start with a simple variant of the Gagliardo-Nirenberg inequality, cf. [28, Chapter 5].

**Lemma 4.3** *Let $0<s_0,s_1<\infty$, $0<p_0,p_1,q_0,q_1\le\infty$ and $0<\theta<1$. Put*

$$s=(1-\theta)s_0+\theta s_1,\quad \frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\quad \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}. \tag{4.5}$$

*Then*

$$\|f|\mathbf{B}^s_{p,q}(\Omega)\|\lesssim\|f|\mathbf{B}^{s_0}_{p_0,q_0}(\Omega)\|^{1-\theta}\cdot\|f|\mathbf{B}^{s_1}_{p_1,q_1}(\Omega)\|^{\theta} \tag{4.6}$$

*for all $f\in\mathbf{B}^{s_0}_{p_0,q_0}(\Omega)\cap\mathbf{B}^{s_1}_{p_1,q_1}(\Omega)$.*

P r o o f : The straightforward proof uses the characterization of $B$-spaces through differences and Hölder's inequality. □

Our approach is based on the classical Whitney decomposition of $\mathbb{R}^n\setminus\Gamma$ and the corresponding decomposition of unity. We summarize the most important properties of this method in the next Lemma and refer to [37, pp.167-170] and [20, pp.21-26] for details and proofs.

**Lemma 4.4** *1. Let $\Gamma\subset\mathbb{R}^n$ be a closed set. Then there exists a collection of cubes $\{Q_i\}_{i\in\mathbb{N}}$, such that*

*(i)* $\mathbb{R}^n\setminus\Gamma=\bigcup_i Q_i$.

*(ii) The interiors of the cubes are mutually disjoint.*

*(iii) The inequality*

$$\mathrm{diam}\ Q_i\le\mathrm{dist}\,(Q_i,\Gamma)\le 4\,\mathrm{diam}\ Q_i$$

*holds for every cube $Q_i$. Here $\mathrm{diam}\ Q_i$ is the diameter of $Q_i$ and $\mathrm{dist}\,(Q_i,\Gamma)$ is its distance from $\Gamma$.*

*(iv) Each point of $\mathbb{R}^n\setminus\Gamma$ is contained in at most $N_0$ cubes $6/5\cdot Q_i$, where $N_0$ depends only on $n$.*

*(v)* *If $\Gamma$ is the boundary of a Lipschitz domain then there is a number $\gamma > 0$, which depends only on $n$, such that $\sigma(\gamma Q_i \cap \Gamma) > 0$ for all $i \in \mathbb{N}$.*

*2. The are $C^\infty$-functions $\{\psi_i\}_{i\in\mathbb{N}}$ such that*

*(i)* $\sum_i \psi_i(x) = 1$ *for every* $x \in \mathbb{R}^n \setminus \Gamma$.

*(ii)* $\operatorname{supp}\psi_i \subset 6/5 \cdot Q_i$.

*(iii)* *For every $\alpha \in \mathbb{N}_0^n$ there is a constant $A_\alpha$ such that $|D^\alpha \psi_i(x)| \le A_\alpha (\operatorname{diam} Q_i)^{-|\alpha|}$ holds for all $i \in \mathbb{N}$ and all $x \in \mathbb{R}^n$.*

If $a$ is a Lipschitz function on the Lipschitz boundary $\Gamma$ of $\Omega$, then the Whitney extension operator Ext is defined by

$$\operatorname{Ext} a(x) = \begin{cases} a(x), & x \in \Gamma, \\ \sum_i \mu_i \psi_i(x), & x \in \Omega, \end{cases} \tag{4.7}$$

where we use the notation of Lemma 4.4 and $\mu_i := \frac{1}{\sigma(\gamma Q_i \cap \Gamma)} \int_{\gamma Q_i \cap \Gamma} a(y) d\sigma(y)$ with the number $\gamma > 0$ as described in Lemma 4.4. It satisfies $\operatorname{Tr} \circ \operatorname{Ext} a = a$ for $a$ Lipschitz continuous on $\Gamma$. This follows directly from the celebrated Whitney's extension theorem (cf. [20, p. 23]) as $\Gamma$ is a closed set if $\Omega$ is a bounded Lipschitz domain.

**Lemma 4.5** *Let $a$ be a Lipschitz function on the Lipschitz boundary $\Gamma$ of $\Omega$. Then* $\operatorname{Ext} a \in C^\infty(\Omega)$ *and*

$$\max_{|\alpha|=k} |D^\alpha \operatorname{Ext} a(x)| \le c_k \delta(x)^{1-k} \cdot \|a|\mathrm{Lip}(\Gamma)\|, \quad k \in \mathbb{N}, \quad x \in \Omega. \tag{4.8}$$

*Here, $\delta(x)$ is the distance of $x$ to $\Gamma$ and $c_k$ depends only on $k$ and $\Omega$.*

Proof: First, let us note that

$$D^\alpha \operatorname{Ext} a(x) = \sum_i \mu_i D^\alpha \psi_i(x), \qquad x \in \Omega, \quad \alpha \in \mathbb{N}_0^n, \quad |\alpha| = k.$$

By Lemma 4.4 we have for every $x \in \Omega$

$$|D^\alpha \psi_i(x)| \le c_k \delta(x)^{-k}, \qquad |\alpha| = k,$$

and

$$\sum_i D^\alpha \psi_i(x) = D^\alpha \sum_i \psi_i(x) = 0.$$

Furthermore, the Lipschitz continuity of $a$ implies

$$|\mu_i - \mu_j| \lesssim \delta(x) \cdot \|a|\mathrm{Lip}(\Gamma)\| \tag{4.9}$$

for $x \in \operatorname{supp}\psi_i \cap \operatorname{supp}\psi_j$. To justify (4.9), we consider natural numbers $i$ and $j$ with $x \in \operatorname{supp}\psi_i \cap \operatorname{supp}\psi_j$, chose any $x_i \in \gamma Q_i \cap \Gamma$ and $x_j \in \gamma Q_j \cap \Gamma$ and calculate

$$\begin{aligned} |\mu_i - \mu_j| &\le \left| \frac{1}{\sigma(\gamma Q_i \cap \Gamma)} \int_{\gamma Q_i \cap \Gamma} a(x) d\sigma(x) - a(x_i) \right| + |a(x_i) - a(x_j)| + \left| a(x_j) - \frac{1}{\sigma(\gamma Q_j \cap \Gamma)} \int_{\gamma Q_j \cap \Gamma} a(x) d\sigma(x) \right| \\ &\le \|a|\mathrm{Lip}(\Gamma)\| \cdot \{\operatorname{diam}(\gamma Q_i \cap \Gamma) + |x_i - x_j| + \operatorname{diam}(\gamma Q_j \cap \Gamma)\} \\ &\lesssim \|a|\mathrm{Lip}(\Gamma)\| \cdot \{\operatorname{diam}(Q_i) + |x_i - x| + |x - x_j| + \operatorname{diam}(Q_j)\} \lesssim \delta(x) \cdot \|a|\mathrm{Lip}(\Gamma)\|. \end{aligned}$$

Let us now fix $x \in \Omega$ and let us denote by $\{i_1, \ldots, i_N\}$, $N \le N_0$, the indices for which $x$ lies in the support of $\psi_i$. Then we write

$$\left|\sum_{j=1}^{N} \mu_{i_j} D^\alpha \psi_{i_j}(x)\right| \le \left|\sum_{j=1}^{N} (\mu_{i_j} - \mu_{i_1}) D^\alpha \psi_{i_j}(x)\right| + \left|\sum_{j=1}^{N} \mu_{i_1} D^\alpha \psi_{i_j}(x)\right|$$
$$\le \sum_{j=1}^{N} |\mu_{i_j} - \mu_{i_1}| \cdot |D^\alpha \psi_{i_j}(x)| \lesssim \delta(x)^{1-k} \cdot \|a|\mathrm{Lip}(\Gamma)\|.$$

□

**Remark 4.6** Let $a$ be a function defined on $\Gamma$ as in Lemma 4.5 with $\operatorname{diam}(\operatorname{supp} a) \le 1$. Then the extension operator from Lemma 4.5 may be combined with a multiplication with a smooth cut-off function. This ensures, that (4.8) still holds and, in addition, $\operatorname{diam}(\operatorname{supp} \operatorname{Ext} a) \lesssim 1$.

The following lemma describes a certain geometrical property of Lipschitz domains, which shall be useful later on. It resembles very much the notion of Minkowski content, cf. [11].

**Lemma 4.7** *Let $\Omega$ be a bounded Lipschitz domain and let $k \in \mathbb{N}$. Let $h \in \mathbb{R}^n$ with $0 < |h| \le 1$ and put $\Omega^h = \{x \in \Omega : [x, x+kh] \subset \Omega\}$. Furthermore, for $j \in \mathbb{N}_0$ we define $\Omega^h_j = \{x \in \Omega^h : 2^{-j} \le \min_{y \in [x, x+kh]} \delta(y) \le 2^{-j+1}\}$, where $\delta(y) = \operatorname{dist}(y, \Gamma)$. Then*

$$|\Omega^h_j| \lesssim 2^{-j} \tag{4.10}$$

*with a constant independent of $j$ and $h$.*

Proof: To simplify the notation, we shall assume that $\Omega$ is a simple Lipschitz domain of the type $\Omega = \{(x', x_n) = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n : x_n > \psi(x'), |x'| < 1\}$, where $\psi$ is a Lipschitz function, and we identify $\Gamma$ with $\{(x', x_n) : x_n = \psi(x'), |x'| < 1\}$.

Step 1: First, let us observe that

$$\operatorname{dist}(x, \Gamma) \approx (x_n - \psi(x')) \quad \text{for} \quad x = (x', x_n) \in \Omega \tag{4.11}$$

and the constants in this equivalence depend only on the Lipschitz constant of $\psi$. The simple proof of this fact is based on the inner cone property of Lipschitz domains. We refer to [37, Chapter VI, Section 3.2, Lemma 2] for details.

Step 2:

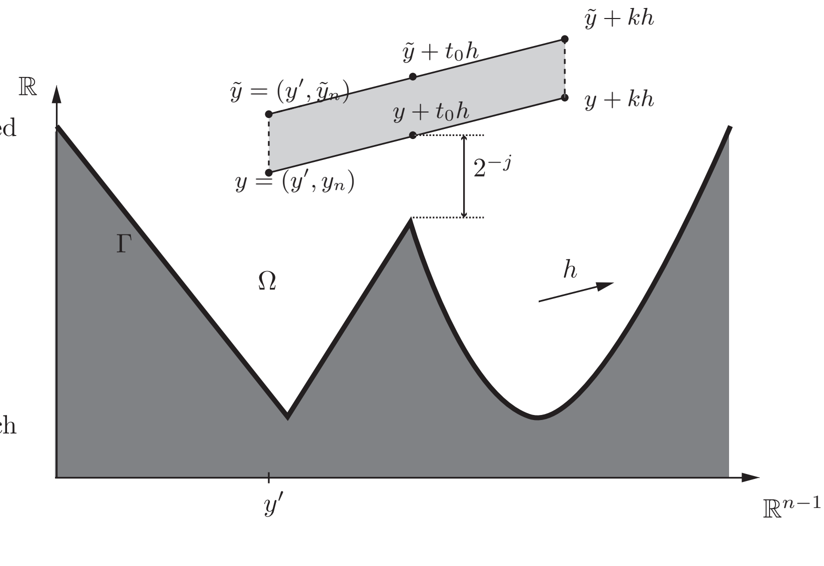
Let $j \in \mathbb{N}_0$ and $0 < |h| \le 1$ be fixed and let

$$y = (y', y_n) \in \Omega^h_j$$

and let also

$$\tilde{y} = (y', \tilde{y}_n) \in \Omega^h_j$$

with $\tilde{y}_n > y_n$.
As $\tilde{y} \in \Omega^h_j$, there is a $t_0 \in [0, k]$ such that $\operatorname{dist}(\tilde{y} + t_0 h, \Gamma) \le 2^{-j+1}$.



Then we use $\psi(y'+t_0h) < t_0h_n + y_n$ (which follows from $y\in\Omega^h$ and $y+t_0h\in\Omega$) and (4.11) to get

$$\begin{aligned}\tilde y_n - y_n &= [\tilde y_n + t_0h_n - \psi(y'+t_0h')] + [\psi(y'+t_0h') - t_0h_n - y_n]\\ &\lesssim \operatorname{dist}(\tilde y + t_0h, \Gamma) \lesssim 2^{-j}.\end{aligned}\tag{4.12}$$

Step 3: Using (4.12), we observe that the set $\Omega(x') = \{x_n\in\mathbb{R} : (x',x_n)\in\Omega_j^h\}$ has for every $|x'|<1$ length smaller then $c\,2^{-j}$. From this, the inequality (4.10) quickly follows. □

We shall use this geometrical observation together with the extension operator (4.7) to prove the following.

**Lemma 4.8** *Let $\Omega$ be a bounded Lipschitz domain and let $\Gamma$ be its boundary. Let $a$ be a Lipschitz function on $\Gamma$. Let $0<p\le\infty$, $0<s<\infty$ and $k\in\mathbb{N}$ with $0<s<k<1/p+1$. Then the extension operator defined by* (4.7) *satisfies*

$$\|\operatorname{Ext} a|\mathbf{B}^s_{p,p}(\Omega)\| \lesssim \|a|\operatorname{Lip}(\Gamma)\| \tag{4.13}$$

*with the constant independent of $a\in\operatorname{Lip}(\Gamma)$.*

Proof : Using the characterization by differences, we obtain

$$\begin{aligned}\|\operatorname{Ext} a|\mathbf{B}^s_{p,p}(\Omega)\| &\lesssim \|\operatorname{Ext} a|\mathbf{B}^{s'}_{p,\infty}(\Omega)\|\\ &\lesssim \|\operatorname{Ext} a|L_p(\Omega)\| + \sup_{0<|h|\le 1} |h|^{-s'}\|\Delta_h^k \operatorname{Ext} a(\cdot,\Omega)|L_p(\Omega)\|,\end{aligned}$$

for $s'>0$ with $s<s'<k$. Furthermore, we observe that one may modify the definition of $\Delta_h^r f(x,\Omega)$ given in (1.10) to be zero also if the whole segment $[x,x+kh]$ is not a subset of $\Omega$. This follows by a detailed inspection of [41, Section 2.5.12] as well as [9] and [8], which are all based on the integration in cones.
Using the definition of $\mu_i$, the first term may be estimated easily as

$$\|\operatorname{Ext} a|L_p(\Omega)\| \lesssim \|\operatorname{Ext} a|L_\infty(\Omega)\| \le \|a|L_\infty(\Gamma)\|.$$

To estimate the second term, we shall need the following relationship between differences and derivatives. If $f\in C^k(\mathbb{R}^n)$ and $x,h\in\mathbb{R}^n$, we put $g(t)=f(x+th)$ for $t\in\mathbb{R}$ and obtain

$$\Delta_h^k f(x) = \Delta_1^k g(0) = \int_0^k g^{(k)}(t)B_k(t)dt, \tag{4.14}$$

where $B_k$ is the standard $B$ spline of order $k$, i.e. the $k$-fold convolution of $\chi_{[0,1]}$ given by $B_k = \chi_{[0,1]} * \cdots * \chi_{[0,1]}$. Although (4.14) is a classical result of approximation theory (c.f. [6, Section 4.7]), let us give a short proof using Fubini's Theorem and induction over $k$:

$$\begin{aligned}\Delta_1^{k+1}g(0) &= \Delta_1^k g(1) - \Delta_1^k g(0) = \int_0^k (g^{(k)}(t+1) - g^{(k)}(t))B_k(t)dt\\ &= \int_0^k B_k(t)\int_t^{t+1} g^{(k+1)}(u)du\,dt = \int_0^{k+1} g^{(k+1)}(u)\int_{u-1}^u B_k(t)dtdu = \int_0^{k+1} g^{(k+1)}(u)B_{k+1}(u)du.\end{aligned}$$

Hence if $[x,x+kh]\subset\Omega$ for some $x\in\Omega$, we obtain

$$|\Delta_h^k \operatorname{Ext} a(x,\Omega)| \lesssim |h|^k\int_0^k \max_{|\alpha|=k}|D^\alpha \operatorname{Ext} a(x+th)|\cdot B_k(t)dt \lesssim |h|^k\cdot\|a|\operatorname{Lip}(\Gamma)\|\cdot\int_0^k \delta(x+th)^{1-k}\cdot B_k(t)dt.$$

Let us fix $h \in \mathbb{R}^n$ with $0 < |h| \le 1$ and let us denote $\Omega^h = \{x \in \Omega : [x, x+kh] \subset \Omega\}$ as in Lemma 4.7. We obtain

$$\begin{aligned}
|h|^{-s'} \|\Delta_h^k \mathrm{Ext}\, a(\cdot,\Omega)|L_p(\Omega)\| &\lesssim |h|^{k-s'} \|a|\mathrm{Lip}(\Gamma)\| \left( \int_{\Omega^h} \left( \int_0^k \delta(x+th)^{1-k} \cdot B_k(t)dt \right)^p dx \right)^{1/p} \\
&\lesssim \|a|\mathrm{Lip}(\Gamma)\| \left( \int_{\Omega^h} \max_{y \in [x,x+kh]} \delta(y)^{(1-k)p} dx \right)^{1/p} \\
&\lesssim \|a|\mathrm{Lip}(\Gamma)\| \left( \sum_{j=0}^{\infty} 2^{-j(1-k)p} |\Omega_j^h| \right)^{1/p} .
\end{aligned}$$

This, together with Lemma 4.7 and with $k < 1/p + 1$ finishes the proof.

$\square$

**Lemma 4.9** *Let $0 < s' < 1$ be fixed. There is a non-linear extension operator (denoted by* **Ext***), which extends $\mathrm{Lip}^\Gamma$-atoms $a_{j,m}$ to $(s'+1/p, p)$-atoms on $\mathbb{R}^n$.*

Proof : As the definition of $\mathrm{Lip}^\Gamma$-atoms as well as the definition of $(s'+1/p, p)$-atoms works with $a_j(2^{-j}\cdot)$, by homogeneity arguments it is enough to prove

$$\|\mathbf{Ext}\, a_{0,m}|\mathbf{B}^{s'+1/p}_{p,p}(\mathbb{R}^n)\| \lesssim \|a_{0,m}|\mathrm{Lip}(\Gamma)\| \tag{4.15}$$

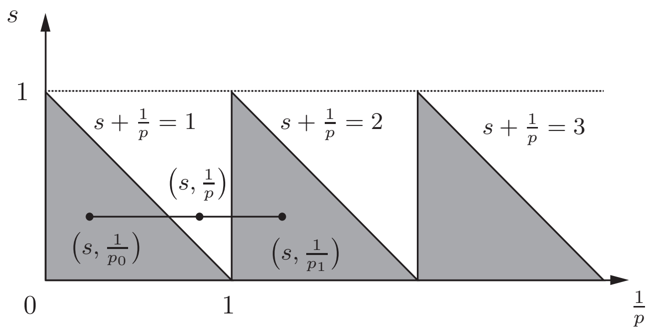
for $\mathrm{Lip}^\Gamma$-atoms $a_{j,m}$ with $j = 0$. First we show that

$$\|\mathrm{Ext}\, a_{0,m}|\mathbf{B}^{s'+1/p}_{p,p}(\Omega)\| \lesssim \|a_{0,m}|\mathrm{Lip}(\Gamma)\| \tag{4.16}$$

for the extension operator constructed in (4.7). Let $0 < s' < 1$ and $0 < p \le \infty$. We observe, that Lemma 4.8 implies (4.16) for all $0 < s' < 1$ for which there is a $k \in \mathbb{N}_0$ with

$$s' + 1/p < k < 1 + 1/p.$$

In the diagram aside these points correspond to all $(s', \frac{1}{p})$ in the gray-shaded triangles. Then Lemma 4.3 yields (4.16) for all $0 < s' < 1$ and $0 < p \le \infty$ with $s_0 = s_1 = s'$ and $p_0 < p < p_1$ chosen in an appropriate way, see the attached diagram.




Finally, by Remark 1.5, we know that there is a function (denoted by $\mathbf{Ext}\, a_{0,m}$), such that

$$\|\mathbf{Ext}\, a_{0,m}|\mathbf{B}^{s'+1/p}_{p,p}(\mathbb{R}^n)\| \lesssim \|\mathrm{Ext}\, a_{0,m}|\mathbf{B}^{s'+1/p}_{p,p}(\Omega)\|.$$

This together with (4.16) finishes the proof of (4.15). $\square$

We are now able to complete the proof of the missing part of the trace theorem.

**Theorem 4.10** *Let $n \ge 2$ and $\Omega$ be a bounded Lipschitz domain with boundary $\Gamma$. Then for $0 < s < 1$ and $0 < p, q \le \infty$ there is a bounded non-linear extension operator*

$$Ext : \mathbf{B}^s_{p,q}(\Gamma) \longrightarrow \mathbf{B}^{s+\frac{1}{p}}_{p,q}(\Omega). \tag{4.17}$$

P r o o f : Let $f \in \mathbf{B}^s_{p,q}(\Gamma)$ with optimal decomposition in the sense of Theorem 3.8

$$f(x) = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} a^{\Gamma}_{j,m}(x), \tag{4.18}$$

where $a^{\Gamma}_{j,m}$ are $\mathrm{Lip}^{\Gamma}$-atoms, (4.18) converges in $L_p(\Gamma)$, and $\|f|\mathbf{B}^s_{p,q}(\Gamma)\| \sim \|\lambda|b^s_{p,q}(\Gamma)\|$.
We use the extension operator constructed in Lemma 4.9 and define by

$$Ext\, f := \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} (\mathbf{Ext}\, a^{\Gamma}_{j,m})|_{\Omega} \tag{4.19}$$

an atomic decomposition of $f$ in the space $\mathbf{B}^{s+1/p}_{p,q}(\Omega)$ with non-smooth $(s'+1/p,p)$-atoms $\mathbf{Ext}\, a^{\Gamma}_{j,m}$, where $s < s' < 1$. The convergence of (4.19) in $L_p(\Omega)$ follows in the same way as in the proof of Step 3 of Theorem 3.8.
Together with $\|\lambda|b^s_{p,q}(\Gamma)\| \sim \|\lambda|b^{s+1/p}_{p,q}(\Omega)\|$, this shows that

$$\|Ext\, f|\mathbf{B}^{s+1/p}_{p,q}(\Omega)\| \lesssim \|\lambda|b^{s+1/p}_{p,q}(\Omega)\| \sim \|\lambda|b^s_{p,q}(\Gamma)\| < \infty$$

is bounded.

Theorems 4.2 and 4.10 together now allow us to state the general result for traces on Lipschitz domains without any restrictions on the parameters $s, p$ and $q$.

**Theorem 4.11** *Let $n \geq 2$ and $\Omega$ be a bounded Lipschitz domain with boundary $\Gamma$. Then for $0 < s < 1$ and $0 < p, q \leq \infty$,*

$$\mathrm{Tr}\, \mathbf{B}^{s+\frac{1}{p}}_{p,q}(\Omega) = \mathbf{B}^s_{p,q}(\Gamma). \tag{4.20}$$

The above Theorem extends the trace results obtained in [34, Th. 2.4] from $C^k$ domains with $k > s$ to Lipschitz domains.
Furthermore, the trace results for spaces of Triebel-Lizorkin type carry over as well to the case of Lipschitz domains. The proof follows [34, Th. 2.6] where the independence of the trace on $q$ was established for F-spaces. Let us mention that the sequence spaces $f^s_{p,q}(\Omega)$ are defined similarly as $b^s_{p,q}(\Omega)$, cf. Definition 1.1, with $\ell_p$ and $\ell_q$ summation interchanged. The corresponding function spaces (denoted by $\mathfrak{F}^s_{p,q}(\Omega)$) are then defined as in Definition 1.4.
The main ingredient in the study of traces for Triebel-Lizorkin spaces $\mathfrak{F}^s_{p,q}(\Omega)$ is then the fact that the corresponding sequence spaces $f^s_{p,q}(\Gamma)$ are independent of $q$,

$$f^s_{p,q}(\Gamma) = b^s_{p,p}(\Gamma). \tag{4.21}$$

A proof may be found in [45, Prop. 9.22, p. 394] for $\Gamma$ being a compact porous set in $\mathbb{R}^n$ with [13] as an important forerunner. In [47, Prop. 3.6] it is shown that the boundaries $\partial\Omega = \Gamma$ of $(\varepsilon, \delta)$-domains $\Omega$ are porous. Therefore, this result is also true for boundaries of Lipschitz domains.
For completeness we state the trace results for F-spaces below.

**Corollary 4.12** *Let $0 < p < \infty$, $0 < q \leq \infty$, $0 < s < 1$, and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with boundary $\Gamma$. Then*

$$\mathrm{Tr}\, \mathfrak{F}^{s+\frac{1}{p}}_{p,q}(\Omega) = \mathbf{B}^s_{p,p}(\Gamma). \tag{4.22}$$

## 4.3 The limiting case

We briefly discuss what happens in the limiting case $s = 0$. In [35, Th. 2.7] traces for Besov and Triebel-Lizorkin spaces on $d$-sets $\Gamma$, $0 < d < n$, were studied. In particular, it was shown that for $0 < p < \infty$ and $0 < q \leq \infty$,

$$\operatorname{Tr} \mathbf{B}^{\frac{n-d}{p}}_{p,q}(\mathbb{R}^n) = L_p(\Gamma), \qquad 0 < q \leq \min(1, p), \tag{4.23}$$

and

$$\operatorname{Tr} \mathfrak{F}^{\frac{n-d}{p}}_{p,q}(\mathbb{R}^n) = L_p(\Gamma), \qquad 0 < p \leq 1. \tag{4.24}$$

Since the boundary $\Gamma$ of a Lipschitz domain $\Omega$ is a $d$-set with $d = n - 1$ the results follow almost immediately from these previous results, using the fact that the B- and F-spaces on domains $\Omega$ are defined as restrictions of the corresponding spaces on $\mathbb{R}^n$, cf. Remark 1.5.

**Corollary 4.13** *Let $\Omega$ be a bounded Lipschitz domain with boundary $\Gamma$. Furthermore, let $0 < p < \infty$ and $0 < q \leq \infty$.*

*(i) Then*

$$\operatorname{Tr} \mathbf{B}^{\frac{1}{p}}_{p,q}(\Omega) = L_p(\Gamma), \qquad 0 < q \leq \min(1, p). \tag{4.25}$$

*(ii) Furthermore,*

$$\operatorname{Tr} \mathfrak{F}^{\frac{1}{p}}_{p,q}(\Omega) = L_p(\Gamma), \qquad 0 < p \leq 1. \tag{4.26}$$

# 5 Pointwise multipliers in function spaces

As an application we now use our results on non-smooth atomic decompositions to deal with pointwise multipliers in the respective function spaces.

A function $m$ in $L^{loc}_{\min(1,p)}(\mathbb{R}^n)$ is called a *pointwise multiplier* for $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ if

$$f \mapsto mf$$

generates a bounded map in $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$. The collection of all multipliers for $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ is denoted by $M(\mathbf{B}^s_{p,q}(\mathbb{R}^n))$. In the following, let $\psi$ stand for a non-negative $C^\infty$ function with

$$\operatorname{supp} \psi \subset \{y \in \mathbb{R}^n : |y| \leq \sqrt{n}\} \tag{5.1}$$

and

$$\sum_{l \in \mathbb{Z}^n} \psi(x - l) = 1, \qquad x \in \mathbb{R}^n. \tag{5.2}$$

**Definition 5.1** *Let $s > 0$ and $0 < p, q \leq \infty$. We define the space $\mathbf{B}^s_{p,q,\mathrm{selfs}}(\mathbb{R}^n)$ to be the set of all $f \in L^{loc}_{\min(1,p)}(\mathbb{R}^n)$ such that*

$$\|f|\mathbf{B}^s_{p,q,\mathrm{selfs}}(\mathbb{R}^n)\| := \sup_{j \in \mathbb{N}_0, l \in \mathbb{Z}^n} \|\psi(\cdot - l) f(2^{-j}\cdot)|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| \tag{5.3}$$

*is finite.*

**Remark 5.2** The study of pointwise multipliers is one of the key problems of the theory of function spaces. As far as classical Besov spaces and (fractional) Sobolev spaces with $p > 1$ are concerned we refer to [23], [24], and [25]. Pointwise multipliers in general spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ have been studied in great detail in [29, Ch. 4].
Selfsimilar spaces were first introduced in [44] and then considered in [45, Sect. 2.3]. Corresponding

results for anisotropic function spaces may be found in [26]. We also mention their forerunners, the uniform spaces $\mathbf{B}^s_{p,q,\mathrm{unif}}(\mathbb{R}^n)$, studied in detail in [29, Sect. 4.9]. As stated in [21], for these spaces it is known that

$$M(\mathbf{B}^s_{p,q}(\mathbb{R}^n)) = \mathbf{B}^s_{p,q,\mathrm{unif}}(\mathbb{R}^n), \qquad 1 \le p \le q \le \infty, \quad s > \frac{n}{p},$$

cf. [36] concerning the proof. Selfsimilar spaces are also closely connected with pointwise multipliers. We shall use the abbreviation

$$\mathbf{B}^s_{p,\mathrm{selfs}}(\mathbb{R}^n) := \mathbf{B}^s_{p,p,\mathrm{selfs}}(\mathbb{R}^n).$$

One can easily show

$$\mathbf{B}^s_{p,q,\mathrm{selfs}}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n). \tag{5.4}$$

To see this applying homogeneity gives

$$\|\psi(\cdot-l)f(2^{-j}\cdot)|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| \sim 2^{j\frac{n}{p}}\|\psi(2^j\cdot-l)f|L_p(\mathbb{R}^n)\| + 2^{-j(s-\frac{n}{p})}\left(\int_0^1 t^{-sq}\omega_r(\psi(2^j\cdot-l)f,t)_p{}^q\frac{\mathrm{d}t}{t}\right)^{1/q}$$

uniformly for all $j\in\mathbb{N}_0$ and $l\in\mathbb{Z}^n$. Consequently,

$$2^{jn}\int_{\mathbb{R}^n}|\psi(2^jy-l)|^p|f(y)|^p\mathrm{d}y \le c\|f|\mathbf{B}^s_{p,q,\mathrm{selfs}}(\mathbb{R}^n)\|^p. \tag{5.5}$$

Thus, the right-hand side of (5.5) is just a uniform bound for $|f(\cdot)|^p$ at its Lebesgue points, cf. [38, Cor. p.13], which proves the desired embedding (5.4).

**Definition 5.3** *Let $s>0$ and $0<p,q\le\infty$. We define*

$$\mathbf{B}^{s+}_{p,q,\mathrm{selfs}}(\mathbb{R}^n) := \bigcup_{\sigma>s}\mathbf{B}^\sigma_{p,q,\mathrm{selfs}}(\mathbb{R}^n).$$

We have the following relation between pointwise multipliers and self-similar spaces.

**Theorem 5.4** *Let $s>0$ and $0<p,q\le\infty$. Then*

*(i)* $\mathbf{B}^{s+}_{p,q,\mathrm{selfs}}(\mathbb{R}^n) \subset M(\mathbf{B}^s_{p,q}(\mathbb{R}^n)) \hookrightarrow \mathbf{B}^s_{p,q,\mathrm{selfs}}(\mathbb{R}^n)$

*(ii) Additionally, if $0<p\le 1$,*

$$M(\mathbf{B}^s_p(\mathbb{R}^n)) = \mathbf{B}^s_{p,\mathrm{selfs}}(\mathbb{R}^n).$$

P r o o f : We first prove the right-hand side embedding in (i). Let $m\in M(\mathbf{B}^s_{p,q}(\mathbb{R}^n))$. An application of the homogeneity property from Theorem 1.8 yields

$$\begin{aligned}
\|\psi(\cdot-l)m(2^{-j}\cdot)|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| &\sim 2^{-j(s-\frac{n}{p})}\|\psi(2^j\cdot-l)m|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| \\
&\lesssim 2^{-j(s-\frac{n}{p})}\|m|M(\mathbf{B}^s_{p,q}(\mathbb{R}^n))\|\cdot\|\psi(2^j\cdot-l)|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| \\
&= 2^{-j(s-\frac{n}{p})}\|m|M(\mathbf{B}^s_{p,q}(\mathbb{R}^n))\|\cdot\|\psi(2^j\cdot)|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| \\
&\sim \|m|M(\mathbf{B}^s_{p,q}(\mathbb{R}^n))\|\|\psi|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| \lesssim \|m|M(\mathbf{B}^s_{p,q}(\mathbb{R}^n))\|
\end{aligned}$$

for all $l\in\mathbb{Z}^n$, $j\in\mathbb{N}_0$, and hence,

$$\begin{aligned}
\|m|\mathbf{B}^s_{p,q,\mathrm{selfs}}(\mathbb{R}^n)\| &= \sup_{j\in\mathbb{N}_0,l\in\mathbb{Z}^n}\|\psi(\cdot-l)m(2^{-j})|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\| \\
&\lesssim \|m|M(\mathbf{B}^s_{p,q}(\mathbb{R}^n))\|.
\end{aligned}$$

We make use of the non-smooth atomic decompositions for $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ from Theorem 2.6 in order to prove the first inclusion in (i). Let $m\in\mathbf{B}^\sigma_{p,q,\mathrm{selfs}}$ with $\sigma>s$. Let $f\in\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ with optimal smooth atomic decomposition

$$f=\sum_{j=0}^\infty\sum_{l\in\mathbb{Z}^n}\lambda_{j,l}a_{j,l} \quad \text{with} \quad \|f|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\|\sim\|\lambda|b^s_{p,q}\|, \tag{5.6}$$

where $a_{j,m}$ are $K$-atoms with $K > \sigma$. Then

$$mf = \sum_{j=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \lambda_{j,l} \, (m a_{j,l}) \, , \tag{5.7}$$

and we wish to prove that, up to normalizing constants, the $m a_{j,l}$ are $(\sigma, p)$-atoms. The support condition is obvious:

$$\operatorname{supp} m a_{j,l} \subset \operatorname{supp} a_{j,l} \subset d Q_{j,l}, \qquad j \in \mathbb{N}_0, l \in \mathbb{Z}^n.$$

If $l = 0$ we put $a_j = a_{j,l}$. Note that

$$\operatorname{supp} a_j(2^{-j}) \subset \{y : |y_i| \le \frac{d}{2}\}$$

and we can assume that

$$\psi(y) > 0 \qquad \text{if} \quad y \in \{x : |x_i| \le d\}.$$

Then – using multiplier assertions from [34, Prop. 2.15(ii)] – we have for any $g \in \mathbf{B}^\sigma_{p,q}(\mathbb{R}^n)$,

$$\begin{aligned} \|a_j(2^{-j})\psi^{-1} g | \mathbf{B}^\sigma_{p,q}(\mathbb{R}^n)\| &\lesssim \|a_j(2^{-j})\psi^{-1} | C^K(\mathbb{R}^n)\| \|g | \mathbf{B}^\sigma_{p,q}(\mathbb{R}^n)\| \\ &\lesssim \|g | \mathbf{B}^\sigma_{p,q}(\mathbb{R}^n)\| \end{aligned}$$

and hence

$$\|a_j(2^{-j})\psi^{-1} | M(\mathbf{B}^\sigma_{p,q}(\mathbb{R}^n))\| \lesssim 1, \qquad j \in \mathbb{N}_0. \tag{5.8}$$

By (5.8) and the homogeneity property we then get, for any $\sigma > \sigma' > s$ and $j \in \mathbb{N}_0$,

$$\begin{aligned} \|(m a_j)(2^{-j}\cdot) | \mathbf{B}^{\sigma'}_p(\mathbb{R}^n)\| &\lesssim \|m(2^{-j}\cdot) a_j(2^{-j}\cdot) | \mathbf{B}^\sigma_{p,q}(\mathbb{R}^n)\| \\ &\lesssim \|a_j(2^{-j}\cdot)\psi^{-1} | M(\mathbf{B}^\sigma_{p,q}(\mathbb{R}^n))\| \|m(2^{-j}\cdot)\psi | \mathbf{B}^\sigma_{p,q}(\mathbb{R}^n)\| \\ &\lesssim \|m(2^{-j}\cdot)\psi | \mathbf{B}^\sigma_{p,q}(\mathbb{R}^n)\|. \end{aligned} \tag{5.9}$$

In the case of $a_{j,l}$ with $l \in \mathbb{Z}^n$ one arrives at (5.9) with $a_{j,l}$ and $\psi(\cdot - l)$ in place of $a_j$ and $\psi$, respectively. Hence

$$\begin{aligned} \|m a_{j,l}(2^{-j}\cdot) | \mathbf{B}^{\sigma'}_p(\mathbb{R}^n)\| &\lesssim \sup_{j,l} \|m(2^{-j}\cdot)\psi(\cdot - l) | \mathbf{B}^\sigma_{p,q}(\mathbb{R}^n)\| \\ &= \|m | \mathbf{B}^\sigma_{p,q,\mathrm{selfs}}(\mathbb{R}^n)\|, \qquad j \in \mathbb{N}_0, l \in \mathbb{Z}^n, \end{aligned} \tag{5.10}$$

and therefore, $m a_{j,l}$ is a $(\sigma', p)$-atom where $\sigma' > s$. By Theorem 2.6, in view of (5.7), $mf \in \mathbf{B}^s_{p,q}(\mathbb{R}^n)$ and

$$\|mf | \mathbf{B}^s_{p,q}(\mathbb{R}^n)\| \le \|\lambda | b^s_{p,q}\| \|m | \mathbf{B}^\sigma_{p,q,\mathrm{selfs}}(\mathbb{R}^n)\| \sim \|f | \mathbf{B}^s_{p,q}\| \|m | \mathbf{B}^\sigma_{p,q,\mathrm{selfs}}(\mathbb{R}^n)\|,$$

which completes the proof of (i).

We now prove (ii). Restricting ourselves to $p = q$, let now $m \in \mathbf{B}^s_{p,\mathrm{selfs}}(\mathbb{R}^n)$. We can modify (5.9) by choosing $\sigma' = \sigma = s$,

$$\begin{aligned} \|(m a_j)(2^{-j}\cdot) | \mathbf{B}^s_p(\mathbb{R}^n)\| &= \|m(2^{-j}\cdot) a_j(2^{-j}\cdot) | \mathbf{B}^s_p(\mathbb{R}^n)\| \\ &\lesssim \|a_j(2^{-j}\cdot)\psi^{-1} | M(\mathbf{B}^s_p(\mathbb{R}^n))\| \|m(2^{-j}\cdot)\psi | \mathbf{B}^s_p(\mathbb{R}^n)\| \\ &\lesssim \|m(2^{-j}\cdot)\psi | \mathbf{B}^s_p(\mathbb{R}^n)\|, \end{aligned} \tag{5.11}$$

yielding for general atoms $a_{j,l}$,

$$\begin{aligned} \|m a_{j,l}(2^{-j}\cdot) | \mathbf{B}^s_{p,}(\mathbb{R}^n)\| &\lesssim \sup_{j,l} \|m(2^{-j}\cdot)\psi(\cdot - l) | \mathbf{B}^s_p(\mathbb{R}^n)\| \\ &= \|m | \mathbf{B}^s_{p,\mathrm{selfs}}(\mathbb{R}^n)\|, \qquad j \in \mathbb{N}_0, l \in \mathbb{Z}^n. \end{aligned} \tag{5.12}$$

Since $p \le 1$, we have that $\mathbf{B}^s_p(\mathbb{R}^n)$ is a $p$-Banach space. From (5.6), using (5.7) and (5.12), we obtain

$$
\begin{aligned}
\|mf|\mathbf{B}^s_p(\mathbb{R}^n)\|^p &\le \sum_{j=0}^{\infty}\sum_{l\in\mathbb{Z}^n} |\lambda_{j,l}|^p 2^{j(s-\frac{n}{p})p} 2^{-j(s-\frac{n}{p})p} \|ma_{j,l}|\mathbf{B}^s_p(\mathbb{R}^n)\|^p \\
&\sim \|\lambda|b^s_{p,p}\|^p \|(ma_{j,l})(2^{-j}\cdot)|\mathbf{B}^s_p(\mathbb{R}^n)\|^p \\
&\lesssim \|\lambda|b^s_{p,p}\|^p \|m|\mathbf{B}^s_{p,\mathrm{selfs}}(\mathbb{R}^n)\|^p. \qquad (5.13)
\end{aligned}
$$

Hence $m \in M(\mathbf{B}^s_p(\mathbb{R}^n))$ and, moreover, $\mathbf{B}^s_{p,\mathrm{selfs}}(\mathbb{R}^n) \hookrightarrow M(\mathbf{B}^s_p(\mathbb{R}^n))$. The other embedding follows from part (i). $\square$

**Remark 5.5** It remains open whether it is possible or not to generalize Theorem 5.4(ii) to the case when $p \neq q$. The problem in the proof given above is the estimate (5.13), which only holds if $p = q$.

**Characteristic functions as multipliers** The final part of this work is devoted to the question in which function spaces the characteristic function $\chi_\Omega$ of a domain $\Omega \subset \mathbb{R}^n$ is a pointwise multiplier. We contribute to this question mainly as an application of Theorem 5.4. The results shed some light on a relationship between some fundamental notion of fractal geometry and pointwise multipliers in function spaces. For complementary remarks and studies in this direction we refer to [44].
There are further considerations of a similar kind in the literature, asking for geometric conditions on the domain $\Omega$ such that the corresponding characteristic function $\chi_\Omega$ provides multiplier properties, cf. [15, 16], [13], and [29, Sect. 4.6.3].

**Definition 5.6** *Let $\Gamma$ be a non-empty compact set in $\mathbb{R}^n$. Let $h$ be a positive non-decreasing function on the interval $(0,1]$. Then $\Gamma$ is called a h-set, if there is a finite Radon measure $\mu \in \mathbb{R}^n$ with*

$$
\operatorname{supp}\mu = \Gamma \quad \text{and} \quad \mu(B(\gamma,r)) \sim h(r), \qquad \gamma \in \Gamma,\ 0 < r \le 1. \qquad (5.14)
$$

**Remark 5.7** A measure $\mu$ with (5.14) satisfies the so-called *doubling condition*, meaning there is a constant $c > 0$ such that

$$
\mu(B(\gamma,2r)) \le c\mu(B(\gamma,r)), \qquad \gamma \in \Gamma,\ 0 < r < 1. \qquad (5.15)
$$

We refer to [44, p. 476] for further explanations.

**Theorem 5.8** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Moreover, let $\sigma > 0$, $0 < p < \infty$, $0 < q \le \infty$, and let $\Gamma = \partial\Omega$ be an $h$-set with*

$$
\sup_{j\in\mathbb{N}_0} \sum_{k=0}^{\infty} 2^{k\sigma q} \left( \frac{h(2^{-j})}{h(2^{-j-k})} 2^{-kn} \right)^{q/p} < \infty, \qquad (5.16)
$$

(*with the usual modifications if $q = \infty$*). *Let $\mathbf{B}^\sigma_{p,q,\mathrm{selfs}}(\mathbb{R}^n)$ be the spaces defined in* (5.3). *Then*

$$
\chi_\Omega \in \mathbf{B}^\sigma_{p,q,\mathrm{selfs}}(\mathbb{R}^n).
$$

Proof: It simplifies the argument, and causes no loss of generality, to assume $\operatorname{diam}\Omega < 1$. We define

$$
\Omega^k = \{x \in \Omega : 2^{-k-2} \le \operatorname{dist}(x,\Gamma) \le 2^{-k}\}, \qquad k \in \mathbb{N}_0.
$$

Moreover, let

$$
\{\varphi^k_l : k \in \mathbb{N}_0,\ l = 1,\ldots,M_k\} \subset C^\infty_0(\Omega)
$$

be a resolution of unity,

$$
\sum_{k\in\mathbb{N}_0} \sum_{l=1}^{M_k} \varphi^k_l(x) = 1 \qquad \text{if } x \in \Omega, \qquad (5.17)
$$

with

$$\operatorname{supp}\varphi_l^k \subset \{x : |x - x_l^k| \le 2^{-k}\} \subset \Omega^k$$

and

$$|D^\alpha \varphi_l^k(x)| \lesssim 2^{|\alpha|k}, \qquad |\alpha| \le K,$$

where $K \in \mathbb{N}$ with $K > \sigma$. It is well known that resolutions of unity with the required properties exist. We now estimate the number $M_k$ in (5.17). Combining the fact that the measure $\mu$ satisfies the doubling condition (5.15) together with (5.14) we arrive at

$$M_k h(2^{-k}) \lesssim 1, \qquad k \in \mathbb{N}_0. \tag{5.18}$$

Since the $\varphi_l^k$ in (5.17) are $K$-atoms according to Definition 1.3, we obtain

$$\|\chi_\Omega|\mathbf{B}^\sigma_{p,q}(\mathbb{R}^n)\|^q \le \sum_{k=0}^{\infty} 2^{k(\sigma-n/p)q} M_k^{q/p} \lesssim \sum_{k=0}^{\infty} 2^{k\sigma q}\left(\frac{2^{-kn}}{h(2^{-k})}\right)^{q/p} < \infty. \tag{5.19}$$

This shows that $\chi_\Omega \in \mathbf{B}^\sigma_{p,q}(\mathbb{R}^n)$. We now prove that $\chi_\Omega \in \mathbf{B}^\sigma_{p,q,\mathrm{selfs}}(\mathbb{R}^n)$. We consider the non-negative function $\psi \in C^\infty(\mathbb{R}^n)$ satisfying (5.1) and (5.2). By the definition of self-similar spaces, it suffices to consider

$$\chi_\Omega(2^{-j}\cdot)\psi,$$

assuming in addition that $0 \in 2^j\Gamma = \{2^j\gamma = (2^j\gamma_1, \ldots, 2^j\gamma_n) : \gamma \in \Gamma\}$, $j \in \mathbb{N}$. Let $\mu^j$ be the image measure of $\mu$ with respect to the dilations $y \mapsto 2^j y$. Then we obtain

$$\mu^j(B(0,\sqrt{n}) \cap 2^j\Gamma) \sim h(2^{-j}), \qquad j \in \mathbb{N}_0.$$

We apply the same argument as above to $B(0,\sqrt{n}) \cap 2^j\Omega$ and $B(0,\sqrt{n}) \cap 2^j\Gamma$ in place of $\Omega$ and $\Gamma$, respectively. Let $M_k^j$ be the counterpart of the above number $M_k$. Then

$$M_k^j h(2^{-j-k}) \lesssim h(2^{-j}), \qquad j \in \mathbb{N}_0,\ k \in \mathbb{N}_0,$$

is the generalization of (5.18) we are looking for, which completes the proof. $\square$

In view of Theorem 5.4 we have the following result.

**Corollary 5.9** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Moreover, let $\sigma > 0$, $0 < p < \infty$, $0 < q \le \infty$, and let $\Gamma = \partial\Omega$ be a $h$-set satisfying* (5.16). *Then*

$$\chi_\Omega \in M(\mathbf{B}^s_{p,q}(\mathbb{R}^n)) \qquad \textit{for} \quad 1 < p < \infty, \quad 0 < s < \sigma,$$

*and*

$$\chi_\Omega \in M(\mathbf{B}^\sigma_p(\mathbb{R}^n)) \qquad \textit{for} \quad 0 < p \le 1.$$

**Remark 5.10** As for the assertion (5.16) we mention that

$$\sup_{j\in\mathbb{N}_0, k\in\mathbb{N}_0} 2^{k\sigma}\left(\frac{h(2^{-j})}{h(2^{-j-k})}2^{-kn}\right)^{1/p} < \infty$$

is the adequate counterpart for $\mathbf{B}^\sigma_{p,\infty}(\mathbb{R}^n)$. In the special case of $d$-sets, which corresponds to $h(t) \sim t^d$, the condition (5.16) therefore corresponds to

$$\sigma < \frac{n-d}{p} \qquad \text{or} \qquad \sigma = \frac{n-d}{p} \quad \text{and} \quad q = \infty.$$

For bounded Lipschitz domains $\Omega$, i.e., $d = n-1$, Theorem 5.8 therefore yields $\chi_\Omega \in \mathbf{B}^\sigma_{p,q,\mathrm{selfs}}(\mathbb{R}^n)$ if

$$\sigma < \frac{1}{p} \qquad \text{or} \qquad \sigma = \frac{1}{p} \quad \text{and} \quad q = \infty. \tag{5.20}$$

These results are sharp since there exists a Lipschitz domain $\Omega$ in $\mathbb{R}^n$ such that

$$\chi_\Omega \in \mathbf{B}^{\frac{1}{p}}_{p,\infty,\mathrm{selfs}}(\mathbb{R}^n) \qquad \text{and} \qquad \chi_\Omega \notin \mathbf{B}^{\frac{1}{p}}_{p,q}(\mathbb{R}^n) \quad \text{if} \quad 0 < q < \infty.$$

In order to see this let $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$ . Observing that

$$\omega_r(\chi_\Omega, t)_p \lesssim t^{\frac{1}{p}}$$

one calculates

$$\left(\int_0^1 t^{-\sigma q} \omega_r(\chi_\Omega, t)_p^q \frac{\mathrm{d}t}{t}\right)^{1/q} \lesssim \left(\int_0^1 t^{(\frac{1}{p}-\sigma)q} \frac{\mathrm{d}t}{t}\right)^{1/q}$$

which is finite if, and only if, $\sigma$ satisfies (5.20). Therefore, in view of Theorem 5.4, concerning Lipschitz domains there is an

*alternative s.t. either the trace of* $\mathbf{B}^\sigma_{p,q}(\mathbb{R}^n)$ *on* $\Gamma$ *exists or* $\chi_\Omega$ *is a pointwise multiplier for* $\mathbf{B}^\sigma_{p,q}(\mathbb{R}^n)$,

as was conjectured for F-spaces in [43, p.36]: For smoothness $\sigma > \frac{1}{p}$ we have traces according to Theorem 4.11 whereas for $\sigma < \frac{1}{p}$ we know that $\chi_\Omega$ is a pointwise multiplier for $\mathbf{B}^\sigma_{p,q}(\mathbb{R}^n)$. The limiting case $\sigma = \frac{1}{p}$ needs to be discussed separately: according to Corollary 4.13 we have traces for B-spaces with $q \le \min(1,p)$, but $\chi_\Omega$ is (possibly) only a multiplier for $\mathbf{B}^{1/p}_{p,\infty}(\mathbb{R}^n)$. There remains a 'gap' for spaces

$$\mathbf{B}^{1/p}_{p,q}(\mathbb{R}^n) \qquad \text{when} \qquad \min(1,p) < q < \infty.$$

**Acknowledgment:**

We thank Winfried Sickel and Hans Triebel for their valuable comments. Jan Vybíral acknowledges the financial support provided by the FWF project Y 432-N15 START-Preis "Sparse Approximation and Optimization in High Dimensions".

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