On the Interplay of Regularity and Decay in Case of Radial Functions I. Inhomogeneous spaces

Winfried Sickel*, Leszek Skrzypczak and Jan Vybiral

July 29, 2010

Abstract

We deal with decay and boundedness properties of radial functions belonging to Besov and Lizorkin-Triebel spaces. In detail we investigate the surprising interplay of regularity and decay. Our tools are atomic decompositions in combination with trace theorems.

1 Introduction

At the end of the seventies Strauss [39] was the first who observed that there is an interplay between the regularity and decay properties of radial functions. We recall his

Radial Lemma: Let $d \geq 2$*. Every radial function* $f \in H^1(\mathbb{R}^d)$ *is almost everywhere equal to a function* $\widetilde{f}$*, continuous for* $x \neq 0$*, such that*

$$|\widetilde{f}(x)| \leq c\,|x|^{\frac{1-d}{2}}\,\|\,f\,|H^1(\mathbb{R}^d)\|\,, \tag{1}$$

where c depends only on d.

Strauss stated (1) with the extra condition $|x| \geq 1$, but this restriction is not needed. The *Radial Lemma* contains three different assertions:

(a) the existence of a representative of f, which is continuous outside the origin;

(b) the decay of f near infinity;

(c) the limited unboundedness near the origin.

These three properties do not extend to all functions in $H^1(\mathbb{R}^d)$, of course. In particular, $H^1(\mathbb{R}^d) \not\subset L_\infty(\mathbb{R}^d)$, $d \geq 2$, and consequently, functions in $H^1(\mathbb{R}^d)$ can be unbounded in the neigborhood of any fixed point $x \in \mathbb{R}^d$. It will be our aim in this

*Corresponding author

paper to investigate the specific regularity and decay properties of radial functions in a more general framework than Sobolev spaces. In our opinion a discussion of these properties in connection with fractional order of smoothness results in a better understanding of the announced interplay of regularity on the one side and local smoothness, decay at infinity and limited unboundedness near the origin on the other side. In the literature there are several approaches to fractional order of smoothness. Probably most popular are Bessel potential spaces $H^s_p(\mathbb{R}^d)$, $s \in \mathbb{R}$, or Slobodeckij spaces $W^s_p(\mathbb{R}^d)$ ($s > 0$, $s \not\in \mathbb{N}$). These scales would be enough to explain the main interrelations. However, for some limiting cases these scales are not sufficient. For that reason we shall discuss generalizations of the *Radial Lemma* in the framework of Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ and Lizorkin-Triebel spaces $F^s_{p,q}(\mathbb{R}^d)$. These scales essentially cover the Bessel potential and the Slobodeckij spaces since

- $W^m_p(\mathbb{R}^d) = F^m_{p,2}(\mathbb{R}^d)$, $m \in \mathbb{N}_0$, $1 < p < \infty$;
- $H^s_p(\mathbb{R}^d) = F^s_{p,2}(\mathbb{R}^d)$, $s \in \mathbb{R}$, $1 < p < \infty$;
- $W^s_p(\mathbb{R}^d) = F^s_{p,p}(\mathbb{R}^d) = B^s_{p,p}(\mathbb{R}^d)$, $s > 0$, $s \not\in \mathbb{N}$, $1 \le p \le \infty$,

where all identities have to be understood in the sense of equivalent norms, see, e.g., [41, 2.2.2] and the references given there.
All three phenomena (a)-(c) extend to a certain range of parameters which we shall characterize exactly. For instance, decay near infinity will take place in spaces with $s \ge 1/p$ (see Theorem 10) and limited unboundedness near the origin in the sense of (1) will happen in spaces such that $1/p \le s \le d/p$ (see Theorem 13). For $s = 1/p$ (or $s = d/p$) always the microscopic parameter q comes into play. We will study the above properties also for spaces with $p < 1$. To a certain extent this is motivated by the fact, that the decay properties of radial functions near infinity are determined by the parameter p and the decay rate increases when p decreases, see Theorem 10. Our main tools here are the following. Based on the atomic decomposition theorem for inhomogeneous Besov and Lizorkin-Triebel spaces, which we proved in [32], we shall deduce a trace theorem for radial subspaces which is of interest on its own. Then this trace theorem will be applied to derive the extra regularity properties of radial functions. To derive the decay estimates and the assertions on controlled unboundedness near zero we shall also employ the atomic decomposition technique. With respect to the decay it makes a difference, whether one deals with inhomogeneous or homogeneous spaces of Besov and Lizorkin-Triebel type. Homogeneous spaces (with a proper interpretation) are larger than their inhomogeneous counterparts (at least if $s > d \max(0, \frac{1}{p} - 1)$). Hence, the decay rate of the elements of inhomogeneous spaces can be better than that one for homogeneous spaces. This turns out to be true. However, here in this article we concentrate on inhomogeneous

spaces. Radial subspaces of homogeneous spaces will be subject to the continuation of this paper, see [33]. In a further paper [34] we shall investigate a few more properties of radial subspaces like complex interpolation and characterization by differences.

The paper is organized as follows.

We add a few comments. In 2.1.1 we state also trace assertions for radial subspaces of Hölder-Zygmund classes. Within the borderline cases in Subsection 2.2.2 the spaces $BV(\mathbb{R}^d)$ show up. In this context we will also deal with the trace problem for the associated radial subspaces. All proofs will be given in Sections 3 and 4. There also additional material is collected, e.g., in Subsection 3.1 we deal with interpolation of radial subspaces, in Subsection 3.3.2 we recall the characterization of radial subspaces by atoms as given in [32], and finally, in Subsection 3.8 we discuss the regularity properties of some families of test functions.

Besov and Lizorkin-Triebel spaces are discussed at various places, we refer, e.g., to the monographs [26, 29, 41, 42, 44]. We will not give definitions here and refer for this to the quoted literature.

The present paper is a continuation of [32], [36] and [21].

Notation

As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ denotes the integers and $\mathbb{R}$ the real numbers. If X and Y are two quasi-Banach spaces, then the symbol $X \hookrightarrow Y$ indicates that the embedding is continuous. The set of all linear and

bounded operators $T : X \to Y$, denoted by $\mathcal{L}(X,Y)$, is equipped with the standard quasi-norm. As usual, the symbol c denotes positive constants which depend only on the fixed parameters s, p, q and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line. Sometimes we will use the symbols " $\lesssim$ " and " $\gtrsim$ " instead of "$\le$" and "$\ge$", respectively. The meaning of $A \lesssim B$ is given by: there exists a constant $c > 0$ such that $A \le c\,B$. Similarly $\gtrsim$ is defined. The symbol $A \asymp B$ will be used as an abbreviation of $A \lesssim B \lesssim A$.
We shall use the following conventions throughout the paper:

- If E denotes a space of functions on $\mathbb{R}^d$ then by RE we mean the subset of radial functions in E and we endow this subset with the same quasi-norm as the original space.
- Inhomogeneous Besov and Lizorkin-Triebel spaces are denoted by $B^s_{p,q}$ and $F^s_{p,q}$, respectively. If there is no reason to distinguish between these two scales we will use the notation $A^s_{p,q}$. Similarly for the radial subspaces.
- If an equivalence class $[f]$ contains a continuous representative then we call the class continuous and speak of values of f at any point (by taking the values of the continuous representative).
- Throughout the paper $\psi \in C_0^\infty(\mathbb{R}^d)$ denotes a specific radial cut-off function, i.e., $\psi(x) = 1$ if $|x| \le 1$ and $\psi(x) = 0$ if $|x| \ge 3/2$.

2 Main results

This section consists of two parts. In Subsection 2.1 we concentrate on trace theorems which are the basis for the understanding of the higher regularity of radial functions outside the origin. Subsection 2.2 is devoted to the study of decay and boundedness properties of radial functions in dependence on their regularity. To begin with we study the decay of radial functions near infinity. Special emphasize is given to the limiting situation which arises for $s = 1/p$. Then we continue with an investigation of the behaviour of radial functions near the origin. Also here we investigate the limiting situations $s = d/p$ and $s = 1/p$ in some detail.

2.1 The characterization of the traces of radial subspaces

Let $d \ge 2$. Let $f : \mathbb{R}^d \to \mathbb{C}$ be a locally integrable radial function. By using a Lebesgue point argument its restriction

$$f_0(t) := f(t, 0, \dots, 0), \qquad t \in \mathbb{R}.$$

is well defined a.e. on $\mathbb{R}$. However, this restriction need not be locally integrable. A simple example is given by the function

$$f(x) := \psi(x)\,|x|^{-1}\,, \qquad x \in \mathbb{R}^d\,,$$

where ψ denotes a smooth cut-off function s.t. $\psi(0) \neq 0$. Furthermore, if we start with a measurable and even function $g : \mathbb{R} \to \mathbb{C}$, s.t. g is locally integrable on all intervals (a,b), $0 < a < b < \infty$, then (again using a Lebesgue point argument) the function

$$f(x) := g(|x|)\,, \qquad x \in \mathbb{R}^d$$

is well-defined a.e. on $\mathbb{R}^d$ and is radial, of course. In what follows we shall study properties of the associated operators

$$\mathrm{tr}\ : f \mapsto f_0 \qquad \text{and} \qquad \mathrm{ext}\ : g \mapsto f\,.$$

Both operators are defined pointwise only. Later on we shall have a short look onto the existence of the trace in the distributional sense, see Subsection 2.1.4. Probably it would be more natural to deal with functions defined on $[0,\infty)$ in this context. However, that would result in more complicated descriptions of the trace spaces. So, our target spaces will be spaces of even functions defined on $\mathbb{R}$.

2.1.1 Traces of radial subspaces with $p = \infty$

The first result is maybe well-known but we did not find a reference for it.

Theorem 1 *Let $d \geq 2$. For $m \in \mathbb{N}_0$ the mapping* tr *is a linear isomorphism of $RC^m(\mathbb{R}^d)$ onto $RC^m(\mathbb{R})$ with inverse* ext *.*

Using real interpolation it is not difficult to derive the following result for the spaces of Hölder-Zygmund type.

Theorem 2 *Let $s > 0$ and let $0 < q \leq \infty$. Then the mapping* tr *is a linear isomorphism of $RB^s_{\infty,q}(\mathbb{R}^d)$ onto $RB^s_{\infty,q}(\mathbb{R})$ with inverse* ext *.*

2.1.2 Traces of radial subspaces with $p < \infty$

Now we turn to the description of the trace classes of radial Besov and Lizorkin-Triebel spaces with $p < \infty$. Again we start with an almost trivial result. We need a further notation. By $L_p(\mathbb{R}, w)$ we denote the weighted Lebesgue space equipped with the norm

$$\|\, f\,|L_p(\mathbb{R},w)\| := \Big(\int_{-\infty}^{\infty} |f(t)|^p\, w(t)\, dt\Big)^{1/p}$$

with usual modification if $p = \infty$.

Lemma 1 *Let $d \geq 2$.*
(i) *Let $0 < p < \infty$. Then* $\mathrm{tr} \,:\, RL_p(\mathbb{R}^d) \to RL_p(\mathbb{R}, |t|^{d-1})$ *is an linear isomorphism with inverse* ext *.*
(ii) *Let $p = \infty$. Then* $\mathrm{tr} \,:\, RL_\infty(\mathbb{R}^d) \to RL_\infty(\mathbb{R})$ *is an linear isomorphism with inverse* ext *.*

In particular this means, that whenever the Besov-Lizorkin-Triebel space $A^s_{p,q}(\mathbb{R}^d)$ is contained in $L_1(\mathbb{R}^d) + L_\infty(\mathbb{R}^d)$, then tr is well-defined on its radial subspace. This is in sharp contrast to the general theory of traces on these spaces. To guarantee that tr is meaningful on $A^s_{p,q}(\mathbb{R}^d)$ one has to require

$$s > \frac{d-1}{p} + \max\Big(0, \frac{1}{p} - 1\Big)\,,$$

cf. e.g. [20], [14], [41, Rem. 2.7.2/4] or [12]. On the other hand we have

$$B^s_{p,q}(\mathbb{R}^d)\,,\; F^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d) + L_\infty(\mathbb{R}^d)$$

if $s > d\, \max(0, \frac{1}{p} - 1)$, see, e.g., [35]. Since

$$d\, \max(0, \frac{1}{p} - 1) < \frac{d-1}{p} + \max\Big(0, \frac{1}{p} - 1\Big)$$

we have the existence of tr with repect to $RA^s_{p,q}(\mathbb{R}^d)$ for a wider range of parameters than for $A^s_{p,q}(\mathbb{R}^d)$.
Below we shall develop a description of the traces of the radial subspaces of $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$ in terms of atoms. To explain this we need to introduce first an appropriate notion of an atom and second, adapted sequence spaces.

Definition 1 *Let $L \geq 0$ be an integer. Let I be a set either of the form $I = [-a, a]$ or of the form $I = [-b, -a] \cup [a, b]$ for some $0 < a < b < \infty$. An even function $g \in C^L(\mathbb{R})$ is called an even L-atom centered at I if*

$$\max_{t\in\mathbb{R}} |b^{(n)}(t)| \leq |I|^{-n}\,, \qquad 0 \leq n \leq L\,.$$

and if either

$$\mathrm{supp}\; g \subset [-\frac{3a}{2}, \frac{3a}{2}] \qquad \textit{in case} \quad I = [-a, a]\,,$$

or

$$\mathrm{supp}\; g \subset [-\frac{3b-a}{2}, -\frac{3a-b}{2}] \cup [\frac{3a-b}{2}, \frac{3b-a}{2}] \qquad \textit{in case} \quad I = [-b, -a] \cup [a, b]\,.$$

Definition 2 *Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. Let*

$$\chi^{\#}_{j,k}(t) := \begin{cases} 1 & \textit{if} \quad 2^{-j}k \leq |t| \leq 2^{-j}(k+1)\,, \\ 0 & \textit{otherwise}\,. \end{cases} \qquad t \in \mathbb{R}\,.$$

Then we define

$$b^s_{p,q,d} := \left\{ s = (s_{j,k})_{j,k} \ : \quad \| \, s \, | b^s_{p,q,d} \| = \left(\sum_{j=0}^{\infty} 2^{j(s-\frac{d}{p})q} \left(\sum_{k=0}^{\infty} (1{+}k)^{d-1} \, |s_{j,k}|^p \right)^{q/p} \right)^{1/q} < \infty \right\} .$$

and

$$f^s_{p,q,d} \ := \ \Big\{ s = (s_{j,k})_{j,k} \ :$$

$$\| \, s \, | f^s_{p,q,d} \| \ = \ \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \sum_{k=0}^{\infty} |s_{j,k}|^q \, \chi^{\#}_{j,k}(\cdot) \right)^{1/q} | L_p(\mathbb{R}, |t|^{d-1}) \right\| < \infty \Big\} ,$$

respectively.

Remark 1 *Observe $b^s_{p,q,d} = f^s_{p,q,d}$ in the sense of equivalent quasi-norms.*

Adapted to these sequence spaces we define now function spaces on $\mathbb{R}$.

Definition 3 *Let $0 < p < \infty$, $0 < q \le \infty$, $s > 0$ and $L \in \mathbb{N}_0$.*
(i) *Then $TB^s_{p,q}(\mathbb{R}, L, d)$ is the collection of all functions $g : \mathbb{R} \to \mathbb{C}$ such that there exists a decomposition*

$$g(t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} s_{j,k} \, g_{j,k}(t) \tag{2}$$

(convergence in $L_{\max(1,p)}(\mathbb{R}, |t|^{d-1})$), where the sequence $(s_{j,k})_{j,k}$ belongs to $b^s_{p,q,d}$ and the functions $g_{j,k}$ are even L-atoms centered at either $[-2^{-j}, 2^{-j}]$ if $k = 0$ or at

$$[-2^{-j}(k+1), -2^{-j}k] \ \cup \ [2^{-j}k, 2^{-j}(k+1)]$$

if $k > 0$. We put

$$\| \, g \, | TB^s_{p,q}(\mathbb{R}, L, d) \| := \inf \Big\{ \| \, (s_{j,k}) \, | b^s_{p,q,d} \| \ : \quad (2) \quad holds \Big\} .$$

(ii) *Then $TF^s_{p,q}(\mathbb{R}, L, d)$ is the collection of all functions $g : \mathbb{R} \to \mathbb{C}$ such that there exists a decomposition (2), where the sequence $(s_{j,k})_{j,k}$ belongs to $f^s_{p,q,d}$ and the functions $g_{j,k}$ are as in (i). We put*

$$\| \, g \, | TF^s_{p,q}(\mathbb{R}, L, d) \| := \inf \Big\{ \| \, (s_{j,k}) \, | f^s_{p,q,d} \| \ : \quad (2) \quad holds \Big\} .$$

We need a few further notation. In connection with Besov and Lizorkin-Triebel spaces quite often the following numbers occur:

$$\sigma_p(d) := d \, \max\left(0, \frac{1}{p} - 1\right) \qquad \text{and} \qquad \sigma_{p,q}(d) := d \, \max\left(0, \frac{1}{p} - 1, \frac{1}{q} - 1\right) . \tag{3}$$

For a real number s we denote by $[s]$ the integer part, i.e. the largest integer m such that $m \le s$.

Theorem 3 *Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$.*
(i) *Suppose $s > \sigma_p(d)$ and $L \geq [s]+1$. Then the mapping* tr *is a linear isomorphism of $RB^s_{p,q}(\mathbb{R}^d)$ onto $TB^s_{p,q}(\mathbb{R}, L, d)$ with inverse* ext .
(ii) *Suppose $s > \sigma_{p,q}(d)$ and $L \geq [s]+1$. Then the mapping* tr *is a linear isomorphism of $RF^s_{p,q}(\mathbb{R}^d)$ onto $TF^s_{p,q}(\mathbb{R}, L, d)$ with inverse* ext .

Remark 2 Let $0 < p \leq 1 < q \leq \infty$. Then the spaces $RB^{\sigma_p}_{p,q}(\mathbb{R}^d)$ contain singular distributions, see [35]. In particular, the Dirac delta distribution belongs to $RB^{\frac{d}{p}-d}_{p,\infty}(\mathbb{R}^d)$, see, e.g., [29, Rem. 2.2.4/3]. Hence, our pointwise defined mapping tr is not meaningful on those spaces, or, with other words, Theorem 3 does not extend to values $s < \sigma_p(d)$.

Outside the origin radial distributions are more regular. We shall discuss several examples for this claim.

Theorem 4 *Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$. Suppose $s > \max(0, \frac{1}{p} - 1)$. Let $f \in RA^s_{p,q}(\mathbb{R}^d)$ s.t. $0 \notin \operatorname{supp} f$. Then f is a regular distribution in $\mathcal{S}'(\mathbb{R}^d)$.*

Remark 3 There is a nice and simple example which explains the sharpness of the restrictions in Thm. 4. We consider the singular distribution f defined by

$$\varphi \mapsto \int_{|x|=1} \varphi(x)\, dx\,, \qquad \varphi \in \mathcal{S}(\mathbb{R}^d)\,.$$

By using the wavelet characterization of Besov spaces, it is not difficult to prove that the spherical mean distribution f belongs to the spaces $B^{\frac{1}{p}-1}_{p,\infty}(\mathbb{R}^d)$ for all p.

Theorem 5 *Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$. Suppose $s > \max(0, \frac{1}{p} - 1)$. Let $f \in RA^s_{p,q}(\mathbb{R}^d)$ s.t. $0 \notin \operatorname{supp} f$. Then $f_0 = \operatorname{tr} f$ belongs to $A^s_{p,q}(\mathbb{R})$.*

Remark 4 As mentioned above

$$A^s_{p,q}(\mathbb{R}) \hookrightarrow L_1(\mathbb{R}) + L_\infty(\mathbb{R}) \qquad \text{if} \qquad s > \sigma_p(1) = \max\Big(0, \frac{1}{p} - 1\Big)\,,$$

which shows again that we deal with regular distributions. However, in Thm. 5 some additional regularity is proved.

2.1.3 Traces of radial subspaces of Sobolev spaces

Clearly, one can expect that the description of the traces of radial Sobolev spaces can be given in more elementary terms. We discuss a few examples without having the complete theory.

Theorem 6 *Let $d \geq 2$ and $1 \leq p < \infty$.*
(i) *The mapping* tr *is a linear isomorphism (with inverse* ext *) of $RW^1_p(\mathbb{R}^d)$ onto the closure of $RC^\infty_0(\mathbb{R})$ with respect to the norm*

$$\| \, g \, | L_p(\mathbb{R}, |t|^{d-1})\| + \| \, g' \, | L_p(\mathbb{R}, |t|^{d-1})\| \, .$$

(ii) *The mapping* tr *is a linear isomorphism (with inverse* ext *) of $RW^2_p(\mathbb{R}^d)$ onto the closure of $RC^\infty_0(\mathbb{R})$ with respect to the norm*

$$\| \, g \, | L_p(\mathbb{R}, |t|^{d-1})\| + \| \, g' \, | L_p(\mathbb{R}, |t|^{d-1})\| + \| \, g'/r \, | L_p(\mathbb{R}, |t|^{d-1})\| + \| \, g'' \, | L_p(\mathbb{R}, |t|^{d-1})\| \, .$$

Remark 5 Both statements have elementary proofs, see (11) for (i). However, the complete extension to higher order Sobolev spaces is open.

There are several ways to define Sobolev spaces on $\mathbb{R}^d$. For instance, if $1 < p < \infty$ we have

$$f \in W^{2m}_p(\mathbb{R}^d) \quad \Longleftrightarrow \quad f \in L_p(\mathbb{R}^d) \quad \text{and} \quad \Delta^m f \in L_p(\mathbb{R}^d) \, . \tag{4}$$

Such an equivalence does not extend to $p = 1$ or $p = \infty$ if $d \geq 2$, see [37, pp. 135/160]. Recall that the Laplace operator Δ applied to a radial function yields a radial function. In particular we have

$$\Delta f(x) = D_r f_0(r) := f_0''(r) + \frac{d-1}{r} \, f_0'(r) \, , \qquad r = |x| \, , \tag{5}$$

in case that f is radial and tr $f = f_0$. Obviously, if $f \in RC^\infty_0(\mathbb{R}^d)$, then

$$\begin{aligned} \| \, f \, | L_p(\mathbb{R}^d)\| \quad &+ \quad \| \, \Delta^m f \, | L_p(\mathbb{R}^d)\| \\ &= \quad \Big(\frac{\pi^{d/2}}{\Gamma(d/2)} \Big)^{1/p} \Big(\| \, f_0 \, | L_p(\mathbb{R}, |t|^{d-1})\| + \| \, D^m_r f_0 \, | L_p(\mathbb{R}, |t|^{d-1})\| \Big) \, . \end{aligned} \tag{6}$$

This proves the next characterization.

Theorem 7 *Let $1 < p < \infty$ and $m \in \mathbb{N}$. Then the mapping* tr *yields a linear isomorphism (with inverse* ext *) of $RW^{2m}_p(\mathbb{R}^d)$ onto the closure of $RC^\infty_0(\mathbb{R})$ with respect to the norm*

$$\| \, f_0 \, | L_p(\mathbb{R}, |t|^{d-1})\| + \| \, D^m_r f_0 \, | L_p(\mathbb{R}, |t|^{d-1})\| \, .$$

Remark 6 By means of Hardy-type inequalities one can simplify the terms $\| \, D^m_r f_0 \, | L_p(\mathbb{R}, |t|^{d-1})\|$ to some extent, see Theorem 6(ii) for a comparison. We do not go into detail.

2.1.4 The trace in $\mathcal{S}'(\mathbb{R})$

Many times applications of traces are connected with boundary value problems. In such a context the continuity of tr considered as a mapping into S' is essential. Again we consider the simple situation of the L_p-spaces first.

Lemma 2 *Let $d \geq 2$ and let $0 < p < \infty$. Then $RL_p(\mathbb{R}, |t|^{d-1}) \subset S'(\mathbb{R})$ if and only if $d < p$.*

From the known embedding relations of $RA^s_{p,q}(\mathbb{R}^d)$ into L_u-spaces one obtains one half of the proof of the following general result.

Theorem 8 *Let $d \geq 2$, $0 < p < \infty$, and $0 < q \leq \infty$.*
(a) *Let $s > \sigma_p(d)$ and $L \geq [s] + 1$. Then the following assertions are equivalent:*
- (i) *The mapping* tr *maps $RB^s_{p,q}(\mathbb{R}^d)$ into $S'(\mathbb{R})$.*
- (ii) *The mapping* tr $:$ *$RB^s_{p,q}(\mathbb{R}^d) \to S'(\mathbb{R})$ is continuous.*
- (iii) *We have $TB^s_{p,q}(\mathbb{R}, L, d) \hookrightarrow S'(\mathbb{R})$.*
- (iv) *We have either $s > d(\frac{1}{p} - \frac{1}{d})$ or $s = d(\frac{1}{p} - \frac{1}{d})$ and $q \leq 1$.*

(b) *Let $s > \sigma_{p,q}(d)$ and $L \geq [s] + 1$. Then following assertions are equivalent:*
- (i) *The mapping* tr *maps $RF^s_{p,q}(\mathbb{R}^d)$ into $S'(\mathbb{R})$.*
- (ii) *The mapping* tr $:$ *$RF^s_{p,q}(\mathbb{R}^d) \to S'(\mathbb{R})$ is continuous.*
- (iii) *We have $TF^s_{p,q}(\mathbb{R}, L, d) \hookrightarrow S'(\mathbb{R})$.*
- (iv) *We have either $s > d(\frac{1}{p} - \frac{1}{d})$ or $s = d(\frac{1}{p} - \frac{1}{d})$ and $0 < p \leq 1$.*

2.1.5 The trace in $\mathcal{S}'(\mathbb{R})$ and weighted function spaces of Besov and Lizorkin-Triebel type

Weighted function spaces of Besov and Lizorkin-Triebel type, denoted by $B^s_{p,q}(\mathbb{R}, w)$ and $F^s_{p,q}(\mathbb{R}, w)$, respectively, are a well-developed subject in the literature, we refer to [5, 6, 30]. Fourier analytic definitions as well as characterizations by atoms are given under various restrictions on the weights, see e.g. [4, 5, 6, 16, 18, 31]. In this subsection we are interested in these spaces with respect to the weights $w_{d-1}(t) := |t|^{d-1}$, $t \in \mathbb{R}$, $d \geq 2$. Of course, these weights belong to the Muckenhoupt class $\mathcal{A}_\infty$, more exactly $w_{d-1} \in \mathcal{A}_r$ for any $r > d$, see [38].

Theorem 9 *Let $d \geq 2$, $0 < p < \infty$, and $0 < q \leq \infty$.*
(i) *Suppose $s > \sigma_p(d)$ and let $L \geq [s] + 1$. If $TB^s_{p,q}(\mathbb{R}, L, d) \hookrightarrow S'(\mathbb{R})$ (see Theorem 8), then $TB^s_{p,q}(\mathbb{R}, L, d) = RB^s_{p,q}(\mathbb{R}, w_{d-1})$ in the sense of equivalent quasi-norms.*
(ii) *Suppose $s > \sigma_{p,q}(d)$ and let $L \geq [s]+1$. If $TF^s_{p,q}(\mathbb{R}, L, d) \hookrightarrow S'(\mathbb{R})$ (see Theorem 8), then $TF^s_{p,q}(\mathbb{R}, L, d) = RF^s_{p,q}(\mathbb{R}, w_{d-1})$ in the sense of equivalent quasi-norms.*

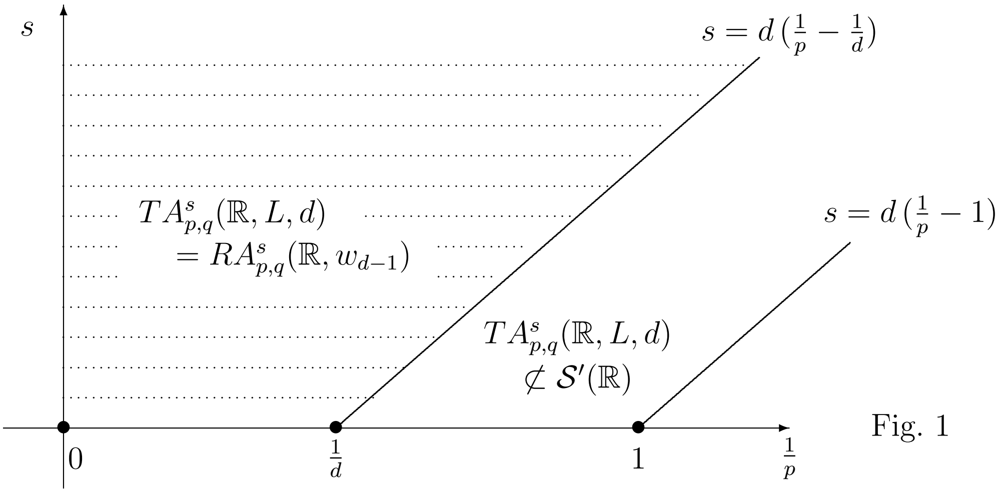

Fig. 1

Remark 7 We add some statements concerning the regularity of the most prominent singular distribution, namely $\delta \ : \ \varphi \to \varphi(0), \ \varphi \in S(\mathbb{R}^d)$. This tempered distribution has the following regularity properties:

- First we deal with the situation on $\mathbb{R}^d$. We have $\delta \in RB^{\frac{d}{p}-d}_{p,\infty}(\mathbb{R}^d)$ (but $\delta \notin RB^{\frac{d}{p}-d}_{p,q}(\mathbb{R}^d)$ for $q < \infty$ and $\delta \notin RF^{\frac{d}{p}-d}_{p,\infty}(\mathbb{R}^d)$), see, e.g., [29, Rem. 2.2.4/3].
- Now we turn to the situation on $\mathbb{R}$. By using more or less the same arguments as on $\mathbb{R}^d$ one can show $\delta \in B^{\frac{d}{p}-1}_{p,\infty}(\mathbb{R}, w_{d-1})$ (but $\delta \notin B^{\frac{d}{p}-1}_{p,q}(\mathbb{R}, w_{d-1})$ for any $q < \infty$ and $\delta \notin F^{\frac{d}{p}-1}_{p,\infty}(\mathbb{R}, w_{d-1})$).

2.1.6 The regularity of radial functions outside the origin

Let f be a radial function such that supp $f \subset \{x \in \mathbb{R}^d \ : \quad |x| \geq \tau\}$ for some $\tau > 0$. Then the following inequality is obvious:

$$\| f_0 \,|L_p(\mathbb{R})\| \leq \tau^{-(d-1)/p} \left(\frac{\Gamma(d/2)}{\pi^{d/2}}\right)^{1/p} \| f \,|L_p(\mathbb{R}^d)\| \,.$$

An extension to first or second order Sobolev spaces can be done by using Theorem 6. However, an extension to all spaces under consideration here is less obvious. Partly it could be done by interpolation, see Proposition 1, but we prefer a different way (not to exclude $p < 1$). We shall compare the atomic decompositions in Theorem 3 with the known atomic and wavelet characterizations of $B^s_{p,q}(\mathbb{R})$ and $F^s_{p,q}(\mathbb{R})$.

Corollary 1 *Let $\tau > 0$. Let $d \geq 2$, $0 < p < \infty$, and $0 < q \leq \infty$.*
(i) *We suppose $s > \sigma_p(d)$. If $f \in RB^s_{p,q}(\mathbb{R}^d)$ such that*

$$\text{supp } f \subset \{x \in \mathbb{R}^d \ : \quad |x| \geq \tau\} \tag{7}$$

then its trace f_0 *belongs to* $B^s_{p,q}(\mathbb{R})$. *Furthermore, there exists a constant* c *(not depending on* f *and* τ*) such that*

$$\| f_0 \,| B^s_{p,q}(\mathbb{R})\| \le c\, \tau^{-(d-1)/p} \, \| f \,| B^s_{p,q}(\mathbb{R}^d)\| \tag{8}$$

holds for all such functions f *and all* $\tau > 0$.
(ii) *We suppose* $s > \sigma_{p,q}(d)$. *If* $f \in RF^s_{p,q}(\mathbb{R}^d)$ *such that (7) holds, then its trace* f_0 *belongs to* $F^s_{p,q}(\mathbb{R})$. *Furthermore, there exists a constant* c *(not depending on* f *and* τ*) such that*

$$\| f_0 \,| F^s_{p,q}(\mathbb{R})\| \le c\, \tau^{-(d-1)/p} \, \| f \,| F^s_{p,q}(\mathbb{R}^d)\| \tag{9}$$

holds for all such functions f *all* $\tau > 0$.

We wish to mention that Corollary 1 has a partial inverse.

Corollary 2 *Let* $d \ge 2$, $0 < p < \infty$, $0 < q \le \infty$ *and* $0 < a < b < \infty$.
(i) *We suppose* $s > \sigma_p(d)$. *If* $g \in RB^s_{p,q}(\mathbb{R})$ *such that*

$$\text{supp } g \subset \{x \in \mathbb{R} \; : \quad a \le |x| \le b\} \tag{10}$$

then the radial function $f := \text{ext } g$ *belongs to* $RB^s_{p,q}(\mathbb{R}^d)$ *and there exist positive constants* A, B *such that*

$$A \, \| g \,| B^s_{p,q}(\mathbb{R})\| \le \| f \,| B^s_{p,q}(\mathbb{R}^d)\| \le B \, \| g \,| B^s_{p,q}(\mathbb{R})\| \, .$$

(ii) *We suppose* $s > \sigma_{p,q}(d)$. *If* $g \in RF^s_{p,q}(\mathbb{R})$ *such that (10) holds, then the radial function* $f := \text{ext } g$ *belongs to* $RF^s_{p,q}(\mathbb{R}^d)$ *and there exist positive constants* A, B *such that*

$$A \, \| g \,| F^s_{p,q}(\mathbb{R})\| \le \| f \,| F^s_{p,q}(\mathbb{R}^d)\| \le B \, \| g \,| F^s_{p,q}(\mathbb{R})\| \, .$$

For our next result we need Hölder-Zygmund spaces. Recall, that $C^s(\mathbb{R}^d) = B^s_{\infty,\infty}(\mathbb{R}^d)$ in the sense of equivalent norms if $s \notin \mathbb{N}_0$. Of course, also the spaces $B^s_{\infty,\infty}(\mathbb{R}^d)$ with $s \in \mathbb{N}$ allow a characterization by differences. We refer to [41, 2.2.2, 2.5.7] and [42, 3.5.3]. We shall use the abbreviation

$$\mathcal{Z}^s(\mathbb{R}^d) = B^s_{\infty,\infty}(\mathbb{R}^d)\, , \qquad s > 0\, .$$

Taking into account the well-known embedding relations for Besov as well as for Lizorkin-Triebel spaces, defined on $\mathbb{R}$, Thm. 5 implies in particular:

Corollary 3 *Let* $d \ge 2$, $0 < p < \infty$, $0 < q \le \infty$, *and* $s > \max(0, \frac{1}{p} - 1)$. *Let* φ *be a smooth radial function, uniformly bounded together with all its derivatives, and such that* $0 \notin \text{supp } \varphi$. *If* $f \in RA^s_{p,q}(\mathbb{R}^d)$, *then* $\varphi \, f \in \mathcal{Z}^{s-1/p}(\mathbb{R}^d)$.

Remark 8 P.L. Lions [23] has proved the counterpart of Corollary 3 for first order Sobolev spaces. We also dealt in [32] with these problems.

Finally, for later use, we would like to know when the radial functions are continuous out of the origin.

Corollary 4 *Let $\tau > 0$. Let $d \geq 2$, $0 < p < \infty$, and $0 < q \leq \infty$.*
(i) *If either $s > 1/p$ or $s = 1/p$ and $q \leq 1$ then $f \in RB^s_{p,q}(\mathbb{R}^d)$ is uniformly continuous on the set $|x| \geq \tau$.*
(ii) *If either $s > 1/p$ or $s = 1/p$ and $p \leq 1$ then $f \in RF^s_{p,q}(\mathbb{R}^d)$ is uniformly continuous on the set $|x| \geq \tau$.*

By looking at the restrictions in Cor. 4 we introduce the following set of parameters.

Definition 4 (i) *We say (s,p,q) belongs to the set $U(B)$ if (s,p,q) satisfies the restrictions in part (i) of Cor. 4.*
(ii) *The triple (s,p,q) belongs to the set $U(F)$ if (s,p,q) satisfies the restrictions in part (ii) of Cor. 4.*

Remark 9 (a) The abbreviation $(s,p,q) \in U(A)$ will be used with the obvious meaning.
(b) Let $1 \leq p = p_0 < \infty$ be fixed. Then there is always a largest space in the set

$$\{B^s_{p_0,q}(\mathbb{R}^d) \,:\, (s,p_0,q) \in U(B)\} \quad \cup \quad \{F^s_{p_0,q}(\mathbb{R}^d) \,:\, (s,p_0,q) \in U(F)\} .$$

This space is given either by $F^1_{1,\infty}(\mathbb{R}^d)$ if $p_0 = 1$ or by $B^{1/p_0}_{p_0,1}(\mathbb{R}^d)$ if $1 < p_0 < \infty$. If $p_0 < 1$, then obviously $B^{1/p_0}_{p_0,1}(\mathbb{R}^d)$ is the largest Besov space and $F^{1/p_0}_{p_0,\infty}(\mathbb{R}^d)$ is the largest Lizorkin-Triebel space in the above family. However, these spaces are incomparable.

2.2 Decay and boundedness properties of radial functions

We deal with improvements of Strauss' *Radial Lemma.* Decay can only be expected if we measure smoothness in function spaces built on $L_p(\mathbb{R}^d)$ with $p < \infty$.
It is instructive to have a short look onto the case of first order Sobolev spaces. Let $f = g(r(x)) \in RC_0^\infty(\mathbb{R}^d)$. Then

$$\frac{\partial f}{\partial x_i}(x) = g'(r)\,\frac{x_i}{r}, \quad r = |x| > 0\,, \qquad i = 1, \ldots, d\,.$$

Hence

$$\| \; |\nabla f(x)| \; |L_p(\mathbb{R}^d)\| = c_d \, \| \, g' \, |L_p(\mathbb{R}, |t|^{d-1})\| \,, \tag{11}$$

where $1 \leq p < \infty$. Next we apply the identity

$$g(r) = - \int_r^\infty g'(t)dt$$

and obtain

$$|g(r)| \le \int_r^\infty |g'(t)|dt \le r^{-(d-1)} \int_r^\infty t^{d-1}|g'(t)|dt.$$

This extends to all functions in $RW_1^1(\mathbb{R}^d)$ by a density argument. On this elementary way we have proved the inequality

$$|x|^{d-1}\, |f(x)| = r^{d-1}\, |g(r)| \le \frac{1}{c_d} \int_{|x|>r} |\nabla f(x)|\, dx \le \frac{1}{c_d} \| \nabla f(x) \|_1 \,. \qquad (12)$$

This inequality can be interpreted in several ways:

- The possible unboundedness in the origin is limited.
- There is some decay, uniformly in f, if $|x|$ tends to $+\infty$.
- We have $\lim_{|x|\to\infty} |x|^{d-1}\, |f(x)| = 0$ for all $f \in RW_1^1(\mathbb{R}^d)$.
- It makes sense to switch to homogeneous function spaces, since in (12) only the norm of the homogeneous Sobolev space occurs (for this, see [7] and [33]).

We shall show that all these phenomena will occur also in the general context of radial subspaces of Besov and Lizorkin-Triebel spaces.

2.2.1 The behaviour of radial functions near infinity

Suppose $(s,p,q) \in U(A)$. Then $f \in RA^s_{p,q}(\mathbb{R}^d)$ is uniformly continuous near infinity and belongs to $L_p(\mathbb{R}^d)$. This implies $\lim_{|x|\to\infty} |f(x)| = 0$. However, much more is true.

Theorem 10 *Let $d \ge 2$, $0 < p < \infty$, and $0 < q \le \infty$.*
(i) *Suppose $(s,p,q) \in U(A)$. Then there exists a constant c s.t.*

$$|x|^{(d-1)/p}\, |f(x)| \le c \,\| f \,|A^s_{p,q}(\mathbb{R}^d)\| \qquad (13)$$

holds for all $|x| \ge 1$ and all $f \in RA^s_{p,q}(\mathbb{R}^d)$.
(ii) *Suppose $(s,p,q) \in U(A)$. Then*

$$\lim_{|x|\to\infty} |x|^{\frac{d-1}{p}}\, |f(x)| = 0 \qquad (14)$$

holds for all $f \in RA^s_{p,q}(\mathbb{R}^d)$.
(iii) *Suppose $(s,p,q) \in U(A)$. Then there exists a constant $c > 0$ such that for all x, $|x| > 1$, there exists a smooth radial function $f \in RA^s_{p,q}(\mathbb{R}^d)$, $\| f \,|RA^s_{p,q}(\mathbb{R}^d)\| = 1$, s.t.*

$$|x|^{\frac{d-1}{p}}\, |f(x)| \ge c\,. \qquad (15)$$

(iv) *Suppose $(s,p,q) \notin U(A)$ and $\frac{1}{p} > \sigma_p(d)$. We assume also that $\frac{1}{p} > \sigma_q(d)$ in the F-case. Then, for all sequences $(x^j)_{j=1}^\infty \subset \mathbb{R}^d \setminus \{0\}$ s.t. $\lim_{j\to\infty} |x^j| = \infty$, there exists a radial function $f \in RA^s_{p,q}(\mathbb{R}^d)$, $\| f \,|RA^s_{p,q}(\mathbb{R}^d)\| = 1$, s.t. f is unbounded in any neighborhood of x^j, $j \in \mathbb{N}$.*

Remark 10 (i) Increasing s (for fixed p) is not improving the decay rate. In the case of Banach spaces, i.e., $p, q \geq 1$, the additional assumptions in point (iv) are always fullfiled. Hence, the largest spaces, guaranteeing the decay rate $(d-1)/p$, are spaces with $s = 1/p$, see Remark 9.
(ii) Observe that in (iii) the function depends on $|x|$. There is no function in $RA^s_{p,q}(\mathbb{R}^d)$ such that (15) holds for all x, $|x| \geq 1$, simultaneously. The naive construction $f(x) := (1-\psi(x))\,|x|^{\frac{1-d}{p}}$, $x \in \mathbb{R}^d$, does not belong to $L_p(\mathbb{R}^d)$.
(iii) If one switches from inhomogeneous spaces to the larger homogeneous spaces of Besov and Lizorkin-Triebel type, then the decay rate becomes smaller. It will depend also on s, see [7] and [33] for details.
(iv) Of course, formula (13) generalizes the estimate (1). Also Coleman, Glazer and Martin [8] have dealt with (1). P.L. Lions [23] proved a p-version of the *Radial Lemma*.
(v) Originally the *Radial Lemma* has been used to prove compactness of embeddings of radial Sobolev spaces into L_p-spaces, see [8], [23]. In the framework of radial subspaces of Besov and of Lizorkin-Triebel spaces compactness of embeddings has been investigated in [32]. There we have given a final answer, i.e., we proved an *if, and only if,* assertion.
(vi) Compactness of embeddings of radial subspaces of homogeneous Besov and of Lizorkin-Triebel spaces will be investigated in [33].

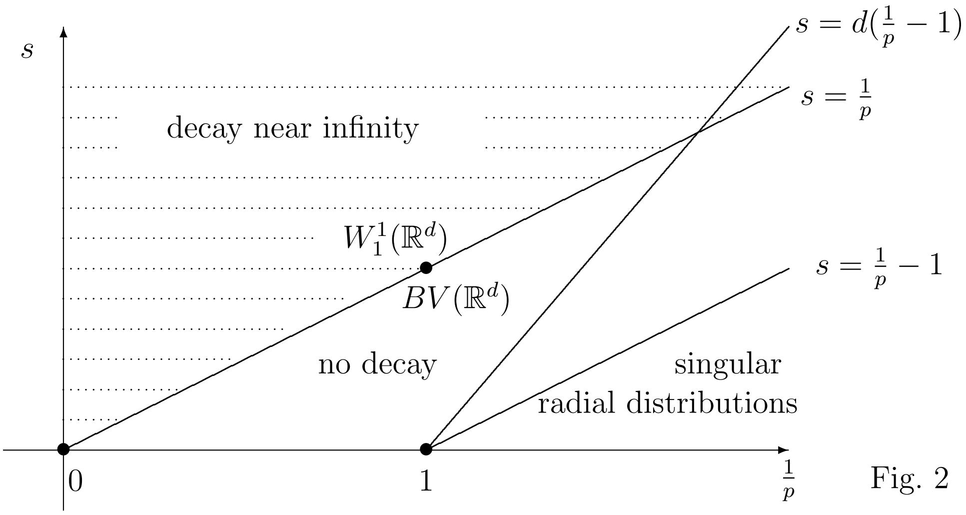

Fig. 2

2.2.2 The behaviour of radial functions near infinity – borderline cases

As indicated in Remark 9, within the scales of Besov and Lizorkin-Triebel spaces the borderline cases for the decay rate $(d-1)/p$ are either $F^1_{1,\infty}(\mathbb{R}^d)$ if $p = 1$ or $B^{1/p}_{p,1}(\mathbb{R}^d)$ if $1 < p < \infty$. Now we turn to spaces which do not belong to these scales and where the elements of the radial subspaces have such a decay rate. Hence, we are looking for spaces of radial functions with a simple norm which satisfy (13). The Sobolev space $RW^1_1(\mathbb{R}^d)$ is such a candidate for which (13) is already known, see [23]. But this is not the end of the story. Also for the radial functions of bounded variation such a decay estimate is true.

Theorem 11 *Let $d \geq 2$. Then there exists constant c s.t.*

$$|x|^{d-1}\,|f(x)| \leq c\,\|\,f\,|BV(\mathbb{R}^d)\| \tag{16}$$

holds for all $|x| > 0$ and all $f \in RBV(\mathbb{R}^d)$. Also

$$\lim_{|x|\to\infty} |x|^{d-1}\,|f(x)| = 0 \tag{17}$$

is true for all $f \in RBV(\mathbb{R}^d)$.

Remark 11 (i) Both assertions, (16) and (17), require an interpretation since, in contrast to the classical definition of $BV(\mathbb{R})$, the spaces $BV(\mathbb{R}^d)$, $d \geq 2$, are spaces of equivalence classes, see Subsection 4.2. Nevertheless, in every equivalence class $[f] \in BV(\mathbb{R}^d)$, there is a representative $\tilde{f} \in [f]$, such that

$$|\tilde{f}(x)| \leq \limsup_{y\to x} |f(y)|$$

(simply take $\tilde{f}(x) := f(x)$ in every Lebesgue point x of f and $\tilde{f}(x) := 0$ otherwise). Hence, (16) and (17) have to be interpreted as follows: whenever we work with values of the equivalence class $[f]$ then we mean the function values of the above representative $\tilde{f}$.
(ii) Notice that $F^1_{1,\infty}(\mathbb{R}^d)$ and $BV(\mathbb{R}^d)$ are incomparable.
(iii) Observe, as in case of the *Radial Lemma*, that (16) holds for $x \neq 0$.

As a preparation for Theorem 11 we shall characterize the traces of radial elements in $BV(\mathbb{R}^d)$. This seems to be of independent interest. For this reason we are forced to introduce weighted spaces of functions of bounded variation on the positive half axis. We put $\mathbb{R}^+ := (0,\infty)$. As usual, $|\nu|$ denotes the total variation of the measure ν, see, e.g., [28, Chapt. 6].

Definition 5 (i) *A function $\varphi \in C([0,\infty))$ belongs to $C^1_c([0,\infty))$ if it is continuously differentiable on $\mathbb{R}^+$, has compact support, satisfies $\varphi(0) = 0$ and $\lim_{t\to 0^+} \varphi'(t) = \varphi'(0) = \lim_{t\to 0^+} \frac{\varphi(t)}{t}$ exists and is finite.*
(ii) *A function $g \in L_1(\mathbb{R}^+, t^{d-1})$ is said to belong to $BV(\mathbb{R}^+, t^{d-1})$ if there is a signed Radon measure ν on $\mathbb{R}^+$ such that*

$$\int_0^\infty g(t)\,[\varphi(s)s^{d-1}]'(t)\,dt = -\int_0^\infty \varphi(t)\,t^{d-1}\,d\nu(t)\,, \qquad \forall \varphi \in C^1_c([0,\infty)) \tag{18}$$

and

$$\| g \,|BV(\mathbb{R}^+, t^{d-1})\| := \| g \,|L_1(\mathbb{R}^+, t^{d-1})\| + \int_0^\infty r^{d-1}\, d|\nu|(r) \tag{19}$$

is finite.

By using these new spaces we can prove the following trace theorem.

Theorem 12 *Let g be a measurable function on $\mathbb{R}^+$. Then* $\mathrm{ext}\, g \in BV(\mathbb{R}^d)$ *if, and only if $g \in BV(\mathbb{R}^+, t^{d-1})$ and*

$$\| \mathrm{ext}\; g \,|BV(\mathbb{R}^d)\| \asymp \| g \,|BV(\mathbb{R}^+, t^{d-1}) \,\| \,.$$

Spaces with $1 < p < \infty$

For $1 < p < \infty$ one could use interpolation between $p = 1$ and $p = \infty$ to obtain spaces with the decay rate $(d-1)/p$. The largest spaces with this respect are obtained by the real method. Let $M_p(\mathbb{R}^d) := (RL_\infty(\mathbb{R}^d), RBV(\mathbb{R}^d))_{\Theta,\infty}$, $\Theta = 1/p$. Then (13) holds for all elements $f \in M_p(\mathbb{R}^d)$. The disadvantage of these classes $M_p(\mathbb{R}^d)$ lies in the fact that elementary descriptions of $M_p(\mathbb{R}^d)$ are not known. However, at least some embeddings are known. From

$$RB^{1/p}_{p,1}(\mathbb{R}^d) = [RB^0_{\infty,1}(\mathbb{R}^d), RB^1_{1,1,}(\mathbb{R}^d)]_\Theta \hookrightarrow (RL_\infty(\mathbb{R}^d), RBV(\mathbb{R}^d))_{\Theta,\infty}\,, \qquad \Theta = 1/p\,,$$

(combine Proposition 1 with [1, Thm. 4.7.1]), we get back Theorem 10 (i), but only in case $1 < p < \infty$.

2.2.3 The behaviour of radial functions near the origin

At first we mention that the embedding relations with respect to $L_\infty(\mathbb{R}^d)$ do not change when we switch from $A^s_{p,q}(\mathbb{R}^d)$ to its radial subspace $RA^s_{p,q}(\mathbb{R}^d)$.

Lemma 3 (i) *The embedding $RB^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d)$ holds if and only if either $s > d/p$ or $s = d/p$ and $q \le 1$.*
(ii) *The embedding $RF^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d)$ holds if and only if either $s > d/p$ or $s = d/p$ and $p \le 1$.*

The explicit counterexamples will be given in Lemma 8 below. Hence, unboundedness can only happen in case $s \le d/p$.

Theorem 13 *Let $d \ge 2$, $0 < p < \infty$ and $0 < q \le \infty$.*
(i) *Suppose $(s,p,q) \in U(A)$ and $s < \frac{d}{p}$. Then there exists a constant c s.t.*

$$|x|^{\frac{d}{p}-s}\, |f(x)| \le c \,\| f \,|RA^s_{p,q}(\mathbb{R}^d)\| \tag{20}$$

holds for all $0 < |x| \le 1$ *and all* $f \in RA^s_{p,q}(\mathbb{R}^d)$.
(ii) *Let* $\sigma_p(d) < s < d/p$. *There exists a constant* $c > 0$ *such that for all* x, $0 < |x| < 1$, *there exists a smooth radial function* $f \in RA^s_{p,q}(\mathbb{R}^d)$, $\| f \,|RA^s_{p,q}(\mathbb{R}^d)\| = 1$, *s.t.*

$$|x|^{\frac{d}{p}-s}\, |f(x)| \ge c\,. \tag{21}$$

Remark 12 (i) In case of $RB^s_{p,\infty}(\mathbb{R}^d)$ we have a function which realizes the extremal behaviour for all $|x| < 1$ simultaneously. It is well-known, see e.g. [29, Lem. 2.3.1/1], that the function

$$f(x) := \psi(x)\, |x|^{\frac{d}{p}-s}\,, \qquad x \in \mathbb{R}^d\,,$$

belongs to $RB^s_{p,\infty}(\mathbb{R}^d)$, as long as $s > \sigma_p(d)$. This function does not belong to $RB^s_{p,q}(\mathbb{R}^d)$, $q < \infty$. Since it is also not contained in $RF^s_{p,q}(\mathbb{R}^d)$, $0 < q \le \infty$ we conclude that in these cases there is no function, which realizes this upper bound for all x simultaneously. In these cases the function f in (21) has to depend on x.
(ii) These estimates do not change by switching to the larger homogeneous spaces $R\dot{A}^s_{p,q}(\mathbb{R}^d)$ of Besov and Lizorkin-Triebel type. In case of $R\dot{H}^s(\mathbb{R}^d) = R\dot{F}^s_{p,2}(\mathbb{R}^d)$ this has been observed in a recent paper by Cho and Ozawa [7], see also Ni [25], Rother [27] and Kuzin, Pohozaev [22, 8.1]. The general case is treated in [33].
(iii) In the literature one can find various types of further inequalities for radial functions. Many times preference is given to a homogeneous context, see the inequalities (1) and (16) as examples. Then one has to deal with the behaviour at infinity and around the origin simultaneously. That would be not appropriate in the context of inhomogeneous spaces. Inequalities like (1) and (16) will be investigated systematically in [33]. However, let us refer to [39], [23], [25], [27], [22, 8.1] and [7] for results in this direction. Sometimes also decay estimates are proved by replacing on the right-hand side the norm in the space $A^s_{p,q}(\mathbb{R}^d)$ ($\dot{A}^s_{p,q}(\mathbb{R}^d)$) by products of norms, e.g., $\| f|L_p(\mathbb{R}^d)\|^{1-\Theta}\, \| f\,|\dot{A}^s_{p,q}(\mathbb{R}^d)\|^{\Theta}$ for some $\Theta \in (0,1)$, see [23], [25], [27] and [7]. Here we will not deal with those modifications (improvements).

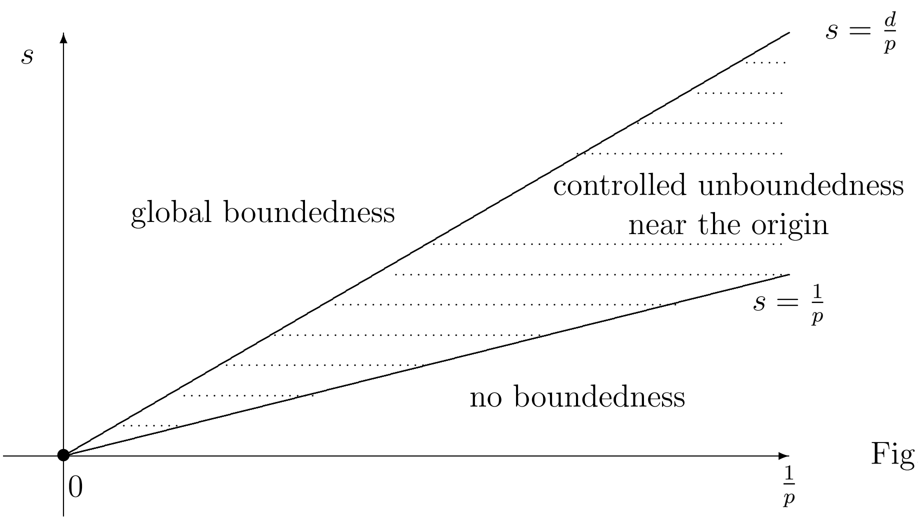

Fig. 3

Finally we have to investigate $s \le 1/p$ and $(s,p,q) \notin U(A)$.

Lemma 4 *Let $d \ge 2$, $0 < p < \infty$ and $0 < q \le \infty$. Suppose $(s,p,q) \notin U(A)$ and $\sigma_p(d) < 1/p$. Moreover let $\sigma_q(d) < 1/p$ in the F-case. Then there exists a radial function $f \in RA^s_{p,q}(\mathbb{R}^d)$, $\| f \,|RA^s_{p,q}(\mathbb{R}^d)\| = 1$, and a sequence $(x^j)_j \subset \mathbb{R}^d \setminus \{0\}$ s.t. $\lim_{j\to\infty} |x^j| = 0$ and f is unbounded in a neighborhood of all x^j.*

2.2.4 The behaviour of radial functions near the origin – borderline cases

Now we turn to the remaining limiting situation. We shall show that there is also controlled unboundedness near the origin if $s = d/p$ and $RA^{d/p}_{p,q}(\mathbb{R}^d) \not\subset L_\infty(\mathbb{R}^d)$.

Theorem 14 *Let $d \ge 2$, $0 < p < \infty$, $0 < q \le \infty$, and suppose $s = d/p$.*
(i) *Let $1 < q \le \infty$. Then there exists constant c s.t.*

$$(-\log|x|)^{-1/q'}\, |f(x)| \le c\, \| f \,|B^{d/p}_{p,q}(\mathbb{R}^d)\| \tag{22}$$

holds for all $0 < |x| \le 1/2$ and all $f \in RB^{d/p}_{p,q}(\mathbb{R}^d)$.
(ii) *Let $1 < p < \infty$. Then there exists constant c s.t.*

$$(-\log|x|)^{-1/p'}\, |f(x)| \le c\, \| f \,|F^{d/p}_{p,q}(\mathbb{R}^d)\| \tag{23}$$

holds for all $0 < |x| \le 1/2$ and all $f \in RF^{d/p}_{p,q}(\mathbb{R}^d)$.

Remark 13 Comparing Lemma 8 below and Theorem 14 we find the following. For the case $q = \infty$ in Theorem 14(i) the function $f_{1,0}$, see (61), realizes the extremal behaviour. In all other cases there remains a gap of order $\log\log$ to some power.

3 Traces of radial subspaces – proofs

The main aim of this section is to prove Theorem 3. It expresses the fact that all information about a radial function is contained in its trace onto a straight line through the origin. However, a few things more will be done here. For later use one subsection is devoted to the study of interpolation of radial subspaces (Subsection 3.1) and another one is devoted to the study of certain test functions (Subsection 3.8).

3.1 Interpolation of radial subspaces

We mention two different results here, one with respect to the complex method and one with respect to the real method of interpolation.

3.1.1 Complex interpolation of radial subspaces

In [36] one of the authors has proved that in case $p, q \geq 1$ the spaces $RB^s_{p,q}(\mathbb{R}^d)$ ($RF^s_{p,q}(\mathbb{R}^d)$) are complemented subspaces of $B^s_{p,q}(\mathbb{R}^d)$ ($F^s_{p,q}(\mathbb{R}^d)$). By means of the method of retraction and coretraction, see, e.g., Theorem 1.2.4 in [40], this allows to transfer the interpolation formulas for the original spaces $B^s_{p,q}(\mathbb{R}^d)$ ($F^s_{p,q}(\mathbb{R}^d)$) to its radial subspaces. However, we prefer to quote a slightly more general result, proved in [34], concerning the complex method. It is based on the results on complex interpolation for Lizorkin-Triebel spaces from [14] and uses the method of [24] for an extension to the quasi-Banach space case.

Proposition 1 *Let $0 < p_0, p_1 \leq \infty$, $0 < q_0, q_1 \leq \infty$, $s_0, s_1 \in \mathbb{R}$, and $0 < \Theta < 1$. Define $s := (1-\Theta)\, s_0 + \Theta\, s_1$,*

$$\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \qquad \textit{and} \qquad \frac{1}{q} := \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}\, .$$

(i) *Let $\max(p_0, q_0) < \infty$. Then we have*

$$RB^s_{p,q}(\mathbb{R}^d) = \Big[RB^{s_0}_{p_0,q_0}(\mathbb{R}^d), RB^{s_1}_{p_1,q_1}(\mathbb{R}^d)\Big]_\Theta\, .$$

(ii) *Let $p_1 < \infty$ and $\min(q_0, q_1) < \infty$. Then we have*

$$RF^s_{p,q}(\mathbb{R}^d) = \Big[RF^{s_0}_{p_0,q_0}(\mathbb{R}^d), RF^{s_1}_{p_1,q_1}(\mathbb{R}^d)\Big]_\Theta\, .$$

3.1.2 Real Interpolation of radial subspaces

For later use we also formulate a result with respect to the real method of interpolation.

Proposition 2 *Let $d \geq 1$, $1 \leq q, q_0, q_1 \leq \infty$, $s_0, s_1 \in \mathbb{R}$, $s_0 \neq s_1$, and $0 < \Theta < 1$.*
(i) *Let $1 \leq p \leq \infty$. Then, with $s := (1-\Theta)\, s_0 + \Theta\, s_1$, we have*

$$RB^s_{p,q}(\mathbb{R}^d) = \Big(RB^{s_0}_{p,q_0}(\mathbb{R}^d), RB^{s_1}_{p,q_1}(\mathbb{R}^d)\Big)_{\Theta,q}\, .$$

(ii) *Let $1 \leq p < \infty$. Then, with $s := (1-\Theta)\, s_0 + \Theta\, s_1$, we have*

$$RB^s_{p,q}(\mathbb{R}^d) = \Big(RF^{s_0}_{p,q_0}(\mathbb{R}^d), RF^{s_1}_{p,q_1}(\mathbb{R}^d)\Big)_{\Theta,q}\, .$$

Proof. As mentioned above, the spaces $RB^s_{p,q}(\mathbb{R}^d)$ ($RF^s_{p,q}(\mathbb{R}^d)$) are complemented subspaces of $B^s_{p,q}(\mathbb{R}^d)$ ($F^s_{p,q}(\mathbb{R}^d)$), see [36]. Using the method of retraction and coretraction, see [41, 1.2.4], the above statements are consequences of the corresponding formulas without R, see e.g. [41, 2.4.2]. ∎

3.2 Proofs of the statements in Subsection 2.1.1

Let $m \in \mathbb{N}_0$. Then $C^m(\mathbb{R}^d)$ denotes the collection of all functions $f : \mathbb{R}^d \to \mathbb{C}$ such that all derivatives $D^\alpha f$ of order $|\alpha| \le m$ exist, are uniformly continuous and bounded. We put

$$\| f \,|C^m(\mathbb{R}^d)\| := \sum_{|\alpha|\le m} \| D^\alpha f \,|L_\infty(\mathbb{R}^d)\| \, .$$

By $RC^m(\mathbb{R}^d)$ we denote its subspace of radial functions.

Proof of Theorem 1

Step 1. Proof in case $m \in \{0,1\}$. The case $m = 0$ is obvious. Hence, we deal with $m = 1$. Let $f \in RC^1(\mathbb{R}^d)$. Obviously,

$$\frac{\partial f}{\partial x_1}(x) = f_0'(t) \, , \qquad x = (x_1, 0, \ldots , 0) \, , \quad t = x_1 \, ,$$

which proves the estimate

$$\| \text{ tr } f \,|C^1(\mathbb{R})\| \le \| f \,|C^1(\mathbb{R}^d)\| \tag{24}$$

and at the same time the continuity of the function tr $f = f_0$ and its derivative. Now we assume that $g \in RC^1(\mathbb{R})$. Let $f :=$ ext g. If $x \neq 0$ we have

$$\frac{\partial f}{\partial x_1}(x) = g'(r)\, \frac{x_1}{r} \, , \qquad r = |x| > 0 \, . \tag{25}$$

This implies

$$\sup_{x\neq 0} |\frac{\partial f}{\partial x_1}(x)| \le \sup_{r>0} |g'(r)| = \sup_{t\in\mathbb{R}} |g'(t)| \, .$$

It remains to deal with the continuity of the derivative at the origin. Since g is even and continuously differentiable we know $g'(0) = 0$. This implies

$$\frac{\partial f}{\partial x_1}(0) = \lim_{h\to 0} \frac{f(h, 0, \ldots , 0) - f(0)}{h} = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = g'(0) = 0 \, .$$

¿From (25), the continuity of g' and $g'(0) = 0$ we conclude $\lim_{x\to 0} \frac{\partial f}{\partial x_1}(x) = 0$. This proves the claim for $m = 1$.

Step 2. We proceed by induction. Our induction hypothesis is as follows. If the assertion of Theorem 1 holds for the pair $(m, m + 1)$, then it holds also for $m + 2$.

Substep 2.1. If $f \in RC^{m+2}(\mathbb{R}^d)$, then, of course, $f_0 =$ tr $f \in RC^{m+2}(\mathbb{R})$ and also the corresponding analogue of (24) follows immediately.

Substep 2.2. Now, let $g \in RC^{m+2}(\mathbb{R})$ and define $f :=$ ext g. Then f is a radial function, which is $m + 2$-times continuously differentiable on $\mathbb{R}^d \setminus \{0\}$. Therefore, it is enough to discuss the regularity properties of f in the origin and to prove the estimate

$$\| \text{ ext } g\,|C^{m+2}(\mathbb{R}^d)\| \;\lesssim\; \| g\,|C^{m+2}(\mathbb{R})\| \,. \tag{26}$$

First, let us state the following fact, which may be easily proved by induction. For every $n \in \mathbb{N}_0$ there is a constant $c > 0$ such that the function $r = r(x)$ satisfies

$$|D^\alpha r(x)| \le c r(x)^{1-|\alpha|} \tag{27}$$

for every multiindex $\alpha \in \mathbb{N}_0^d$ and all $x \in \mathbb{R}^d \setminus \{0\}$.

First we deal with a simplified situation. We assume that

$$g(0) = g'(0) = \cdots = g^{(m+2)}(0) = 0. \tag{28}$$

This clearly implies for $0 \le \ell \le m+2$

$$g^{(\ell)}(t) = o(|t|^{m+2-\ell}) \qquad \text{if} \quad t \to 0\,. \tag{29}$$

Then, using chain rule and the estimates (27), (29) we find

$$\begin{aligned} |\,(D^\alpha f)(x)\,| \quad &\lesssim \quad \sum_{\ell=1}^{|\alpha|} |g^{(\ell)}(r)| \sum_{\beta^1+\ldots+\beta^\ell=\alpha} |D^{\beta^1} r(x)| \,\ldots\, |D^{\beta^\ell} r(x)| \\ &\lesssim \quad \sum_{\ell=1}^{|\alpha|} o(r^{m+2-\ell})\, r^{\ell-|\alpha|} = o(r^{m+2-|\alpha|}), \qquad r \downarrow 0\,, \end{aligned} \tag{30}$$

where $|\alpha| \le m+2$. Using the induction hypothesis we immediately get $D^\alpha f(0) = 0$ if $|\alpha| \le m+1$. Now let $|\alpha| = m+2$. For simplicity we concentrate on $\alpha = (m+2, 0, \ldots, 0)$. Then, as a consequence of (30), we find

$$\begin{aligned} &\frac{D^{(m+1,0,\ldots,0)} f(h,0,\ldots,0) - D^{(m+1,0,\ldots,0)} f(0,0,\ldots,0)}{h} \\ &\qquad = \quad \frac{D^{(m+1,0,\ldots,0)} f(h,0,\ldots,0)}{h} = o(1) \qquad \text{if} \quad h \to 0\,. \end{aligned}$$

This yields $D^{(m+2,0,\ldots,0)} f(0) = 0$ and with the same type of argument $D^\alpha f(0) = 0$ for all derivatives of order $|\alpha| = m+2$. This and (30) prove the continuity of $D^\alpha f$, $|\alpha| \le m+2$, in the origin. Observe, that the inequality (26) follows as in (30) by using the chain rule.

Finally, we wish to remove the restriction (28). Suppose that m is even. Hence

$$g'(0) = g'''(0) = \;\ldots\; = g^{(m+1)}(0) = 0,$$

but $g(0), g''(0), \ldots, g^{(m+2)}(0)$ can be arbitrary. Let $\psi_0 = \text{tr } \psi$. We introduce the function

$$h(t) := g(t) - g_1(t)\, \psi_0(t)\,, \qquad t \in \mathbb{R}\,,$$

where

$$g_1(t) := g(0) + \frac{g''(0)}{2!} t^2 + \cdots + \frac{g^{(m+2)}(0)}{(m+2)!} t^{m+2}, \qquad t \in \mathbb{R}.$$

The extension of $g_1 \psi_0$ is a radial function with compact support and continuous derivatives of arbitrary order. Furthermore, we have the obvious estimate

$$\begin{aligned} \| \operatorname{ext} g_1 \psi_0 \, | C^{m+2}(\mathbb{R}^d) \| &\le \sum_{j=0}^{\frac{m}{2}+1} \frac{|g^{(2j)}(0)|}{(2j)!} \| \, |x|^{2j} \, \psi(x) \, | C^{m+2}(\mathbb{R}^d) \| \\ &\lesssim \| g \, | C^{m+2}(\mathbb{R}) \| \, . \end{aligned}$$

The function h satisfies (28). Hence, ext h belongs to $RC^{(m+2)}(\mathbb{R}^d)$ and

$$\begin{aligned} \| \operatorname{ext} h \, | C^{m+2}(\mathbb{R}^d) \| &\lesssim \| h \, | C^{m+2}(\mathbb{R}) \| \\ &\lesssim \| g \, | C^{m+2}(\mathbb{R}) \| + \| g_1 \psi_0 \, | C^{m+2}(\mathbb{R}) \| \\ &\lesssim \| g \, | C^{m+2}(\mathbb{R}) \| \, . \end{aligned}$$

This shows that ext $g =$ ext $h +$ ext $(g_1 \psi_0) \in RC^{(m+2)}(\mathbb{R}^d)$ and in addition we also get the estimate

$$\| \operatorname{ext} g \, | C^{m+2}(\mathbb{R}^d) \| \lesssim \| g \, | C^{m+2}(\mathbb{R}) \| \, .$$

For odd m the proof is similar. ■

Proof of Theorem 2

For tr $\in \mathcal{L}(B^s_{\infty,q}(\mathbb{R}^d), B^s_{\infty,q}(\mathbb{R}))$ we refer to [41, 2.7.2]. This immediately gives tr $\in \mathcal{L}(RB^s_{\infty,q}(\mathbb{R}^d), RB^s_{\infty,q}(\mathbb{R}))$. Concerning ext we argue by using real interpolation. Observe, that ext $\in \mathcal{L}(RC^m(\mathbb{R}), RC^m(\mathbb{R}^d))$ for all $m \in \mathbb{N}_0$, see Theorem 1. From the interpolation property of the real interpolation method we derive

$$\operatorname{ext} \in \mathcal{L}\Big((RC^m(\mathbb{R}), RC(\mathbb{R}))_{\Theta,q}, (RC^m(\mathbb{R}^d), RC(\mathbb{R}^d))_{\Theta,q} \Big) \, .$$

Using Proposition 2 the claim follows. ■

3.3 Proofs of the assertions in Subsection 2.1.2

3.3.1 Proof of Lemma 1

Recall, for $f \in RL_p(\mathbb{R})$ we have

$$\int_{\mathbb{R}^d} |f(x)|^p \, dx = 2 \, \frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty |f_0(r)|^p \, r^{d-1} \, dr \, .$$

Using

$$\int_0^\infty |f_0(r)|^p \, r^{d-1} \, dr = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty |f_0(r)|^p \, r^{d-1} \, dr \, ,$$

which implies the density of the test functions in $L_p([0,\infty), r^{d-1})$, we can read this formula also from the other side, it means

$$\int_{\mathbb{R}^d} |\operatorname{ext}\, g(x)|^p\, dx = 2\, \frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty |g(r)|^p\, r^{d-1}\, dr$$

for all $g \in L_p([0,\infty), r^{d-1})$. This proves (i). Part (ii) is obvious. ■

3.3.2 Characterizations of radial subspaces by atoms

As mentioned above our proof of the trace theorem relies on atomic decompositions of radial distributions on $\mathbb{R}^d$. We recall our characterizations of $RA^s_{p,q}(\mathbb{R}^d)$ from [32], see also [21].

In this paper we shall consider two different versions of atoms. They are not related to each other. We hope that it will be always clear from the context with which type of atoms we are working. For the following definition of an atom we refer to [14] or [42, 3.2.2]. For an open set Q and $r > 0$ we put $r\,Q = \{x \in \mathbb{R}^d \,:\, \operatorname{dist}(x, Q) < r\}$. Observe that Q is always a subset of $r\,Q$ whatever r is.

Definition 6 *Let $s \in \mathbb{R}$ and let $1 \le p \le \infty$. Let L and M be integers such that $L \ge 0$ and $M \ge -1$. Let $Q \subset \mathbb{R}^d$ be an open connected set with* $\operatorname{diam} Q = r$.

(a) *A smooth function $a(x)$ is called an 1_L-atom centered in Q if*

$$\begin{aligned} \operatorname{supp} a &\subset \frac{r}{2} Q\,, \\ \sup_{y \in \mathbb{R}^d} |D^\alpha a(y)| &\le 1\,, \qquad |\alpha| \le L\,. \end{aligned}$$

(b) *A smooth function $a(x)$ is called an $(s,p)_{L,M}$-atom centered in Q if*

$$\begin{aligned} \operatorname{supp} a &\subset \frac{r}{2} Q\,, \\ \sup_{y \in \mathbb{R}^d} |D^\alpha a(y)| &\le r^{s-|\alpha|-\frac{d}{p}}\,, \qquad |\alpha| \le L\,, \\ \int_{\mathbb{R}^d} a(y)\, y^\alpha\, dy &= 0\,, \qquad |\alpha| \le M\,. \end{aligned}$$

Remark 14 If $M = -1$, then the interpretation is that no moment condition is required.

In [32] and [21] we constructed a regular sequence of coverings with certain special properties which we now recall. Consider the annuli (balls if $k = 0$)

$$P_{j,k} := \left\{ x \in \mathbb{R}^d \,:\quad k\, 2^{-j} \le |x| < (k+1)\, 2^{-j} \right\}, \qquad j = 0, 1, \ldots\,, \quad k = 0, 1, \ldots\,.$$

Then there is a sequence $(\Omega_j)_{j=0}^\infty = ((\Omega_{j,k,\ell})_{k,\ell})_{j=0}^\infty$ of coverings of $\mathbb{R}^d$ such that

(a) all $\Omega_{j,k,\ell}$ are balls with center in $x_{j,k,\ell}$ s.t. $x_{j,0,1} = 0$ and $|x_{j,k,\ell}| = 2^{-j}(k+1/2)$ if $k \geq 1$;

(b) diam $\Omega_{j,k,\ell} = 12 \cdot 2^{-j}$ for all k and all ℓ;

(c) $P_{j,k} \subset \bigcup_{\ell=1}^{C(d,k)} \Omega_{j,k,\ell}\,, \quad j = 0,1,\ldots\,, \quad k = 0,1,\ldots\,,$ where the numbers $C(d,k)$ satisfy the relations $C(d,k) \leq (2k+1)^{d-1}$, $C(d,0) = 1$.

(d) the sums $\sum_{k=0}^{\infty} \sum_{\ell=1}^{C(d,k)} \mathcal{X}_{j,k,\ell}(x)$ are uniformly bounded in $x \in \mathbb{R}^d$ and $j = 0,1,\ldots$ (here $\mathcal{X}_{j,k,\ell}$ denotes the characteristic function of $\Omega_{j,k,\ell}$);

(e) $\Omega_{j,k,\ell} = \{x \in \mathbb{R}^d \,:\, 2^j\, x \in \Omega_{0,k,\ell}\}$ for all j,k and ℓ;

(f) There exists a natural number K (independent of j and k) such that

$$\{(x_1, 0, \ldots, 0) : \quad x_1 \in \mathbb{R}\} \cap \frac{\mathrm{diam}\,(\Omega_{j,k,\ell})}{2}\, \Omega_{j,k,\ell} = \emptyset \qquad \text{if} \quad \ell > K \tag{31}$$

(with an appropriate enumeration).

We collect some properties of related atomic decompositions. To do this it is convenient to introduce some sequence spaces.

Definition 7 *Let $0 < q \leq \infty$.*
(i) *If $0 < p \leq \infty$, then we define*

$$b_{p,q,d} := \left\{ s = (s_{j,k})_{j,k} \,: \quad \|\, s\,|b_{p,q,d}\| = \left(\sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} (1+k)^{d-1}\, |s_{j,k}|^p \right)^{q/p} \right)^{1/q} < \infty \right\}.$$

(ii) *By $\widetilde{\chi}_{j,k}$ we denote the characteristic function of the set $P_{j,k}$. If $0 < p < \infty$ we define*

$$\begin{aligned} f_{p,q,d} \;&:=\; \Bigg\{ s = (s_{j,k})_{j,k} \,: \\ \|\, s\,|f_{p,q,d}\| \;&=\; \left\| \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |s_{j,k}|^q\, 2^{\frac{jdq}{p}}\, \widetilde{\chi}_{j,k}(\cdot) \right)^{1/q} \Big| L_p(\mathbb{R}^d) \right\| < \infty \Bigg\}. \end{aligned}$$

Remark 15 Observe $b_{p,p,d} = f_{p,p,d}$ in the sense of equivalent quasi-norms.

Atoms have to satisfy moment and regularity conditions. With this respect we suppose

$$L \geq \max(0, [s]+1)\,, \qquad M \geq \max([\sigma_p(d) - s], -1) \tag{32}$$

in case of Besov spaces and

$$L \geq \max(0, [s]+1)\,, \qquad M \geq \max([\sigma_{p,q}(d) - s], -1) \tag{33}$$

in case of Lizorkin-Triebel spaces. Under these restrictions the following assertions are known to be true:

(i) Each $f \in RB^s_{p,q}(\mathbb{R}^d)$ ($f \in RF^s_{p,q}(\mathbb{R}^d)$) can be decomposed into

$$f = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k}\, a_{j,k,\ell} \qquad (\text{convergence in} \quad \mathcal{S}'(\mathbb{R}^d)\,), \tag{34}$$

where the functions $a_{j,k,\ell}$ are $(s,p)_{L,M}$-atoms with respect to $\Omega_{j,k,\ell}$ $(j \ge 1)$, and the functions $a_{0,k,\ell}$ are 1_L-atoms with respect to $\Omega_{0,k,\ell}$.

(ii) Any formal series $\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k}\, a_{j,k,\ell}$ converges in $\mathcal{S}'(\mathbb{R}^d)$ with limit in $B^s_{p,q}(\mathbb{R}^d)$ if the sequence $s = (s_{j,k})_{j,k}$ belongs to $b_{p,q,d}$ and if the $a_{j,k,\ell}$ are $(s,p)_{L,M}$-atoms with respect to $\Omega_{j,k,\ell}$ $(j \ge 1)$, and the $a_{0,k,\ell}$ are 1_L-atoms with respect to $\Omega_{0,k,\ell}$. There exists a universal constant such that

$$\| \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k}\, a_{j,k,\ell}\, |B^s_{p,q}(\mathbb{R}^d)\| \le c\, \|\, s\, |b_{p,q,d}\| \tag{35}$$

holds for all sequences $s = (s_{j,k})_{j,k}$.

(iii) There exists a constant c such that for any $f \in RB^s_{p,q}(\mathbb{R}^d)$ there exists an atomic decomposition as in (34) satisfying

$$\| \,(s_{j,k})_{j,k}\, |b_{p,q,d}\| \le c\, \|\, f\, |B^s_{p,q}(\mathbb{R}^d)\|\,. \tag{36}$$

(iv) The infimum on the left-hand side in (35) with respect to all admissible representations (34) yields an equivalent norm on $RB^s_{p,q}(\mathbb{R}^d)$.

(v) Any formal series $\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k}\, a_{j,k,\ell}$ converges in $\mathcal{S}'(\mathbb{R}^d)$ with limit in $F^s_{p,q}(\mathbb{R}^d)$ if the sequence $s = (s_{j,k})_{j,k}$ belongs to $f_{p,q,d}$ and if the functions $a_{j,k,\ell}$ are $(s,p)_{L,M}$-atoms with respect to $\Omega_{j,k,\ell}$ $(j \ge 1)$, and the functions $a_{0,k,\ell}$ are 1_L-atoms with respect to $\Omega_{0,k,\ell}$. There exists a universal constant such that

$$\| \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k}\, a_{j,k,\ell}\, |F^s_{p,q}(\mathbb{R}^d)\| \le c\, \|\, s\, |f_{p,q,d}\| \tag{37}$$

holds for all sequences $s = (s_{j,k})_{j,k}$.

(vi) There exists a constant c such that for any $f \in RF^s_{p,q}(\mathbb{R}^d)$ there exists an atomic decomposition as in (34) satisfying

$$\| \,(s_{j,k})_{j,k}\, |f_{p,q,d}\| \le c\, \|\, f\, |F^s_{p,q}(\mathbb{R}^d)\|\,. \tag{38}$$

(vii) The infimum on the left-hand side in (37) with respect to all admissible representations (34) yields an equivalent norm on $RF^s_{p,q}(\mathbb{R}^d)$. Such decompositions as in (36) and (38) we shall call optimal.

Remark 16 For proofs of all these facts (even with respect to more general decompositions of $\mathbb{R}^d$) we refer to [32] and [36]. A different approach to atomic decompositions of radial subspaces has been given by Epperson and Frazier [10].

3.3.3 Proof of Theorem 3

Step 1. Let $f \in RB^s_{p,q}(\mathbb{R}^d)$. Then there exists an optimal atomic decomposition, i.e.

$$\begin{aligned} f &= \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{\ell=1}^{C(d,k)} s_{j,k}\, a_{j,k,\ell} \qquad (39) \\ \| \, f \, |B^s_{p,q}(\mathbb{R}^d)\| &\asymp \| \, (s_{j,k})_{j,k} \, |b_{p,q,d}\| \, , \end{aligned}$$

see (34) - (36). Since f is even we obtain

$$f(x) = \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{\ell=1}^{C(d,k)} s_{j,k} \, \frac{a_{j,k,\ell}(x) + a_{j,k,\ell}(-x)}{2} \, . \qquad (40)$$

We define

$$g_{j,k,\ell}(t) := 2^{j(s-d/p)} \Big(\mathrm{tr} \, \frac{a_{j,k,\ell}(\,\cdot\,) + a_{j,k,\ell}(-\,\cdot\,)}{2} \Big)(t) \, , \qquad t \in \mathbb{R} \, ,$$

and $d_{j,k} := 2^{-j(s-d/p)}\, s_{j,k}$. Of course, $a_{j,k,\ell}(\,\cdot\,) + a_{j,k,\ell}(-\,\cdot\,)$ is not a radial function. But it is an even and continuous. So, tr means simply the restriction to the x_1-axis. Clearly,

$$f_N(x) := \sum_{j=0}^N \sum_{k=0}^N \sum_{\ell=1}^{C(d,k)} s_{j,k} \, \frac{a_{j,k,\ell}(x) + a_{j,k,\ell}(-x)}{2} \, , \qquad x \in \mathbb{R}^d, \quad N \in \mathbb{N} \, ,$$

is an even (not necessarily radial) function in $C^L(\mathbb{R}^d)$. By means of property (f) of the particular coverings of $\mathbb{R}^d$, stated in the previous subsection, we obtain

$$\mathrm{tr} \, f_N = \sum_{j=0}^N \sum_{k=0}^N \sum_{\ell=1}^{\min(C(d,k),K)} d_{j,k} \, g_{j,k,\ell}$$

(here K is the natural number in (31)). Furthermore

$$\max_{0 \le n \le L} \, \sup_{t \in \mathbb{R}} \, |(g_{j,k,\ell})^{(n)}(t)| \le 12^{s-d/p} \, 2^{jn} \, .$$

Obviously

$$\begin{aligned} \| \, (s_{j,k})_{j,k} \, |b_{p,q,d}\| &= \left(\sum_{j=0}^\infty \left(\sum_{k=0}^\infty (1+k)^{d-1} \, |s_{j,k}|^p \right)^{q/p} \right)^{1/q} \\ &= \left(\sum_{j=0}^\infty 2^{j(s-\frac{d}{p})q} \left(\sum_{k=0}^\infty (1+k)^{d-1} \, |d_{j,k}|^p \right)^{q/p} \right)^{1/q} . \end{aligned}$$

This implies

$$\| \operatorname{tr} f_N \,|TB^s_{p,q}(\mathbb{R},L,d)\| \le K\, c \,\| f \,|B^s_{p,q}(\mathbb{R}^d)\|$$

where c and K are independent of f and N.

Next we comment on the convergence of the sequences $(f_N)_N$ and $(\operatorname{tr} f_N)_N$. Of course, f_N converges in $\mathcal{S}'(\mathbb{R}^d)$ to f. For the investigation of the convergence of $(\operatorname{tr} f_N)_N$ we choose s' such that $s > s' > \sigma_p(d)$ and conclude with $N > M$

$$\begin{aligned}
\| \operatorname{tr} f_N - \operatorname{tr} f_M \,|TB^{s'}_{p,p}(\mathbb{R},L,d)\| & \lesssim \Big\| \sum_{j=M+1}^{N} \sum_{k=0}^{N} \sum_{\ell=1}^{\min(C(d,k),K)} d_{j,k}\, g_{j,k,\ell} \Big| TB^{s'}_{p,p}(\mathbb{R}) \Big\| \\
& \quad + \Big\| \sum_{j=0}^{M} \sum_{k=M+1}^{N} \sum_{\ell=1}^{\min(C(d,k),K)} d_{j,k}\, g_{j,k,\ell} \Big| TB^{s'}_{p,p}(\mathbb{R}) \Big\| \\
& \lesssim \Big(\sum_{j=M+1}^{\infty} \sum_{k=0}^{\infty} (1+k)^{d-1} |2^{j(s'-s)}\, s_{j,k}|^p \Big)^{1/p} \\
& \quad + \Big(\sum_{j=0}^{M} \sum_{k=M+1}^{\infty} (1+k)^{d-1} |2^{j(s'-s)}\, s_{j,k}|^p \Big)^{1/p} \Big),
\end{aligned}$$

by taking into account the different normalization of the atoms in $RB^{s'}_{p,p}(\mathbb{R}^d)$ and in $RB^s_{p,q}(\mathbb{R}^d)$, respectively. The right-hand side in the previous inequality tends to zero if M tends to infinity since $\| (s_{j,k})_{j,k} \,|b_{p,q,d}\| < \infty$. Lemma 1 in combination with $B^s_{p,q}(\mathbb{R}^d) \subset L_{\max(1,p)}(\mathbb{R}^d)$ implies the continuity of $\operatorname{tr} : RB^s_{p,1}(\mathbb{R}^d) \to L_{\max(1,p)}(\mathbb{R}, t^{d-1})$ as well as the existence of $\operatorname{tr} f \in L_{\max(1,p)}(\mathbb{R}, |t|^{d-1})$. Consequently

$$\lim_{N\to\infty} \operatorname{tr} f_N = \operatorname{tr} \big(\lim_{N\to\infty} f_N\big) = \operatorname{tr} f$$

with convergence in $L_{\max(1,p)}(\mathbb{R}, |t|^{d-1})$. This proves that tr maps $RB^s_{p,q}(\mathbb{R}^d)$ into $TB^s_{p,q}(\mathbb{R},L,d)$ if L satisfies (32). Observe, that M can be chosen -1 in (32).

Step 2. The same type of arguments proves that tr maps $RF^s_{p,q}(\mathbb{R}^d)$ into $TF^s_{p,q}(\mathbb{R})$, in particular the convergence analysis is the same. Furthermore, observe

$$\begin{aligned}
& \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} \sum_{k=0}^{\infty} |s_{j,k}|^q \, \chi^{\#}_{j,k}(\cdot) \Big)^{1/q} |L_p(\mathbb{R}, t^{d-1}) \Big\| \\
& \qquad = c_d \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} \sum_{k=0}^{\infty} |s_{j,k}|^q \, \widetilde{\chi}_{j,k}(\cdot) \Big)^{1/q} |L_p(\mathbb{R}^d) \Big\| .
\end{aligned}$$

This proves that tr maps $RF^s_{p,q}(\mathbb{R}^d)$ into $TF^s_{p,q}(\mathbb{R},L,d)$ if L satisfies (33) (again we use $M = -1$).

Step 3. Properties of ext. Let g be an even function with a decomposition as in (2) and

$$\| g \,|TB^s_{p,q}(\mathbb{R},L,d)\| \asymp \| (s_{j,k}) \,|b^s_{p,q,d}\| .$$

We define

$$a_{j,k}(x) := g_{j,k}(|x|) \, , \qquad x \in \mathbb{R}^d .$$

The functions $a_{j,k}$ are compactly supported, continuous, and radial. Obviously

$$\mathrm{supp}\ a_{j,k} \subset \{x : \quad 2^{-j}k - 2^{-j-1} \le |x| \le 2^{-j}(k+1) + 2^{-j-1}\}\,, \qquad k \in \mathbb{N}\,,$$

and

$$\mathrm{supp}\ a_{j,0} \subset \{x : \quad |x| \le 3 \cdot 2^{-j-1}\}\,.$$

From Theorem 1 we derive

$$|D^\alpha a_{j,k}(x)| \le \|\, a_{j,k}\, |C^{|\alpha|}(\mathbb{R}^d)\| \ \lesssim\ \|\, g_{j,k}\, |C^{|\alpha|}(\mathbb{R})\| \ \lesssim\ 2^{j|\alpha|}\,, \tag{41}$$

if $|\alpha| \le L$. Here the constants behind $\lesssim$ do not depend on j, k and $g_{j,k}$. We continue with an investigation of the sequence

$$h_N(x) := \sum_{j=0}^{N}\sum_{k=0}^{\infty} s_{j,k}\, a_{j,k}(x)\,, \qquad x \in \mathbb{R}^d\,, \quad N \in \mathbb{N}\,. \tag{42}$$

Related to our decomposition $(\Omega_{j,k,\ell})_{j,k,\ell}$ of $\mathbb{R}^d$, see Subsection 3.3.2, there is a sequence of decompositions of unity $(\psi_{j,k,\ell})_{j,k,\ell}$, i.e.

$$\sum_{k=0}^{\infty}\sum_{\ell=1}^{C(d,k)} \psi_{j,k,\ell}(x) \quad = \quad 1 \qquad \text{for all} \quad x \in \mathbb{R}^d\,, \quad j = 0, 1, \ldots\,, \tag{43}$$

$$\mathrm{supp}\ \psi_{j,k,\ell} \quad \subset \quad \Omega_{j,k,\ell}\,, \tag{44}$$

$$|D^\alpha \psi_{j,k,\ell}| \quad \le \quad C_L\, 2^{j|\alpha|} \qquad |\alpha| \le L\,, \tag{45}$$

see [32]. Hence

$$\begin{aligned} h_N(x) &= \sum_{j=0}^{N}\sum_{k=0}^{\infty} s_{j,k}\, a_{j,k}(x) \left(\sum_{m=0}^{\infty}\sum_{\ell=1}^{C(d,m)} \psi_{j,m,\ell}(x)\right) \\ &= \sum_{j=0}^{N}\sum_{m=0}^{\infty} \left(\sum_{\ell=1}^{C(d,m)}\sum_{k=-7}^{7} s_{j,m+k}\, a_{j,m+k}(x)\psi_{j,m,\ell}(x)\right) \\ &= \sum_{j=0}^{N}\sum_{m=0}^{\infty} \lambda_{j,m} \sum_{\ell=1}^{C(d,k+m)} e_{j,m,\ell}(x)\,. \end{aligned}$$

where

$$\begin{aligned} \lambda_{j,m} &:= 2^{j(s-\frac{d}{p})} \max_{-1\le k\le 1} |s_{j,m+k}| \\ e_{j,m,\ell}(x) &:= 2^{-j(s-\frac{d}{p})}\, t_{j,m} \sum_{k=-7}^{7} s_{j,m+k}\, a_{j,m+k}(x)\, \psi_{j,m,\ell}(x) \\ t_{j,m} &:= \begin{cases} 1 & \text{if} \quad \max_{k=-7,\ldots,7} |s_{j,m+k}| = 0, \\ \\ (\max_{k=-7,\ldots,7} |s_{j,m+k}|)^{-1} & \text{otherwise}\,. \end{cases} \end{aligned}$$

We claim that the functions $e_{j,m,\ell}$ are $(s,p)_{L,-1}$-atoms (1_L-atoms if $j=0$) on $\mathbb{R}^d$ related to the covering $(\Omega_{j,k,\ell})_{j,k,\ell}$ (up to a universal constant). But this follows immediately from (44), (45), and (41). Finally we show that the sequence $\lambda = (\lambda_{j,m})_{j,m}$ belongs to $b_{p,q,d}$. The estimate

$$\begin{aligned} \| \lambda \, | b_{p,q,d} \| &= \Big(\sum_{j=0}^{N} \Big(\sum_{m=0}^{\infty} (1+m)^{d-1} \, |\lambda_{j,m}|^p \Big)^{q/p} \Big)^{1/q} \\ &\lesssim \Big(\sum_{j=0}^{N} 2^{j(s-\frac{d}{p})q} \Big(\sum_{m=0}^{\infty} (1+m)^{d-1} \, |s_{j,m}|^p \Big)^{q/p} \Big)^{1/q} \end{aligned}$$

is obvious. Hence ext maps $TB^s_{p,q}(\mathbb{R}, L, d)$ into $RB^s_{p,q}(\mathbb{R}^d)$. Here we need that the pair $(L,-1)$ satisfies (32).

Step 4. The proof of the F-case is similar. Here we need that the pair $(L,-1)$ satisfies (33). The proof is complete. ■

3.3.4 Proof of Theorem 4

Since $RF^s_{p,q}(\mathbb{R}^d) \hookrightarrow RB^s_{p,\infty}(\mathbb{R}^d)$ it will be enough to deal with radial Besov spaces.

Step 1. Let $1 \le p < \infty$. Then $\sigma_p(d) = \sigma_p(1) = 0$. ¿From $s > 0$ we derive $B^s_{p,q}(\mathbb{R}^d) \subset L_p(\mathbb{R}^d)$. Hence f is a regular distribution.

Step 2. Let $0 < p < 1$. Since $0 \notin \operatorname{supp} f$ there exists some $\varepsilon > 0$ s.t. the ball with radius ε and centre in the origin has an empty intersection with supp f. Let $\lambda > 0$. Since f is a regular distribution if, and only if $f(\lambda \, \cdot \,)$ is a regular distribution we may assume $\varepsilon = 2$. Let $\varphi \in RC^\infty(\mathbb{R}^d)$ be a function s.t. $\varphi(x) = 1$ if $|x| \ge 2$ and $\varphi(x) = 0$ if $|x| \le 1$. Again we shall work with an optimal atomic decomposition of $f \in RB^s_{p,q}(\mathbb{R}^d)$, see (39). Obviously

$$f = f\varphi = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k} \, (\varphi \, a_{j,k,\ell}) \, .$$

By checking the various support conditions we obtain

$$f = \sum_{j=0}^{3} s_{j,0} \, (\varphi \, a_{j,0,1}) + \sum_{j=0}^{\infty} \sum_{k=\max(1,2^j-9)}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k} \, (\varphi \, a_{j,k,\ell}) \, .$$

We consider the following splitting

$$\begin{aligned} f_1(x) &:= \sum_{j=0}^{3} s_{j,0} \, \varphi(x) \, \frac{a_{j,0,1}(x) + a_{j,0,1}(-x)}{2} \\ f_2(x) &:= \sum_{j=0}^{\infty} \sum_{k=\max(1,2^j-9)}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k} \, \varphi(x) \, \frac{a_{j,k,\ell}(x) + a_{j,k,\ell}(-x)}{2} \, . \end{aligned}$$

Concerning the first part f_1 we observe that tr f_1 is a compactly supported even C^L function. Now we concentrate on f_2. Let

$$f_{2,N}(x) := \sum_{j=0}^{N} \sum_{k=\max(1,2^j-9)}^{2^N} \sum_{\ell=1}^{C(d,k)} s_{j,k}\, \varphi(x)\, \frac{a_{j,k,\ell}(x) + a_{j,k,\ell}(-x)}{2}\, , \qquad N \in \mathbb{N}\, .$$

We put

$$g_{j,k,\ell}(t) := \mathrm{tr}\, \Big(\varphi(\,\cdot\,)\, \frac{a_{j,k,\ell}(\,\cdot\,) + a_{j,k,\ell}(-\,\cdot\,)}{2}\Big)(t)\, , \qquad t \in \mathbb{R}\, ,$$

by using the same convention concerning tr as in Step 1 of the proof of Thm. 3. Since

$$\sup_{t\in\mathbb{R}} |g^{(n)}_{j,k,\ell}(t)| \le c_\varphi\, (12\,\cdot\, 2^{-j})^{s-n-d/p}\, , \qquad 0 \le n \le L\, ,$$

and $|\,\mathrm{supp}\; g_{j,k,\ell}| \lesssim 2^{-j}$ we obtain for a natural number m

$$\begin{aligned}
\int_m^{m+1} |\,\mathrm{tr}\; f_{2,N}(t)|\, dt &\lesssim \sum_{j=0}^{N} \sum_{k=\max(1,2^j-9)}^{2^N} \sum_{\ell=1}^{\min(C(d,k),K)} |s_{j,k}| \int_m^{m+1} |g_{j,k,\ell}(t)|\, dt \\
&\lesssim \sum_{j=0}^{N} 2^{-j}\, 2^{-j(s-d/p)} \sum_{k=2^j m-9}^{2^j(m+1)+6} |s_{j,k}| \\
&\lesssim m^{-(d-1)/p} \sum_{j=0}^{N} 2^{-j}\, 2^{-j(s-d/p)}\, 2^{-j(d-1)/p} \Big(\sum_{k=2^j m-9}^{2^j(m+1)+6} (1+k)^{d-1} |s_{j,k}|^p\Big)^{1/p} .
\end{aligned}$$

Hence

$$\begin{aligned}
\|\ \mathrm{tr}\; f_N\, |L_1(\mathbb{R})\| &= 2 \int_1^\infty |\,\mathrm{tr}\; f_N(t)|\, dt \lesssim \|\,(s_{j,k})_{j,k}\, |b_{p,\infty,d}\| \sum_{m=1}^{\infty} m^{-(d-1)/p} \\
&\lesssim \|\, f\, |B^s_{p,q}(\mathbb{R}^d)\|
\end{aligned}$$

since $s > 0$ and $0 < p < 1$. Let $M \le N$. Then the same type of argument yields

$$\begin{aligned}
\int_m^{m+1} |\,\mathrm{tr}\; f_{2,N}(t) &- \mathrm{tr}\; f_{2,M}(t)|\, dt \\
&\lesssim \sum_{j=M+1}^{N} \sum_{k=\max(1,2^j-9)}^{2^N} \sum_{\ell=1}^{\min(C(d,k),K)} |s_{j,k}| \int_m^{m+1} |g_{j,k,\ell}(t)|\, dt \\
&\quad + \sum_{j=0}^{N} \sum_{k=\max(2^M,2^j m-9)}^{2^N} \sum_{\ell=1}^{\min(C(d,k),K)} |s_{j,k}| \int_m^{m+1} |g_{j,k,\ell}(t)|\, dt \\
&\lesssim 2^{-Ms} \sup_{j=M+1,\ldots} 2^{j\frac{d-1}{p}} \sum_{k=2^j m-9}^{2^j(m+1)+6} |s_{j,k}| \\
&\quad + \sum_{j=0}^{N} 2^{-j}\, 2^{-j(s-d/p)} \sum_{k=\max(2^M,2^j m-9)}^{2^j(m+1)+6} |s_{j,k}|
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \quad m^{-(d-1)/p} \Big(2^{-Ms} \, \| \, (s_{j,k})_{j,k} \, | b_{p,\infty,d} \| \\
&\qquad\qquad + \quad \sup_{j=0,1,\ldots} \Big(\sum_{k=\max(2^M, 2^j m - 9)}^{2^j (m+1)+6} (1+k)^{d-1} |s_{j,k}|^p \Big)^{1/p} \Big) .
\end{aligned}
$$

Since

$$
\lim_{M\to\infty} \sup_{j=0,1,\ldots} \Big(\sum_{k=\max(2^M, 2^j m - 9)}^{2^j (m+1)+6} (1+k)^{d-1} |s_{j,k}|^p \Big)^{1/p} = 0
$$

for all $m \in \mathbb{N}$ we conclude

$$
\| \, \mathrm{tr} \; f_{2,N} - \mathrm{tr} \; f_{2,M} \, | L_1(\mathbb{R}) \| \longrightarrow 0 \qquad \text{if} \quad M \to \infty \, .
$$

Hence

$$
\sum_{j=0}^{\infty} \sum_{k=\max(1,2^j-9)}^{\infty} \sum_{\ell=1}^{\min(C(d,k),K)} s_{j,k} \, g_{j,k,\ell} \in L_1(\mathbb{R}) \, .
$$

Let $\theta \in \mathbb{R}^d$, $|\theta| = 1$. We denote by Tr_θ the restriction of a continuous function to the line $\Theta := \{ t \, \theta : \quad t \in \mathbb{R} \}$. Now we repeat, what we have done with respect to the x_1-axis, for such a line. As the outcome we obtain

$$
\mathrm{Tr}_\theta \Big(\sum_{j=0}^{3} s_{j,0} \, (\varphi \, a_{j,0,1}) + \sum_{j=0}^{N} \sum_{k=\max(1,2^j-9)}^{2^N} \sum_{\ell=1}^{C(d,k)} s_{j,k} \, (\varphi \, a_{j,k,\ell}) \Big) \, , \qquad N \in \mathbb{N} \, ,
$$

is a Cauchy sequence in $L_1(\Theta)$ and the limit satisfies

$$
\begin{aligned}
\Big\| \mathrm{Tr}_\theta \Big(\sum_{j=0}^{3} s_{j,0} \, (\varphi \, a_{j,0,1}) \quad &+ \quad \sum_{j=0}^{\infty} \sum_{k=\max(1,2^j-9)}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k} \, (\varphi \, a_{j,k,\ell}) \Big) \Big| L_1(\Theta) \Big\| \\
&\lesssim \quad \| \, f \, | B^s_{p,q}(\mathbb{R}^d) \|
\end{aligned}
$$

with a constant independent of θ (of course, here, by a slight abuse of notation, Tr_θ denotes the continuous extension of the previously defined mapping). Using spherical coordinates this yields

$$
\begin{aligned}
\int_{\mathbb{R}^d} |f(x)| \, dx \quad &= \quad \int_{|\theta|=1} \int_0^\infty \, |f(t \, \theta)| \, dt \, d\theta \\
&\lesssim \quad \| \, f \, | B^s_{p,q}(\mathbb{R}^d) \| \, .
\end{aligned}
$$

But this means f is a regular distribution. ■

Remark 17 We have proved a bit more than stated. Under the given restrictions the pointwise trace tr f of a distribution $f \in RB^s_{p,q}(\mathbb{R}^d)$, $0 \not\in \mathrm{supp}\, f$, makes sense and belongs to $L_1(\mathbb{R})$.

3.3.5 Proof of Remark 3

We shall argue by using the wavelet characterization of $B^{\frac{1}{p}-1}_{p,\infty}(\mathbb{R}^d)$, see, e.g., [44, Thm. 1.20]. Let ϕ denote an appropriate univariate scaling function and Ψ an associated Daubechies wavelet of sufficiently high order. The tensor product ansatz yields $(d-1)$ generators $\Psi_1, \ldots, \Psi_{2^d-1}$ for the wavelet basis in $L_2(\mathbb{R}^d)$. Let Φ denote the d-fold tensor product of the univariate scaling function. We shall use the abbreviations

$$\Phi_k(x) := \Phi(x-k)\,, \qquad k \in \mathbb{Z}^d\,,$$

and

$$\Psi_{i,j,k}(x) := 2^{jd/2}\,\Psi_i(2^j x - k)\,, \qquad k \in \mathbb{Z}^d\,, \quad j \in \mathbb{N}_0\,, \quad i = 1, \ldots, 2^d - 1\,.$$

An equivalent norm in $B^{\frac{1}{p}-1}_{p,\infty}(\mathbb{R}^d)$ is given by

$$\|\, f \,|B^{\frac{1}{p}-1}_{p,\infty}(\mathbb{R}^d)\| = \Big(\sum_{k\in\mathbb{Z}^d} |\langle f, \Phi_k\rangle|^p\Big)^{1/p} + \sup_{j=0,1,\ldots} 2^{j(\frac{1}{p}-1+d(\frac{1}{2}-\frac{1}{p}))}\Big(\sum_{i=1}^{2^d-1}\sum_{k\in\mathbb{Z}^d} |\langle f, \Psi_{i,j,k}\rangle|^p\Big)^{1/p}\,.$$

Daubechies wavelets have compact support. This implies

$$\mathrm{supp}\ \Psi_{i,j,k} \subset C\,\{x \in \mathbb{R}^d :\ 2^{-j}(k_\ell - 1) \le x_\ell \le 2^{-j}(k_\ell + 1)\,,\ \ell = 1, \ldots d\}$$

and

$$\mathrm{supp}\ \Phi_k \subset C\,\{x \in \mathbb{R}^d :\ (k_\ell - 1) \le x_\ell \le (k_\ell + 1)\,,\ \ell = 1, \ldots d\}$$

for an appropriate $C > 1$. By employing these relations we conclude that for fixed j the cardinality of the set of those functions $\Psi_{i,j,k}$, which do not vanish identically on $|x| = 1$ is $\lesssim 2^{j(d-1)}$. There is the general estimate

$$|\langle f, \Psi_{i,j,k}\rangle| = \Big|\int_{|x|=1} 2^{jd/2}\,\Psi_i(2^j x - k)\,dx\Big| \lesssim 2^{jd/2}\,2^{-j(d-1)}\,,$$

by using the information on the size of the support. Inserting this we find

$$\begin{aligned} 2^{j(\frac{1}{p}-1+d(\frac{1}{2}-\frac{1}{p}))}\Big(\sum_{i=1}^{2^d-1}\sum_{k\in\mathbb{Z}^d} |\langle f, \Psi_{i,j,k}\rangle|^p\Big)^{1/p} &\lesssim 2^{j(\frac{1}{p}-1+d(\frac{1}{2}-\frac{1}{p}))}\,2^{j(d-1)/p}\,2^{jd/2}\,2^{-j(d-1)} \\ &\lesssim 1\,. \end{aligned}$$

This proves the claim. ■

3.3.6 Proof of Theorem 5

¿From Thm. 4 we already know that for $f \in RA^s_{p,q}(\mathbb{R}^d)$, $0 \not\in \mathrm{supp}\ f$, the trace $\mathrm{tr}\ f$ makes sense and that $\mathrm{tr}\ f \in L_1(\mathbb{R})$.

Step 1. Let $f \in RB^s_{p,q}(\mathbb{R}^d)$. Since $0 \notin \operatorname{supp} f$ there exists some $\varepsilon > 0$ s.t. the ball with radius ε and centre in the origin has an empty intersection with $\operatorname{supp} f$. Without loss of generality we assume $\varepsilon < 1$. Let $\varphi \in RC^\infty(\mathbb{R}^d)$ be a function s.t. $\varphi(x) = 1$ if $|x| \ge \varepsilon$ and $\varphi(x) = 0$ if $|x| \le \varepsilon/2$. Again we shall work with an optimal atomic decomposition of f, see (39). It follows

$$f = \sum_{j=0}^{m} s_{j,0}\,(\varphi\, a_{j,0,1}) + \sum_{j=0}^{\infty}\sum_{k=k_j}^{\infty}\sum_{\ell=1}^{C(d,k)} s_{j,k}\,(\varphi\, a_{j,k,\ell})$$

where

$$m := 1 + [\log_2(18\,\varepsilon^{-1})] \qquad \text{and} \qquad k_j := \max(1, [2^{j-1}\,\varepsilon] - 10)\,.$$

As in the previous proof we introduce the splitting $f = f_1 + f_2$, where

$$f_1(x) := \sum_{j=0}^{m} s_{j,0}\,\varphi(x)\,\frac{a_{j,0,1}(x) + a_{j,0,1}(-x)}{2}\,.$$

Obviously, $\operatorname{tr} f_1$ is a compactly supported even C^L function. Let

$$f_{2,N}(x) := \sum_{j=0}^{N}\sum_{k=k_j}^{2^N}\sum_{\ell=1}^{C(d,k)} s_{j,k}\,\varphi(x)\,\frac{a_{j,k,\ell}(x) + a_{j,k,\ell}(-x)}{2}\,, \qquad N \in \mathbb{N}\,.$$

As above we use the notation

$$g_{j,k,\ell}(t) := \operatorname{tr}\Big(\varphi(\,\cdot\,)\,\frac{a_{j,k,\ell}(\,\cdot\,) + a_{j,k,\ell}(-\,\cdot\,)}{2}\Big)(t)\,, \qquad t \in \mathbb{R}\,.$$

Hence

$$\operatorname{tr} f_{2,N}(t) = \sum_{j=0}^{N}\sum_{k=k_j}^{2^N}\sum_{\ell=1}^{\min(C(d,k),K)} s_{j,k}\,g_{j,k,\ell}(t)$$

Since

$$\sup_{t\in\mathbb{R}} |g^{(n)}_{j,k,\ell}(t)| \le c_\varphi\,(12\,\cdot\,2^{-j})^{s-n-d/p} = c_\varphi\,(12\,\cdot\,2^{-j})^{-(d-1)/p}\,(12\,\cdot\,2^{-j})^{s-n-1/p}$$

the functions $2^{-j(d-1)/p}12^{(d-1)/p}\,g_{j,k,\ell}/c_\varphi$ are $(s,p)_{L,-1}$-atoms in the sense of Subsection 3.3.2 (in the one-dimensional context). Applying property (ii) from this subsection we find

$$\begin{aligned}
\|\operatorname{tr} f_{2,N}\,|B^s_{p,q}(\mathbb{R})\| &\lesssim \Big(\sum_{j=0}^{N} 2^{j(d-1)q/p}\Big(\sum_{k=k_j}^{2^N}|s_{j,k}|^p\Big)^{q/p}\Big)^{1/q}\\
&\lesssim \Big(\sum_{j=0}^{N}\Big(\sum_{k=k_j}^{2^N}(1+k)^{d-1}|s_{j,k}|^p\Big)^{q/p}\Big)^{1/q}\\
&\lesssim \|\,f\,|RB^s_{p,q}(\mathbb{R}^d)\|\,.
\end{aligned}$$

Now we consider convergence of the sequence tr f_N. Let $\sigma_p(1) < s' < s$. Arguing as before (but taking into account the different normalization of the atoms with respect to $B^{s'}_{p,p}(\mathbb{R})$) we find

$$\begin{aligned} \| \text{ tr } f_{2,N} - \text{tr } f_{2,M} \, | L_1(\mathbb{R})\| &\le \| \text{ tr } f_{2,N} - \text{tr } f_{2,M} \, | B^{s'}_{p,p}(\mathbb{R})\| \\ &\lesssim \Big(\sum_{j=M+1}^{N} \sum_{k=k_j}^{2^N} (1+k)^{d-1} |2^{j(s'-s)} s_{j,k}|^p \Big)^{1/p} \\ &\quad + \Big(\sum_{j=0}^{M} \sum_{k=\max(2^M,k_j)}^{2^N} (1+k)^{d-1} |2^{j(s'-s)} s_{j,k}|^p \Big)^{1/p} \end{aligned}$$

Since $\|(s_{j,k})_{j,k} \, |b_{p,q,d}\| < \infty$ it follows that the right-hand side tends to 0 if $M \to \infty$. The uniform boundedness of (tr $f_{2,N})_N$ in $B^s_{p,q}(\mathbb{R})$ in combination with the weak convergence of this sequence yields $\lim_{N\to\infty}$ tr $f_{2,N} \in B^s_{p,q}(\mathbb{R})$ by means of the so-called Fatou property, see [3, 13]. Hence, tr $f_2 \in B^s_{p,q}(\mathbb{R})$. In combination with our knowledge about f_1 the claim in case of Besov spaces follows.
Step 2. Let $f \in RF^s_{p,q}(\mathbb{R}^d)$. One can argue as in Step 1. For the Fatou property of the spaces $F^s_{p,q}(\mathbb{R})$ we refer to [13]. ■

3.4 Proofs of the statements in Subsection 2.1.3

Proof of Theorem 6

Step 1. The proof of Theorem 6(i) follows from formula (11) and the density of $RC^\infty_0(\mathbb{R}^d)$ in $RW^1_p(\mathbb{R}^d)$.
Step 2. Let $f \in RC^\infty_0(\mathbb{R}^d)$. This is equivalent to tr $f = f_0 \in RC^\infty_0(\mathbb{R})$, see Thm. 1. Observe, that

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \begin{cases} f_0''(r) \cdot \frac{x_i \cdot x_j}{r^2} - f_0'(r) \, \frac{x_i \cdot x_j}{r^3}, & \text{if} \quad i \neq j \,, \\ \\ f_0''(r) \cdot \frac{x_i^2}{r^2} - f_0'(r) \cdot \frac{r^2 - x_i^2}{r^3}, & \text{if} \quad i = j. \end{cases}$$

We fix $j \in \{1, 2, \ldots, d\}$ and sum up

$$\sum_{i=1}^{d} \Big(\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \Big)^2 = \frac{f_0''(r)^2}{r^2} \, x_j^2 + \frac{f_0'(r)^2}{r^4} \cdot (r^2 - x_j^2) \,.$$

Now we sum up with respect to j and find

$$\sum_{i,j=1}^{d} \Big(\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \Big)^2 = f_0''(r)^2 + \frac{d-1}{r^2} \cdot f_0'(r)^2.$$

Since the terms on the right-hand side are nonnegative this proves the claim for smooth f. As above the density argument completes the proof. ■

Proof of Theorem 7

The formulas (4)-(6) have to be combined with the density of $RC_0^\infty(\mathbb{R}^d)$ in $RW_p^{2m}(\mathbb{R}^d)$. ■

3.5 Proof of the statements in Subsection 2.1.4

3.5.1 Proof of Lemma 2

Step 1. Necessity of $p > d$. Let $\varphi \in C_0^\infty(\mathbb{R})$ be an even function s.t. $\varphi(0) \neq 0$ and $\operatorname{supp} \varphi \subset [-1/2, 1/2]$. Since $d \geq 2$ the function $g_1(t) := \varphi(t)\,|t|^{-1}$, $t \in \mathbb{R}$, belongs to $RL_p(\mathbb{R}, |t|^{d-1})$ if $p < d$. Hence $RL_p(\mathbb{R}, |t|^{d-1}) \not\subset S'(\mathbb{R})$ if $p < d$. Let $p = d$ and take $g_2(t) := \varphi(t)\,|t|^{-1}\,(-\log|t|)^{-\alpha}$, $t \in \mathbb{R} \setminus \{0\}$, for $\alpha > 0$. In case $\alpha\, d > 1$ we have $g_2 \in RL_d(\mathbb{R}, |t|^{d-1})$. However, if $\alpha < 1$ then $g_2 \notin S'(\mathbb{R})$. With $1/d < \alpha < 1$ the claim follows.
Step 2. Sufficiency of $p > d$. Using Hölder's inequality we find

$$\int_{-1}^{1} |g(t)|\, dt \leq \Big(\int_{-1}^{1} |g(t)|^p\, |t|^{d-1}\, dt\Big)^{1/p} \Big(\int_{-1}^{1} |t|^{-\frac{(d-1)p'}{p}}\, dt\Big)^{1/p'} .$$

The second factor on the right-hand side is finite if, and only if,

$$(d-1)(p'-1) < 1 \qquad \Longleftrightarrow \qquad d < p\,.$$

Complemented by the obvious inequality

$$\int_{|t|>1} |g(t)|^p\, dt \leq \int_{|t|>1} |g(t)|^p\, |t|^{d-1} dt$$

we conclude $L_p(\mathbb{R}, |t|^{d-1}) \hookrightarrow L_1(\mathbb{R}) + L_p(\mathbb{R}) \subset S'(\mathbb{R})$. ■

3.5.2 Proof of Theorem 8

Thm. 3 implies the equivalence of (ii) and (iii). Also (i) and (ii) are obviously equivalent.
Step 1. We shall prove that (iv) implies (i) and (ii).
Substep 1.1. The B-case. It will be enough to deal with the limiting case. Let $s = d(\frac{1}{p} - \frac{1}{d}) > 0$ ($s > \sigma_p(d)$) and $q = 1$. In addition we assume $1 \leq p < d$, where the upper bound results from the previous restriction on s, see Fig. 1 in Subsection 2.1.5. For $f \in RB^s_{p,q}(\mathbb{R}^d)$ we select an optimal atomic decomposition of the trace in

the sense of Theorem 3. Let $\varphi \in S(\mathbb{R})$. Then

$$\begin{aligned}
\Big| \int_{-\infty}^{\infty} & \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} s_{j,k}\, b_{j,k}(t)\, \varphi(t)\, dt \Big| \\
& \le 4 \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |s_{j,k}|\, 2^{-j}\, \| \, b_{j,k} \, |L_\infty(\mathbb{R})\| \, \| \, \varphi \, |L_\infty(\mathbb{R})\| \\
& \le 4 \| \, \varphi \, |L_\infty(\mathbb{R})\| \sum_{j=0}^{\infty} 2^{j(s-d/p)} \sum_{k=0}^{\infty} |s_{j,k}| \\
& \le 4 \| \, \varphi \, |L_\infty(\mathbb{R})\| \, \Big(\sum_{k=0}^{\infty} (1+k)^{-(d-1)\frac{p'}{p}} \Big)^{1/p'} \\
& \qquad \times \sum_{j=0}^{\infty} 2^{j(s-d/p)} \Big(\sum_{k=0}^{\infty} (1+k)^{d-1} |s_{j,k}|^p \Big)^{1/p} .
\end{aligned}$$

Since

$$\Big(\sum_{k=0}^{\infty} (1+k)^{-(d-1)\frac{p'}{p}} \Big)^{1/p'} < \infty$$

if $1 \le p < d$, we obtain

$$\begin{aligned}
\Big| \int_{-\infty}^{\infty} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} s_{j,k}\, b_{j,k}(t)\, \varphi(t)\, dt \Big| & \le c_1 \, \| \, \varphi \, |L_\infty(\mathbb{R})\| \, \| \, (s_{j,k})_{j,k} \, |b^s_{p,1,d}\| \\
& \le c_2 \, \| \, \varphi \, |L_\infty(\mathbb{R})\| \, \| \, f \, |B^s_{p,1}(\mathbb{R}^d)\| ,
\end{aligned}$$

see Theorem 3. This proves sufficiency for $1 \le p < d$ and $s = d(\frac{1}{p} - \frac{1}{d})$. Now, let $0 < p < 1$. Then it is enough to apply the continuous embedding

$$B^{\frac{d}{p}-1}_{p,1}(\mathbb{R}^d) \hookrightarrow B^{d-1}_{1,1}(\mathbb{R}^d) ,$$

see e.g. [41, 2.7.1] or [35].

Substep 1.2. Now we turn to the same implication in case of the F-spaces. Also here an embedding argument turns out to be sufficient. For $0 < p \le 1$ and $p < p_1 < \infty$ we have

$$F^{\frac{d}{p}-1}_{p,\infty}(\mathbb{R}^d) \hookrightarrow B^{\frac{d}{p_1}-1}_{p_1,1}(\mathbb{R}^d) ,$$

see [19] or [35]. Now the claim follows from Substep 1.1.

Step 2. Since tr is an isomorphism of $RA^s_{p,q}(\mathbb{R}^d)$ onto $TA^s_{p,q}(\mathbb{R}, L, d)$ we deduce from Step 1 the implication (iv) $\Longrightarrow$ (iii).

Step 3. It remains to prove the implication (i) $\Longrightarrow$ (vi). We argue by contradiction.

Substep 3.1. The B-case. Let $s = \frac{d}{p} - 1$ and suppose $q > 1$. Oriented at our investigations in Lemma 5 we will use as test functions

$$f_\alpha(x) := \varphi(|x|)\, |x|^{-1} (-\log|x|)^{-\alpha} , \qquad x \in \mathbb{R}^d . \tag{46}$$

It is known, see e.g. [29, Lem. 2.3.1], that

$$f_\alpha \in B^{\frac{d}{p}-1}_{p,q}(\mathbb{R}^d) \qquad \text{if, and only if,} \qquad q\,\alpha > 1\,.$$

Since tr $f_\alpha \notin S'(\mathbb{R})$ if $\alpha < 1$, we obtain that tr does not map into $S'(\mathbb{R})$ as long as $1/q < \alpha < 1$.

Substep 3.2. The F-case. This time it holds

$$f_\alpha \in F^{\frac{d}{p}-1}_{p,\infty}(\mathbb{R}^d) \qquad \text{if, and only if,} \qquad p\,\alpha > 1\,,$$

see [29, Lem. 2.3.1]. Choosing $1/p < \alpha < 1$ we obtain that tr does not map $F^{\frac{d}{p}-1}_{p,\infty}(\mathbb{R}^d)$ into $S'(\mathbb{R})$. ■

3.6 Proof of the assertions in Subsection 2.1.5

Proof of Theorem 9

Comparing our atomic decomposition with that one for weighted spaces obtained in [16] it is essentially a question of renormalization of the atoms. This is enough to prove $TA^s_{p,q}(\mathbb{R}, L, d) \hookrightarrow RA^s_{p,q}(\mathbb{R}, w_{d-1})$. To see the converse one has to start with the fact that $f \in RA^s_{p,q}(\mathbb{R}, w_{d-1})$ is even. This allows to decompose f into sum of atoms that are even as well, see (39) and (40) for this argument. ■

Proof of Remark 7

The regularity of the δ distribution is calculated at several places, see e.g. [29, Remark 2.2.4/3]. The argument, used in this reference, comes from Fourier analysis and transfers to the weighted case. For the Fourier analytic characterization of $A^s_{p,q}(\mathbb{R}, w_{d-1})$ we refer to [5, 6] and [16]. ■

3.7 Proof of the assertions in Subsection 2.1.6

Proof of Corollary 1

We shall only prove part (i). The proof for the Lizorkin-Triebel spaces is similar. By our trace theorem we have

$$\|\, f_0\,|TB^s_{p,q}(\mathbb{R}, L, d)\| \;\lesssim\; \|\, f\,|B^s_{p,q}(\mathbb{R}^d)\|$$

if $L > [s] + 1$, cf. Theorem 3. Thus, it is sufficient to prove that

$$\|\, f_0\,|B^s_{p,q}(\mathbb{R})\| \;\lesssim\; \tau^{-(d-1)/p}\,\|\, f_0\,|TB^s_{p,q}(\mathbb{R}, L, d)\|\,. \tag{47}$$

The trace $f_0 \in TB^s_{p,q}(\mathbb{R}, L, d)$ can be represented in the form

$$f_0(t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} s_{j,k}\, g_{j,k}(t) \tag{48}$$

(convergence in $L_{\max(1,p)}(\mathbb{R}, |t|^{d-1})$), where the sequence $(s_{j,k})_{j,k}$ belongs to $b^s_{p,q,d}$, cf. (2). Let $\varphi \in C^\infty(\mathbb{R})$ be an even function such that $\varphi(t) = 0$ if $|t| \le \frac{1}{2}$ and $\varphi(t) = 1$ if $|t| \ge 1$. For any $\tau > 0$ we define $\varphi_\tau(t) = \varphi(\tau^{-1}t)$. We will consider two cases: $\tau \ge 2$ and $0 < \tau < 2$.

Case 1. Let $\tau \ge 2$. Under this assumption any function $\varphi_\tau\, g_{j,k}$ is an even L-atom centered at the same interval as $g_{j,k}$ itself (up to a general constant depending on φ), see Definition 1. For any $j \in \mathbb{N}_0$ we define a nonnegative integer k_j by

$$k_j := \max\{k \in \mathbb{N}_0 : \quad 2^{-j}\,(k+1) \le \tau/2\}\,.$$

Hence, $\varphi_\tau\, g_{j,k} = 0$ if $k < k_j$. Furthermore, the functions $2^{-j(s-1/p)}\,\varphi_\tau\, g_{j,k}$, $k \ge 1$, restricted either to the positive or negative half axis, are $(s,p)_{L,-1}$-atoms in the sense of Definition 6 up to a universal constant c. The functions $2^{-j(s-1/p)}\,\varphi_\tau\, g_{j,0}$ are $(s,p)_{L,-1}$-atoms as well (again up to a universal constant). We obtain

$$f_0(t) = \varphi_\tau(t)\, f_0(t) = \sum_{j=0}^{\infty} \sum_{k=k_j}^{\infty} s_{j,k}\, \varphi_\tau(t)\, g_{j,k}(t)$$

and applying (35) (which is also valid for $d = 1$) we arrive at the estimate

$$\begin{aligned}
\| f_0 \,|B^s_{p,q}(\mathbb{R})\| &\lesssim \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{1}{p})q} \Big(\sum_{k=k_j}^{\infty} |s_{j,k}|^p \Big)^{q/p} \Big)^{1/q} \\
&\lesssim \tau^{\frac{1-d}{p}} \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{d}{p})q} \Big(\sum_{k=0}^{\infty} (1+k)^{d-1}\, |s_{j,k}|^p \Big)^{q/p} \Big)^{1/q} \\
&= \tau^{\frac{1-d}{p}}\, \| s \,|b^s_{p,q,d}\| \qquad (49)
\end{aligned}$$

since $k_j \sim 2^j\,\tau$. Taking the infimum with respect to all atomic representations of f_0 we have proved (47).

Case 2. Let $0 < \tau < 2$.

Step 1. We assume $s < d/p$. Then we define $j_0 \in \mathbb{N}_0$ via the relation $2^{-j_0} \le \tau < 2^{-j_0+1}$. Further, we put $K_j := \max(1, 2^{j-j_0-1} - 1)$. Now we decompose f_0 into four sums

$$\begin{aligned}
f_0(t) = \varphi_\tau(t)\, f_0(t) &= \sum_{j=0}^{j_0+1} s_{j,0}\, \varphi_\tau(t)\, g_{j,0}(t) \quad + \quad \sum_{j=j_0}^{\infty} \sum_{k=K_j}^{2^{j-j_0+1}} s_{j,k}\, \varphi_\tau(t)\, g_{j,k}(t) \\
&+ \sum_{j=j_0}^{\infty} \sum_{k=2^{j-j_0+1}+1}^{\infty} s_{j,k}\, g_{j,k}(t) \quad + \quad \sum_{j=0}^{j_0-1} \sum_{k=1}^{\infty} s_{j,k}\, g_{j,k}(t) \\
&= f_1(t) + \ldots + f_4(t)\,,
\end{aligned}$$

with $f_4 = 0$ if $j_0 = 0$. Observe

$$\mathrm{supp}\, f_i \subset \{t:\ |t| \ge \tau\}\,, \qquad i = 3,4\,,$$

whereas the supports of the functions $\varphi_\tau\, g_{j,k}$, occuring in the defintions of f_1 and f_2, may have nontrivial intersections with the interval $(\tau/2, \tau)$. The function f_1 belongs to C^L and has compact support. The functions f_2, f_3, and f_4 are supported on $\{t:\ |t| \ge \tau/2\}$. Thus, the known convergence in $L_{\max(1,p)}(\mathbb{R}, |t|^{d-1})$ implies the convergence in $\mathcal{S}'(\mathbb{R})$. As in *Case 1* the functions $2^{-j(s-1/p)}\, g_{j,k}$, $k \ge 1$, restricted either to the positive or negative half axis, are $(s,p)_{L,-1}$-atoms in the sense of Definition 6. An easy calculation shows that also the functions $2^{-j(s-1/p)}\, \varphi_\tau\, g_{j,k}$, $j \ge j_0$, are $(s,p)_{L,-1}$-atoms (up to a universal constant). Hence we may employ (35) and obtain

$$\begin{aligned} \|\, f_2 + f_3\, | B^s_{p,q}(\mathbb{R})\| \quad & \lesssim \quad \Big(\sum_{j=j_0}^\infty 2^{j(s-\frac{1}{p})q} \Big(\sum_{k=K_j}^\infty |s_{j,k}|^p \Big)^{q/p} \Big)^{1/q} \\ & \lesssim \quad \tau^{\frac{1-d}{p}} \Big(\sum_{j=j_0}^\infty 2^{j(s-\frac{d}{p})q} \Big(\sum_{k=K_j}^\infty (1+k)^{d-1}\, |s_{j,k}|^p \Big)^{q/p} \Big)^{1/q} \end{aligned}$$

as well as

$$\begin{aligned} \|\, f_4\, | B^s_{p,q}(\mathbb{R})\| \quad & \lesssim \quad \Big(\sum_{j=0}^{j_0-1} 2^{j(s-\frac{1}{p})q} \Big(\sum_{k=1}^\infty |s_{j,k}|^p \Big)^{q/p} \Big)^{1/q} \\ & \lesssim \quad 2^{j_0 \frac{d-1}{p}} \Big(\sum_{j=0}^\infty 2^{j(s-\frac{d}{p})q} \Big(\sum_{k=0}^\infty (1+|k|)^{d-1}\, |s_{j,k}|^p \Big)^{q/p} \Big)^{1/q} \\ & \lesssim \quad \tau^{\frac{1-d}{p}} \Big(\sum_{j=0}^\infty 2^{j(s-\frac{d}{p})q} \Big(\sum_{k=0}^\infty (1+|k|)^{d-1}\, |s_{j,k}|^p \Big)^{q/p} \Big)^{1/q} . \end{aligned}$$

Now we turn to the estimate of f_1. First we deal with the estimate of the quasi-norm of the functions $\varphi_\tau\, g_{j,0}$. Let in addition $s \ge 1/p$. Employing the Moser-type estimate of Lemma 5.3.7/1 in [29] (applied with $r = \infty$) we obtain

$$\begin{aligned} \|\, \varphi_\tau\, g_{j,0}\, | B^s_{p,q}(\mathbb{R})\| \quad & \lesssim \quad \|\, \varphi_\tau\, | B^s_{p,q}(\mathbb{R})\|\, \|\, g_{j,0}\, | L_\infty(\mathbb{R})\| + \|\, \varphi_\tau\, | L_\infty(\mathbb{R})\|\, \|\, g_{j,0}\, | B^s_{p,q}(\mathbb{R})\| \\ & \lesssim \quad \tau^{-(s-1/p)}\, \|\, \varphi\, | B^s_{p,q}(\mathbb{R})\| + \|\, \varphi\, | L_\infty(\mathbb{R})\|\, \|\, g_{j,0}\, | B^s_{p,q}(\mathbb{R})\| \\ & \lesssim \quad \tau^{-(s-1/p)} + 2^{j(s-1/p)} \\ & \lesssim \quad 2^{j_0(s-1/p)}\,, \end{aligned} \qquad (50)$$

since the functions $2^{-j(s-1/p)}\, g_{j,0}$ are atoms and $j \le j_0 + 1$. If $s < 1/p$, we argue by using real interpolation. Because of

$$B^s_{p,q}(\mathbb{R}) = \Big(B^{s_0}_{p,q_0}(\mathbb{R}), L_p(\mathbb{R}) \Big)_{\Theta,q}\,, \qquad s = (1-\Theta)s_0 > \sigma_p(1)\,,$$

see [9], an application of the interpolation inequality

$$\| \varphi_\tau \, g_{j,0} \, |B^s_{p,q}(\mathbb{R})\| \lesssim \| \varphi_\tau \, g_{j,0} \, |B^{s_0}_{p,q}(\mathbb{R})\|^{1-\Theta} \, \| \varphi_\tau \, g_{j,0} \, |L_p(\mathbb{R})\|^{\Theta}$$

yields (50) for all $s > \sigma_p(1)$. With $r := \min(1, p, q)$ and $\sigma_p(d) < s < d/p$ we conclude

$$\begin{aligned}
\| f_1 \, |B^s_{p,q}(\mathbb{R})\|^r & \le \sum_{j=0}^{j_0+1} |s_{j,0}|^r \, \| \varphi_\tau \, b_{j,0} \, |B^s_{p,q}(\mathbb{R})\|^r \\
& \lesssim 2^{j_0(s-\frac{1}{p})r} \sum_{j=0}^{j_0+1} |s_{j,0}|^r \, . \\
& \lesssim \tau^{\frac{r(1-d)}{p}} \sum_{j=0}^{j_0+1} 2^{r(j_0-j)(s-\frac{d}{p})} 2^{j(s-\frac{d}{p})r} \, |s_{j,0}|^r \\
& \lesssim \tau^{\frac{r(1-d)}{p}} \Big(\sup_{j=0,\ldots,j_0+1} 2^{j(s-\frac{d}{p})} \, |s_{j,0}| \Big)^r \\
& \lesssim \tau^{\frac{r(1-d)}{p}} \, \| f_1 \, |TB^s_{p,\infty}(\mathbb{R})\|^r \, .
\end{aligned}$$

This proves the claim for $s < d/p$.

Step 2. Let $s \ge d/p$. As in Step 1 we define $j_0 \in \mathbb{N}$ via the relation $2^{-j_0} \le \tau < 2^{-j_0+1}$. We shall use $TA^s_{p,q}(\mathbb{R}, L, d) = RA^s_{p,q}(\mathbb{R}, w_{d-1})$, cf. Theorem 9. Alternatively one could use interpolation, see Propositions 1, 2. The spaces $RA^s_{p,q}(\mathbb{R}, w_{d-1})$ allow a characterization by Daubechies wavelets, see [17] for Besov spaces and [18] for Lizorkin-Triebel spaces. The same is true with respect to the ordinary spaces $A^s_{p,q}(\mathbb{R})$, see e.g. [44, Thm. 1.20]. Let ϕ denote an appropriate scaling function and Ψ an associated Daubechies wavelet of sufficiently high order. Let

$$\phi_{0,\ell}(t) := \phi(t-\ell) \qquad \text{and} \qquad \Psi_{j,\ell}(t) := 2^{j/2} \, \Psi_{j,\ell}(2^j t - \ell) \, , \qquad \ell \in \mathbb{Z} \, , \quad j \in \mathbb{N}_0 \, .$$

Since Ψ has compact support, say supp $\Psi \subset [-2^N, 2^N]$ for some $N \in \mathbb{N}$, and supp $f_0 \subset \{t \in \mathbb{R} : |t| \ge \tau\}$ we find that

$$\langle f_0, \Psi_{j,\ell} \rangle = 0 \qquad \text{if} \qquad j - j_0 \ge N \qquad \text{and} \qquad |\ell| \le 2^{j-j_0} - 2^N \, .$$

Hence, f_0 has a wavelet expansion given by

$$\begin{aligned}
f_0 & = \sum_{\ell \in \mathbb{Z}} \langle f_0, \phi_{0,\ell} \rangle \, \phi_{0,\ell} + \sum_{j=0}^{j_0+N-1} \sum_{\ell \in \mathbb{Z}} \langle f_0, \Psi_{j,\ell} \rangle \, \Psi_{j,\ell} + \sum_{j=j_0+N}^{\infty} \sum_{|\ell| > 2^{j-j_0} - 2^N} \langle f_0, \Psi_{j,\ell} \rangle \, \Psi_{j,\ell} \, . \\
& = f_1 + f_2 + f_3 \, .
\end{aligned}$$

By the references given above it follows

$$\| f_1 \,|B^s_{p,q}(\mathbb{R}, w_{d-1})\| \asymp \Big(\sum_{\ell\in\mathbb{Z}} |\langle f_0, \phi_{0,\ell}\rangle|^p (1+|\ell|)^{d-1}\Big)^{1/p}$$

$$\| f_2 \,|B^s_{p,q}(\mathbb{R}, w_{d-1})\| \asymp \Big(\sum_{j=0}^{j_0+N-1} 2^{j(s+\frac{1}{2}-\frac{d}{p})q} \Big(\sum_{\ell\in\mathbb{Z}} |\langle f_0, \Psi_{j,\ell}\rangle|^p (1+|\ell|)^{d-1}\Big)^{q/p}\Big)^{1/q}$$

$$\| f_3 \,|B^s_{p,q}(\mathbb{R}, w_{d-1})\| \asymp \Big(\sum_{j=j_0+N}^{\infty} 2^{j(s+\frac{1}{2}-\frac{d}{p})q} \Big(\sum_{|\ell|\ge 2^{j-j_0}-2^N} |\langle f_0, \Psi_{j,\ell}\rangle|^p (1+|\ell|)^{d-1}\Big)^{q/p}\Big)^{1/q}.$$

The quasi-norm in the unweighted spaces is obtained by deleting the factor $2^{-j(d-1)/p}\,(1+|\ell|)^{d-1}$, see [44, Thm. 1.20]. This immediately implies

$$\| f_1 \,|B^s_{p,q}(\mathbb{R})\| \lesssim \| f_1 \,|B^s_{p,q}(\mathbb{R}, w_{d-1})\|\,,$$
$$\| f_2 \,|B^s_{p,q}(\mathbb{R})\| \lesssim 2^{(j_0+N)(d-1)/p}\, \| f_2 \,|B^s_{p,q}(\mathbb{R}, w_{d-1})\|\,.$$

Moreover, we also obtain

$$\| f_3 \,|B^s_{p,q}(\mathbb{R})\| \lesssim 2^{(j_0+N)(d-1)/p}\, \| f_3 \,|B^s_{p,q}(\mathbb{R}, w_{d-1})\|\,.$$

This proves (47) in case $s \ge d/p$ and $0 < \tau < 2$. ■

Proof of Corollary 2

We concentrate on the proof in case of Besov spaces. The proof for Lizorkin-Triebel spaces is similar.

Step 1. We claim that $g \in TB^s_{p,q}(\mathbb{R}, L, d)$. We argue as in *Case 2, Step 2* of the proof of Corollary 1.

Under the given restrictions $g \in RB^s_{p,q}(\mathbb{R})$ has a wavelet expansion of the form

$$g = \sum_{|\ell|\le c_1} \langle g, \phi_{0,\ell}\rangle\, \phi_{0,\ell} + \sum_{j=0}^{\infty} \sum_{|\ell|\le c_1 2^j} \langle g_0, \Psi_{j,\ell}\rangle\, \Psi_{j,\ell}$$

with an appropriate constant c_1. Since g is even we obtain

$$g = \sum_{|\ell|\le c_1} \langle g, \phi_{0,\ell}\rangle\, \frac{\phi_{0,\ell}(t)+\phi_{0,\ell}(-t)}{2} + \sum_{j=0}^{\infty} \sum_{|\ell|\le c_1 2^j} \langle g_0, \Psi_{j,\ell}\rangle\, \frac{\Psi_{j,\ell}(t)+\Psi_{j,\ell}(-t)}{2}\,.$$

The functions $2^{-j/2}(\Psi_{j,\ell}(t)+\Psi_{j,\ell}(-t))$ are even L-atoms (up to a universal constant) centered at $c_2\, I_{j,\ell}$, where

$$I_{j,k} := [-2^{-j}(\ell+1), -2^{-j}\ell] \cup [2^{-j}\ell, 2^{-j}(\ell+1)]$$

(modification if $\ell = 0$, see Definition 3). The constant $c_2 > 1$ depends on the size of the supports of the generators ϕ and Ψ. Without proof we mention that Theorem 3 remains true also for those more general decompositions. This implies

$$\begin{aligned} \| \, g \, | TB^s_{p,q}(\mathbb{R}, L, d) \| &\asymp \Big(\sum_{|\ell| \le c_1 b} (1 + |\ell|)^{d-1} |\langle g, \phi_{0,\ell} \rangle|^p \Big)^{1/p} \\ &\quad + \Big(\sum_{j=0}^{\infty} 2^{j(s-d/p)q} \Big(\sum_{|\ell| \le c_1 2^j} (1 + |\ell|)^{d-1} |2^{j/2} \, \langle g, \Psi_{j,\ell} \rangle|^p \Big)^{q/p} \Big)^{1/q} \\ &\lesssim \Big(\sum_{|\ell| \le c_1 b} |\langle g, \phi_{0,\ell} \rangle|^p \Big)^{1/p} + \Big(\sum_{j=0}^{\infty} 2^{j(s+\frac{1}{2}-\frac{1}{p})q} \Big(\sum_{|\ell| \le c_1 2^j} |\langle g, \Psi_{j,\ell} \rangle|^p \Big)^{q/p} \Big)^{1/q} \\ &\lesssim \| \, g \, | B^s_{p,q}(\mathbb{R}) \| \, , \end{aligned}$$

see e.g. [44, Thm. 1.20] for the last step. This proves the claim.
Step 2. Since g belongs to $TB^s_{p,q}(\mathbb{R}, L, d)$ we derive by means of Theorem 3 that $f := \operatorname{ext} g$ is an element of $RB^s_{p,q}(\mathbb{R}^d)$ and

$$\| \, f \, | RB^s_{p,q}(\mathbb{R}^d) \| \ \lesssim \ \| \, g \, | TB^s_{p,q}(\mathbb{R}, L, d) \| \ \lesssim \ \| \, g \, | B^s_{p,q}(\mathbb{R}) \| \, .$$

Since $\operatorname{supp} f \subset \{x : \ |x| \ge a\}$ Corollary 1 yields

$$\| \, g \, | B^s_{p,q}(\mathbb{R}) \| \ \lesssim \ a^{-(d-1)/p} \, \| \, f \, | B^s_{p,q}(\mathbb{R}^d) \| \, ,$$

because of $f_0 = g$. This completes the proof. ■

Remark 18 A closer look onto the proof shows that

$$a^{(d-1)/p} \, \| \, g \, | A^s_{p,q}(\mathbb{R}) \| \ \lesssim \ \| \, f \, | RA^s_{p,q}(\mathbb{R}^d) \| \ \lesssim \ b^{(d-1)/p} \, \| \, g \, | A^s_{p,q}(\mathbb{R}) \|$$

with constants independent of g, $a > 0$ and $b \ge 1$.

Proof of Corollaries 3, 4

Step 1. Proof of Cor. 3. The function φ is a pointwise multiplier for the spaces $A^s_{p,q}(\mathbb{R}^d)$, see e.g. [29, 4.8]. Hence, with f also the product $\varphi \, f$ belongs to $RA^s_{p,q}(\mathbb{R}^d)$ and we can apply Thm. 5 with respect to this product. Concerning the sharp embedding relations for the spaces $A^s_{p,q}(\mathbb{R})$ into Hölder-Zygmund spaces we refer to [35] and the references given there. This proves the assertion for $\varphi_0 \, f_0$. A further application of Theorem 2 finishes the proof.
Observe, that we do not need the assumption $s > \sigma_{p,q}(d)$ in case of Lizorkin-Triebel spaces. We may argue with $RF^s_{p,\infty}(\mathbb{R}^d)$ first and use the elementary embedding $RF^s_{p,q}(\mathbb{R}^d) \hookrightarrow RF^s_{p,\infty}(\mathbb{R}^d)$ afterwards.
Step 2. Proof of Cor. 4. The arguments are as above. Concerning te embedding

relations of the spaces $A^s_{p,q}(\mathbb{R})$ into the space of uniformly continuous and bounded functions we also refer to [35] and the references given there. ■

Remark 19 A different proof of Cor. 4, restricted to Besov spaces, has been given in [32].

3.8 Test functions

Using our previous results, in particular Corollary 2, we shall investigate the regularity of certain families of radial test functions.

Lemma 5 *Let* $0 < \alpha < \min(1, 1/p)$. *Let* $\varphi \in C_0^\infty(\mathbb{R})$ *be an even function such that* $\operatorname{supp} \varphi \subset [-2, -1/2] \cup [1/2, 2]$ *and* $\varphi(1) \neq 0$.
(i) *The function*

$$f_\alpha(x) := \varphi(|x|) \, |\,|x| - 1\,|^{-\alpha} , \qquad x \in \mathbb{R}^d , \tag{51}$$

belongs to $B^{\frac{1}{p}-\alpha}_{p,\infty}(\mathbb{R}^d)$ *if*

$$\alpha < \frac{1}{p} - \sigma_p(d). \tag{52}$$

(ii) *Suppose* $\frac{1}{p} - \alpha > \sigma_p(1)$. *Then* f_α *does not belong to* $B^{\frac{1}{p}-\alpha}_{p,q}(\mathbb{R}^d)$ *for any* $q < \infty$.
(iii) *Under the same restriction as in (ii) we have that* f_α *does not belong to* $F^{\frac{1}{p}-\alpha}_{p,\infty}(\mathbb{R}^d)$.

Proof. *Step 1.* Proof of (i). Let $\widetilde{\varphi} \in C_0^\infty(\mathbb{R})$ be a function such that $\operatorname{supp} \widetilde{\varphi} \subset [1/2, 2]$. Then the regularity of

$$g_\alpha(t) := \widetilde{\varphi}(t) \, |t - 1|^{-\alpha} , \qquad t \in \mathbb{R} ,$$

is well understood, cf. e.g. [29, Lem. 2.3.1/1]. One has $g_\alpha \in B^{\frac{1}{p}-\alpha}_{p,\infty}(\mathbb{R})$ as long as $0 < \alpha < \min(1, 1/p)$. An application of Corollary 2 yields the claim.
Step 2. Proof of (ii) and (iii). It is also known, see again [29, Lem. 2.3.1/1], that

$$g_\alpha \notin (B^{\frac{1}{p}-\alpha}_{p,q}(\mathbb{R}) \cup F^{\frac{1}{p}-\alpha}_{p,\infty}(\mathbb{R})) , \qquad 0 < q < \infty , \quad 0 < \alpha < \min(1, 1/p) .$$

These properties do not change when we "add" the reflection of g_α to the left half of the real axis. With other words, if we replace $\widetilde{\varphi}$ by φ itself we do not change the regularity properties. Now we use Thm. 5. ■

Remark 20 Let $\delta > 0$. Then also the regularity of functions like

$$f_{\alpha,\delta}(x) := \varphi(|x|) \, |\,|x| - 1|^{-\alpha} \, (-\log |\,|x| - 1|)^{-\delta}, \qquad x \in \mathbb{R}^d , \tag{53}$$

can be checked in this way. With the help of the parameter δ one can see the microscopic index q. We refer to [37, 5.6.9] or [29, Lem. 2.3.1/1] for details.

Lemma 6 *Let $\alpha > 0$.*
(i) *Then the function*

$$\Phi_\alpha(x) := \max(0, (1 - |x|^2))^\alpha\,, \qquad x \in \mathbb{R}^d\,, \tag{54}$$

belongs to $B^{\frac{1}{p}+\alpha}_{p,\infty}(\mathbb{R}^d)$ if

$$\frac{1}{p} + \alpha > \sigma_p(d)\,. \tag{55}$$

(ii) *Suppose $\frac{1}{p} + \alpha > \sigma_p(1)$. Then Φ_α does not belong to $B^{\frac{1}{p}+\alpha}_{p,q}(\mathbb{R}^d)$ for any $q < \infty$.*
(iii) *Under the same restrictions as in (ii) we have that Φ_α does not belong to $F^{\frac{1}{p}+\alpha}_{p,\infty}(\mathbb{R}^d)$.*

Proof. *Step 1.* Proof of (i). First we investigate the one-dimensional case. Let $\psi_1, \psi_2 \in C_0^\infty(\mathbb{R})$ be such that

$$\mathrm{supp}\ \psi_1 \subset [-1/2, \infty), \quad \mathrm{supp}\ \psi_2 \subset (-\infty, 1/2] \quad \text{and} \quad \psi_1(t) + \psi_2(t) = 1$$

for all $t \in \mathbb{R}$. We put $\Phi_{i,\alpha} := \psi_i\, \Phi_\alpha$, $i = 1, 2$. Then $\Phi_{1,\alpha}$ behaves near 1 like

$$\phi_\alpha(t) := \begin{cases} t^\alpha & \text{if} \quad t > 0\,, \\ 0 & \text{if} \quad t < 0\,, \end{cases}$$

near the origin. The regularity of ϕ_α is well understood, we refer to [29, Lem. 2.3.1]. As above the transfer to general dimensions $d > 1$ is done by Corollary 2.
Step 2. To prove the statements in (ii) and (iii) we argue by contradiction. If Φ_α belongs to $RA^s_{p,q}(\mathbb{R}^d)$, then also $\varphi\, \Phi_\alpha$ belongs to $RA^s_{p,q}(\mathbb{R}^d)$ for any smooth radial φ. Choosing φ s.t. $0 \not\in \mathrm{supp}\ \varphi$, we my apply Thm. 5 to conclude that $\mathrm{tr}\,(\varphi\, \Phi_\alpha) \in A^s_{p,q}(\mathbb{R})$. But this implies $\Phi_{2,\alpha} \in A^s_{p,q}(\mathbb{R})$. In the one-dimensional case necessary and sufficient conditions are known, we refer again to [29, Lem. 2.3.1]. ∎

Next we shall consider smooth functions supported in thin annuli.

Lemma 7 *Let $d \geq 2$, $0 < p, q \leq \infty$ and $s > \sigma_p(d)$. Let $\varphi \in C_0^\infty(\mathbb{R})$ be an even function such that $\varphi(1) = 1$ and* $\mathrm{supp}\ \varphi \subset [-2, -1/2] \cup [1/2, 2]$. *Then the functions*

$$f_{j,\lambda}(y) := \varphi(2^j|y| - \lambda)\,, \qquad y \in \mathbb{R}^d\,, \quad j \in \mathbb{N}\,, \quad \lambda > 0\,.$$

have the following properties:

$$\mathrm{supp}\ f_{j,\lambda} \ \subset \ \{y : \quad (\lambda - 2)\, 2^{-j} \leq |y| \leq (2 + \lambda)\, 2^{-j}\}\,, \tag{56}$$

$$\|\, f_{j,\lambda}\, |RB^s_{p,q}(\mathbb{R}^d)\| \ \asymp \ 2^{j(s-\frac{d}{p})}\, \lambda^{(d-1)/p} \tag{57}$$

with constants in $\asymp$ independent of $\lambda > 1$ and $j \in \mathbb{N}$.

Proof. *Step 1.* Estimate from above in (57). It will be convenient to use the atomic characterizations described in Subsection 3.3.2. Therefore we shall use the decompositions of unity from (43)-(45). Thanks to the support restrictions for the functions $\psi_{j,k,\ell}$ we obtain

$$f_{j,\lambda}(y) = \sum_{\max(0,\lambda-2-n_0)\le k\le \lambda+2+n_0} \sum_{\ell=1}^{C(d,k)} (\varphi(2^j|y|-\lambda)\,\psi_{j,k,\ell}(y))$$

where n_0 is a fixed number ($n_0 \ge 18$ would be sufficient). The functions

$$a_{j,k,\ell}(y) := 2^{-j(s-\frac{d}{p})}\,\varphi(2^j|y|-\lambda)\,\psi_{j,k,\ell}(y)$$

are $(s,p)_{M,-1}$-atoms for any M (up to a universal constant). Hence

$$\begin{aligned} \|\, f_{j,\lambda}\,|RB^s_{p,q}(\mathbb{R}^d)\| &\lesssim \Big(\sum_{\max(0,\lambda-2-n_0)\le k\le\lambda+2+n_0} k^{d-1} 2^{j(s-\frac{d}{p})p}\Big)^{1/p} \\ &\lesssim 2^{j(s-\frac{d}{p})}\lambda^{(d-1)/p}\,. \end{aligned}$$

Step 2. Estimate from below.
Substep 2.1. First we deal with $p=\infty$. By construction $f_{j,\lambda}(y) = 1$ if $|y| = (1+\lambda)\,2^{-j}$. Furthermore, calculating the derivatives of $f_{j,\lambda}(y_1,0,\ldots,0)$, $y_1\in\mathbb{R}$, it is immediate that

$$\|\, f_{j,\lambda}\,|C^m(\mathbb{R}^d)\| \asymp 2^{jm} \tag{58}$$

for all $m\in\mathbb{N}_0$. Now we argue by contradiction. We fix $s>1$, $q_1\in(0,\infty]$ and assume that

$$\|\, f_{j,\lambda}\,|B^s_{\infty,q_1}(\mathbb{R}^d)\| \le \phi(j,\lambda)\,2^{js}\,,$$

where $\phi:\ \mathbb{N}\times[1,\infty)\to(0,1)$ and $\lim_{\ell\to\infty}\phi(j_\ell,\lambda_\ell)=0$ for some sequence $(j_\ell,\lambda_\ell)_\ell\subset \mathbb{N}\times[1,\infty)$. We choose $\Theta\in(0,1)$ s.t. $m=\Theta\, s$ and $q=1$. Real interpolation between $C(\mathbb{R}^d)$ and $B^s_{\infty,q_1}(\mathbb{R}^d)$ yields

$$\|\, f_{j,\lambda}\,|B^m_{\infty,1}(\mathbb{R}^d)\| \le c\,2^{jm}\,(\phi(j,\lambda))^{\Theta}\,,$$

where c is independent of j and λ, see the proof of Thm. 2. The continuous embedding $B^m_{\infty,1}(\mathbb{R}^d)\hookrightarrow C^m(\mathbb{R}^d)$ leads to a contradiction with (58).
Now let $0<s<1$. We interpolate between $B^s_{\infty,q_1}(\mathbb{R}^d)$ and $B^2_{\infty,q}(\mathbb{R}^d)$. By arguing as above we could improve the estimate from above with respect to the space $B^1_{\infty,1}(\mathbb{R}^d)$. Since $B^1_{\infty,1}(\mathbb{R}^d)\hookrightarrow C^1(\mathbb{R}^d)$ this contradicts again (58). Hence the claim is proved with $p=\infty$, $0<q\le\infty$, and $s>0$.
Substep 2.2. Also obvious is the behaviour in $L_p(\mathbb{R}^d)$. For $0<p\le\infty$ we have

$$\|\, f_{j,\lambda}\,|L_p(\mathbb{R}^d)\| \asymp 2^{-jd/p}\,\lambda^{(d-1)/p}\,. \tag{59}$$

A few more calculations yield

$$\| f_{j,\lambda} \, | W^1_p(\mathbb{R}^d)\| \asymp 2^{j(1-d/p)} \, \lambda^{(d-1)/p} \, , \tag{60}$$

as long as $1 \le p \le \infty$.

Substep 2.3. Let $p_1 < \infty$. We assume that for some fixed $s_1 > \sigma_{p_1}(d)$ and $q_1 \in (0,\infty]$

$$\| f_{j,\lambda} \, | B^{s_1}_{p_1,q_1}(\mathbb{R}^d)\| \le \phi(j,\lambda) \, 2^{j(s_1-d/p_1)} \, \lambda^{(d-1)/p_1}$$

holds, where ϕ is as above. Complex interpolation between $B^{s_1}_{p_1,q_1}(\mathbb{R}^d)$ and $B^{s_2}_{\infty,q_0}(\mathbb{R}^d)$, $s_2 > 0$, yields an improvement of our estimate with respect to $B^s_{p,q}(\mathbb{R}^d)$, where $p > p_1$ is at our disposal. For s_2 large we can choose $p > 1$ s.t. $s = (1-\Theta)s_2 + \Theta \, s_1 > 1$. Now we need a further interpolation, this time real, between $B^s_{p,q}(\mathbb{R}^d)$ and $L_p(\mathbb{R}^d)$, improving the estimate for $B^1_{p,1}(\mathbb{R}^d)$ in this way. But $B^1_{p,1}(\mathbb{R}^d) \hookrightarrow W^1_p(\mathbb{R}^d)$ and so we found a contradiction to (60). ■

Remark 21 Obviously there is no q-dependence in Lemma 7. As an immediate consequence of the elemetary embeddings

$$B^s_{p,\min(p,q)}(\mathbb{R}^d) \hookrightarrow F^s_{p,q}(\mathbb{R}^d) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^d) \, ,$$

see [41,], and (57) we obtain

$$\| f_{j,\lambda} \, | F^s_{p,q}(\mathbb{R}^d)\| \asymp 2^{j(s-\frac{d}{p})} \, \lambda^{(d-1)/p} \, , \qquad \lambda > 1, \quad j \in \mathbb{N} \, .$$

Some extremal functions in $A^{d/p}_{p,q}(\mathbb{R}^d)$ have been investigated by Bourdaud [2], for $B^s_{p,p}(\mathbb{R}^d)$ see also Triebel [43]. We recall the result obtained in [2]. For $(\alpha,\sigma) \in \mathbb{R}^2$ we define

$$f_{\alpha,\sigma}(x) := \psi(x) \Big| \log|x| \Big|^{\alpha} \Big| \log|\log|x|| \Big|^{-\sigma} \, , \qquad x \in \mathbb{R}^d \, . \tag{61}$$

Furthermore we define a set $U_t \subset \mathbb{R}^2$ as follows:

$$U_t := \begin{cases} (\alpha = 0 \quad \text{and} \quad \sigma > 0) \quad \text{or} \quad \alpha < 0 & \text{if} \quad t = 1 \, , \\ \\ (\alpha = 1 - 1/t \quad \text{and} \quad \sigma > 1/t) \quad \text{or} \quad \alpha < 1 - 1/t & \text{if} \quad 1 < t < \infty \, , \\ \\ (\alpha = 1 \quad \text{and} \quad \sigma \ge 0) \quad \text{or} \quad \alpha < 1 & \text{if} \quad t = \infty \, , \end{cases}$$

Lemma 8 (i) *Let $0 < p \le \infty$ and $1 < q \le \infty$. Then $f_{\alpha,\sigma}$ belongs to $RB^{d/p}_{p,q}(\mathbb{R}^d)$ if, and only if $(\alpha,\sigma) \in U_q$.*
(ii) *Let $1 < p < \infty$. Then $f_{\alpha,\sigma}$ belongs to $RF^{d/p}_{p,q}(\mathbb{R}^d)$ if, and only if $(\alpha,\sigma) \in U_p$.*

Remark 22 Let us mention that in [2] the result is stated for $p \ge 1$ only. However, the proof extends to $p < 1$ nearly without changes (in his argument which follows formula (9) in [2] one has to choose $k > d/(2p)$).

4 Decay properties of radial functions – proofs

4.1 Proof of Theorem 10

Step 1. Proof of (i). Following Remark 9 it will be enough to prove the decay estimate (13) for $RB^{1/p}_{p,1}(\mathbb{R}^d)$, $0 < p < \infty$, and for $RF^{1/p}_{p,\infty}(\mathbb{R}^d)$, $0 < p \le 1$. A proof in case $RB^{1/p}_{p,1}(\mathbb{R}^d)$ has been given in [32]. So we are left with the proof for the Lizorkin-Triebel spaces. We will follow the ideas of the proof of Cor. 4. Let $f \in RF^{1/p}_{p,\infty}(\mathbb{R}^d)$. Let

$$f = \sum_{j=0}^{\infty} s_{j,0}\, a_{j,0} + \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \sum_{\ell=1}^{C_{d,k}} s_{j,k}\, a_{j,k,\ell},$$

be an atomic decomposition such that $\|s_{j,k}|f_{p,\infty,d}\| \asymp \| f\, |F^{1/p}_{p,\infty}(\mathbb{R}^d)\|$. We fix x, $|x| > 1$. Observe, that for all $j \ge 0$ there exists $k_j \ge 1$ such that

$$k_j\, 2^{-j} \le |x| < (k_j + 1)\, 2^{-j}\, . \tag{62}$$

Then the main part of f near $(|x|, 0, \ldots, 0)$ is given by the function

$$f^M(y) = \sum_{j=0}^{\infty} s_{j,k_j}\, a_{j,k_j,0}(y)\, , \qquad y \in \mathbb{R}^d\, , \tag{63}$$

(in fact, f is a finite sum of functions of type

$$\sum_{j=0}^{\infty} s_{j,k_j+r_j}\, a_{j,k_j+r_j,t_j}(y)\, ,$$

and $|r_j|$ and $|t_j|$ are uniformly bounded). For convenience we shall derive an estimate of the main part f^M only. Because of (62) and the normalization of the atoms we obtain

$$|f^M(y)| \lesssim \sum_{j=0}^{\infty} |s_{j,k_j}|\, 2^{j\frac{d-1}{p}} \lesssim |x|^{\frac{1-d}{p}} \Big(\sum_{j=0}^{\infty} |s_{j,k_j}|^p\, k_j^{d-1}\Big)^{1/p}\, , \tag{64}$$

since $p \le 1$. On the other hand

$$\begin{aligned} \| s\, |f_{p,\infty,d}\| &= \| \sup_{j=0,1,\ldots} \sup_{k\in\mathbb{N}_0} |s_{j,k}|\, 2^{\frac{jd}{p}}\, \widetilde{\chi}_{j,k}(\cdot)\, |L_p(\mathbb{R}^d)\| \\ &\ge \| \sup_{j=0,1,\ldots} |s_{j,k_j}|\, 2^{\frac{jd}{p}}\, \widetilde{\chi}_{j,k_j}(\cdot)\, |L_p(\mathbb{R}^d)\|\, . \end{aligned} \tag{65}$$

Using $P_{j+1,k_{j+1}} \subset P_{j,k_j}$ we obtain the identity

$$\sup_j |s_{j,k_j}|\, 2^{\frac{jd}{p}}\, \widetilde{\chi}_{j,k_j}(\cdot) = \sum_{j=0}^{\infty} \max_{i=0,\ldots,j} |s_{i,k_i}| \big(\widetilde{\chi}_{j,k_j}(\cdot) - \widetilde{\chi}_{j+1,k_{j+1}}(\cdot)\big)\, .$$

By the pairwise disjointness of the sets $P_{j,k_j} \setminus P_{j+1,k_{j+1}}$ this implies

$$\| \sup_{j=0,1,\ldots} |s_{j,k_j}|\, 2^{\frac{jd}{p}}\, \widetilde{\chi}_{j,k_j}(\cdot)\, |L_p(\mathbb{R}^d)\| \asymp \Big(\sum_{j=0}^{\infty} \max_{i=0,\ldots,j} \big(|s_{i,k_i}|^p 2^{id}\big)\, 2^{-jd} k_j^{d-1}\Big)^{1/p}\, . \tag{66}$$

Obviously

$$\Big(\sum_{j=0}^{\infty} |s_{j,k_j}|^p \, k_j^{d-1}\Big)^{1/p} \le \Big(\sum_{j=0}^{\infty} \max_{i=0,\ldots,j} \big(|s_{i,k_i}|^p 2^{id}\big)\, 2^{-jd} k_j^{d-1}\Big)^{1/p} . \qquad (67)$$

Combining (64) - (67) we have proved (13) in case of Lizorkin-Triebel spaces.
Step 2. Proof of (ii). As in Step 1 it will be sufficient to deal with the limiting cases.
Substep 2.1. Let $f \in RF^{1/p}_{p,\infty}(\mathbb{R}^d)$. Let $B_r(0)$ be the ball in $\mathbb{R}^d$ with center in the origin and radius r. Then (64)-(67) yield

$$\begin{aligned} |f^M(x)| &\lesssim |x|^{\frac{1-d}{p}} \, \| \sup_{j=0,1,\ldots} |s_{j,k_j}| \, 2^{\frac{jd}{p}} \, \widetilde{\chi}_{j,k_j}(\cdot) \, |L_p(\mathbb{R}^d \setminus B_r(0))\| \\ &\lesssim |x|^{\frac{1-d}{p}} \, \| \sup_{j=0,1,\ldots} \sup_{k\in\mathbb{N}_0} |s_{j,k}| \, 2^{\frac{jd}{p}} \, \widetilde{\chi}_{j,k}(\cdot) \, |L_p(\mathbb{R}^d \setminus B_r(0))\| \end{aligned}$$

where $r = |x| - 18 > 0$, see property (b) of the covering $(\Omega_{j,k,\ell})$ in Subsection 3.3.2. In view of this inequality an application of Lebesgue's theorem on dominated convergences proves (14).
Substep 2.2. Let $f \in RB^{1/p}_{p,1}(\mathbb{R}^d)$. We argue as in Substep 2.1 by using the notation from Step 1. Since

$$\lim_{r\to\infty} \quad \sum_{j=0}^{\infty} \Big(\sum_{k\ge r} |s_{j,k}|^p \, (1+k)^{d-1}\Big)^{1/p} = 0$$

we conclude from (64) that (14) holds in this case as well.
Step 3. Proof of (iii). We shall use the test functions constructed in Lemma 7. We choose $s_o > \max\{\sigma_p(d), s\}$. For simplicity we consider $|x| = 2^r$ with $r \in \mathbb{N}$. We choose λ s.t. $|x| = (1+\lambda)/2$. Hence $f_{1,\lambda}(x) = 1$. This implies

$$|x|^{\frac{d-1}{p}} \, |2^{-r(d-1)/p} \, f_{1,\lambda}(x)| = 1 \, ,$$

and

$$\| \, 2^{-r(d-1)/p} \, f_{1,\lambda} \, |A^s_{p,q}(\mathbb{R}^d)\| \lesssim \| \, 2^{-r(d-1)/p} \, f_{1,\lambda} \, |B^{s_o}_{p,q}(\mathbb{R}^d)\| \asymp 1 \, ,$$

see Rem. 21, which proves the claim.
Step 4. Proof of (iv). It will be enough to study the case $s = 1/p$.
Substep 4.1 Let $q > 1$. According to Lemma 8(i) there exists a compactly supported function g_0 which belongs to $RB^{1/p}_{p,q}(\mathbb{R})$ and is unbounded near the origin. By multiplying with a smooth cut-off function if necessary we can make the support of this functions as small as we want. For the given sequence $(x^j)_j$ we define

$$g(t) := \sum_{j=1}^{\infty} \frac{1}{\max(|x^j|, j)^{\alpha}} \, g_0(t - |x^j|) \, , \qquad t \in \mathbb{R} \, ,$$

where we will choose $\alpha > 0$ in dependence on p. The function g is unbounded near $|x^j|$ and by means of the translation invariance of the Besov spaces $B^{1/p}_{p,q}(\mathbb{R})$ we obtain

$$\| g \,|B^{1/p}_{p,q}(\mathbb{R})\|^{\min(1,p)} \le \| g_0 \,|B^{1/p}_{p,q}(\mathbb{R})\|^{\min(1,p)} \sum_{j=1}^\infty j^{-\alpha \, \min(1,p)} \lesssim \| g_0 \,|B^{1/p}_{p,q}(\mathbb{R})\|^{\min(1,p)} ,$$

if $\alpha \cdot \min(1,p) > 1$. We employ Rem. 18 (Cor. 2) with respect to each summand. This yields

$$\begin{aligned} & \| \text{ ext } g \,|B^{1/p}_{p,q}(\mathbb{R}^d)\|^{\min(1,p)} \\ \le \quad & \sum_{j=1}^\infty \max(|x^j|, j)^{-\alpha \, \min(1,p)} \, \| \text{ ext } (g_0(\,\cdot\, - |x^j|)) \,|B^{1/p}_{p,q}(\mathbb{R}^d)\|^{\min(1,p)} \\ \lesssim \quad & \| g_0 \,|B^{1/p}_{p,q}(\mathbb{R})\|^{\min(1,p)} \sum_{j=1}^\infty \Big(\max(|x^j|, j)^{-\alpha} \, |x^j|^{(d-1)/p}\Big)^{\min(1,p)} \\ \lesssim \quad & \| g_0 \,|B^{1/p}_{p,q}(\mathbb{R})\|^{\min(1,p)} \, , \end{aligned}$$

if $(\alpha - (d-1)/p) \, \min(1,p) > 1$.

Substep 4.2. We turn to the F-case. It will be enough to study the situation $s = 1/p$ and $1 < p < \infty$. We argue as above using this time Lemma 8(ii). An application of Rem. 18 (Cor. 2) yields the result but with the extra condition $1/p > \sigma_q(d)$. ■

4.2 Traces of BV-functions and consequences for the decay

We recall a definition of the space $BV(\mathbb{R}^d)$, $d \ge 2$, which will be convenient for us, see [11, 5.1] or [45, 5.1].

Definition 8 *Let $g \in L_1(\mathbb{R}^d)$. We say, that $g \in BV(\mathbb{R}^d)$ if for every $i = 1, \ldots, d$ there is a signed Radon measure μ_i of finite total variation such that*

$$\int_{\mathbb{R}^d} g(x) \frac{\partial}{\partial x_i} \phi(x) dx = - \int_{\mathbb{R}^d} \phi(x) d\mu_i(x), \qquad \phi \in C^1_c(\mathbb{R}^d),$$

where $C^1_c(\mathbb{R}^d)$ denotes the set of all continuously differentiable functions on $\mathbb{R}^d$ with compact support. The space $BV(\mathbb{R}^d)$ is equipped with the norm

$$\| g \,|BV(\mathbb{R}^d)\| = \| g \,|L_1(\mathbb{R}^d)\| + \sum_{i=1}^d \| \mu_i \,|\mathcal{M}\|,$$

where $\| \mu_i \,|\mathcal{M}\|$ is the total variation of μ_i.

4.2.1 Proof of Theorem 12

We need some preparations. Recall, the space $C^1_c([0,\infty))$ has been defined in Definition 5. By ω_{d-1} we denote the surface area of the unit sphere in $\mathbb{R}^d$ and by σ the $(d-1)$-dimensional Hausdorff measure in $\mathbb{R}^d$, i.e. $\omega_{d-1} := \sigma(\{x \in \mathbb{R}^d : |x| = 1\})$. As above $r = r(x) := |x|$.

Lemma 9 *(i) If $\varphi \in C_c^1([0,\infty))$, then all the functions*

$$\phi_i(x) := \begin{cases} \varphi(r(x)) \cdot \frac{x_i}{r(x)} & \text{if} \quad x = (x_1, \ldots, x_d) \neq 0\,, \\ \\ 0 & \text{if} \quad x = 0\,, \end{cases} \tag{68}$$

$i = 1, \ldots, d$, belong to $C_c^1(\mathbb{R}^d)$.
(ii) If $\phi \in C_c^1(\mathbb{R}^d)$, then all functions

$$\varphi_i(t) := \begin{cases} \frac{1}{\omega_{d-1}\, t^{d-1}} \int_{|x|=t} \phi(x) \cdot \frac{x_i}{r(x)}\, d\sigma(x) & \text{if} \quad t > 0\,, \\ \\ 0 & \text{if} \quad t = 0\,, \end{cases} \tag{69}$$

$i = 1, \ldots, d$, belong to $C_c^1([0,\infty))$.

Proof. *Step 1.* Proof of (i). Under the given assumption we immediately get $\phi_i \in C^1(\mathbb{R}^d \setminus \{0\})$ and supp ϕ_i is compact. Hence, we have to study the regularity properties in the origin. Obviously, $\phi_i(0) = 0$ and $\lim_{x \to 0} \phi_i(x) = 0$. We claim that

$$\frac{\partial \phi_i}{\partial x_j}(0) = \begin{cases} \varphi'(0) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $e_1, \ldots e_d$ denote the elements of the canonical basis of $\mathbb{R}^d$. If $i = j$, then

$$\lim_{t \to 0} \frac{\phi_i(te_i)}{t} = \lim_{t \to 0} \frac{\varphi(|t|)}{|t|} = \varphi'(0)\,.$$

Hence $\frac{\partial \phi_i}{\partial x_i}(0) = \varphi'(0)$. The cases $i \neq j$ are obvious.
Next, we show, that the functions $\frac{\partial \phi_i}{\partial x_j}$ are continuous in the origin. To begin with we investigate the case $i = j$. Then

$$\begin{aligned} \left| \frac{\partial \phi_i}{\partial x_i}(x) - \varphi'(0) \right| &= \left| \varphi'(r(x)) \cdot \frac{x_i^2}{r^2(x)} + (\varphi(r(x)) - \varphi(0)) \cdot \frac{r^2(x) - x_i^2}{r^3(x)} - \varphi'(0) \right| \\ &= \left| \varphi'(r(x)) \cdot \frac{x_i^2}{r^2(x)} + \varphi'(\theta r(x)) \cdot \frac{r^2(x) - x_i^2}{r^2(x)} - \varphi'(0) \right|, \end{aligned}$$

where we have used the Mean Value Theorem with a suitable $0 < \theta = \theta(x) < 1$. The continuity of φ' implies, that the expression on the right-hand side tends to zero if $x \to 0$. If $i \neq j$, we write

$$\begin{aligned} \frac{\partial \phi_i}{\partial x_j}(x) &= \varphi'(r(x)) \frac{x_i x_j}{r^2(x)} - \varphi(r(x)) \frac{x_i x_j}{r^3(x)} \\ &= \varphi'(r(x)) \frac{x_i x_j}{r^2(x)} + (\varphi(0) - \varphi(r(x))) \frac{x_i x_j}{r^3(x)} \\ &= \frac{x_i x_j}{r^2(x)} \left[\varphi'(r(x)) - \varphi'(\theta r(x)) \right]\,, \end{aligned} \tag{70}$$

with some $0 < \theta < 1$. Again the continuity of φ' implies, that the expression in (70) tends to zero as $x \to 0$. Hence, $\phi_i \in C_c^1(\mathbb{R}^d)$.

Step 2. Proof of (ii). The regularity and support properties of φ_i on $(0, \infty)$ are obvious. Hence, we are left with the study of the behaviour near 0. We shall use the identity

$$\int_{|x|=t} \frac{x_i}{r(x)}\, d\sigma(x) = 0\,, \qquad t > 0\,.$$

In case $t > 0$ this leads to the estimate

$$\begin{aligned} |\varphi_i(t)| &\le \frac{1}{\omega_{d-1} t^{d-1}} \Big(\Big| \int_{|x|=t} \phi(0)\, \frac{x_i}{r(x)} d\sigma(x) \Big| + \Big| \int_{|x|=t} (\phi(x) - \phi(0))\, \frac{x_i}{r(x)}\, d\sigma(x) \Big| \Big) \\ &\le 0 + \sup_{|x|=t} |\phi(x) - \phi(0)|. \end{aligned}$$

Hence, $\varphi_i(t)$ tends to 0 if $t \to 0^+$. Furthermore,

$$\begin{aligned} \frac{\varphi_i(t)}{t} &= \frac{1}{\omega_{d-1}\, t^d} \int_{|x|=t} (\phi(x) - \phi(0))\, \frac{x_i}{r(x)}\, d\sigma(x) \\ &= \frac{1}{\omega_{d-1}\, t^{d+1}} \int_{|x|=t} (\nabla \phi(\eta_x x) \cdot x)\, x_i\, d\sigma(x) \\ &= \frac{1}{\omega_{d-1}\, t^{d+1}} \sum_{j=1}^d \int_{|x|=t} \left(\frac{\partial \phi}{\partial x_j}(\eta_x x) - \frac{\partial \phi}{\partial x_j}(0) \right) x_j\, x_i\, d\sigma(x) \\ &\qquad + \quad \frac{1}{\omega_{d-1}\, t^{d+1}} \sum_{j=1}^d \frac{\partial \phi}{\partial x_j}(0) \int_{|x|=t} x_j\, x_i\, d\sigma(x) \end{aligned}$$

with some $0 < \eta_x < 1$. The first term on the right-hand side tends always to zero as $t \to 0^+$ (since $\dfrac{\partial \phi}{\partial x_j}$ is continuous). From the second term those summands with $i \neq j$ are vanishing for all t. If $i = j$, then the integrand is homogeneous. We obtain, by taking the limit with respect to t,

$$\varphi_i'(0) = \lim_{t \to 0^+} \frac{\varphi_i(t)}{t} = \frac{1}{\omega_{d-1}} \frac{\partial \phi}{\partial x_i}(0) \int_{|y|=1} y_i^2\, d\sigma(y)\,. \tag{71}$$

It remains to check the limit of $\varphi_i'(t)$ if t tends to 0^+. Observe

$$\varphi_i'(t) = \frac{1}{\omega_{d-1}} \frac{d}{dt} \int_{|y|=1} \phi(ty)\, y_i\, d\sigma(y) = \frac{1}{\omega_{d-1}} \sum_{j=1}^d \int_{|y|=1} \frac{\partial \phi}{\partial y_j}(ty)\, y_j\, y_i\, d\sigma(y)\,.$$

Since $\displaystyle\int_{|y|=1} y_j\, y_i\, d\sigma(y) = 0$ if $i \neq j$ those summands (with $i \neq j$) tend to 0 if t tends to 0^+. Hence

$$\lim_{t \to 0^+} \varphi_i'(t) = \lim_{t \to 0^+} \frac{1}{\omega_{d-1}} \int_{|y|=1} \frac{\partial \phi}{\partial y_i}(ty)\, y_i^2\, d\sigma(y) = \varphi'(0)\,,$$

see (71). The proof is complete. ■

Proof of Theorem 12.

Step 1. Let $f(x) = g(|x|) \in BV(\mathbb{R}^d)$. We claim that $g \in BV(\mathbb{R}^+, t^{d-1})$. Let $\mu_1, \ldots, \mu_d$ denote the corresponding signed Radon measures according to Definition 8. By means of

$$d\nu := \sum_{i=1}^{d} \frac{x_i}{r(x)}\, d\mu_i$$

we define the measure ν on $\mathbb{R}^d$. Since $g(r(x))$ is radial we conclude $\mu_i(\{0\}) = 0$, $i = 1, \ldots, d$. Hence the measure $d\nu$ is well defined. In addition we introduce a measure ν^+ on $\mathbb{R}^+$ by

$$\omega_{d-1} \int_A t^{d-1}\, d\nu^+(t) := \int_{\{|x| \in A\}} d\nu(x),$$

for any Lebesgue measurable subset $A \subset \mathbb{R}^+$. We fix $\varphi \in C_c^1([0,\infty))$. Since $\varphi(r(x))\frac{x_i}{r(x)} \in C_c^1(\mathbb{R}^d)$, cf. Lemma 9(i), we calculate

$$\begin{aligned}
\omega_{d-1} \int_0^\infty g(t)\ \ [\varphi(s)\, s^{d-1}]'(t)\, dt &= \omega_{d-1} \int_0^\infty t^{d-1}\, g(t) \left[\varphi'(t) + \frac{d-1}{t}\varphi(t)\right] dt \\
&= \int_{\mathbb{R}^d} g(r(x)) \left[\varphi'(r(x)) + \varphi(r(x)) \cdot \frac{d-1}{r(x)}\right] dx \\
&= \sum_{i=1}^{d} \int_{\mathbb{R}^d} g(r(x)) \left[\varphi'(r(x))\frac{x_i^2}{r^2(x)} + \varphi(r(x)) \cdot \frac{r^2(x) - x_i^2}{r^3(x)}\right] dx \\
&= \sum_{i=1}^{d} \int_{\mathbb{R}^d} g(r(x))\, \frac{\partial}{\partial x_i} \left[\varphi(r(x)) \cdot \frac{x_i}{r(x)}\right] dx \\
&= -\sum_{i=1}^{d} \int_{\mathbb{R}^d} \varphi(r(x))\, \frac{x_i}{r(x)}\, d\mu_i = -\int_{\mathbb{R}^d} \varphi(r(x))\, d\nu(x) \\
&= -\omega_{d-1} \int_0^\infty t^{d-1}\, \varphi(t)\, d\nu^+(t)\,.
\end{aligned}$$

This proves (18). Moreover, (19) follows from

$$\|\, g(r(x))\, |L_1(\mathbb{R}^d)\| = \omega_{d-1}\, \|\, g\, |L_1(\mathbb{R}, |t|^{d-1})\|$$

and

$$\begin{aligned}
\omega_{d-1} \int_0^\infty t^{d-1}\, d|\nu^+|(t) &= \int_{\mathbb{R}^d} d|\nu|(x) \le \sum_{i=1}^{d} \int_{\mathbb{R}^d} \frac{|x_i|}{r(x)}\, d|\mu_i| \\
&\le \sum_{i=1}^{d} \int_{\mathbb{R}^d} d|\mu_i| \le \|\, g(r(x))\, |BV(\mathbb{R}^d)\|\,.
\end{aligned}$$

Step 2. Let g be a function in $BV(\mathbb{R}^+, t^{d-1})$. We claim, that $g(r(x)) \in BV(\mathbb{R}^d)$. Let ν^+ be the signed Radon measure associated to g according to (18). We define

$$\nu(A) := \int_0^\infty \sigma(\{x : |x| = t\} \cap A)\, d\nu^+(t)$$

for any Lebesgue measurable set $A \subset \mathbb{R}^d$. Further we put $\mu_i := \frac{x_i}{r(x)}\nu$, $i = 1, \ldots d$. Let χ_A denote the characteristic function of A. Then

$$\nu(A) = \int_{\mathbb{R}^d} \chi_A(x)\, d\nu(x) = \int_0^\infty \Big[\int_{|x|=t} \chi_A(x)\, d\sigma(x) \Big]\, d\nu^+(t)$$

and this identity can be extended to

$$\int_{\mathbb{R}^d} \phi(x)\, d\nu(x) = \int_0^\infty \Big[\int_{|x|=t} \phi(x)\, d\sigma(x) \Big]\, d\nu^+(t)\,, \qquad \phi \in L_1(\mathbb{R}^d)\,,$$

by using some standard arguments. Next we want to show, that $\mu_i, i = 1, \ldots, d$, are the weak derivatives of $g(r(x))$. Let $\phi \in C_c^1(\mathbb{R}^d)$ and let φ_i be the associated functions, see (69). According to Lemma 9 (ii) we know that $\varphi_i \in C_c^1([0, \infty))$. Using the normalized outer normal with respect to the surface $\{x : |x| = T\}$, which is obviously given by

$$n(x) = (n_1(x), \ldots, n_d(x)) = \frac{1}{r(x)}\ (x_1, \ldots, x_d)\ ,$$

and the Gauss Theorem, we obtain

$$\begin{aligned} -\varphi_i(T)\, \omega_{d-1}\, T^{d-1} &= - \int_{|x|=T} \phi(x)\, \frac{x_i}{r(x)}\, d\sigma(x) = - \int_{|x|=T} \phi(x)\, n_i(x)\, d\sigma(x) \\ &= - \int_{|x|\le T} \frac{\partial \phi}{\partial x_i}(x)\, dx = \int_{|x|\ge T} \frac{\partial \phi}{\partial x_i}(x)\, dx \\ &= \int_T^\infty \Big[\int_{|x|=t} \frac{\partial \phi}{\partial x_i}(x)\, d\sigma(x) \Big]\, dt. \end{aligned}$$

Hence

$$\omega_{d-1}\ \left[\varphi_i(t)\, t^{d-1}\right]'(T) = \int_{|x|=T} \frac{\partial \phi}{\partial x_i}(x)\, d\sigma(x)\,, \qquad T > 0.$$

This formula justifies the identity

$$\begin{aligned} \int_{\mathbb{R}^d} g(r(x))\, \frac{\partial \phi(x)}{\partial x_i}\, dx &= \int_0^\infty g(t) \int_{|x|=t} \frac{\partial \phi(x)}{\partial x_i}\, d\sigma(x)\, dt \\ &= \omega_{d-1} \int_0^\infty g(t) [\varphi_i(s)\, s^{d-1}]'(t) dt\,. \end{aligned}$$

Next we use $g \in BV(\mathbb{R}^+, t^{d-1})$. This implies

$$\begin{aligned} \int_{\mathbb{R}^d} g(r(x)) \frac{\partial \phi(x)}{\partial x_i} dx &= -\omega_{d-1} \int_0^\infty \varphi_i(t)\, t^{d-1}\, d\nu^+(t) \\ &= - \int_0^\infty \int_{|x|=t} \phi(x) \cdot \frac{x_i}{r(x)}\, d\sigma(x)\, d\nu^+(t) \\ &= - \int_{\mathbb{R}^d} \phi(x)\, \frac{x_i}{r(x)}\, d\nu(x) = - \int_{\mathbb{R}^d} \phi(x)\, d\mu_i(x)\,, \end{aligned}$$

which proves that the μ_i are the weak derivatives of $g(r(x))$.
It remains to prove the estimates for the related norms. The required estimate follows easily by

$$\begin{aligned}\int_{\mathbb{R}^d} d|\mu_i| &= \int_{\mathbb{R}^d} \frac{|x_i|}{r(x)}\, d|\nu|(x) = \int_0^\infty \Big[\int_{|x|=t} \frac{|x_i|}{r(x)}\, d\sigma(x)\Big]\, d|\nu^+|(t)\\ &\le \omega_{d-1} \int_0^\infty t^{d-1}\, d|\nu^+|(t)\,.\end{aligned}$$

The proof is complete. ■

4.2.2 Proof of Theorem 11

Recall, that we will work with the particular representative $\tilde f$ of the equivalence class $[f]$, see Remark 11. For convenience we will drop the tilde. We shall apply standard mollifiers. Let $\varphi \in C_0^\infty(\mathbb{R})$ be a function such that $\varphi \ge 0$, supp $\varphi \subset [0,1]$, and $\int \varphi(t)\, dt = 1$. For $R > 0$ and $\varepsilon > 0$ we define

$$\varphi_\varepsilon(t) := \varepsilon^{-1} \int_R^\infty \varphi\Big(\frac{t-y}{\varepsilon}\Big)\, dy = \int_{-\infty}^{\frac{t-R}{\varepsilon}} \varphi(z)\, dz \tag{72}$$

(which is nothing but the mollification of the characteristic function of the interval (R,∞)). In addition we need a cut-off function. Let $\eta \in C_0^\infty(\mathbb{R})$ s.t. $\eta(t) = 1$ if $|t| \le 1$ and $\eta(t) = 0$ if $|t| \ge 2$. For $M \ge 1$ we define $\eta_M(t) := \eta(t/M)$, $t \in \mathbb{R}$. It is easily checked that the functions

$$\phi_{M,\varepsilon}(t) := t^{1-d}\, \varphi_\varepsilon(t)\, \eta_M(t)\,, \qquad t \in \mathbb{R}\,,$$

belong to $C_c^1([0,\infty))$. For $g \in BV(\mathbb{R}^+, t^{d-1})$ this implies

$$\int_0^\infty g(t)\, [\phi_{M,\varepsilon}(s)\, s^{d-1}]'(t)\, dt = - \int_0^\infty \varphi_\varepsilon(t)\, \eta_M(t)\, d\nu^+(t)\,, \tag{73}$$

see (18). Since for $M > R + \varepsilon$ we have

$$\begin{aligned}\int_0^\infty g(t)\, [\phi_{M,\varepsilon}(s)\, s^{d-1}]'(t)\, dt &= \int_R^{R+\varepsilon} g(t)\, \varepsilon^{-1}\, \varphi\Big(\frac{t-R}{\varepsilon}\Big)\, dt\\ &\quad + M^{-1} \int_M^\infty g(t)\, \varphi_\varepsilon(t)\, \eta'(t/M)\, dt\end{aligned}$$

and

$$\lim_{M\to\infty} M^{-1} \int_M^\infty g(t)\, \varphi_\varepsilon(t)\, \eta'(t/M)\, dt = 0$$

($g \in L_1(\mathbb{R}^+, t^{d-1})$), we get

$$\lim_{\varepsilon\downarrow 0, M\to\infty} \int_0^\infty g(t)\, [\phi_{M,\varepsilon}(s)\, s^{d-1}]'(t)\, dt = g(R)\,, \tag{74}$$

if R is a Lebesgue point of g. But

$$\Big| \int_0^\infty \varphi_\varepsilon(t)\, \eta_M(t)\, d\nu^+(t) \Big| \le \int_R^\infty \varphi_\varepsilon(t)\, d|\nu^+|(t) \le R^{1-d} \int_R^\infty t^{d-1}\, d|\nu^+|(t)\,.$$

Combining (74), (73) with this estimate we have proved (16) and (17) simultaneously. ■

4.3 Proof of the assertions in Subsection 2.2.3

4.3.1 Proof of Lemma 3

Sufficiency of the conditions is obvious, see e.g. [35]. Necessity follows from the examples investigated in Lemma 8. ■

4.3.2 Proof of Theorem 13

We argue by using the atomic characterizations in Subsection 3.3.2.

Step 1. Proof of (i). For simplicity let $|x| = 2^{-r}$, $r \in \mathbb{N}$. If y satisfies $2^{-r-1} \le |y| \le 2^{-r+1}$, then, using the support condition for atoms, we know that f allows an optimal atomic decomposition such that

$$f(y) = \sum_{j=0}^\infty \sum_{k=\max(0,[2^{j-r}]-n_0)}^{[2^{j-r}]+n_0} \sum_{\ell=1}^{C(d,k)} s_{j,k}\, a_{j,k,\ell}(y)\,. \tag{75}$$

Here n_0 is a general natural number depending on the decomposition Ω, but not on r.

Substep 1.1. We assume first that $\frac{1}{p} < s < \frac{d}{p}$. ¿From the L_∞-estimate of the atoms, property (f) of the coverings $(\Omega_{j,k,\ell})_{j,k,\ell}$, and the inequality (36) we derive

$$\begin{aligned}
|f(y_1,0,\ldots,0)| &\le \sum_{j=0}^\infty \sum_{k=\max(0,[2^{j-r}]-n_0)}^{[2^{j-r}]+n_0} \sum_{\ell=1}^{K} |s_{j,k}|\, |a_{j,k,\ell}(y_1,0,\ldots,0)| \\
&\lesssim \Big(\sum_{j=0}^{r+n_1} \sum_{k=0}^{n_2} |s_{j,k}|\, 2^{-j(s-d/p)} + \sum_{j=r+n_1+1}^\infty \sum_{k=2^{j-r}-n_0}^{2^{j-r}+n_0} |s_{j,k}|\, 2^{-j(s-d/p)} \Big) \\
&\lesssim \|\, f\, |RB^s_{p,\infty}(\mathbb{R}^d)\| \Big(\sum_{j=0}^{r+n_1} 2^{-j(s-d/p)} + \sum_{j=r+n_1+1}^\infty 2^{-(j-r)(d-1)/p}\, 2^{-j(s-d/p)} \Big) \\
&\lesssim 2^{r(\frac{d}{p}-s)}\, \|\, f\, |RB^s_{p,\infty}(\mathbb{R}^d)\|\ ,
\end{aligned}$$

for appropriate natural numbers n_1, n_2 (independent of r). For the last two steps of the estimate we used $1/p < s < d/p$. Taking into account the elementary embedding $A^s_{p,q}(\mathbb{R}^d) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^d)$ we obtain the inequality (20).

Substep 1.2. Now, let $s = \frac{1}{p}$. For the Besov spaces $RB^{1/p}_{p,1}(\mathbb{R}^d)$ the inequality (20) was proved in [32]. So it remains to consider the Lizorkin -Triebel spaces $RF^{1/p}_{p,\infty}(\mathbb{R}^d)$ with $0 < p \le 1$. For simplicity we regard the main part of $f \in RF^{1/p}_{p,\infty}(\mathbb{R}^d)$, cf. (75) and compare with (63). Now $k_j = 1$ if $0 \le j \le r + n_1$ and $k_j \sim |x| 2^j$ if $j > r + n_1$. Hence, we obtain

$$
\begin{aligned}
|f^M(|x|, 0, \ldots, 0)| \le \sum_{j=0}^{\infty} |s_{j,k_j}|\, |a_{j,k_j,0}(y)| \; &\lesssim \; \Big(\sum_{j=0}^{\infty} |s_{j,k_j}|^p\, 2^{j(d-1)} \Big)^{1/p} \\
\lesssim \quad |x|^{\frac{1-d}{p}} \Big(\sum_{j=0}^{r+n_1} |s_{j,k_j}|^p + \sum_{j=r+n_1+1}^{\infty} |s_{j,k}|^p\, k_j^{d-1} \Big)^{1/p} \; &\lesssim \; |x|^{\frac{1-d}{p}}\, \| \, f \, | RF^{1/p}_{p,\infty}(\mathbb{R}^d) \| \; ,
\end{aligned}
$$

where we used $p \le 1$ for the second step and (65)-(66) for the last one.
Step 2. Proof of (ii). By using elementary embeddings it will be enough to prove (21) with $RA^s_{p,q}(\mathbb{R}^d) = RB^s_{p,q}(\mathbb{R}^d)$ and q small. Again we concentrate on $|x| = 2^{-r}$, $r \in \mathbb{N}$. Our test function is taken to be $2^{-r(s-d/p)}\, f_{j,\lambda}$, see Lemma 7, where we choose $j := 1 + r$ and $\lambda := 1$. Then it follows

$$
\| \, 2^{-r(s-d/p)}\, f_{j,\lambda} \, | B^s_{p,q}(\mathbb{R}^d) \| \asymp 1 \qquad \text{and} \qquad f_{j,\lambda}(x) = 2^{-r(s-d/p)} = |x|^{s-d/p} \, .
$$

The proof is complete. ■

4.3.3 Proof of Lemma 4

The arguments are the same as in proof of Theorem 10(iv). ■

4.4 Proof of Theorem 14

We shall use the notation from the proof of Theorem 13, Step 1. Again we employ the formula (75) and obtain

$$
|f(y_1, 0, \ldots, 0)| \; \lesssim \; \Big(\sum_{j=0}^{r+n_1} \sum_{k=0}^{n_2} |s_{j,k}| + \sum_{j=r+n_1+1}^{\infty} \sum_{k=2^{j-r}-n_0}^{2^{j-r}+n_0} |s_{j,k}| \Big)
$$

since $s = d/p$.
Step 1. Proof of (i). We shall use the standard abbreviation $q' := q/(q-1)$. Since $q > 1$ we can use Hölder's inequality and conclude

$$
\sum_{j=0}^{r+n_1} \sum_{k=0}^{n_2} |s_{j,k}| \; \lesssim \; r^{1/q'} \Big(\sum_{j=0}^{r+n_1} \sum_{k=0}^{n_2} |s_{j,k}|^q \Big)^{1/q} \; \lesssim \; (-\log |x|)^{1/q'}\, \| \, f \, | B^{d/p}_{p,q}(\mathbb{R}^d) \| \qquad (76)
$$

as well as

$$
\begin{aligned}
\sum_{j=r+n_1+1}^{\infty} \sum_{k=2^{j-r}-n_0}^{2^{j-r}+n_0} |s_{j,k}| \ & \lesssim \ \sum_{j=r+n_1+1}^{\infty} 2^{-(j-r)(d-1)/p} \Big(\sum_{k\ge 2^{j-r}-n_0} (1+|k|)^{d-1}\, |s_{j,k}|^p \Big)^{1/p} \\
\lesssim \ & \Big(\sum_{j=r+n_1+1}^{\infty} \Big(\sum_{k\ge 2^{j-r}-n_0} (1+|k|)^{d-1}\, |s_{j,k}|^p \Big)^{q/p} \Big)^{1/q} \Big(\sum_{j=r+n_1+1}^{\infty} 2^{-(j-r)q'(d-1)/p} \Big)^{1/q'} \\
\lesssim \ & \| \, f \, | B^{d/p}_{p,q}(\mathbb{R}^d) \| \, .
\end{aligned}
\tag{77}
$$

The inequalities (76) and (77) yield (22).

Step 2. Proof of (ii). Let $1 < p < p_0 < \infty$. The Jawerth-Franke embedding $F^{d/p}_{p,1}(\mathbb{R}^d) \hookrightarrow B^{d/p_0}_{p_0,p}(\mathbb{R}^d)$, see [19] or [35], combined with (22) proves (23). ■

References

[1] J. Bergh and J. Löfström, *Interpolation spaces. An Introduction*, Springer Verlag, 1976.

[2] G. Bourdaud, *A sharpness result for powers of Besov functions*, J. Function Spaces Appl. **2** (2004), 267-277.

[3] G. Bourdaud and Y. Meyer, *Fonctions qui opèrent sur les espaces de Sobolev.* J. Funct. Anal. **97** (1991), 351–360.

[4] M. Bownik and Kwok-Pun Ho, *Atomic and molecular decompositions of anisotropic Triebel-Lizorkin spaces.* Trans. Amer. Math. Soc. **358** (2006), no. 4, 1469–1510.

[5] H.-Q. Bui, *Weighted Besov and Triebel spaces : Interpolation by the real method.* Hiroshima Math. J., **12** (1982), no. 3, 581–605.

[6] H.-Q. Bui, *Characterizations of weighted Besov and Triebel-Lizorkin spaces via temperatures.* J. Funct. Anal., **55** (1984), no. 1, 39–62.

[7] Y. Cho and T. Ozawa, *Sobolev inequalities with symmetry*, Comm. Contemp. Math. **11** (2009), 355-365.

[8] S. Coleman, V. Glazer and A. Martin, *Action minima among solutions to a class of Euclidean scalar field equations*, Comm. in Math. Phys. **58** (1978), 211-221.

[9] R. DeVore and P. Sharpley, *Besov spaces on domains in* $\mathbb{R}^d$, Trans. AMS **335** (1993), 843-864.

[10] J. Epperson and M. Frazier, *An almost orthogonal radial wavelet expansion for radial distributions,* J. Fourier Analysis Appl. **1** (1995), 311-353.

[11] L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton, 1992.

[12] W. Farkas, J. Johnsen, and W. Sickel, *Traces of anisotropic Besov–Lizorkin–Triebel spaces —a complete treatment of the borderline cases*, Math. Bohemica **125** (2000), 1-37.

[13] J. Franke, *On the space $F^s_{p,q}$ of Triebel-Lizorkin type: Pointwise multipliers and spaces on domains*. Math. Nachr. **125** (1986), 29-68.

[14] M. Frazier and B. Jawerth, *Decomposition of Besov spaces.* Indiana Univ. Math. J. **34** (1985), 777-799.

[15] M. Frazier and B. Jawerth, *A discrete transform and decomposition of distribution spaces,* J. Funct. Anal. **93** (1990), 34-170.

[16] D. Haroske and I. Piotrowska, *Atomic decompositions of function spaces with Muckenhoupt weights, and some relations to fractal geometry.* Math. Nachr. **281** (2008), 1476-1494.

[17] D. Haroske and L. Skrzypczak, *Entropy and approximation numbers of embeddings of function spaces with Muckenhoupt weights, I.* Rev. Mat. Complut., **21** (2008), 135–177.

[18] M. Izuki and Y. Sawano, *Wavelet bases in the weighted Besov and Triebel-Lizorkin spaces with A^{loc}_p-weights*, JAT **161** (2009), 656-673.

[19] B. Jawerth, *Some observations on Besov and Lizorkin-Triebel spaces.* Math. Scand. **40** (1977), 94–104.

[20] B. Jawerth, *The trace of Sobolev and Besov spaces if* $0 < p < 1$, Studia Math. **62** (1978), 65-71.

[21] T. Kühn, H.-G. Leopold, W. Sickel, and L. Skrzypczak, *Entropy numbers of Sobolev embeddings of radial Besov spaces*, JAT **121** (2003), 244-268.

[22] I. Kuzin, S. Pohozaev, *Entire solutions of semilinear elliptic equations.* Birkhäuser, Basel 1997.

[23] P.L. Lions, *Symmetrie et compacité dans les espaces de Sobolev,* J. Funct. Anal. **49** (1982), 315-334.

[24] O. Mendez and M. Mitrea, *The Banach envelopes of Besov and Triebel–Lizorkin spaces and applications to partial differential equations*, J. Four. Anal. and Appl. 6(5) (2000), 503-531.

[25] W.M. Ni, *A nonlinear Dirichlet problem on the unit ball and its applications*, Indiana Univ. Math. J. **31** (1982), 801-807.

[26] J. Peetre, *New thoughts on Besov spaces,* Duke Univ. Press, Durham, 1976.

[27] W. Rother, *Some existence results for the equation* $\Delta U + K(x)\, U^p = 0$, Comm. in PDE **15** (1990), 1461-1473.

[28] W. Rudin, *Real and complex analysis*, (third ed.) McGraw-Hill Book Co., New York, 1987.

[29] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators and nonlinear partial differential equations,* de Gruyter, Berlin, 1996.

[30] V.S. Rychkov, *Littlewood-Paley theory and function spaces with* $\mathcal{A}_p^{\ell oc}$ *weights*. Math. Nachr. **224** (2001), 145-180.

[31] H.-J. Schmeisser and H. Triebel, *Topics in Fourier analysis and function spaces*. Wiley, Chichester, 1987.

[32] W. Sickel and L. Skrzypczak, *Radial subspaces of Besov and Lizorkin-Triebel spaces: extended Strauss lemma and compactness of embeddings,* J. Fourier Anal. Appl. **106** (2000), 639-662.

[33] W. Sickel and L. Skrzypczak, *On the interplay of regularity and decay in case of radial functions II. Homogeneous spaces*, Preprint, Jena, 2010.

[34] W. Sickel, L. Skrzypczak, and J. Vybiral, *Radial subspaces of Besov- and Lizorkin-Triebel spaces - complex interpolation and characterization by differences*, (work in progress), Jena, Poznan, Linz 2010.

[35] W. Sickel and H. Triebel, *Hölder inequalities and sharp embeddings in function spaces of* $B^s_{p,q}$ *and* $F^s_{p,q}$ *type*, ZAA **14** (1995), 105-140.

[36] L. Skrzypczak, *Rotation invariant subspaces of Besov and Triebel-Lizorkin spaces: compactness of embeddings, smoothness and decay properties*. Revista Mat. Iberoamericana **18**, 267-299 (2002).

[37] E.M. Stein, *Singular integrals and differentiability properties of functions.* Princeton Univ. Press, Princeton, 1970.

[38] E.M. Stein, *Harmonic analysis, real-variable methods, orthogonality, and oscillatory integrals.* Princeton Univ. Press, Princeton, 1993.

[39] W.A. Strauss, *Existence of solitary waves in higher dimensions.* Comm. in Math. Physics **55** (1977), 149-162.

[40] H. Triebel, *Interpolation theory, function spaces, differential operators*, North Holland, Amsterdam, 1978.

[41] H. Triebel, *Theory of function spaces.* Birkhäuser, Basel, 1983.

[42] H. Triebel, *Theory of Function Spaces II.* Birkhäuser, Basel, 1992.

[43] H. Triebel, *Approximation numbers and entropy numbers of embeddings of fractional Besov-Sobolev spaces in Orlicz spaces.* Proc. London Math. Soc. **66** (1993), 589-618.

[44] H. Triebel, *Function spaces and wavelets on domains*, EMS Publishing House, Zürich, 2008.

[45] W.P. Ziemer, *Weakly differentiable functions*, Springer, New York, 1989.

Leszek Skrzypczak
Adam Mikiewicz University Poznan
Faculty of Mathematics and Computer Science
Ul. Umultowska 87
61-614 Poznań, Poland
e–mail: `lskrzyp@amu.edu.pl`

Winfried Sickel
Friedrich-Schiller-Universität Jena
Mathematisches Institut
Ernst-Abbe-Platz 2
07743 Jena, Germany
e-mail: `winfried.sickel@uni-jena.de`

Jan Vybiral
Austrian Academy of Sciences
Johann Radon Institute
for Computational and Applied Mathematics
Altenberger Str. 69
A-4040 Linz, Austria
e-mail: `jan.vybiral@oeaw.ac.at`