

# Johnson-Lindenstrauss lemma for circulant matrices

Aicke Hinrichs

Mathematisches Institut, Universität Jena  
Ernst-Abbe-Platz 2, 07740 Jena, Germany  
email: a.hinrichs@uni-jena.de

Jan Vybíral

Radon Institute for Computational and Applied Mathematics (RICAM)  
Austrian Academy of Sciences  
Altenbergerstraße 69  
A-4040 Linz, Austria  
email: jan.vybiral@oeaw.ac.at

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## Abstract

We prove a variant of a Johnson-Lindenstrauss lemma for matrices with circulant structure. This approach allows to minimise the randomness used, is easy to implement and provides good running times. The price to be paid is the higher dimension of the target space  $k = O(\varepsilon^{-2} \log^3 n)$  instead of the classical bound  $k = O(\varepsilon^{-2} \log n)$ .

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## 1 Introduction

The classical Johnson-Lindenstrauss lemma may be formulated as follows.

**Theorem 1.1.** *Let  $\varepsilon \in (0, \frac{1}{2})$  and let  $x_1, \dots, x_n \in \mathbb{R}^d$  be arbitrary points. Let  $k = O(\varepsilon^{-2} \log n)$  be a natural number. Then there exists a (linear) mapping  $f : \mathbb{R}^d \rightarrow \mathbb{R}^k$  such that*

$$(1 - \varepsilon) \|x_i - x_j\|_2^2 \leq \|f(x_i) - f(x_j)\|_2^2 \leq (1 + \varepsilon) \|x_i - x_j\|_2^2$$

for all  $i, j \in \{1, \dots, n\}$ . Here  $\|\cdot\|_2$  stands for the Euclidean norm in  $\mathbb{R}^d$  or  $\mathbb{R}^k$ , respectively.

The original proof of Johnson and Lindenstrauss [11] uses (up to a scaling factor) an orthogonal projection onto a random  $k$ -dimensional subspace of  $\mathbb{R}^d$ . We refer also to [7] for a beautiful and self-contained proof. Later on, this lemma found many applications, especially in design of algorithms, where it sometimes allows to reduce the dimension of the underlying problem essentially and break the so-called “curse of dimension”, cf. [9] or [10].

The evaluation of  $f(x)$ , where  $f$  is a projection onto a random  $k$  dimensional subspace, is a very time-consuming operation. Therefore, a significant effort was devoted to

- minimize the running time of  $f(x)$ ,
- minimize the memory used,

- minimize the number of random bits used,
- simplify the algorithm to allow an easy implementation.

Achlioptas observed in [1], that the mapping may also be realised by a matrix, where each component is selected independently at random with a fixed distribution. This decreases the time for evaluation of  $f(x)$  essentially.

An important breakthrough was achieved by Ailon and Chazelle in [3]. Let us briefly describe their *Fast Johnson-Lindenstrauss transform* (FJLT). The FJLT is the product of three matrices  $f(x) = PHDx$ , where

- $P$  is a  $k \times d$  matrix, where each component is generated independently at random. In particular,  $P_{i,j} \approx N(0, 1)$  with probability

$$q = \min \left\{ \Theta \left( \frac{\log^2 n}{d} \right), 1 \right\}$$

and  $P_{i,j} = 0$  with probability  $1 - q$ ,

- $H$  is the  $d \times d$  normalised Hadamard matrix,
- $D$  is a random  $d \times d$  diagonal matrix, with each  $D_{i,i}$  drawn independently from  $\{-1, 1\}$  with probability  $1/2$ .

It follows, that with high probability,  $f(x)$  may be calculated in time  $O(d \log d + qd\varepsilon^{-2} \log n)$ .

We refer to [14] for a historical overview as well as for an extensive description of the present “state of the art”.

In this note we propose another direction to approach the Johnson-Lindenstrauss lemma, namely we investigate the possibility of taking a partial circulant matrix for  $f$  combined with a random  $\pm 1$  diagonal matrix, see the next section for exact definitions.

This transform has a running time of  $O(d \log d)$ , requires less randomness ( $2d$  instead of  $kd$  or  $(k+1)d$  used in [1, 2, 3]) and allows a simpler implementation.

Unfortunately, up to now, we were only able to prove the statement with  $k = O(\varepsilon^{-2} \log^3 n)$ , compared to the standard value  $k = O(\varepsilon^{-2} \log n)$ . We leave the possible improvements of this bound open for further investigations.

## 2 Circulant matrices

We study the question (which to our knowledge has not been addressed in the literature before), whether  $f$  in the Johnson-Lindenstrauss lemma may be chosen as a circulant matrix. Let us give the necessary notation.

Let  $a = (a_0, \dots, a_{d-1})$  be independent identically distributed random variables. We denote by  $M_{a,k}$  the partial circulant matrix

$$M_{a,k} = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{d-1} \\ a_{d-1} & a_0 & a_1 & \dots & a_{d-2} \\ a_{d-2} & a_{d-1} & a_0 & \dots & a_{d-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{d-k+1} & a_{d-k+2} & a_{d-k+3} & \dots & a_{d-k} \end{pmatrix}.$$

Furthermore, if  $\varkappa = (\varkappa_0, \dots, \varkappa_{d-1})$  are independent Bernoulli variables, we put

$$D_{\varkappa} = \begin{pmatrix} \varkappa_0 & 0 & \dots & 0 \\ 0 & \varkappa_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \varkappa_{d-1} \end{pmatrix}.$$

**Theorem 2.1.** *Let  $x_1, \dots, x_n$  be arbitrary points in  $\mathbb{R}^d$ , let  $\varepsilon \in (0, \frac{1}{2})$  and let  $k = O(\varepsilon^{-2} \log^3 n)$  be a natural number. Let  $a = (a_0, \dots, a_{d-1})$  be independent Bernoulli variables or independent normally distributed variables. Let  $M_{a,k}$  and  $D_{\varkappa}$  be as above and put  $f(x) = \frac{1}{\sqrt{k}} M_{a,k} D_{\varkappa} x$ .*

*Then with probability at least  $2/3$  the following holds*

$$(1 - \varepsilon) \|x_i - x_j\|_2^2 \leq \|f(x_i) - f(x_j)\|_2^2 \leq (1 + \varepsilon) \|x_i - x_j\|_2^2, \quad i, j = 1, \dots, n.$$

The preconditioning of  $x$  using  $D_{\varkappa}$  seems to be necessary and we shall comment on this point later on. Its role may be compared with the use of the random Fourier transform in [3].

In contrast to the above mentioned variants of the Johnson-Lindenstrauss lemma, the coordinates of  $f(x)$  are now no longer independent random variables. Our approach “decouples” the dependence caused by the circulant structure. It resembles in some aspects the methods used recently in compressed sensing, cf. [4, 5, 15].

First, we recall the Lemma 1 from Section 4.1 of [13] (cf. also Lemma 2.2 of [14]), which shall be useful later on.

**Lemma 2.2.** *Let*

$$Z = \sum_{i=1}^D \alpha_i (a_i^2 - 1),$$

*where  $a_i$  are i.i.d. normal variables and  $\alpha_i$  are nonnegative real numbers. Then for any  $t > 0$*

$$\begin{aligned} \mathbb{P}(Z \geq 2\|\alpha\|_2 \sqrt{t} + 2\|\alpha\|_{\infty} t) &\leq \exp(-t), \\ \mathbb{P}(Z \leq -2\|\alpha\|_2 \sqrt{t}) &\leq \exp(-t). \end{aligned}$$

Furthermore, we shall use the decoupling lemma of [6, Proposition 1.9].

**Lemma 2.3.** *Let  $\xi_0, \dots, \xi_{d-1}$  be independent random variables with  $\mathbb{E} \xi_0 = \dots = \mathbb{E} \xi_{d-1} = 0$  and let  $\{x_{i,j}\}_{i,j=0}^{d-1}$  be a double sequence of real numbers. Then for  $1 \leq p < \infty$*

$$\mathbb{E} \left| \sum_{i \neq j} x_{i,j} \xi_i \xi_j \right|^p \leq 4^p \mathbb{E} \left| \sum_{i \neq j} x_{i,j} \xi_i \xi'_j \right|^p,$$

*where  $(\xi'_0, \dots, \xi'_{d-1})$  denotes an independent copy of  $(\xi_0, \dots, \xi_{d-1})$ .*

The key role in the proof of the Johnson-Lindenstrauss lemma is played by the following estimates.

**Lemma 2.4.** *Let  $k \leq d$  be natural numbers and let  $\varepsilon \in (0, \frac{1}{2})$ . Let  $a = (a_0, \dots, a_{d-1})$ ,  $M_{a,k}$  and  $D_{\varkappa}$  be as in Theorem 2.1 and let  $x \in \mathbb{R}^d$  be a unit vector. Put  $f(x) = M_{a,k} D_{\varkappa} x$ .*

*Then there is a constant  $c$ , independent on  $k, d, \varepsilon$  and  $x$ , such that*

$$\mathbb{P}_{a,\varkappa} \left( \|f(x)\|_2^2 \geq (1 + \varepsilon)k \right) \leq \exp(-c(k\varepsilon^2)^{1/3})$$

*and*

$$\mathbb{P}_{a,\varkappa} \left( \|f(x)\|_2^2 \leq (1 - \varepsilon)k \right) \leq \exp(-c(k\varepsilon^2)^{1/3}).$$

*Proof.* Let  $S : \mathbb{R}^d \rightarrow \mathbb{R}^d$  denote the shift operator

$$S(x_0, x_1, \dots, x_{d-1}) = (x_{d-1}, x_0, x_1, \dots, x_{d-2}), \quad x \in \mathbb{R}^d.$$

Then

$$\|f(x)\|_2^2 = \|M_{a,k} D_{\varkappa} x\|_2^2 = \sum_{j=0}^{k-1} |\langle S^j a, D_{\varkappa} x \rangle|^2 = \sum_{j=0}^{k-1} \left( \sum_{i=0}^{d-1} a_i \varkappa_{j+i} x_{j+i} \right)^2 = I + II,$$

where

$$I = \sum_{i=0}^{d-1} a_i^2 \cdot \sum_{j=0}^{k-1} x_{j+i}^2$$

and

$$II = \sum_{j=0}^{k-1} \sum_{i \neq i'} a_i a_{i'} \varkappa_{j+i} \varkappa_{j+i'} x_{j+i} x_{j+i'}.$$

Here (and any time later) the summation in the index is to be understood modulo  $d$ .

The decoupling of the circulant matrix is based on

$$\mathbb{P}_{a,\varkappa} \left( \|M_{a,k} D_{\varkappa} x\|_2^2 \geq (1 + \varepsilon)k \right) \leq \mathbb{P}_a(I \geq (1 + \varepsilon/2)k) + \mathbb{P}_{a,\varkappa}(II \geq \varepsilon k/2) \quad (2.1)$$

and

$$\mathbb{P}_{a,\varkappa} \left( \|M_{a,k} D_{\varkappa} x\|_2^2 \leq (1 - \varepsilon)k \right) \leq \mathbb{P}_a(I \leq (1 - \varepsilon/2)k) + \mathbb{P}_{a,\varkappa}(II \leq -\varepsilon k/2). \quad (2.2)$$

We use Lemma 2.2 to estimate the diagonal term  $I$ .

We choose  $\alpha_i = \sum_{j=0}^{k-1} x_{j+i}^2$  and get  $\|\alpha\|_1 = k$ ,  $\|\alpha\|_\infty \leq 1$  and hence  $\|\alpha\|_2 \leq \sqrt{k}$ . This leads to

$$\mathbb{P}_a(I \leq k - 2\sqrt{kt}) \leq \exp(-t) \quad (2.3)$$

and

$$\mathbb{P}_a(I \geq k + 2\sqrt{kt} + 2t) \leq \exp(-t). \quad (2.4)$$

We set  $\varepsilon k/2 = 2\sqrt{kt}$ , i.e.  $t = \varepsilon^2 k/16$ , in (2.3) and obtain

$$\mathbb{P}_a(I \leq (1 - \varepsilon/2)k) \leq \exp(-\varepsilon^2 k/16). \quad (2.5)$$

On the other hand, if  $c = 5/2 - \sqrt{6} > 1/20$ , then  $\sqrt{c} + c/2 = 1/4$  and

$$2\sqrt{kt} + 2t \leq \varepsilon k/2$$

for  $t = c\varepsilon^2 k$ , which finally gives

$$\mathbb{P}_a(I \geq (1 + \varepsilon/2)k) \leq \exp(-c\varepsilon^2 k). \quad (2.6)$$

Next, we estimate the moments of the off-diagonal part  $II$ . We use Lemma 2.3 twice, which gives

$$\mathbb{E}_{a,\varkappa} |II|^p \leq 16^p \mathbb{E}_{a,a',\varkappa,\varkappa'} |II'|^p := 16^p \mathbb{E}_{a,a',\varkappa,\varkappa'} \left| \sum_{j=0}^{k-1} \sum_{i \neq i'} a_i a_{i'} \varkappa_{j+i} \varkappa'_{j+i'} x_{j+i} x_{j+i'} \right|^p,$$

where  $a'$  and  $\varkappa'$  are independent copies of  $a$  and  $\varkappa$ , respectively.

First, we make a substitution  $v = j + i, v' = j + i'$  and use the Khintchine inequality with the optimal constant  $c_p \leq \sqrt{p}$  and the random variable  $\varkappa$  to obtain

$$\begin{aligned} \mathbb{E}_{\varkappa} \left| \sum_{j=0}^{k-1} \sum_{i \neq i'} a_i a'_{i'} \varkappa_{j+i} \varkappa'_{j+i'} x_{j+i} x_{j+i'} \right|^p &= \mathbb{E}_{\varkappa} \left| \sum_{v=0}^{d-1} \varkappa_v x_v \sum_{v' \neq v} \varkappa'_{v'} x_{v'} \sum_{j=0}^{k-1} a_{v-j} a'_{v-j'} \right|^p \\ &\leq c_p^p \left( \sum_{v=0}^{d-1} x_v^2 \left( \sum_{v' \neq v} \varkappa'_{v'} x_{v'} \sum_{j=0}^{k-1} a_{v-j} a'_{v-j'} \right)^2 \right)^{p/2}. \end{aligned}$$

Next, we involve Minkowski's inequality with respect to  $p/2 \geq 1$  and Khintchine's inequality for the random variable  $\varkappa'$ .

$$\begin{aligned} \mathbb{E}_{\varkappa, \varkappa'} |II'|^p &\leq c_p^p \mathbb{E}_{\varkappa'} \left( \sum_{v=0}^{d-1} x_v^2 \left( \sum_{v' \neq v} \varkappa'_{v'} x_{v'} \sum_{j=0}^{k-1} a_{v-j} a'_{v-j'} \right)^2 \right)^{p/2} \\ &\leq c_p^p \left( \sum_{v=0}^{d-1} x_v^2 \left( \mathbb{E}_{\varkappa'} \left| \sum_{v' \neq v} \varkappa'_{v'} x_{v'} \sum_{j=0}^{k-1} a_{v-j} a'_{v-j'} \right|^p \right)^{2/p} \right)^{p/2} \\ &\leq c_p^{2p} \left( \sum_{v \neq v'} x_v^2 x_{v'}^2 \left( \sum_{j=0}^{k-1} a_{v-j} a'_{v-j'} \right)^2 \right)^{p/2}. \end{aligned}$$

Furthermore, the Minkowski inequality for  $a$  and  $a'$  gives

$$\mathbb{E}_{a, a', \varkappa, \varkappa'} |II'|^p \leq c_p^{2p} \left( \sum_{v \neq v'} x_v^2 x_{v'}^2 \left( \mathbb{E}_{a, a'} \left| \sum_{j=0}^{k-1} a_{v-j} a'_{v-j'} \right|^p \right)^{2/p} \right)^{p/2}.$$

If  $a_0, \dots, a_{d-1}$  are Bernoulli variables, then Khintchine's inequality gives

$$\left( \mathbb{E}_{a, a'} \left| \sum_{j=0}^{k-1} a_{v-j} a'_{v-j'} \right|^p \right)^{1/p} \leq \sqrt{kp},$$

as the product of two independent Bernoulli variables is again of this type.

For normal variables, we use first Khintchine's inequality and spherical coordinates to obtain

$$\begin{aligned} \mathbb{E}_{a, a'} \left| \sum_{j=0}^{k-1} a_{v-j} a'_{v-j'} \right|^p &= \mathbb{E}_{a, a'} \left| \sum_{j=0}^{k-1} a_j a'_j \right|^p \leq c_p^p \mathbb{E}_a \left( \sum_{j=0}^{k-1} |a_j|^2 \right)^{p/2} \\ &= c_p^p \mathbb{E}_a \|a\|_2^p = \frac{c_p^p}{(2\pi)^{k/2}} \int_{\mathbb{R}^k} e^{-\|a\|_2^2/2} \|a\|_2^p da \\ &= \frac{c_p^p}{(2\pi)^{k/2}} \cdot A_k \cdot \int_0^\infty e^{-r^2/2} r^{p+k-1} dr, \end{aligned} \tag{2.7}$$

where

$$A_k = \frac{2\pi^{k/2}}{\Gamma(k/2)}$$

is the area of the unit ball in  $\mathbb{R}^k$ .

We combine (2.7) with Stirling's inequality and obtain

$$\left( \mathbb{E}_{a, a'} \left| \sum_{j=0}^{k-1} a_{v-j} a'_{v-j'} \right|^p \right)^{1/p} \leq \sqrt{2} c_p \left[ \frac{\Gamma((k+p)/2)}{\Gamma(k/2)} \right]^{1/p} \leq c \sqrt{p(k+p)}.$$

Hence, if  $a_0, \dots, a_{d-1}$  are independent Bernoulli or normally distributed variables, we may estimate

$$\left(\mathbb{E}_{a, a', \varkappa, \varkappa'} |II'|^p\right)^{1/p} \leq cp \cdot \sqrt{(k+p)p} \cdot \|x\|^2 = cp^{3/2} \sqrt{k+p}. \quad (2.8)$$

Markov's inequality then gives

$$\mathbb{P}_{a, a', \varkappa, \varkappa'}(|II'| > k\varepsilon/2) = \mathbb{P}_{a, a', \varkappa, \varkappa'}\left(\frac{2^p |II'|^p}{k^p \varepsilon^p} \geq 1\right) \leq \frac{2^p \mathbb{E}_{a, a', \varkappa, \varkappa'} |II'|^p}{k^p \varepsilon^p} \leq \left(\frac{cp^{3/2} \sqrt{k+p}}{k\varepsilon}\right)^p.$$

We choose  $p$  by the condition  $\frac{\sqrt{2}cp^{3/2}}{\sqrt{k\varepsilon}} = e^{-1}$ . We may assume  $c \geq 1$ , which ensures that  $p \leq k$  and  $\frac{\sqrt{k+p}}{k} \leq \frac{\sqrt{2}}{\sqrt{k}}$ , which leads to

$$\mathbb{P}_{a, a', \varkappa, \varkappa'}(|II'| > k\varepsilon/2) \leq \exp(-c'(k\varepsilon^2)^{1/3}). \quad (2.9)$$

The proof then follows by (2.1) and (2.2) combined with (2.5), (2.6) and (2.9).  $\square$

The proof of Theorem 2.1 follows from Lemma 2.4 by the union bound over all  $\binom{n}{2}$  pairs of points.

*Remark 2.5.* (i) We note that (2.8) follows directly by very well known estimates of moments of Gaussian chaos, cf. [8, 12]. We preferred to give a simple and direct proof.

(ii) Let us also mention that Lemma 2.4 fails, if the multiplication with  $D_\varkappa$  is omitted. Namely, let  $k \leq d$  be natural numbers, let  $a_0, \dots, a_{d-1}$  be independent normal variables and let  $x = \frac{1}{\sqrt{d}}(1, \dots, 1)$ . If  $f(x) = M_{a, k}x$ , then

$$\|f(x)\|_2^2 = k \left( \sum_{j=0}^{d-1} \frac{a_j}{\sqrt{d}} \right)^2.$$

Due to the 2-stability of the normal distribution, the variable

$$b := \sum_{j=0}^{d-1} \frac{a_j}{\sqrt{d}}$$

is again normally distributed, i.e.  $b \approx N(0, 1)$ . Hence

$$\mathbb{P}_a\left(\|f(x)\|_2^2 > (1 + \varepsilon)k\right) = \mathbb{P}_b\left(b^2 > (1 + \varepsilon)\right)$$

depends neither on  $k$  nor on  $d$  and Lemma 2.4 cannot hold.

(iii) The statement of Theorem 2.1 holds also for matrices with Toeplitz structure. The proof is literally the same, only notational changes are necessary.

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