Characterisations of function spaces with dominating mixed smoothness properties

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Abstract

We investigate function spaces with dominating mixed smoothness properties of Besov and Triebel - Lizorkin type. Equivalent quasi-norms in terms of local means are derived. Also theorems on atomic and subatomic decomposition are proved.

Keywords: function space, dominating mixed smoothness, local means, atomic decompositions, subatomic decompositions

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Preface

In this paper we deal with function spaces of dominating mixed smoothness properties. First spaces of this type were defined by S. M. Nikol'skij in [**N1**] and [**N2**]. He introduced the spaces of Sobolev type

$$S_p^{\overline{r}}W(\mathbb{R}^2)=\Big\{f|f\in L_p(\mathbb{R}^2),||f|S_p^{\overline{r}}W(\mathbb{R}^2)||=||f|L_p||+\Big|\Big|\ \frac{\partial^{r_1}f}{\partial x_1^{r_1}}\ |\ L_p\ \Big|\Big|+$$
$$+\Big|\Big|\ \frac{\partial^{r_2}f}{\partial x_2^{r_2}}\ |\ L_p\ \Big|\Big|+\Big|\Big|\ \frac{\partial^{r_1+r_2}f}{\partial x_1^{r_1}\partial x_2^{r_2}}\ |\ L_p\ \Big|\Big|<\infty\Big\},$$

where $1<p<\infty, r_i=0,1,2,\dots;(i=1,2)$. The mixed derivative $\frac{\partial^{r_1+r_2}f}{\partial x_1^{r_1}\partial x_2^{r_2}}$ plays the dominant part here and gave the name to this class of spaces. The detailed study of spaces of such type was performed by many authors, for example T. I. Amanov, O. V. Besov, K. K. Golovkin, P. I. Lizorkin, S. M. Nikol'skij, M. K. Potapov and H.–J. Schmeisser. We refer to [**Am**] for a systematic treatment of this topic. As in the theory of classical Sobolev spaces an alternative definition in terms of Fourier transform may be given (see (1.6) and (1.7)). This definition is based on a decomposition

$$f=\sum_{\overline{k}\in\mathbb{N}_0^d}(\varphi_{k_1}\otimes\cdots\otimes\varphi_{k_d}\hat{f})^\vee,\quad \text{convergence in } S'(\mathbb{R}^d),$$

where $\{\varphi_k\}_{k\in\mathbb{N}_0}$ is a decomposition of unity on $\mathbb{R}$ known from the theory of classical Besov spaces and $\varphi_{\overline{k}}=\varphi_{k_1}\otimes\cdots\otimes\varphi_{k_d},\overline{k}=(k_1,\dots,k_d)$ is a tensor product.

We refer mainly to [**SchT**], as far as the Fourier-analytic approach to these spaces is considered. In Chapter 2 of this book the classical theory of spaces with dominating mixed smoothness properties is developed. Several types of equivalent quasinorms, embedding and trace theorems and characterisation of these spaces by differences are proven there. One studies also basic properties of crucial operators on these spaces, namely of lifting and maximal operators and Fourier multipliers. We recall some facts from this book, which shall be useful later on, in Chapter 1. As we don't restrict the dimension of the underlying Euclidean space to $d=2$, we state these results formulated for general dimension $d\ge 2$. As mentioned in [**SchT**] this generalisation is obvious.

The second Chapter is devoted to local means, atomic and subatomic decompositions of spaces with dominating mixed derivative. We prove our results only for spaces of Triebel-Lizorkin type. The counterparts of our results for Besov-type spaces are formulated in Chapter 3. Their proofs are omitted as they are very similar to the proofs presented here. First of all, we characterise this class of spaces by so-called local means. See Theorem 2.4 for details. This fundamental characterisation serve us as a basis for atomic (and subatomic) decompositions.

By atomic decomposition of a function f one usually means a decomposition of type

$$f(x)=\sum_\nu\sum_m\lambda_{\nu m}a_{\nu m}(x),\qquad \text{convergence in}\quad S'(\mathbb{R}^d),$$

where $a_{\nu m}$ are some simple building blocks, called *atoms*, and $\lambda_{\nu m}$ are complex numbers. A function f then belongs to some function space if and only if the sequence of coefficients $\{\lambda_{\nu m}\}_{\nu m}$ belongs to some sequence space. For the exact formulation see Theorem 2.11. Let us mention that the atoms are specified only implicitly - a function a is an atom if and only if it satisfies some properties (see Definition 2.9).

By a subatomic decomposition we mean a decomposition of a type

$$f(x)=\sum_{\beta}\sum_{\nu}\sum_{m}\lambda^{\beta}_{\nu m}(\beta\mathrm{qu})_{\nu m}(x),\qquad \text{convergence in}\quad S'(\mathbb{R}^d),$$

where $(\beta\mathrm{qu})_{\nu m}(x)$ are so-called *quarks* and $\lambda^{\beta}_{\nu m}$ are complex numbers. A quark is a special type of atom defined explicitly by (2.85). Hence the basic building blocks, quarks, are much more specific in this kind of decomposition. The price one has to pay for that is a more complicated connection between f and $\{\lambda^{\beta}_{\nu m}\}$. It is described in detail in Theorem 2.13. In this sence each of these decompositions has its advantages and disadvantages. But both of them have something in common : they build a connection between function spaces and sequence spaces. As the sequence spaces are simpler to deal with, it turns out that this connection is very useful in many situations (embeddings, traces, entropy numbers, ...). On this place we have to mention another important way how to switch from function spaces to sequence spaces — namely the so-called φ-transform of M. Frazier and B. Jawerth. We refer to [**Ho**] as this topic is considered.

The classical theory of atomic decompositions of Besov and Triebel-Lizorkin spaces was developed mainly in the works M. Frazier and B. Jawerth ([**FrJ1**], [**FrJ2**]) and H. Triebel ([**Tr1**], [**Tr2**]). The subatomic decomposition of these spaces is due to H. Triebel ([**Tr3**], [**T03**]). We follow their ideas and prove similar decomposition theorems for spaces with dominating mixed derivative. This is done in Chapter 2 and is the focus of this work.

In Chapter 3 we give some remarks and comments to Chapter 2. Namely, we present an improved version of Theorem 2.1 based on [**Rych**], we give an alternative proof of the existence of optimal atomic decomposition using some ideas from [**HN**] and we formulate our results also for spaces of Besov type.

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CHAPTER 1

Introduction

1. Notation

First of all, we shall introduce some basic notation. The dimension of the underlying Euclidean space will be denoted by d. d-dimensional indices will be denoted by $\overline{k}, \overline{l}, \overline{m}, \ldots$ and d-dimensional variables by $x, y, z, \ldots$. Their components are numbered from 1 to d. Hence we write $\overline{k} = (k_1, \ldots, k_d)$. We use a standard vector notation in this connection, namely

$$
\begin{aligned}
&\overline{k} + \overline{r} = (k_1 + r_1, \ldots, k_d + r_d) && \overline{k}, \overline{r} \in \mathbb{R}^d,\\
&\overline{k} \cdot \overline{r} = \sum_{i=1}^{d} k_i r_i, && \overline{k}, \overline{r} \in \mathbb{R}^d,\\
&\lambda \overline{k} = (\lambda k_1, \ldots, \lambda k_d), && \lambda \in \mathbb{R}, \overline{k} \in \mathbb{R}^d,\\
&x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_d^{\alpha_d}, && x, \alpha \in \mathbb{R}^d,\\
&\alpha + \lambda = (\alpha_1 + \lambda, \ldots, \alpha_d + \lambda), && \lambda \in \mathbb{R}, \alpha \in \mathbb{R}^d,\\
&\lambda^\alpha = (\lambda^{\alpha_1}, \ldots, \lambda^{\alpha_d}), && \lambda \in \mathbb{R}, \alpha \in \mathbb{R}^d,\\
&\alpha^\lambda = (\alpha_1^\lambda, \ldots, \alpha_d^\lambda), && \lambda \in \mathbb{R}, \alpha \in \mathbb{R}^d.
\end{aligned}
$$

We write $x > y$ for $x, y \in \mathbb{R}^d$ if and only if $x_i > y_i$ for all $i = 1, \ldots, d$. Similarly we define $x < y, x \le y$ and $x \ge y$. In the same sense we define $x > \lambda$ for $x \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$.

We denote the $d-$dimensional Fourier transform by F or $\,^\wedge$ and its inverse by F^{-1} or $\,^\vee$. Sometimes we need also the one-dimensional Fourier transform. This will be denoted by F_1 or $\,^{\wedge_1}$ and its inverse by F_1^{-1} or $\,^{\vee_1}$.

2. Prerequisites

Definition 1.1. Let $\Phi(\mathbb{R})$ be the collection of all systems $\{\varphi_j(t)\}_{j=0}^\infty \subset S(\mathbb{R})$ such that

$$
\left\{
\begin{array}{l}
\operatorname{supp} \varphi_0 \subset \{t \in \mathbb{R} : \ |t| \le 2\}\\
\operatorname{supp} \varphi_j \subset \{t \in \mathbb{R} : \ 2^{j-1} \le |t| \le 2^{j+1}\} \quad \text{if} \quad j = 1, 2, \ldots;
\end{array}
\right. \tag{1.1}
$$

for every $\alpha \in \mathbb{N}_0$ there exists a positive constant c_α such that

$$2^{j\alpha} |D^\alpha \varphi_j(t)| \le c_\alpha \quad \text{for all} \quad j = 0, 1, 2, \ldots \text{ and all} \quad t \in \mathbb{R}, \tag{1.2}$$

and

$$\sum_{j=0}^{\infty} \varphi_j(t) = 1 \quad \text{for every} \quad t \in \mathbb{R}. \tag{1.3}$$

For $\overline{k} = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we define $\varphi_{\overline{k}}(x) = \varphi_{k_1}(x_1) \cdot \ldots \cdot \varphi_{k_d}(x_d)$. Using this kind of notation, we can give a definition of spaces $l_q(L_p)$, $L_p(l_q)$, $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ and $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$.

Definition 1.2. Let $0 < p \le \infty$ and $0 < q \le \infty$. Having a sequence of complex-valued functions $\{f_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d}$ on $\mathbb{R}^d$ we put

$$||f_{\overline{k}}|l_q(L_p)|| = \Big(\sum_{\overline{k}\in\mathbb{N}_0^d} ||f_{\overline{k}}|L_p||^q\Big)^{1/q} \tag{1.4}$$

and

$$||f_{\overline{k}}|L_p(l_q)|| = \bigg|\bigg|\Big(\sum_{\overline{k}\in\mathbb{N}_0^d} |f_{\overline{k}}(x)|^q\Big)^{1/q}|L_p\bigg|\bigg|, \tag{1.5}$$

where the L_p norm is taken with respect to $x \in \mathbb{R}^d$. When $q = \infty$, the usual change is necessary.

Definition 1.3. Let $\overline{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$, $0 < q \le \infty$ and $\{\varphi_j\}_{j=0}^{\infty} \in \Phi(\mathbb{R})$.
(i) Let $0 < p \le \infty$. Then $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|S^{\overline{r}}_{p,q}B(\mathbb{R}^d)||_{\varphi} = \Big(\sum_{\overline{k}\in\mathbb{N}_0^d} 2^{q\overline{k}\cdot\overline{r}}||(\varphi_{\overline{k}}\hat{f})^{\vee}|L_p||^q\Big)^{1/q} = ||2^{\overline{k}\cdot\overline{r}}(\varphi_{\overline{k}}\hat{f})^{\vee}|l_q(L_p)|| \tag{1.6}$$

is finite.
(ii) Let $0 < p < \infty$. Then $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)||_{\varphi} = \bigg|\bigg|\Big(\sum_{\overline{k}\in\mathbb{N}_0^d} |2^{\overline{k}\cdot\overline{r}}(\varphi_{\overline{k}}\hat{f})^{\vee}(\cdot)|^q\Big)^{1/q}|L_p(\mathbb{R}^d)\bigg|\bigg| = ||2^{\overline{k}\cdot\overline{r}}(\varphi_{\overline{k}}\hat{f})^{\vee}|L_p(l_q)|| \tag{1.7}$$

is finite.

Remark 1.4. All function spaces considered in this paper are defined on $\mathbb{R}^d$. Hence we write $S^{\overline{r}}_{p,q}B$ and $S^{\overline{r}}_{p,q}F$ instead of $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ and $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$.

Remark 1.5. According to (1.3), we have

$$\sum_{k\in\mathbb{N}_0^d} \varphi_{\overline{k}}(x) = \Big(\sum_{k_1=0}^{\infty} \varphi_{k_1}(x_1)\Big)\cdot\ldots\cdot\Big(\sum_{k_d=0}^{\infty}\varphi_{k_d}(x_d)\Big) = 1 \quad \text{for all } x = (x_1, \cdots, x_d) \in \mathbb{R}^d.$$

In this sense, $\{\varphi_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d}$ is also a decomposition of unity, in this case on $\mathbb{R}^d$.

Let us recall some very well known notation, for details see [**SchT**].
Let $\Omega = \{\Omega_{\overline{k}}\}_{k\in\mathbb{N}_0^d}$ be a sequence of compact subset of $\mathbb{R}^d$. Then we denote by $l_q^{\Omega}(L_p)$, resp. $L_p^{\Omega}(l_q)$ a set of all sequences $f = \{f_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d}$ for which

$$f_{\overline{k}} \in S'(\mathbb{R}^d) \quad \text{and} \quad \operatorname{supp}\hat{f}_{\overline{k}} \subset \Omega_{\overline{k}} \quad \text{for all} \quad \overline{k} \in \mathbb{N}_0^d \tag{1.8}$$

$$||f_{\overline{k}}|l_q(L_p)|| < \infty, \quad \text{resp.} \quad ||f_{\overline{k}}|L_p(l_q)|| < \infty. \tag{1.9}$$

Next, we recall some known facts from the theory of these spaces. Their proves may be found in [**SchT**] for $d = 2$.

Theorem 1.6. (Nikol'skij inequality) *Let $0 < p \le u \le \infty$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$. Let $\overline{b} = (b_1, \dots, b_d) > 0$ and $Q_{\overline{b}} = [-b_1, b_1] \times \cdots \times [-b_d, b_d] \subset \mathbb{R}^d$. Then there exists a positive constant c, which is independent of $\overline{b}$, such that*

$$||D^{\alpha}f|L_u|| \le c\, b_1^{\alpha_1+\frac{1}{p}-\frac{1}{u}}\cdot\ldots\cdot b_d^{\alpha_d+\frac{1}{p}-\frac{1}{u}}||f|L_p|| = c\, b^{\alpha+\frac{1}{p}-\frac{1}{u}}||f|L_p||$$

holds for every $f \in S'(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)$ *with* $\operatorname{supp} \hat{f} \subset Q_{\overline{b}}$.

THEOREM 1.7. *Let* $\{\varphi_j\}_{j=0}^{\infty}, \{\psi_j\}_{j=0}^{\infty} \in \Phi(\mathbb{R})$. *Let* $\overline{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ *and* $0 < q \leq \infty$.

(i) *Let* $0 < p \leq \infty$. *Then* $||f|S^{\overline{r}}_{p,q}B||_{\varphi}$ *and* $||f|S^{\overline{r}}_{p,q}B||_{\psi}$ *are equivalent quasi-norms. Furthermore,* $S^{\overline{r}}_{p,q}B$ *is a quasi-Banach space (Banach space if* $\min(p,q) \geq 1$) *and*

$$S \subset S^{\overline{r}}_{p,q}B \subset S'.$$

(ii) *Let* $0 < p < \infty$. *Then* $||f|S^{\overline{r}}_{p,q}F||_{\varphi}$ *and* $||f|S^{\overline{r}}_{p,q}F||_{\psi}$ *are equivalent quasi-norms. Furthermore,* $S^{\overline{r}}_{p,q}F$ *is a quasi-Banach space (Banach space if* $\min(p,q) \geq 1$) *and*

$$S \subset S^{\overline{r}}_{p,q}F \subset S'.$$

For the proof in the case $d = 2$, see [**SchT**, pages 87, 93]. So, we may write $||f|S^{\overline{r}}_{p,q}B||$ and $||f|S^{\overline{r}}_{p,q}F||$ without any index φ or ψ meaning one of these equivalent quasi-norms.

As in the case of classical Besov and Triebel-Lizorkin spaces, we can define a lifting operator.

DEFINITION 1.8. Let $\overline{\rho} = (\rho_1, \ldots, \rho_d) \in \mathbb{R}^d$. Then we define the so-called lifting operator $I_{\overline{\rho}}$ by

$$I_{\overline{\rho}}f = F^{-1}(1+x_1^2)^{\rho_1/2} \cdot \ldots \cdot (1+x_d^2)^{\rho_d/2}Ff = F^{-1}(1+x^2)^{\overline{\rho}/2}Ff, \quad f \in S'(\mathbb{R}^d). \tag{1.10}$$

THEOREM 1.9. *Let ,* $0 < q \leq \infty$, $\overline{\rho}, \overline{r} \in \mathbb{R}^d$.

(i) *Let* $0 < p \leq \infty$. *Then* $I_{\overline{\rho}}$ *maps* $S^{\overline{r}}_{p,q}B$ *isomorphically onto* $S^{\overline{r}-\overline{\rho}}_{p,q}B$ *and* $||I_{\overline{\rho}}f|S^{\overline{r}-\overline{\rho}}_{p,q}B||$ *is an equivalent quasi-norm in* $S^{\overline{r}}_{p,q}B$.

(ii) *Let* $0 < p < \infty$. *Then* $I_{\overline{\rho}}$ *maps* $S^{\overline{r}}_{p,q}F$ *isomorphically onto* $S^{\overline{r}-\overline{\rho}}_{p,q}F$ *and* $||I_{\overline{\rho}}f|S^{\overline{r}-\overline{\rho}}_{p,q}F||$ *is an equivalent quasi-norm in* $S^{\overline{r}}_{p,q}F$.

The proof may be again found in [**SchT**, page 98].

Next we collect some useful maximal theorems which play a crucial role in the further theory.

For every function $f(x) \in L_1^{loc}(\mathbb{R}^d)$ we define the classical Hardy-Littlewood maximal operator

$$(Mf)(x) = \sup_Q \frac{1}{|Q|}\int_Q |f(y)|\mathrm{d}y, \qquad x \in \mathbb{R}^d, \tag{1.11}$$

where the supremum is taken over all cubes Q cantered at x with sides parallel with coordinate axes. The famous Hardy-Littlewood inequality tells that for every p with $1 < p \leq \infty$ there is a c such that

$$||Mf|L_p(\mathbb{R}^d)|| \leq c\,||f|L_p(\mathbb{R}^d)||, \qquad f \in L_p(\mathbb{R}^d). \tag{1.12}$$

The following theorem is a vector-valued generalisation of (1.12) and is due to C. Fefferman and E. M. Stein [**FS**].

THEOREM 1.10. *Let* $1 < p < \infty$ *and* $1 < q \leq \infty$. *There exists a constant c such that*

$$||Mf_{\overline{k}}|L_p(l_q)|| \leq c\,||f_{\overline{k}}|L_p(l_q)|| \tag{1.13}$$

holds for all sequences $\{f_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d}$ *of locally Lebesgue-integrable functions on* $\mathbb{R}^d$.

Next we define one-dimensional version of (1.11).

$$(M_1f)(x) = \sup_{s>0} \frac{1}{2s}\int_{x_1-s}^{x_1+s} |f(t,x_2,\ldots,x_d)|\mathrm{d}t \tag{1.14}$$

and in a similar way for other variables. We denote the composition of these operators by $\overline{M} = M_d \circ \cdots \circ M_1$. The following maximal theorem is due to R. J. Bagby [**Ba**] (it is a special case of much more general theorem given there).

THEOREM 1.11. *Let $1 < p < \infty$ and $1 < q \leq \infty$. There exists a constant c such that*

$$||M_i f_{\overline{k}}|L_p(l_q)|| \leq c\,||f_{\overline{k}}|L_p(l_q)||, \quad i = 1, \dots, d \tag{1.15}$$

holds for all sequences $\{f_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d} \subset L_p(l_q)$ of functions on $\mathbb{R}^d$.

Iteration of this Theorem shows that the estimate (1.15) holds also for the operator $\overline{M}$.

THEOREM 1.12. *Let $\Omega = \{\Omega_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d}$ be the sequence of compact subset of $\mathbb{R}^d$ with following properties*

$$\Omega_{\overline{k}} = \{x \in \mathbb{R}^d : |x_1| \leq a_{1,k_1}, \dots, |x_d| \leq a_{d,k_d}\} \quad \text{with} \quad a_{1,k_1}, \dots, a_{d,k_d} > 0.$$

Let $0 < p < \infty$, $0 < q \leq \infty$ and $0 < r_1, \dots, r_d < \min(p, q)$. Then there is a positive constant c such that

$$\left|\left|\sup_{y\in\mathbb{R}^d} \frac{|f_{\overline{k}}(\cdot - y)|}{(1 + |a_{1,k_1}y_1|^{1/r_1}) \cdot \ldots \cdot (1 + |a_{d,k_d}y_d|^{1/r_d})} |L_p(l_q)\right|\right| \leq c||f_{\overline{k}}|L_p(l_q)||$$

holds for all systems $\{f_{\overline{k}}\} \in L_p^{\Omega}(l_q)$.

For $1 < p < \infty$ and $\overline{r} \in \mathbb{R}^d$ we denote

$$S_p^{\overline{r}}H = S_{p,2}^{\overline{r}}F = \{f | f \in S', ||(1 + x^2)^{\overline{r}/2}Ff|L_2|| < \infty\}.$$

For more details, see [**SchT**, Theorem 2.3.1].

THEOREM 1.13. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $\overline{r} = (r_1, \dots, r_d) > \frac{1}{\min(p,q)} + \frac{1}{2}$. Let $\Omega = \{\Omega_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d}, a_{1,k_1}, \dots, a_{d,k_d} > 0$ be the same sequences as above. Then there is a positive constant c such that*

$$||(\varrho_{\overline{k}}\hat{f}_{\overline{k}})^{\vee}|L_p(l_q)|| \leq c\Big(\sup_{\overline{k}\in\mathbb{N}_0^d} ||\varrho_{\overline{k}}(a_{1,k_1}\cdot, \dots, a_{d,k_d}\cdot)|S_2^{\overline{r}}H||\Big) \cdot ||f_{\overline{k}}|L_p(l_q)||$$

holds for all systems $\{f_{\overline{k}}\} \in L_p^{\Omega}(l_q)$ and all systems $\{\varrho_{\overline{k}}\} \subset S_2^{\overline{r}}H$.

DEFINITION 1.14. Let $\{\Omega_j\}_{j=0}^{\infty}$ be a sequence of compact subset of $\mathbb{R}$ such that

$$\Omega_0 \subset [-2, 2], \quad \Omega_j \subset \{t \in \mathbb{R} : 2^{j-1} \leq |t| \leq 2^{j+1}\}; \quad j \in \mathbb{N}$$

Let $\{\psi_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d} \subset S(\mathbb{R}^d)$ with

$$\operatorname{supp} \psi_{\overline{k}} \subset \Omega_{k_1} \times \ldots \times \Omega_{k_d}; \quad k \in \mathbb{N}_0^d.$$

If $f \in S'(\mathbb{R}^d)$, $\overline{a} = (a_1, \dots, a_d) > 0$ then we put

$$(\psi_{\overline{k}}^* f)_{\overline{a}}(x) = \sup_{y\in\mathbb{R}^d} \frac{|(\psi_{\overline{k}}\hat{f})^{\vee}(x - y)|}{(1 + |2^{k_1}y_1|^{a_1}) \cdot \ldots \cdot (1 + |2^{k_d}y_d|^{a_d})}, \quad x \in \mathbb{R}^d, k \in \mathbb{N}_0^d. \tag{1.16}$$

As usual for any $\varkappa \in \mathbb{R}$ we put $\varkappa_+ = \max(\varkappa, 0)$ and $[\varkappa]$ stands for the largest integer smaller then or equal to $\varkappa$.

Theorem 1.15. *Let* $\{\Omega_j\}_{j=0}^{\infty}$ *and* $\{\psi_k\}_{k\in\mathbb{N}_0^d}$ *be as in the above definition. For every* $\overline{L} = (L_1, \ldots, L_d) \in \mathbb{N}_0^d$ *let us put*

$$C_{\psi,\overline{L}} = \sup_{\overline{k}\in\mathbb{N}_0^d} \sup_{x\in\mathbb{R}^d} \sum_{0\le\alpha\le\overline{L}} (1+x_1^2)^{\alpha_1/2} \cdot \ldots \cdot (1+x_d^2)^{\alpha_d/2} |D^\alpha \psi_{\overline{k}}(x)|.$$

(i) *Let* $\overline{r} \in \mathbb{R}^d$, $0 < p < \infty$, $0 < q \le \infty$. *If*

$$\overline{a} > \frac{1}{p}, \quad \overline{L} > \left[\frac{1}{\min(1,p)} - \frac{1}{2}\right]$$

and $C_{\psi,\overline{L}}$ *is finite, then there is a positive constant* c *such that*

$$||2^{\overline{r}\cdot\overline{k}}(\psi_{\overline{k}}^* f)_{\overline{a}}|l_q(L_p)|| \le c\, C_{\psi,\overline{L}} ||f|S^{\overline{r}}_{p,q}B||, \quad f \in S^{\overline{r}}_{p,q}B.$$

(ii) *Let* $\overline{r} \in \mathbb{R}^d$, $0 < p < \infty$, $0 < q \le \infty$. *If*

$$\overline{a} > \frac{1}{\min(p,q)}, \quad \overline{L} > \left[\frac{1}{\min(p,q)} + \frac{1}{2}\right]$$

and $C_{\psi,\overline{L}}$ *is finite, then there is a positive constant* c *such that*

$$||2^{\overline{r}\cdot\overline{k}}(\psi_{\overline{k}}^* f)_{\overline{a}}|L_p(l_q)|| \le c\, C_{\psi,\overline{L}} ||f|S^{\overline{r}}_{p,q}F||, \quad f \in S^{\overline{r}}_{p,q}F.$$

Furthermore, let

$$\sigma_{pq} = \left(\frac{1}{\min(p,q)} - 1\right)_+ \quad \text{and} \quad \sigma_p = \left(\frac{1}{p} - 1\right)_+ \tag{1.17}$$

for every $0 < p < \infty$ and $0 < q \le \infty$.

CHAPTER 2

Local means, Atomic and Subatomic decompositions

1. Local means

THEOREM 2.1. *Let $0 < p < \infty$, $0 < q \leq \infty$, $\overline{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$. Let $\overline{N} = (N_1, \ldots, N_d) \in \mathbb{N}_0^d$ be d even numbers with $\overline{r} < \overline{N}$.*
Let ψ_0 and $\psi^1, \ldots, \psi^d$ be $d+1$ complex-valued functions from $S(\mathbb{R})$, which satisfy the Tauberian conditions

$$|\psi_0(x)| > 0 \quad \text{if} \quad |x| \leq 2, \qquad \text{and} \qquad |\psi^i(x)| > 0 \quad \text{if} \quad \frac{1}{2} \leq |x| \leq 2, \quad i = 1, \ldots, d. \tag{2.1}$$

Let us also suppose that

$$D^\alpha \psi^i(0) = 0, \qquad 0 \leq \alpha \leq N_i - 1, \qquad i = 1, \ldots, d. \tag{2.2}$$

Let $\psi_0^i = \psi_0$ and $\psi_j^i(t) = \psi^i(2^{-j}t)$ if $t \in \mathbb{R}$, $j \in \mathbb{N}$ and $i = 1, \ldots, d$. Further let $\psi_{\overline{k}}(x) = \psi_{k_1}^1(x_1) \cdot \ldots \cdot \psi_{k_d}^d(x_d)$ whenever $\overline{k} = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Then

$$\Big|\Big|\Big(\sum_{\overline{k} \in \mathbb{N}_0^d} |2^{\overline{k}\cdot\overline{r}}(\psi_{\overline{k}}\hat{f})^\vee(\cdot)|^q\Big)^{1/q} |L_p(\mathbb{R}^d)\Big|\Big| \tag{2.3}$$

is an equivalent quasi-norm in $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$.

First of all we derive some properties of the functions involved in this theorem.

LEMMA 2.2. *Let $h, H \in S(\mathbb{R})$ be two functions with*

$$\operatorname{supp} h \subset \{y \in \mathbb{R} : |y| \leq 2\}, \qquad \operatorname{supp} H \subset \{y \in \mathbb{R} : \frac{1}{4} \leq |y| \leq 4\},$$

$$h(x) = 1 \quad \text{if} \quad |x| \leq 1, \qquad H(x) = 1 \quad \text{if} \quad \frac{1}{2} \leq |x| \leq 2.$$

Further let $\overline{a}, \overline{b} \in \mathbb{R}^d$. Then

$$\sup_{K \in \mathbb{N}_0} \int_{\mathbb{R}} \left| \left(\frac{\psi^i(\cdot)h(2^{-K}\cdot)}{|\cdot|^{N_i}} \right)^{\vee_1} (t) \right| (1+|t|)^{a_i} \mathrm{d}t < \infty, \quad i = 1, \ldots, d, \tag{2.4}$$

$$\sup_{m \in \mathbb{N}} 2^{-mb_i} \int_{\mathbb{R}} |[\psi^i(2^m\cdot)H(\cdot)]^{\vee_1}(t)|(1+|t|)^{a_i} \mathrm{d}t < \infty, \quad i = 1, \ldots, d, \tag{2.5}$$

$$\sup_{m \in \mathbb{N}} 2^{-mb_i} \int_{\mathbb{R}} |[\psi_0(2^m\cdot)H(\cdot)]^{\vee_1}(t)|(1+|t|)^{a_i} \mathrm{d}t < \infty, \quad i = 1, \ldots, d. \tag{2.6}$$

PROOF. Step 1. We show that

$$\sup_{m \in \mathbb{N}} 2^{m\gamma} \int_R |(\psi(\xi)\Psi(2^{-m}\xi))^{\vee_1}(t)|(1+|t|)^\mu \mathrm{d}t < \infty \tag{2.7}$$

for all real numbers $\gamma, \mu \in \mathbb{R}$ and for all functions $\psi, \Psi \in S(\mathbb{R})$ with $\operatorname{supp} \Psi \subset \{t \in \mathbb{R} : 1/4 \leq |t| \leq 4\}$.

Using the Hölder inequality and elementary properties of Fourier transform, we get for $\nu > \mu + 1$

$$
\begin{aligned}
||(\psi(\xi)\Psi(2^{-m}\xi))^{\vee_1}(t)(1+|t|)^{\mu}|L_1|| &\le c\,||(\psi(\xi)\Psi(2^{-m}\xi))^{\vee_1}(t)(1+|t|)^{\nu}|L_\infty|| \\
&\le c \max_{0\le\lambda\le\nu+1} ||[D^\lambda(\psi(\xi)\Psi(2^{-m}\xi))]^{\vee_1}(t)|L_\infty|| \\
&\le c \max_{0\le\lambda\le\nu+1} ||D^\lambda(\psi(\xi)\Psi(2^{-m}\xi))(t)|L_1||.
\end{aligned} \tag{2.8}
$$

Next we use the support property of Ψ and the decay of ψ and get

$$
\begin{aligned}
|D^\lambda[\psi(\xi)\Psi(2^{-m}\xi)](t)| &\le c \max_{0\le\eta\le\lambda} |D^\eta\psi(t)|2^{-m(\lambda-\eta)}|(D^{\lambda-\eta}\Psi)(2^{-m}t)| \\
&\le c\,(1+|t|)^{-\gamma}\chi_{\{t\in\mathbb{R}:1/4\le 2^{-m}|t|\le 4\}}(t).
\end{aligned} \tag{2.9}
$$

Putting (2.9) into (2.8), we obtain (2.7).

Step 2. To prove (2.5) put the identity

$$
\int_{\mathbb{R}} |(\psi^i(2^m\cdot)H(\cdot))^{\vee_1}(t)|(1+|t|)^{a_i}\mathrm{d}t = \int_{\mathbb{R}} |(\psi^i(\cdot)H(2^{-m}\cdot))^{\vee_1}(t)|(1+|2^m t|)^{a_i}\mathrm{d}t
$$

into (2.7). In the same way one can prove (2.6).

Step 3. To prove (2.4) use the decomposition $h(2^{-K}\cdot) = h(\cdot) + \sum_{\mu=1}^{K}[h(2^{-\mu}\cdot) - h(2^{-\mu+1}\cdot)]$, (2.7) and the fact that $\frac{\psi^i(\cdot)}{|\cdot|^{N_i}} \in S(\mathbb{R})$. □

Proof. We follow the proof of Theorem 2.4.1 in [**Tr1**].

Part 1.

Let $f \in S^{\overline{r}}_{p,q}F$. In the first part we prove that the quasi-norm (2.3) can be estimated from above by $c||f|S^{\overline{r}}_{p,q}F||$. Let $\{\varphi_j\}_{j=0}^\infty \in \Phi(\mathbb{R})$. Let $\varphi_j(x) = 0$ if $-j \in \mathbb{N}$. Let us write $\varphi_{\overline{l}}(x) = \varphi_{l_1}(x_1)\cdot\ldots\cdot\varphi_{l_d}(x_d)$ for $l \in \mathbb{Z}^d, x \in \mathbb{R}^d$. We shall deal with the decomposition

$$
\begin{aligned}
2^{\overline{k}\cdot\overline{r}}(\psi_{\overline{k}}\hat{f})^\vee(x) &= \sum_{\overline{l}\in\mathbb{Z}^d} 2^{\overline{k}\cdot\overline{r}}(\psi_{\overline{k}}\varphi_{\overline{k}+\overline{l}}\hat{f})^\vee(x) = \\
&\left(\sum_{l_1=-\infty}^{K} + \sum_{l_1=K+1}^{\infty}\right)\times\ldots\times\left(\sum_{l_d=-\infty}^{K} + \sum_{l_d=K+1}^{\infty}\right)\left(2^{\overline{k}\cdot\overline{r}}(\psi_{\overline{k}}\varphi_{\overline{k}+\overline{l}}\hat{f})^\vee(x)\right).
\end{aligned} \tag{2.10}
$$

The decomposition is given in the formal way. To obtain an exact expression, containing 2^d terms, one has to proceed through the formal multiplication of sums.

The natural number K will be chosen later on.

In the first step we estimate the term with $-\infty < \overline{l} \le K$, in the second step we deal with the term with $K+1 \le \overline{l} < \infty$. The third step is devoted to the other terms and in the fourth step we discuss the convergence of (2.10). In the first three steps we take it for granted that this decomposition converges pointwise and in $S'(\mathbb{R}^d)$ to the same limit and that $(\psi_{\overline{k}}\hat{f})^\vee$ is a regular distribution.

Step 1. $-\infty < \overline{l} \le K$.

We fix a vector $\overline{k} \in \mathbb{N}_0^d$ and suppose that $\overline{k} \ge 1$. The other cases are discussed later.

Let $\tilde{\varphi}_{\overline{m}}(x) = |2^{-m_1}x_1|^{N_1}\cdot\ldots\cdot|2^{-m_d}x_d|^{N_d}\varphi_{\overline{m}}(x), \overline{m} \in \mathbb{Z}^d$. Then we obtain

$$
2^{\overline{k}\cdot\overline{r}}|(\psi_{\overline{k}}\varphi_{\overline{k}+\overline{l}}\hat{f})^\vee(x)| = 2^{\overline{l}\cdot(\overline{N}-\overline{r})}\left|\left(\frac{\psi_{\overline{k}}(z)}{|2^{-k_1}z_1|^{N_1}\cdot\ldots\cdot|2^{-k_d}z_d|^{N_d}}2^{\overline{r}\cdot(\overline{k}+\overline{l})}\tilde{\varphi}_{\overline{k}+\overline{l}}\hat{f}\right)^\vee(x)\right| \tag{2.11}
$$

Using (1.1) we have $\tilde{\varphi}_{\overline{k}+\overline{l}}(z) = \tilde{\varphi}_{\overline{k}+\overline{l}}(z)h(2^{-k_1-K}z_1)\cdot\ldots\cdot h(2^{-k_d-K}z_d)$ for every $\overline{l} \le K$, every $\overline{k} \in \mathbb{N}^d$ and every $z \in \mathbb{R}^d$, where h is the function from Lemma 2.2.

We have also $\psi_{\overline{k}}(x) = \psi^1_{k_1}(x_1)\cdot\ldots\cdot\psi^d_{k_d}(x_d) = \psi^1(2^{-k_1}x_1)\cdot\ldots\cdot\psi^d(2^{-k_d}x_d)$. Then we may estimate the absolute value on the right-hand side of (2.11) from above by

$$c\,2^{\overline{r}\cdot(\overline{k}+\overline{l})}\int_{\mathbb{R}^d}\left|\prod_{i=1}^d\left(\frac{\psi^i(2^{-k_i}\cdot)}{|2^{-k_i}\cdot|^{N_i}}h(2^{-k_i-K}\cdot)\right)^{\vee_1}(y_i)\right||(\tilde{\varphi}_{\overline{k}+\overline{l}}\hat{f})^\vee(x-y)|\mathrm{d}y \tag{2.12}$$

Next we apply the formula $(\lambda(\alpha^{-1}\cdot))^{\vee_1}(t) = \alpha(\lambda^{\vee_1})(\alpha t)$ to every term in the product in this integral and substitute $v_i = 2^{k_i}y_i$. Then we use a maximal function from (1.16) and obtain a formula

$$|(\tilde{\varphi}_{\overline{k}+\overline{l}}\hat{f})^\vee(x_1-2^{-k_1}v_1,\ldots,x_d-2^{-k_d}v_d)| \le (\tilde{\varphi}^*_{\overline{k}+\overline{l}}f)_{\overline{a}}(x)\left(1+|2^{l_1}v_1|^{a_1}\right)\cdot\ldots\cdot\left(1+|2^{l_d}v_d|^{a_d}\right), \tag{2.13}$$

where $a = (a_1,\ldots,a_d) \in \mathbb{R}^d$ is greater than $\frac{1}{\min(p,q)}$.

The indicated operations together with the formula (2.13) allows to estimate the right-hand side of (2.11) by

$$2^{\overline{l}\cdot(\overline{N}-\overline{r})}2^{\overline{r}\cdot(\overline{k}+\overline{l})}(\tilde{\varphi}^*_{\overline{k}+\overline{l}}f)_{\overline{a}}(x)\times$$
$$\times\int_{\mathbb{R}^d}\left|\prod_{i=1}^d\left(\frac{\psi^i(\cdot)}{|\cdot|^{N_i}}h(2^{-K}\cdot)\right)^{\vee_1}(v_i)\right|\left(1+|2^{l_1}v_1|^{a_1}\right)\cdot\ldots\cdot\left(1+|2^{l_d}v_d|^{a_d}\right)\mathrm{d}v.$$

We use (2.4) to estimate the last integral and finally we arrive at

$$\left|\sum_{-\infty<\overline{l}\le K}2^{\overline{k}\cdot\overline{r}}(\psi_{\overline{k}}\varphi_{\overline{k}+\overline{l}}\hat{f})^\vee(x)\right| \le c\,2^{K\cdot\overline{a}}\sum_{-\infty<\overline{l}\le K}2^{\overline{l}\cdot(\overline{N}-\overline{r})}2^{\overline{r}\cdot(\overline{k}+\overline{l})}(\tilde{\varphi}^*_{\overline{k}+\overline{l}}f)_{\overline{a}}(x). \tag{2.14}$$

We apply the l_q-quasi-norm with respect to $\overline{k}$ and the L_p-quasi-norm with respect to x. We obtain

$$\left\|\left(\sum_{\overline{k}\in\mathbb{N}^d}\left|\sum_{-\infty<\overline{l}\le K}2^{\overline{k}\cdot\overline{r}}(\psi_{\overline{k}}\varphi_{\overline{k}+\overline{l}}\hat{f})^\vee(\cdot)\right|^q\right)^{1/q}|L_p\right\| \le \tag{2.15}$$
$$\le c\,2^{K\cdot\overline{a}+K\cdot(\overline{N}-\overline{r})}\left\|\left(\sum_{\overline{m}\in\mathbb{N}_0^d}2^{q\overline{m}\cdot\overline{r}}(\tilde{\varphi}^*_{\overline{m}}f)^q_{\overline{a}}(\cdot)\right)^{1/q}|L_p\right\|.$$

There are still two problems left. We have to incorporate terms with (at least) one k_i equal to zero and we have to replace $\tilde{\varphi}^*_m f$ by $(\varphi_m\hat{f})^\vee$.

As for the second problem, we may use theorem 1.12 to replace $\tilde{\varphi}^*_m f$ by $(\tilde{\varphi}_m\hat{f})^\vee$. To use the multiplier theorem 1.13 we write $\sigma_0 = h, \sigma_j(t) = H(2^{-j}t), j \in \mathbb{N}, t \in \mathbb{R}$ and define

$$\varrho_{\overline{m}}(x) = |2^{-m_1}x_1|^{N_1}\ldots|2^{-m_d}x_d|^{N_d}\sigma_{m_1}(x_1)\ldots\sigma_{m_d}(x_d).$$

We have to be sure that these functions belongs to the space $S_2^{(s,\ldots,s)}H(\mathbb{R}^d)$, $s > \sigma_{pq}+1/2$ and that the norms $||\varrho_{\overline{m}}(2^{m_1}x_1,\ldots,2^{m_d}x_d)|S_2^{(s,\ldots,s)}H(\mathbb{R}^d)||$ are uniformly bounded for all $\overline{m} \in \mathbb{N}_0^d$. But this fact is trivial as all N_i are even non-negative numbers.

As for the first problem, one can modify the approach given above for $\overline{k} \ge 1$. One has to replace our definition of $\tilde{\varphi}$ by $\tilde{\varphi}_{(m_1,\ldots,m_d)}(z) = |2^{-m_1}z_1|^{N_1}\varphi_{(m_1,\ldots,m_d)}(z)$ (in the case when $k_2 = \cdots = k_d = 0, k_1 > 0$ and similarly in other cases).

Altogether, we obtain

$$\Big|\Big|\Big(\sum_{\overline{k}\in\mathbb{N}_0^d}\Big|\sum_{-\infty<\overline{l}\le K} 2^{\overline{k}\cdot\overline{r}}(\psi_{\overline{k}}\varphi_{\overline{k}+\overline{l}}\hat{f})^\vee(\cdot)\Big|^q\Big)^{1/q}|L_p\Big|\Big| \le c\,2^{K\cdot\overline{a}+K\cdot(\overline{N}-\overline{r})}||f|S^{\overline{r}}_{p,q}F||$$

with c independent of K and $\overline{r}$.

Step 2. $\overline{l}\ge K+1$. We proceed in a similar way as in the Step 1. Let $\varphi'_{\overline{m}}(x) = |2^{-m_1}x_1|^{r_1^0}\cdot\ldots\cdot|2^{-m_d}x_d|^{r_d^0}\varphi_{\overline{m}}(x), \overline{m}\in\mathbb{N}^d, x\in\mathbb{R}^d$. The numbers r_i^0 are to be specified later (in general, they have to be sufficiently small). Let us again fix a $\overline{k}\in\mathbb{N}_0^d$. In this step we don't have to distinguish between $k_i=0$ and $k_i\ge 1$. But we have to take care about the dependence of constants on K and $\overline{r}^0=(r_1^0,\ldots,r_d^0)\in\mathbb{R}^d$ as well.

So we have a counterpart of (2.11):

$$2^{\overline{k}\cdot\overline{r}}|(\psi_{\overline{k}}\varphi_{\overline{k}+\overline{l}}\hat{f})^\vee(x)| = 2^{\overline{l}\cdot(\overline{r}^0-\overline{r})}\Big|\Big(\frac{\psi_{\overline{k}}(z)}{|2^{-k_1}z_1|^{r_1^0}\cdot\ldots\cdot|2^{-k_d}z_d|^{r_d^0}}2^{\overline{r}\cdot(\overline{k}+\overline{l})}\varphi'_{\overline{k}+\overline{l}}(z)\hat{f}\Big)^\vee(x)\Big|. \tag{2.16}$$

An as in the Step 1 we use a cut-off function, in this case the function H from Lemma 2.2. Namely, the last expression doesn't change if we replace $\psi_{\overline{k}}(z)$ with $\psi_{\overline{k}}(z)H(2^{-k_1-l_1}z_1)\cdot\ldots\cdot H(2^{-k_d-l_d}z_d)$.

Instead of (2.12) we obtain that the left-hand side of (2.16) can be estimated from above by

$$c2^{\overline{l}\cdot(\overline{r}^0-\overline{r})}2^{\overline{r}\cdot(\overline{k}+\overline{l})}\int_{\mathbb{R}^d}\Big|\prod_{i=1}^d\Big(\frac{\psi^i_{k_i}(\cdot)}{|2^{-k_i}\cdot|^{r_i^0}}H(2^{-k_i-l_i}\cdot)\Big)^{\vee_1}(y_i)\Big||(\varphi'_{\overline{k}+\overline{l}}\hat{f})^\vee(x-y)|\mathrm{d}y. \tag{2.17}$$

The c is now independent of K and $\overline{r}^0$. Let us now suppose that $\overline{k}\ge 1$. As we are now working with expression vanishing identically around 0 and the conditions (2.5) and (2.6) are the same for ψ_0 and ψ^i, this only simplifies the notation. Then we may substitute $\psi^i_{k_i}(\cdot)=\psi^i(2^{-k_i}\cdot)$. We apply again the formula $(\lambda(\alpha^{-1}\cdot))^{\vee_1}(t)=\alpha(\lambda^{\vee_1})(\alpha t)$ to every term in the product, this time with the coefficient $\alpha=2^{k_i+l_i}$. Then we substitute $v_i=2^{k_i+l_i}y_i$ and use the maximal function (1.16) to get an analogy of (2.13).

When we proceed through this indicated calculation we obtain in the same way as in the Step 1, that the left-hand side of (2.16) can be estimated from above by

$$c\,2^{\overline{l}\cdot(\overline{r}^0-\overline{r})+\overline{r}\cdot(\overline{k}+\overline{l})}(\varphi'^*_{\overline{k}+\overline{l}}f)_{\overline{a}}(x)\prod_{i=1}^d\int_{\mathbb{R}}\Big|\Big(\frac{\psi^i(2^{l_i}\cdot)}{|2^{l_i}\cdot|^{r_i^0}}H(\cdot)\Big)^{\vee_1}(v_i)\Big|(1+|v_i|^{a_i})\mathrm{d}v_i. \tag{2.18}$$

When some $k_i=0$ just substitute ψ^i with ψ_0. We may trivially interchange $(1+|v_i|^{a_i})$ for $(1+|v_i|)^{a_i}$, but we have to show that each of this n integrals can be estimated from above by

$$c2^{-l_ir_i^0}\int_{\mathbb{R}}|[\psi^i(2^{l_i}\cdot)H(\cdot)]^{\vee_1}(v_i)|(1+|v_i|)^{a_i}\mathrm{d}v_i. \tag{2.19}$$

To see that, let $\chi_{l_i}(t)=\psi^i(2^{l_i}t)H(t)$ and introduce $\rho\in S(\mathbb{R})$ with

$$\rho(t)=1\quad\text{for}\quad\{t\in\mathbb{R}:1/4<|t|<4\}\qquad\text{and}\qquad\operatorname{supp}\rho\subset\{t\in\mathbb{R}:1/8<|t|<8\}.$$

Then we may rewrite and estimate the $i-$th integral in (2.18) from above by

$$2^{-l_i r_i^0}\int_{\mathbb{R}}\bigg|\bigg(\frac{\chi_{l_i}(t)\rho(t)}{|t|^{r_i^0}}\bigg)^{\vee_1}(v_i)\bigg|(1+|v_i|)^{a_i}\mathrm{d}v_i\le$$
$$c2^{-l_i r_i^0}\int_{\mathbb{R}}\int_{\mathbb{R}}\bigg|\chi_{l_i}^{\vee_1}(z_i)\bigg(\frac{\rho(t)}{|t|^{r_i^0}}\bigg)^{\vee_1}(v_i-z_i)\bigg|\mathrm{d}z_i(1+|v_i|)^{a_i}\mathrm{d}v_i$$

We use a Fubini theorem for non-negative functions, make a shift in the inner integral and apply the formula $(1+|v_i+z_i|)^{a_i}\le(1+|v_i|)^{a_i}(1+|z_i|)^{a_i}$. And using the fact, that $\rho(t)|t|^{-r_i^0}\in S(\mathbb{R})$, we obtain that the integrals in (2.18) can be estimated from above by (2.19), hence by a constant which depends on r_i^0 , see (2.5) and (2.6).

Altogether, we arrive at

$$\sum_{\overline{l}\ge K+1}|2^{\overline{k}\cdot\overline{r}}(\psi_{\overline{k}}\varphi_{\overline{k}+\overline{l}}\hat{f})^\vee(x)|\le c\sum_{\overline{l}\ge K+1}2^{\overline{l}\cdot(\overline{r}^0-\overline{r})+\overline{r}\cdot(\overline{k}+\overline{l})}(\varphi'^*_{\overline{k}+\overline{l}}f)_{\overline{a}}(x). \tag{2.20}$$

Assume that $\overline{r}^0<\overline{r}$ and apply the l_q-quasi-norm and after that the L_p-quasi-norm. We get

$$\bigg|\bigg|\bigg(\sum_{k\in\mathbb{N}_0^d}\bigg|\sum_{\overline{l}\ge K+1}2^{\overline{k}\cdot\overline{r}}(\psi_{\overline{k}}\varphi_{\overline{k}+\overline{l}}\hat{f})^\vee(x)\bigg|^q\bigg)^{1/q}|L_p\bigg|\bigg|\le c\,2^{K\cdot(\overline{r}^0-\overline{r})}\bigg|\bigg|\bigg(\sum_{\overline{m}\in\mathbb{N}^d}2^{q\overline{m}\cdot\overline{r}}(\varphi'^*_{\overline{m}}f)^q_{\overline{a}}(x)\bigg)^{1/q}|L_p\bigg|\bigg|$$

As the terms with $m_i=0$ are now not present on the right-hand side, we can use Theorem 1.15 and estimate the last expression by $c\,2^{K\cdot(\overline{r}^0-\overline{r})}||f|S^{\overline{r}}_{p,q}F||$. Let us stress that c is independent on K and depends on $\overline{r}^0<\overline{r}$ in the Step 2.

Step 3. In this step we shall discuss the remaining 2^d-2 terms of (2.10). We shall estimate only one of them, the others being very similar. Let us choose a term with $l_1\le K$ and $l_i\ge K+1$ for $i=2,\dots,d$. Let us define $\overline{s}=(s_1,\dots,s_d)\in\mathbb{R}^d$ as a combination of $\overline{r}^0$ and $\overline{N}$:

$$s_1=N_1,\quad s_i=r_i^0, i=2,\dots,d. \tag{2.21}$$

Suppose that $k_1\ge 1$. Let $\varphi^{\#}_{\overline{m}}(x)=|2^{-m_1}x_1|^{s_1}\cdot\ldots\cdot|2^{-m_d}x_d|^{s_d}\varphi_{\overline{m}}(x)$. Then we get in analogy with (2.11)

$$2^{\overline{k}\cdot\overline{r}}|(\psi_{\overline{k}}\varphi_{\overline{k}+\overline{l}}\hat{f})^\vee(x)|=2^{\overline{l}\cdot(\overline{s}-\overline{r})}\bigg|\bigg(\bigg(\prod_{i=1}^d\frac{\psi^i_{k_i}(z_i)}{|2^{-k_i}z_i|^{s_i}}\bigg)2^{\overline{r}\cdot(\overline{k}+\overline{l})}\varphi^{\#}_{\overline{k}+\overline{l}}(z)\hat{f}(z)\bigg)^\vee(x)\bigg|. \tag{2.22}$$

The last expression again doesn't change if we use the information about support of $\varphi^{\#}_{\overline{k}+\overline{l}}$ and introduce the cut-off functions h,H. Hence, we may replace $\psi^1_{k_1}(z_1)$ with $\psi^1_{k_1}(z_1)h(2^{-k_1-K}z_1)$ and $\psi^i_{k_i}(z_i)$ with $\psi^i_{k_i}(z_i)H(2^{-k_i-l_i}z_i)$ for $i=2,\dots,d$.

Using the elementary properties of Fourier transform, we obtain a counterpart of (2.12) and (2.17)

$$2^{\overline{k}\cdot\overline{r}}|(\psi_{\overline{k}}\varphi_{\overline{k}+\overline{l}}\hat{f})^\vee(x)|\le$$
$$c\,2^{\overline{l}\cdot(\overline{s}-\overline{r})}\int_{\mathbb{R}^d}\bigg|\bigg(\frac{\psi^1(2^{-k_1}z_1)}{|2^{-k_1}z_1|^{s_1}}h(2^{-k_1-K}z_1)\bigg)^{\vee_1}(y_1)\bigg|\cdot\bigg|\prod_{i=2}^d\bigg(\frac{\psi^i_{k_i}(z_i)}{|2^{-k_i}z_i|^{s_i}}H(2^{-k_i-l_i}z_i)\bigg)^{\vee_1}(y_i)\bigg|\cdot$$
$$\cdot|2^{\overline{r}\cdot(\overline{k}+\overline{l})}(\varphi^{\#}_{\overline{k}+\overline{l}}\hat{f})^\vee(x-y)|\mathrm{d}y$$

We can now deal with the first term as in the Step 1. and with the other terms as in the Step 2. Together with the analogy of (2.13)

$$|(\varphi^{\#}_{\overline{k}+\overline{l}}\hat{f})^{\vee}(x_1-2^{-k_1}v_1,x_2-2^{-k_2-l_2}v_2,\dots)|\le(\varphi^{\#*}_{\overline{k}+\overline{l}}f)_{\overline{a}}(x)\left(1+|2^{l_1}v_1|^{a_1}\right)\prod_{i=2}^{d}(1+|v_i|^{a_i})$$

we get

$$2^{\overline{k}\cdot\overline{r}}|(\psi_{\overline{k}}\varphi_{\overline{k}+\overline{l}}\hat{f})^{\vee}(x)|\le c\,2^{\overline{l}\cdot(\overline{s}-\overline{r})+\overline{r}\cdot(\overline{k}+\overline{l})}(\varphi^{\#*}_{\overline{k}+\overline{l}}f)_{\overline{a}}(x)\times$$
$$\times\int_{\mathbb{R}}\left(\frac{\psi^1(\cdot)}{|\cdot|^{s_1}}h(2^{-K}\cdot)\right)^{\vee_1}(v_1)(1+|2^{l_1}v_1|)^{a_1}\mathrm{d}v_1\prod_{i=2}^{d}\int_{\mathbb{R}}\left(\frac{\psi^i(\cdot)}{|\cdot|^{s_i}}H(\cdot)\right)^{\vee_1}(v_i)(1+|v_i|)^{a_i}\mathrm{d}v_i.$$

When $k_i=0$ for some $2\le i\le d$, just replace ψ^i with ψ^i_0.

We estimate these integral as in the Step 1, resp. Step 2, and get finally

$$2^{\overline{k}\cdot\overline{r}}|(\psi_{\overline{k}}\varphi_{\overline{k}+\overline{l}}\hat{f})^{\vee}(x)|\le c\,2^{Ka_1+\overline{l}\cdot(\overline{s}-\overline{r})+\overline{r}\cdot(\overline{k}+\overline{l})}(\varphi^{\#*}_{\overline{k}+\overline{l}}f)_{\overline{a}}(x),$$

Applying l_q-and L_p-quasi-norm, we get an analogy of (2.15). We now proceed in the same way as in the end of Step 1 to incorporate the term with $k_1=0$ and to replace $(\varphi^{\#*}_{\overline{m}}f)_{\overline{a}}$ with $(\varphi_{\overline{m}}\hat{f})^{\vee}$. Hence all the middle 2^d-2 terms in (2.10) can be estimated from above by $c\,2^{K\cdot\overline{a}}2^{K\cdot(\overline{s}-\overline{r})}||f|S^{\overline{r}}_{p,q}F||$, where $\overline{s}$ is defined by (2.21). Recall that this definition changes slightly from one term to another and that c is again independent of K and depends on $\overline{r}^0$.

Step 4. In the last step of the first part we shall discuss the convergence of (2.10). In the first three steps we have used that $(\psi_{\overline{k}}\hat{f})^{\vee}$ is a regular distribution and that the decomposition (2.10) converges pointwise to this function.

We know that this decomposition converges to $(\psi_{\overline{k}}\hat{f})^{\vee}$ in S'. If we prove that this decomposition forms a fundamental sequence in some $L_p, 1\le p<\infty$, then we obtain that $(\psi_{\overline{k}}\hat{f})^{\vee}\in L_p$ is a regular distribution. And when we prove that its also pointwise convergent, then its pointwise limit must coincide with the L_p-limit, hence with $(\psi_{\overline{k}}\hat{f})^{\vee}$ and the pointwise convergence of (2.10) to $(\psi_{\overline{k}}\hat{f})^{\vee}$ follows.

We shall restrict ourselves to the sum considered in the second step ($\overline{l}\ge K+1$). The other cases are completely the same. We start with the pointwise convergence. First recall that the estimate (2.20) was obtained for each considered $\overline{l}$ independently. So we may rewrite it as

$$\left|\sum_{L\le\overline{l}\le M}2^{\overline{k}\cdot\overline{r}}(\psi_{\overline{k}}\varphi_{\overline{k}+\overline{l}}\hat{f})^{\vee}(x)\right|\le c\sum_{L\le\overline{l}\le M}2^{\overline{l}\cdot(\overline{r}^0-\overline{r})+\overline{r}\cdot(\overline{k}+\overline{l})}(\varphi'^{*}_{\overline{k}+\overline{l}}f)_{\overline{a}}(x),\tag{2.23}$$

for every $K+1<L<M$. Using $\overline{r}^0<\overline{r}$ and $l_q\hookrightarrow l_1$ if $0<q<1$ or the Hölder inequality if $1<q\le\infty$, we conclude that the right-hand side of (2.23) may be further estimated by

$$\left(\sum_{L\le\overline{l}}2^{q\overline{l}\cdot\overline{r}}(\varphi'^{*}_{\overline{l}}f)^q_{\overline{a}}(x)\right)^{1/q}.$$

According to the Theorem 1.15 and the discussion in the second step, this expression is finite a.e. and arbitrary small for L large. Hence the decomposition (2.10) has some pointwise limit.

Let $0 < p < 1$ and put $\overline{\sigma} = \overline{r} - \sigma_p$. Then we have the embedding $S^{\overline{r}}_{p,q}F \hookrightarrow S^{\overline{\sigma}}_{1,1}F$, see [**SchT**]. We use (2.23) and get

$$|| \sum_{L\le\overline{l}\le M} 2^{\overline{k}\cdot\overline{r}}(\psi_{\overline{k}}\varphi_{\overline{k}+\overline{l}}\hat{f})^\vee(x)|L_1|| \le c\, 2^{\overline{\sigma}_p\cdot\overline{k}}|| \sum_{L\le\overline{l}\le M} 2^{\overline{l}\cdot(\overline{r}^0-\overline{\sigma})+\overline{\sigma}\cdot(\overline{k}+\overline{l})}(\varphi'^*_{\overline{k}+\overline{l}}f)_{\overline{a}}(x)|L_1||$$
$$\le c\, 2^{\overline{\sigma}_p\cdot\overline{k}}|| \sum_{L\le\overline{l}} 2^{\overline{l}\cdot\overline{\sigma}}(\varphi'^*_{\overline{l}}f)_{\overline{a}}(x)|L_1||.$$

We need $\overline{r}^0$ sufficiently small ($\overline{r}^0 < \overline{\sigma}$) for the last estimate. Because of the embedding mentioned above, the last expression is arbitrary small as L increases. Hence the decomposition (2.10) forms a fundamental sequence in L_1 and its limit must coincide with the S' limit : $(\psi_{\overline{k}}\hat{f})^\vee$.

When $1 \le p < \infty$, we get the L_p convergence in a similar way.

Part 2.

We prove that $||f|S^{\overline{r}}_{p,q}F||$ may be estimated from above by (2.3). Unfortunately, we have to use that $f \in S^{\overline{r}}_{p,q}F$. First of all let $\chi(t) = \sum_{j=0}^K \varphi_j(t), t \in \mathbb{R}$. Hence

$$\text{(2.24)} \qquad \operatorname{supp}\chi \subset \{t \in \mathbb{R} : |t| \le 2^{K+1}\} \quad \text{and} \quad \chi(t) = 1 \text{ if } |t| \le 2^K.$$

We write again $\chi_{\overline{m}}(x) = \chi(2^{-m_1}x_1)\cdot\ldots\cdot\chi(2^{-m_d}x_d)$ for $\overline{m} \in \mathbb{N}^d_0$ and $x \in \mathbb{R}^d$.

Using (2.1) and the support properties of $\varphi_{\overline{k}}$, we get for every $\overline{k} \in \mathbb{N}^d_0$

$$\text{(2.25)} \qquad |(\varphi_{\overline{k}}\hat{f})^\vee(x)| = |(\varphi_{\overline{k}}\chi_{\overline{k}}\hat{f})^\vee(x)| \le c \int_{\mathbb{R}^d}\left|\left(\frac{\varphi_{\overline{k}}}{\psi_{\overline{k}}}\right)^\vee(y)(\psi_{\overline{k}}\chi_{\overline{k}}\hat{f})^\vee(x-y)\right|\mathrm{d}y.$$

For fixed $x \in \mathbb{R}^d$, the Fourier transform of the y-function in the last integral has a support contained in a cube $cQ_{\overline{k}+K} = \{y \in \mathbb{R}^d : |y_i| \le c2^{k_i+K}, i = 1,\ldots,d\}$. Let $0 < s < \min(1,p,q)$. We use the Nikol'skij inequality and obtain

$$\text{(2.26)} \qquad |(\varphi_{\overline{k}}\hat{f})^\vee(x)|^s \le c\, 2^{(\overline{k}+K)\cdot(1-s)}\int_{\mathbb{R}^d}\left|\left(\frac{\varphi_{\overline{k}}}{\psi_{\overline{k}}}\right)^\vee(y)(\psi_{\overline{k}}\chi_{\overline{k}}\hat{f})^\vee(x-y)\right|^s\mathrm{d}y$$

Both functions $\varphi_{\overline{k}}, \psi_{\overline{k}}$ are assumed to have a product structure and the dyadic structure of $\varphi_j(t) = \varphi(2^{-j}t), j \ge 1$ can be assumed without loss of generality. Hence we may estimate

$$\text{(2.27)} \qquad \left|\left(\frac{\varphi_{\overline{k}}}{\psi_{\overline{k}}}\right)^\vee(y)\right|^s = 2^{\overline{k}\cdot s}\prod_{i=1}^d\left|\left(\frac{\varphi}{\psi^i}\right)(2^{k_i}y_i)\right|^s \le c\, 2^{\overline{k}\cdot s}\prod_{i=1}^d(1+|2^{k_i}y_i|)^{-b}, \quad \overline{k} \ge 1,$$

where $b > 0$ is at our disposal. A similar estimate holds when one or more of the numbers $k_i, i = 1,\ldots,d$ is equal to zero. We now split the integration over $\mathbb{R}^d$ in (2.26) into $\sum_{\overline{l}\in\mathbb{N}^d_0}\int_{I_{\overline{l}}}$, where $I_0 = [-1,1] \subset \mathbb{R}, I_\nu = \{t \in \mathbb{R} : 2^{\nu-1} \le |t| \le 2^\nu\}, \nu \ge 1$ and $I_{\overline{l}} = I_{l_1}\times\cdots\times I_{l_d}$. Then we use (2.27) in each of these integrals and then we replace the integration over $I_{\overline{l}}$ by integration over $Q_{\overline{l}}$, where $Q_\nu = [-2^\nu, 2^\nu] \subset \mathbb{R}$ and $Q_{\overline{l}} = Q_{l_1}\times\cdots\times Q_{l_d}$. We get

$$|(\varphi_{\overline{k}}\hat{f})^\vee(x)|^s \le c\, 2^{(\overline{k}+K)\cdot(1-s)+\overline{k}\cdot s}\sum_{\overline{l}\in\mathbb{N}^d_0}2^{-b\cdot(\overline{k}+\overline{l})}\int_{Q_{\overline{l}}}|(\psi_{\overline{k}}\chi_{\overline{k}}\hat{f})^\vee(x-y)|^s\mathrm{d}y.$$

These integrals may be estimated from above by

$$c\, 2^{l_1+\cdots+l_d}\overline{M}(|(\psi_{\overline{k}}\chi_{\overline{k}}\hat{f})^\vee|^s)(x).$$

If we use this estimate and suppose that $b > 1$ we obtain finally

$$\text{(2.28)} \qquad |(\varphi_{\overline{k}}\hat{f})^\vee(x)|^s \le c\, 2^{Kd(1-s)}\overline{M}(|(\psi_{\overline{k}}\chi_{\overline{k}}\hat{f})^\vee|^s)(x).$$

We multiply (2.28) by $2^{s\overline{k}\cdot\overline{r}}$, apply the $l_{q/s}$ and $L_{p/s}$ norm and use Theorem 1.11 to obtain

$$
\begin{aligned}
&\left|\left|\left(\sum_{\overline{k}\in\mathbb{N}_0^d}|2^{\overline{k}\cdot\overline{r}}(\varphi_{\overline{k}}\hat{f})^\vee(\cdot)|^q\right)^{1/q}|L_p\right|\right|^s \\
&\le c\,2^{Kd(1-s)}\left|\left|\left[\sum_{\overline{k}\in\mathbb{N}_0^d}(\overline{M}(|(2^{\overline{k}\cdot\overline{r}}\psi_{\overline{k}}\chi_{\overline{k}}\hat{f})^\vee|^s)(\cdot))^{q/s}\right]^{s/q}|L_{p/s}\right|\right| \qquad (2.29)\\
&\le c\,2^{Kd(1-s)}\left|\left|\left(\sum_{\overline{k}\in\mathbb{N}_0^d}|(2^{\overline{k}\cdot\overline{r}}\psi_{\overline{k}}\chi_{\overline{k}}\hat{f})^\vee(\cdot)|^q\right)^{1/q}|L_p\right|\right|^s.
\end{aligned}
$$

We have used that $1 < \frac{p}{s} < \infty$ and $1 < \frac{q}{s} \le \infty$. Next we insert the decomposition

$$\psi_{\overline{k}}\chi_{\overline{k}} = \psi_{\overline{k}} - \psi_{\overline{k}}(1-\chi_{\overline{k}})$$

into (2.29). We see that $||f|S^{\overline{r}}_{p,q}F||^s$ can be estimated from above by the s-th power of (2.3) and the additional term

$$c\,2^{Kd(1-s)}\left|\left|\left(\sum_{\overline{k}\in\mathbb{N}_0^d}|(2^{\overline{k}\cdot\overline{r}}\psi_{\overline{k}}(1-\chi_{\overline{k}})\hat{f})^\vee(\cdot)|^q\right)^{1/q}|L_p\right|\right|^s. \qquad (2.30)$$

We may assume that

$$\varphi_0(2^{-\mu}t) = \sum_{\nu=0}^{\mu}\varphi_{\nu(t)}, \quad \mu\in\mathbb{N}, t\in\mathbb{R}$$

and obtain

$$\chi_{k_i}(x_i) = \sum_{\nu=0}^{K}\varphi_\nu(2^{-k_i}x_i) = \sum_{\nu=-\infty}^{K}\varphi_{\nu+k_i}(x_i).$$

Hence the expression (2.30) is exactly that one which we estimated in the second and third Step in Part 1. From the calculation done there, it follows that, if $\overline{r}^0$ is sufficiently small, then there is an $\epsilon > 0$ such that (2.30) may be estimated from above by $C\,2^{-K\epsilon}||f|S^{\overline{r}}_{p,q}F||$. Let us remark that C is now independent of K (but depends of $\overline{r}^0$) and that the suitable choice of $\overline{r}^0$ can be obtained by $\overline{r}^0 < -d(1-s) - \sum_{i=1}^d(a_i+N_i-r_i)$.

Recall that the natural number K is still at our disposal. We choose K large enough, so that $C2^{-K\epsilon} < 1/2$. Then (2.29) and the splitting mentioned above give the result. $\square$

Remark 2.3. Let us stress that the Tauberian conditions (2.1) were necessary only in the second part of the proof.

Next we reformulate Theorem 2.1 using the local means.

Theorem 2.4. *Let $0 < p < \infty, 0 < q \le \infty, \overline{r}\in\mathbb{R}^d$. Let $\overline{N}\in\mathbb{N}_0^d$ be d even nonnegative integers with $\overline{N} > \overline{r}$. Further let $k_0, k^1, \dots, k^d$ be $d+1$ complex-valued functions from $S(\mathbb{R})$ whose supports lie in the set $\{t\in\mathbb{R} : |t| < 1\}$. Let us assume that*

$$F_1(k_0)(0)\neq 0, \qquad F_1(k^i)(0)\neq 0,\ i=1,\dots,d. \qquad (2.31)$$

Let us denote

$$k_0^i(t) = k_0(t) \qquad \textit{and} \qquad k_\nu^i(t) = 2^\nu\left(\frac{\mathrm{d}^{N_i}}{\mathrm{d}t^{N_i}}k^i\right)(2^\nu t),\ i=1,\dots,d,\ \nu\in\mathbb{N},\ t\in\mathbb{R}.$$

As usually, we denote by $k_{\overline{m}}(x) = k^1_{m_1}(x_1)\cdot\ldots\cdot k^d_{m_d}(x_d)$ the tensor product of these functions.

The corresponding local means are defined by

$$k_{\overline{m}}(f)(x)=\int_{\mathbb{R}^d} k_{\overline{m}}(y)f(x+y)\mathrm{d}y, \tag{2.32}$$

appropriately interpreted for any $f\in S'(\mathbb{R}^d)$. Then

$$\Big\|\Big(\sum_{\overline{m}\in\mathbb{N}_0^d} 2^{q\overline{m}\cdot\overline{r}}|k_{\overline{m}}(f)(\cdot)|^q\Big)^{1/q}|L_p\Big\| \tag{2.33}$$

is an equivalent quasi-norm in $S^{\overline{r}}_{p,q}F$.

Proof. Put $\psi_0=F_1^{-1}k_0$ and $\psi^i=F_1^{-1}(\frac{\mathrm{d}^{N_i}}{\mathrm{d}t^{N_i}}k^i)$. Then the Tauberian condition are satisfied (maybe after some dilation in the arguments of k_0,k^i) and (2.2) is also true. If we define $\psi_{\overline{m}},\overline{m}\in\mathbb{R}^d$ as in Theorem 2.1, we get

$$\begin{aligned}(\psi_{\overline{m}}\hat{f})^\vee(x)&=c\int_{\mathbb{R}^d}(\psi_{\overline{m}})^\vee(y)f(x-y)\mathrm{d}y=c\int_{\mathbb{R}^d}(F\psi_{\overline{m}})(y)f(x+y)\mathrm{d}y\\&=c\int_{\mathbb{R}^d}\Big(\prod_{i=1}^d(F_1\psi^i_{m_i})(y_i)\Big)f(x+y)\mathrm{d}y.\end{aligned} \tag{2.34}$$

Finally, if $m_i=0$ we get $(F_1\psi_0^i)(y_i)=(F_1\psi_0)(y_i)=k_0^i(y_i)$ and if $m_i\ge 1$ we obtain in a similar way

$$(F_1\psi^i_{m_i})(y_i)=(F_1(\psi^i(2^{-m_i}\cdot)))(y_i)=2^{m_i}(F_1\psi^i)(2^{m_i}y_i)=2^{m_i}\big(\frac{\mathrm{d}^{N_i}}{\mathrm{d}t^{N_i}}k^i\big)(2^{m_i}y_i)=k^i_{m_i}(y_i).$$

Using this calculation and (2.34) we get

$$(\psi_{\overline{m}}\hat{f})^\vee(x)=\int_{\mathbb{R}^d}k_{\overline{m}}(y)f(x+y)\mathrm{d}y$$

and the theorem follows. □

We shall need some other modifications of Theorem 2.1. The first of them is interesting on its own, the second will be useful later on.

For $\overline{m}\in\mathbb{N}_0^d,\overline{l}\in\mathbb{Z}^d$ we denote by $Q_{\overline{m}\,\overline{l}}$ the cube with the centre at the point $2^{-\overline{m}}\overline{l}=(2^{-m_1}l_1,\dots,2^{-m_d}l_d)$ with sides parallel to coordinate axes and of lengths $2^{-m_1},\dots,2^{-m_d}$. Hence

$$Q_{\overline{m}\,\overline{l}}=\{x\in\mathbb{R}^d:|x_i-2^{-m_i}l_i|\le 2^{-m_i-1},i=1,\dots,d\},\qquad \overline{m}\in\mathbb{N}_0^d,\ \overline{l}\in\mathbb{Z}^d. \tag{2.35}$$

If $\gamma>0$ then $\gamma Q_{\overline{m}\,\overline{l}}$ denotes a cube concentric with $Q_{\overline{m}\,\overline{l}}$ with sides also parallel to coordinate axes and of lengths $\gamma 2^{-m_1},\dots,\gamma 2^{-m_d}$.

Theorem 2.5. *Let $\overline{r}\in\mathbb{R}^d$, $0<p<\infty$, $0<q\le\infty$. Let $\overline{N}\in\mathbb{N}_0^d>\overline{r}$ and $k_{\overline{m}}$ be as in Theorem 2.4. Then for any $\gamma>0$*

$$\Big\|\Big(\sum_{\overline{m}\in\mathbb{N}_0^d}2^{q\overline{m}\cdot\overline{r}}\sup_{x-y\in\gamma Q_{\overline{m},0}}|k_{\overline{m}}(f)(y)|^q\Big)^{1/q}|L_p\Big\| \tag{2.36}$$

is an equivalent quasi-norm in $S^{\overline{r}}_{p,q}F$.

Proof. The proof is the combination of the approach described in the proof of Theorem 2.1 and the following Lemma.

LEMMA 2.6. *Let $\overline{m} \in \mathbb{N}_0^d$, $\overline{a}, \overline{\alpha} \in \mathbb{R}_+^d$ and $f \in S'(\mathbb{R}^d)$. Let $(\varphi^*_{\overline{m}} f)_{\overline{a}}$ be defined by (1.16) and let $x, y \in \mathbb{R}^d$ be two points with $|x_i - y_i| \le \alpha_i 2^{-m_i}$. Then*

$$(\varphi^*_{\overline{m}} f)_{\overline{a}}(y) \le c_{\overline{a}}\, \overline{\alpha}^{\overline{a}} (\varphi^*_{\overline{m}} f)_{\overline{a}}(x), \tag{2.37}$$

where the $c_{\overline{a}}$ depends only on $\overline{a}$.

The proof of this Lemma involves only the definition (1.16) and some trivial algebraic identities.

Now we proceed in the same way as in the proof of Theorem 2.1. For $x, y \in \mathbb{R}^d$ with $|x_i - y_i| \le \gamma 2^{-k_i} \le \gamma 2^{-k_i - l_i} 2^K$ we get in analogy to (2.14)

$$\begin{aligned} |2^{\overline{k}\cdot\overline{r}} (\psi_{\overline{k}} \varphi_{\overline{k}+\overline{l}} \hat{f})^\vee(y)| &\le c 2^{K\cdot\overline{a} + \overline{l}\cdot(\overline{N} - \overline{r}) + \overline{r}\cdot(\overline{k}+\overline{l})} (\tilde{\varphi}^*_{\overline{k}+\overline{l}})_{\overline{a}}(y) \\ &\le c 2^{2K\cdot\overline{a} + \overline{l}\cdot(\overline{N} - \overline{r}) + \overline{r}\cdot(\overline{k}+\overline{l})} (\tilde{\varphi}^*_{\overline{k}+\overline{l}})_{\overline{a}}(x), \end{aligned} \tag{2.38}$$

which gives us the analogy of (2.15) with the supremum involved on the left-hand side and additional constant $2^{K\cdot\overline{a}}$ on the right-hand side.

The modification necessary in the second step is very similar. Namely, we obtain for $x, y \in \mathbb{R}^d$ with $|x_i - y_i| \le \gamma 2^{-k_i} = \gamma 2^{-k_i - l_i} 2^{l_i}$ in analogy to (2.20)

$$|2^{\overline{k}\cdot\overline{r}} (\psi_{\overline{k}} \varphi_{\overline{k}+\overline{l}} \hat{f})^\vee(y)| \le c\, 2^{\overline{l}\cdot(\overline{r}^0 - \overline{r}) + \overline{r}\cdot(\overline{k}+\overline{l})} (\varphi'^*_{\overline{k}+\overline{l}} f)_{\overline{a}}(y) \le c\, 2^{\overline{l}\cdot(\overline{r}^0 - \overline{r} + \overline{a}) + \overline{r}\cdot(\overline{k}+\overline{l})} (\varphi'^*_{\overline{k}+\overline{l}} f)_{\overline{a}}(x). \tag{2.39}$$

So, we have to choose $\overline{r}^0 < \overline{r} - \overline{a}$. The modification of Step 3. is just the combination of (2.38) and (2.39). Hence the expression (2.36) may be estimated from above by $c||f|S^{\overline{r}}_{p,q}F||$. The reverse estimate is trivial, as (2.36) is smaller than (2.33). □

The second modification is rather technical and deals with 'directional' local means, namely with local means of the form ($d = 2$):

$$\int_{\mathbb{R}} k^1_{\nu_1}(y_1) f(x_1 + y_1, x_2) \mathrm{d}y_1.$$

To introduce these local means in the general dimension, we define for every $A \subset \{1, \dots, d\}$

$$k_{\overline{m},A}(f)(x) = \int_{\mathbb{R}^{|A|}} \Big(\prod_{i\in A} k^i_{m_i}(y_i)\Big) f(x_1 + y_1\chi_A(1), \dots, x_d + y_d\chi_A(d)) \Big(\prod_{i\in A} \mathrm{d}y_i\Big). \tag{2.40}$$

We simply restrict the integration in (2.32) to those variables y_i for which $i \in A$. The others are left untouched.

Using this notation, we may state our next Lemma.

LEMMA 2.7. *Let $0 < p < \infty, 0 < q \le \infty, A \subset \{1, \dots, d\}$ and $\gamma > 0$. Let $\overline{r} \in \mathbb{R}^d$ be such that $r_i > \frac{1}{\min(p,q)}$ for $i \notin A$. Let $N_i \in \mathbb{N}_0$ and $k^i_{\overline{m}}$ be as in Theorem 2.4 for every $i \in A$. Further let $k_{\overline{m},A}(f)$ be defined by (2.40). Then*

$$\Bigg|\Bigg|\Bigg(\sum_{\substack{\overline{m}\in\mathbb{N}_0^d \\ m_i = 0, i\notin A}} 2^{q\overline{m}\cdot\overline{r}} \sup_{x - y \in \gamma Q_{\overline{m},0}} |k_{\overline{m},A}(f)(y)|^q \Bigg)^{1/q} |L_p \Bigg|\Bigg| \le c||f|S^{\overline{r}}_{p,q}F|| \tag{2.41}$$

holds for every $f \in S^{\overline{r}}_{p,q}F$. The sum is taken over all $\overline{m} = (m_1, \dots, m_d) \in \mathbb{N}_0^d$ with $m_i = 0$ whenever $i \notin A$. The L_p-quasi-norm is then taken with respect to x.

PROOF. For every direction $i \notin A$, we use the decomposition (1.3). This gives the following equality for all $\overline{m}$ from the sum in (2.41)

$$k_{\overline{m},A}(f)(y) = \sum_{\substack{\overline{\nu}\in\mathbb{N}_0^d \\ \nu_i=m_i, i\in A}} \int_{\mathbb{R}^d} \Big(\prod_{i\in A} k^i_{\nu_i}(z_i)\Big)\Big(\prod_{i\notin A} F_1\varphi_{\nu_i}(z_i)\Big) f(y+z)\mathrm{d}z. \tag{2.42}$$

Denoting the product of all $k^i_{\nu_i}(z_i)$ and $\varphi_{\nu_i}(z_i)$ in the last integral by $\tilde{k}_{\overline{\nu}}(z)$ and using Hölder inequality (if $q>1$) we get for every $\epsilon>0$

$$|k_{\overline{m},A}(f)(y)|^q \le c \sum_{\substack{\overline{\nu}\in\mathbb{N}_0^d \\ \nu_i=m_i, i\in A}} 2^{q\epsilon\sum_{i\notin A}\nu_i} |\tilde{k}_{\overline{\nu}}(f)(y)|^q. \tag{2.43}$$

Using this estimate in the left-hand side of (2.41), we get

$$\sum_{\substack{\overline{m}\in\mathbb{N}_0^d \\ m_i=0, i\notin A}} 2^{q\overline{m}\cdot\overline{r}} \sup |k_{\overline{m},A}(f)(y)|^q \le c \sum_{\overline{m}\in\mathbb{N}_0^d} 2^{q\sum_{i\in A}(m_i r_i)+q\epsilon\sum_{i\notin A} m_i} \sup |\tilde{k}_{\overline{m}}(f)(y)|^q. \tag{2.44}$$

The supremum on both sides is taken with respect to the same set as in (2.41). Now we stand in the same position as in the beginning of the proof of Theorem 2.5 with only two little changes.

The first is that we may use the decomposition (2.10) only for those directions x_i for which $i\in A$. The local means $\tilde{k}_{\overline{m}}(f)(y)$ are already based on the functions φ_{m_i} in the remaining directions. The second is that the supremum is now taken over larger set.

To deal with the second problem, we just use Lemma 2.6 with $\alpha_i = 2^K$ for $i\in A$ and with $\alpha_i = 2^{m_i}$ for $i\notin A$ (in the Step 1) and with obvious modifications in Steps 2 and 3. This change will result into the factor $2^{q\sum_{i\notin A}(m_i a_i)}$. As $r_i > \frac{1}{\min(p,q)}$ for $i\notin A$ and $\epsilon>0$ may be chosen arbitrary small, we may assume that $r_i > a_i+\epsilon$. Hence

$$q\sum_{i\in A}(m_i r_i) + q\epsilon\sum_{i\notin A} m_i + q\sum_{i\notin A}(m_i a_i) \le q\overline{m}\cdot\overline{r} \tag{2.45}$$

and the Lemma follows.

We stress only that the convergence of (2.42) is covered by the Step 4. of the proof of Theorem 2.1. □

2. Atomic decomposition

In this section we shall describe an atomic decomposition for spaces $S^{\overline{r}}_{p,q}F$. We follow again the approach given in [**Tr1**]. First of all, we give the necessary definitions.

DEFINITION 2.8. Recall that for $\overline{\nu}\in\mathbb{N}_0^d, \overline{m}\in\mathbb{Z}^d$ we denote by $Q_{\overline{\nu}\,\overline{m}}$ the cube with centre at the point $2^{-\overline{\nu}}\overline{m} = (2^{-\nu_1}m_1,\dots,2^{-\nu_d}m_d)$ with sides parallel to coordinate axes and of lengths $2^{-\nu_1},\dots,2^{-\nu_d}$, see (2.35).

By $\chi^{(p)}_{\overline{\nu}\,\overline{m}}$ we denote a p-normalised characteristic function of $Q_{\overline{\nu}\,\overline{m}}$, it means that $\chi^{(p)}_{\overline{\nu}\,\overline{m}}(x) = 2^{\overline{\nu}\cdot\frac{1}{p}}\chi_{Q_{\overline{\nu}\,\overline{m}}}(x)$. Finally, if $0<p\le\infty$, $0<q\le\infty$ and

$$\lambda = \{\lambda_{\overline{\nu}\,\overline{m}}\in\mathbb{C} : \overline{\nu}\in\mathbb{N}_0^d, \overline{m}\in\mathbb{Z}^d\}$$

then we define

$$s_{pq}b = \left\{\lambda : ||\lambda|s_{pq}b|| = \left(\sum_{\overline{\nu}\in\mathbb{N}_0^d}\Big(\sum_{\overline{m}\in\mathbb{Z}^d}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\right)^{1/q} < \infty\right\} \tag{2.46}$$

and

$$s_{pq}f = \left\{ \lambda : ||\lambda|s_{pq}f|| = \left|\left| \left(\sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in\mathbb{Z}^d} |\lambda_{\overline{\nu}\,\overline{m}} \chi^{(p)}_{\overline{\nu}\,\overline{m}}(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^d) \right|\right| < \infty \right\} \tag{2.47}$$

with the usual modification for p and/or q equal to ∞.

In the next definition we introduce the normalised building blocks called *atoms*.

DEFINITION 2.9. Let $\overline{r} \in \mathbb{R}^d, 0 < p \le \infty, \overline{K} \in \mathbb{N}_0^d, \overline{L}+1 \in \mathbb{N}_0^d$, and $\gamma > 1$. A $\overline{K}$-times differentiable complex-valued function $a(x)$ is called $(\overline{r},p)_{\overline{K}\,\overline{L}}$-atom centred at $Q_{\overline{\nu}\,\overline{m}}$ if

$$\operatorname{supp} a \subset \gamma Q_{\overline{\nu}\,\overline{m}}, \tag{2.48}$$

$$|D^\alpha a(x)| \le 2^{-\overline{\nu}\cdot(\overline{r}-\frac{1}{p})+\alpha\cdot\overline{\nu}} \qquad \text{for} \quad 0 \le \alpha \le \overline{K} \tag{2.49}$$

and

$$\int_{\mathbb{R}} x_i^j a(x) \mathrm{d}x_i = 0 \qquad \text{if} \quad i = 1,\dots,d;\ j = 0,\dots,L_i \quad \text{and} \quad \nu_i \ge 1. \tag{2.50}$$

REMARK 2.10. The condition (2.50) is void if $\nu_i = 0$ or if $\nu_i \ge 1$ but $L_i = -1$.

THEOREM 2.11. *Let $0 < p < \infty$, $0 < q \le \infty$ and $\overline{r} \in \mathbb{R}^d$. Fix $\overline{K} \in \mathbb{N}_0^d$ and $\overline{L}+1 \in \mathbb{N}_0^d$ with*

$$K_i \ge (1 + [r_i])_+ \quad \text{and} \quad L_i \ge \max(-1, [\sigma_{pq} - r_i]), \qquad i = 1,\dots,d. \tag{2.51}$$

(i) *If $\lambda \in s_{pq}f$ and $\{a_{\overline{\nu}\,\overline{m}}(x)\}_{\overline{\nu}\in\mathbb{N}_0^d, \overline{m}\in\mathbb{Z}^d}$ are $(\overline{r},p)_{\overline{K},\overline{L}}$-atoms centred at $Q_{\overline{\nu}\,\overline{m}}$, then the sum*

$$\sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}} a_{\overline{\nu}\,\overline{m}}(x) \tag{2.52}$$

converges in $S'(\mathbb{R}^d)$, its limit f belongs to the space $S^{\overline{r}}_{p,q}F$ and

$$||f|S^{\overline{r}}_{p,q}F|| \le c\, ||\lambda|s_{pq}f||, \tag{2.53}$$

where the constant c is universal for all admissible λ and $a_{\overline{\nu}\,\overline{m}}$.

(ii) *For every $f \in S^{\overline{r}}_{p,q}F$ there is a $\lambda \in s_{pq}f$ and $(\overline{r},p)_{\overline{K},\overline{L}}$-atoms centred at $Q_{\overline{\nu}\,\overline{m}}$ (denoted again by $\{a_{\overline{\nu}\,\overline{m}}(x)\}_{\overline{\nu}\in\mathbb{N}_0^d, \overline{m}\in\mathbb{Z}^d}$) such that the sum (2.52) converges in $S'(\mathbb{R}^d)$ to f and*

$$||\lambda|s_{pq}f|| \le c\, ||f|S^{\overline{r}}_{p,q}F||. \tag{2.54}$$

The constant c is again universal for every $f \in S^{\overline{r}}_{p,q}F$.

PROOF. Step 1.

First of all we prove the convergence of (2.52) in $S'(\mathbb{R}^d)$. Let $\varphi \in S(\mathbb{R}^d)$. We use the Taylor expansion of φ with respect to the first variable

$$\begin{aligned} \varphi(y) = \sum_{\alpha_1 \le L_1} & \frac{D^{(\alpha_1,0,\dots,0)}\varphi(2^{-\nu_1}m_1, y_2, \dots, y_d)}{\alpha_1!}(y_1 - 2^{\nu_1}m_1)^{\alpha_1} \\ & + \frac{1}{L_1!}\int_{2^{-\nu_1}m_1}^{y_1} (y_1 - 2^{-\nu_1}m_1)^{L_1} D^{(L_1+1,0,\dots,0)}\varphi(t_1, y_2, \dots, y_d)\mathrm{d}t_1 \end{aligned} \tag{2.55}$$

and (2.50) to obtain

(2.56)

$$\int_{\mathbb{R}^d} a_{\overline{\nu}\,\overline{m}}(y)\varphi(y)\mathrm{d}y = \int_{\mathbb{R}^d} \frac{a_{\overline{\nu}\,\overline{m}}(y)}{L_1!} \int_{2^{-\nu_1}m_1}^{y_1} (y_1 - 2^{-\nu_1}m_1)^{L_1} D^{(L_1+1,0,\dots,0)}\varphi(t_1, y_2, \dots, y_d)\mathrm{d}t_1\mathrm{d}y.$$

Using an analogy of (2.55) iteratively for the remaining $d-1$ variables we see that the left hand side (2.56) is equal to

$$\int_{\mathbb{R}^d} \frac{a_{\overline{\nu}\,\overline{m}}(y)}{\overline{L}!} \int_{2^{-\nu_1}m_1}^{y_1} \dots \int_{2^{-\nu_d}m_d}^{y_d} \prod_{i=1}^d (y_i - 2^{-\nu_i}m_i)^{L_i} D^{\overline{L}+1}\varphi(t_1,\dots,t_d)\mathrm{d}t\mathrm{d}y. \tag{2.57}$$

Using the support property (2.48) of $a_{\overline{\nu}\,\overline{m}}$ we may estimate the absolute value of the inner $d-$dimensional integration from above by

$$c2^{-\overline{\nu}\cdot(\overline{L}+1)} \sup_{x\in\gamma Q_{\overline{\nu}\,\overline{m}}} |(D^{\overline{L}+1}\varphi)(x)| \le c2^{-\overline{\nu}\cdot(\overline{L}+1)}\langle y\rangle^{-M} \sup_{x\in\gamma Q_{\overline{\nu}\,\overline{m}}} \langle x\rangle^M |(D^{\overline{L}+1}\varphi)(x)|, \tag{2.58}$$

where M is at our disposal. Let us now suppose that $p\ge 1$ and use (2.49) and Hölder inequality to get for M large enough

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} \sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}} a_{\overline{\nu}\,\overline{m}}(y)\varphi(y)\mathrm{d}y\right| \\
&\le c\,2^{-\overline{\nu}\cdot(\overline{r}+\overline{L}+1)} \sup_{x\in\mathbb{R}^d}\langle x\rangle^M |(D^{\overline{L}+1}\varphi)(x)| \int_{\mathbb{R}^d}\left(\sum_{\overline{m}\in\mathbb{Z}^d} 2^{\overline{\nu}\cdot\frac{1}{p}}|\lambda_{\overline{\nu}\,\overline{m}}|\chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(y)\right)\langle y\rangle^{-M}\mathrm{d}y \\
&\le c\,2^{-\overline{\nu}\cdot(\overline{r}+\overline{L}+1)} \left(\sum_{\overline{m}\in\mathbb{Z}^d}|\lambda_{\overline{\nu}\,\overline{m}}|^p\right)^{1/p} \sup_{x\in\mathbb{R}^d}\langle x\rangle^M |(D^{\overline{L}+1}\varphi)(x)|.
\end{aligned}$$

As $\lambda\in s_{p,q}f\subset s_{p,\infty}b$ and $\overline{r}+\overline{L}+1>0$, the convergence of (2.52) now follows.
If $p<1$ then we get a similar estimate

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} \sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}} a_{\overline{\nu}\,\overline{m}}(y)\varphi(y)\mathrm{d}y\right| \\
&\le c\,2^{-\overline{\nu}\cdot(\overline{r}+\overline{L}+1-1/p+1)} \sup_{x\in\mathbb{R}^d} |(D^{\overline{L}+1}\varphi)(x)| \int_{\mathbb{R}^d} \sum_{\overline{m}\in\mathbb{Z}^d} 2^{\overline{\nu}\cdot 1}|\lambda_{\overline{\nu}\,\overline{m}}|\chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(y)\mathrm{d}y \\
&\le c\,2^{-\overline{\nu}\cdot(\overline{r}+\overline{L}+1-1/p+1)} \sum_{\overline{m}\in\mathbb{Z}^d}|\lambda_{\overline{\nu}\,\overline{m}}| \sup_{x\in\mathbb{R}^d} |(D^{\overline{L}+1}\varphi)(x)|.
\end{aligned}$$

In this case we use the fact that $\overline{r}+\overline{L}+1-1/p+1>0$ and the embedding $s_{pq}f\subset s_{1,\infty}b$.
Step 2.
Next we prove (2.53). We use the equivalent quasi-norms in $S^{\overline{r}}_{p,q}F$ given by (2.33). Let us choose $\overline{N}>\overline{K}$ and define the functions $k_{\overline{l}}$ for $\overline{l}\in\mathbb{N}_0^d$ as in Theorem 2.4. Then we have for all $\overline{l},\overline{\nu}\in\mathbb{N}_0^d$ and all $\overline{m}\in\mathbb{Z}^d$

$$2^{\overline{l}\cdot\overline{r}}k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x) = 2^{\overline{l}\cdot\overline{r}}\int_{\mathbb{R}^d} k^1_{l_1}(y_1)\cdot\ldots\cdot k^d_{l_d}(y_d)a_{\overline{\nu}\,\overline{m}}(x+y)\mathrm{d}y. \tag{2.59}$$

Further calculation depends on the size of the supports of $k_{\overline{l}}$ and $a_{\overline{\nu}\,\overline{m}}$. Hence we have to distinguish between $l_i\ge\nu_i$ and $l_i<\nu_i$. This leads again to 2^d cases. We describe the first one ($\overline{l}\ge\overline{\nu}$) and the last one ($\overline{l}<\overline{\nu}$) in the full detail and then we discuss the 'mixed' cases.
I. $\quad\overline{l}\ge\overline{\nu}$.
We suppose that $\overline{l}>0$. This only simplifies the notation, the terms with $l_i=\nu_i=0$ may be incorporated afterwards. We use the definition of $k^i_{l_i}$ and make partial integration (K_i-times

in the i^{th} variable) to obtain

$$
\begin{aligned}
2^{\overline{l}\cdot\overline{r}}k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x) &= 2^{\overline{l}\cdot(\overline{r}+1)}\int_{\mathbb{R}^d}\prod_{i=1}^{d}\left(\frac{\mathrm{d}^{N_i}}{\mathrm{d}t^{N_i}}k^i\right)(2^{l_i}y_i)a_{\overline{\nu}\,\overline{m}}(x+y)\mathrm{d}y\\
&= 2^{\overline{l}\cdot\overline{r}}\int_{\mathbb{R}^d}\prod_{i=1}^{d}\left(\frac{\mathrm{d}^{N_i}}{\mathrm{d}t^{N_i}}k^i\right)(y_i)a_{\overline{\nu}\,\overline{m}}(x_1+2^{-l_1}y_1,\ldots,x_d+2^{-l_d}y_d)\mathrm{d}y\\
&= 2^{\overline{l}\cdot(\overline{r}-\overline{K})}\int_{\mathbb{R}^d}\prod_{i=1}^{d}\left(\frac{\mathrm{d}^{N_i-K_i}}{\mathrm{d}t^{N_i-K_i}}k^i\right)(y_i)(D^{\overline{K}}a_{\overline{\nu}\,\overline{m}})(x_1+2^{-l_1}y_1,\ldots,x_d+2^{-l_d}y_d)\mathrm{d}y.
\end{aligned}
$$

Next we use smoothness of k^i, the boundedness of their supports and the properties (2.48) and (2.49) to estimate the absolute value of this expression.

$$
\begin{aligned}
2^{\overline{l}\cdot\overline{r}}|k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x)| \le c\,&2^{\overline{l}\cdot(\overline{r}-\overline{K})}2^{-\overline{\nu}\cdot(\overline{r}-\frac{1}{p})+\overline{\nu}\cdot\overline{K}}\cdot\\
&\cdot\int_{\mathbb{R}^d}\left(\prod_{i=1}^{d}\chi_{\operatorname{supp}k^i}(y_i)\right)\chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x_1+2^{-l_1}y_1,\ldots,x_d+2^{-l_d}y_d)\mathrm{d}y.
\end{aligned}
$$

As $\operatorname{supp}k^i\subset\{t\in\mathbb{R}:|t|\le 1\}, i=1,\ldots,d$, it follows that

$$
2^{\overline{l}\cdot\overline{r}}|k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x)| \le c\,2^{-(\overline{K}-\overline{r})(\overline{l}-\overline{\nu})}2^{\overline{\nu}\cdot\frac{1}{p}}\chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x). \tag{2.60}
$$

II. $\overline{l}<\overline{\nu}$.

The integration in (2.59) may be restricted to $\{y:|y_i|\le 2^{-l_i}\}$. We use the Taylor expansion of functions $k^i_{l_i}(y_i)$ with respect to the off-points $2^{-\nu_i}m_i-x_i$ up to order L_i

$$
2^{-l_i}k^i_{l_i}(y_i)=\sum_{0\le\beta_i\le L_i}c^i_{\beta_i}(x_i)(y_i-2^{-\nu_i}m_i+x_i)^{\beta_i}+2^{l_i(L_i+1)}O(|x_i+y_i-2^{-\nu_i}m_i|^{L_i+1}) \tag{2.61}
$$

and (2.50) to get

$$
2^{\overline{l}\cdot\overline{r}}k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x)=2^{\overline{l}\cdot(\overline{r}+1)}\int_{\{y:|y_i|\le 2^{-l_i}\}}a_{\overline{\nu}\,\overline{m}}(x+y)\prod_{i=1}^{d}2^{l_i(L_i+1)}O(|x_i+y_i-2^{-\nu_i}m_i|^{L_i+1})\mathrm{d}y.
$$

Since

$$
|a_{\overline{\nu}\,\overline{m}}(x+y)|\le 2^{-\overline{\nu}\cdot(\overline{r}-\frac{1}{p})}\chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x+y)
$$

we obtain

$$
2^{\overline{l}\cdot\overline{r}}|k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x)|\le c\,2^{\overline{l}\cdot(\overline{r}+1)}2^{-\overline{\nu}\cdot(\overline{r}-\frac{1}{p})}2^{(\overline{l}-\overline{\nu})\cdot(\overline{L}+1)}\int_{\{y:|y_i|\le 2^{-l_i}\}}\chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x+y)\mathrm{d}y. \tag{2.62}
$$

The last integral is always smaller then $2^{-\overline{\nu}\cdot 1}$ and is zero if $\{y:x+y\in\gamma Q_{\overline{\nu}\,\overline{m}}\}\cap\{y:|y_i|\le 2^{-l_i}\}=\emptyset$. Hence

$$
\int_{\{y:|y_i|\le 2^{-l_i}\}}\chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x+y)\mathrm{d}y\le c\,2^{-\overline{\nu}\cdot 1}\chi_{c2^{\overline{\nu}-\overline{l}}Q_{\overline{\nu}\,\overline{m}}}(x). \tag{2.63}
$$

But the last expression may be estimated from above with the use of maximal operators M_i defined by (1.14).

$$
2^{(\overline{l}-\overline{\nu})\cdot 1}\chi_{c2^{\overline{\nu}-\overline{l}}Q_{\overline{\nu}\,\overline{m}}}(x)\le c\,(\overline{M}\chi_{\overline{\nu}\,\overline{m}})(x). \tag{2.64}
$$

Let $0<\omega<\min(1,p,q)$. Taking the $1/\omega$-power of (2.64) and inserting it in (2.63) we obtain

$$\int_{\{y:|y_i|\le 2^{-l_i}\}} \chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x+y)\mathrm{d}y \le c\,2^{-\overline{\nu}\cdot 1}2^{(\overline{\nu}-\overline{l})\cdot\frac{1}{\omega}}(\overline{M}\chi_{\overline{\nu}\,\overline{m}})^{\frac{1}{\omega}}(x). \tag{2.65}$$

Next we replace $\chi_{\overline{\nu}\,\overline{m}}$ by $\chi^{(p)}_{\overline{\nu}\,\overline{m}}$ in (2.65) and insert it in (2.62).

$$2^{\overline{l}\cdot\overline{r}}|k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x)| \le c\,2^{(\overline{l}-\overline{\nu})\cdot(\overline{r}+1+\overline{L}+1-\frac{1}{\omega})}(\overline{M}\chi^{(p)\omega}_{\overline{\nu}\,\overline{m}})^{\frac{1}{\omega}}(x).$$

By (2.51) and (1.17) one may choose the number ω such that $\overline{\varkappa}=(\overline{r}+1+\overline{L}+1-\frac{1}{\omega})>0$.

III. Mixed terms.

We estimate for example the term with $l_1\ge\nu_1$, $l_i<\nu_i, i=2,\dots,d$.

First we apply (2.61) for $i=2,\dots,d$ and use (2.50) to leave out the terms with $\beta\le\overline{L}$. Then we use K_1 partial integration in the first variable. In the expression we get we use again the support properties of the functions involved and (2.49) to obtain

$$\begin{aligned}2^{\overline{l}\cdot\overline{r}}|k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x)| \le{}& 2^{\overline{\nu}\cdot\frac{1}{p}}2^{(l_1-\nu_1)(r_1-K_1)}2^{\sum_{i=2}^d l_i(r_i+1)+(l_i-\nu_i)(L_i+1)-\nu_i r_i}\\ &\int_{A_{\overline{l}}}\chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x_1+2^{-l_1}y_1,x_2+y_2,\dots,x_d+y_d)\mathrm{d}y,\end{aligned}$$

where $A_{\overline{l}}=\{y\in\mathbb{R}^d:|y_1|\le 1,|y_i|\le 2^{-l_i}, i=2,\dots,d\}$. Due to the product structure of the integrated function we may split the last integral into a one-dimensional integral with respect to $\mathrm{d}y_1$ and $d-1$ dimensional integral with respect to the remaining variables. The first integral then may be estimated from above by $c\chi_{\{t:|t-2^{-\nu_1}m_1|\le 2^{-\nu_i}\}}(x_1)$. Finally we use the maximal operators $M_i, i=2,\dots,d$ to estimate the second integral. And, exactly as in the second step, it turns out, that there is some vector $\overline{\varrho}>0$ such that

$$2^{\overline{l}\cdot\overline{r}}|k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x)| \le c\,2^{-\sum_{i=1}^d|l_i-\nu_i|\varrho_i}(\overline{M}\chi^{(p)\omega}_{\overline{\nu}\,\overline{m}})^{\frac{1}{\omega}}(x). \tag{2.66}$$

Let us observe that also (2.60) may be estimated from above by the right-hand side of(2.66). Hence the estimate (2.66) is valid for all $\overline{l},\overline{\nu}\in\mathbb{N}_0^d$.

Using this estimate, we get for $q\le 1$

$$\left|2^{\overline{l}\cdot\overline{r}}k_{\overline{l}}\Big(\sum_{\overline{\nu},\overline{m}}\lambda_{\overline{\nu}\,\overline{m}}a_{\overline{\nu}\,\overline{m}}\Big)(x)\right|^q \le c\sum_{\overline{\nu},\overline{m}}|\lambda_{\overline{\nu}\,\overline{m}}|^q 2^{-q\sum_{i=1}^d|l_i-\nu_i|\varrho_i}(\overline{M}\chi^{(p)\omega}_{\overline{\nu}\,\overline{m}})^{\frac{q}{\omega}}(x).$$

We sum over $\overline{l}$, take the $\frac{1}{q}-$power and then we apply the L_p-quasi-norm with respect to x. Denoting $g_{\overline{\nu}\,\overline{m}}=\lambda_{\overline{\nu}\,\overline{m}}\chi^{(p)}_{\overline{\nu}\,\overline{m}}$ we arrive at

$$\begin{aligned}&\left\|\Bigg(\sum_{\overline{l}\in\mathbb{N}_0^d}\left|2^{\overline{l}\cdot\overline{r}}k_{\overline{l}}\Big(\sum_{\overline{\nu},\overline{m}}\lambda_{\overline{\nu}\,\overline{m}}a_{\overline{\nu}\,\overline{m}}\Big)(x)\right|^q\Bigg)^{\frac{1}{q}}|L_p(\mathbb{R}^d)\right\|\\ &\qquad\le c\left\|\Bigg(\sum_{\overline{\nu},\overline{m}}|\lambda_{\overline{\nu}\,\overline{m}}|^q(\overline{M}\chi^{(p)\omega}_{\overline{\nu}\,\overline{m}})^{\frac{q}{\omega}}(x)\Bigg)^{\frac{1}{q}}|L_p(\mathbb{R}^d)\right\|\\ &\qquad= c\left\|\Bigg(\sum_{\overline{\nu},\overline{m}}(\overline{M}g^{\omega}_{\overline{\nu}\,\overline{m}})^{\frac{q}{\omega}}(x)\Bigg)^{\frac{\omega}{q}}|L_{\frac{p}{\omega}}(\mathbb{R}^d)\right\|^{\frac{1}{\omega}}.\end{aligned}$$

Using Theorem 1.11 and the definition of ω, we see that this expression may be estimated from above by $||\lambda|s_{pq}f||$. On the other hand, from the improved version of Theorem 2.1, namely from Theorem 3.1, we see that this already ensures that f belongs to $S^{\overline{r}}_{pq}F$ and proves (2.53).

Step 3.

It remains to prove (ii). Let us assume first that

$$\overline{L} = -1, \quad \overline{K} > \overline{r}, \quad \overline{r} > \sigma_{pq}, \quad 0 < p < \infty, \quad 0 < q \leq \infty. \tag{2.67}$$

Furthermore, let $\overline{N} \in \mathbb{N}_0^d$ be vector of even numbers with $\overline{N} > \overline{r}$. According to the construction given at [**Tr2**, page 68], we may find functions $k_0, k^1, \dots, k^d$ such that

(2.68) $k_0, k^1, \dots, k^d \in S(\mathbb{R})$;

(2.69) $\operatorname{supp} k_0, \operatorname{supp} k^i \subset \{t \in \mathbb{R} : |t| \leq 1\}, i = 1, \dots, d$;

(2.70) $1 = F_1(k_0)(\xi) + \sum_{\nu_i=1}^{\infty} F_1(\mathrm{d}^{N_i} k^i)(2^{-\nu_i}\xi), \quad \xi \in \mathbb{R}, i = 1, \dots, d$;

(2.71) $F_1 k_0(0) = 1$;

(2.72) $F_1(\mathrm{d}^{N_i} k^i)(\xi) = (F_1 k_0)(\xi) - (F_1 k_0)(2\xi), \quad \xi \in \mathbb{R}, i = 1, \dots, d.$

Further we may assume that no dilations mentioned in the beginning of the proof of Theorem 2.4 are necessary.

We define $k_{\overline{l}}(x)$ and $k_{\overline{l}}(f)(x)$ as in Theorem 2.4.

We claim that then

$$f = \sum_{\overline{l} \in \mathbb{N}_0^d} k_{\overline{l}}(f)(x) = \lim_{P \to \infty} \sum_{\overline{l} \leq P} k_{\overline{l}}(f), \quad \text{convergence in } S'(\mathbb{R}^d). \tag{2.73}$$

To prove this, fix $\varphi \in S(\mathbb{R}^d)$. Since the Fourier transform is isomorphic mapping from $S'(\mathbb{R}^d)$ onto itself and

$$(k_{\overline{l}}(f))^\wedge(\xi) = \Big(\prod_{i=1}^d F_1(k^i_{l_i})(-\xi_i)\Big)\hat{f}(\xi).$$

it is enough to show that

$$\varphi(\xi) \sum_{\overline{l} \leq P} \Big(\prod_{i=1}^d F_1(k^i_{l_i})(-\xi_i)\Big) \to \varphi(\xi) \quad \text{in } S(\mathbb{R}^d). \tag{2.74}$$

The last sum may be rewritten using (2.72) as

$$\sum_{\overline{l} \leq P} \Big(\prod_{i=1}^d F_1(k^i_{l_i})(-\xi_i)\Big) = \prod_{i=1}^d \Big((F_1 k_0)(-\xi_i) + \sum_{l_i=1}^{P} (F_1(\mathrm{d}^{N_i} k^i))(-2^{-\nu_i}\xi_i)\Big) = \prod_{i=1}^d (F_1 k_0)(-2^{-P}\xi_i).$$

We denote the last expression by $1 - \Phi(2^{-P}\xi)$ and fix $M \in \mathbb{N}$. Using the fact that $\varphi \in S(\mathbb{R}^d)$ we obtain

$$\begin{aligned} p_M(\varphi(\xi)\Phi(2^{-P}\xi)) &\leq c \sup_{\substack{0 \leq \overline{\alpha}, \overline{\beta} \leq M \\ \xi \in \mathbb{R}^d}} 2^{-P \cdot \overline{\beta}} (D^{\overline{\alpha}}\varphi)(\xi)(D^{\overline{\beta}}\Phi)(2^{-P}\xi) \prod_{i=1}^d \langle \xi_i \rangle^M \\ &\leq c \sup_{\substack{0 \leq \overline{\beta} \leq M \\ \xi \in \mathbb{R}^d}} 2^{-P \cdot \overline{\beta}} (D^{\overline{\beta}}\Phi)(2^{-P}\xi) \prod_{i=1}^d \langle \xi_i \rangle^{-1} \end{aligned}$$

where the constant c doesn't depend on P (but depends on M).

If at least one of $\beta_i > 0$, then this expression tends to zero if $P \to \infty$. If $\overline{\beta} = 0$, then we split the supremum into $\sup_{|\xi| \geq 2^P}$ and $\sup_{|\xi| < 2^P}$. The first supremum may be estimated from above by $c2^{-P}$. To estimate the second one, we notice that $|\Phi(\xi)| \le c|\xi|$ in $\{\xi : |\xi| \le 1\}$. Hence

$$c \sup_{|\xi|\le 2^P} \Phi(2^{-P}\xi) \prod_{i=1}^{d} \langle \xi_i \rangle^{-1} \le c \sup_{\xi\in\mathbb{R}^d} \frac{2^{-P}|\xi|}{\langle \xi \rangle}.$$

Hence $\varphi(\xi)\Phi(2^{-P}\xi) \to 0$ for $P \to \infty$. This proves (2.74) and, consequently, also (2.73).

Next we suppose that

$$\psi \in S(\mathbb{R}^d), \quad \operatorname{supp}\psi \text{ is compact and } \sum_{\overline{m}\in\mathbb{Z}^d} \psi(x-\overline{m}) = 1 \text{ for } x \in \mathbb{R}^d. \tag{2.75}$$

Then we define for $\overline{\nu} \in \mathbb{N}_0^d$ and $\overline{m} \in \mathbb{Z}^d$ the function $\psi_{\overline{\nu}\,\overline{m}}(x) = \psi(2^{\overline{\nu}}x - \overline{m})$. Then there is a γ such that

$$\operatorname{supp}\psi_{\overline{\nu}\,\overline{m}} \subset \gamma Q_{\overline{\nu}\,\overline{m}}, \quad \overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d. \tag{2.76}$$

Then we multiply (2.73) by these decompositions of unity and obtain

$$f = \sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in\mathbb{Z}^d} \psi_{\overline{\nu}\,\overline{m}}(x) k_{\overline{\nu}}(f)(x) = \sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}} a_{\overline{\nu}\,\overline{m}}(x), \tag{2.77}$$

where

$$\lambda_{\overline{\nu}\,\overline{m}} = 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})} \sum_{\alpha\le\overline{K}} \sup_{y\in\gamma Q_{\overline{\nu}\,\overline{m}}} |D^\alpha[k_{\overline{\nu}}(f)](y)|$$

and

$$a_{\overline{\nu}\,\overline{m}}(x) = \lambda_{\overline{\nu}\,\overline{m}}^{-1} \psi_{\overline{\nu}\,\overline{m}}(x) k_{\overline{\nu}}(f)(x).$$

(If some $\lambda_{\overline{\nu}\,\overline{m}} = 0$, then we take $a_{\overline{\nu}\,\overline{m}}(x) = 0$ as well). It follows that $a_{\overline{\nu}\,\overline{m}}$ are $(\overline{r},p)_{\overline{K},\overline{L}}$–atoms centred at $Q_{\overline{\nu}\,\overline{m}}$. The properties (2.48) and (2.50) are satisfied trivially (recall that $\overline{L} = -1$), and the property (2.49) is fulfilled up to some constant c independent of $\overline{\nu}, \overline{m}$ and x. To prove that this decomposition satisfies (2.54), write

$$||\lambda|s_{pq}f|| \le c \sum_{0\le\alpha\le\overline{K}} \left|\left| \left(\sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in\mathbb{Z}^d} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q} 2^{\overline{\nu}\cdot\frac{q}{p}} \sup_{x-y\in\gamma Q_{\overline{\nu}\,\overline{m}}} |D^\alpha[k_{\overline{\nu}}(f)(y)]|^q \right)^{1/q} |L_p \right|\right|$$

and use Theorem 2.5 with $D^{\alpha_i}k_0$ and $D^{\alpha_i}k_i$ in the place of k_0 and k^i. We lose the Tauberian condition for these new functions but according to Remark 2.3, this fact is rather harmless.

Step 4.

Now we prove the existence of the optimal decomposition for all $\overline{r} \in \mathbb{R}^d$ and $\overline{L}$ restricted by (2.51). To simplify the notation, we restrict ourselves in this step to $d = 2$. So, let us take $f \in S^{\overline{r}}_{p,q}F(\mathbb{R}^2)$. In Definition 1.8 we may substitute $(1+x^2)^{\overline{\rho}}$ by $(1+x_1^{2\rho_1})(1+x_2^{2\rho_2})$ for $\overline{\rho} \in \mathbb{N}_0^2$ and (using twice Theorem 1.13) we obtain the respective counterpart of Theorem 1.9. Hence f can be decomposed as

$$f = g + \frac{\partial^{2M_1} g}{\partial x_1^{2M_1}} + \frac{\partial^{2M_2} g}{\partial x_2^{2M_2}} + \frac{\partial^{2M_1+2M_2} g}{\partial x_1^{2M_1} x_2^{2M_2}}, \tag{2.78}$$

where $\overline{M} = (M_1, M_2) \in \mathbb{N}_0^2$ is at our disposal and may be chosen arbitrary large, $g \in S^{\overline{r}+2\overline{M}}_{p,q}F(\mathbb{R}^2)$ and $||g|S^{\overline{r}+2\overline{M}}_{p,q}F(\mathbb{R}^2)|| \approx ||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^2)||$.

The optimal decomposition of f will be obtained as a sum of decompositions of these four terms.

To decompose the first term, choose $\overline{M}$ such that

$$||g|S^{\overline{K}}\mathcal{C}(\mathbb{R}^2)|| \le c\,||g|S^{\overline{r}+2\overline{M}}_{p,q}F(\mathbb{R}^2)||.$$

This is possible according to [**SchT**, Theorem 2.4.1.]. Then we decompose

$$g(x) = \sum_{\overline{m}} \psi(x-\overline{m})g(x) = \sum_{\overline{m}} \lambda^1_{0\,\overline{m}} a^1_{0\,\overline{m}},$$

where

$$\lambda^1_{0\,\overline{m}} = c_1 \sum_{0\le\alpha\le\overline{K}} \sup_{|y-\overline{m}|\le c_2} |(D^\alpha g)(y)|$$

and

$$a^1_{0\,\overline{m}} = \frac{1}{\lambda^1_{0\,\overline{m}}}\psi(x-\overline{m})g(x)$$

for c_1, c_2 sufficiently large and for ψ with (2.75) and (2.76). Then $a^1_{0\,\overline{m}}$ are $(\overline{r},p)_{\overline{K},\overline{L}}$-atoms centred at $Q_{0\,\overline{m}}$. Furthermore, according to Lemma 2.7, we have

$$\begin{aligned}||\lambda^1|s_{pq}f|| = \left(\sum_{\overline{m}\in\mathbb{Z}^d} |\lambda^1_{0\,\overline{m}}|^p\right)^{1/p} &\le c_1 \sum_{\alpha\le\overline{K}} \left|\left| \sup_{\cdot-y\in\gamma Q_{0\,0}} |(D^\alpha g)(y)|\;|L_p\right|\right| \\ &\le c||g|S^{\overline{r}+2\overline{M}}_{p,q}F(\mathbb{R}^2)|| \le c||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^2)||.\end{aligned}$$

We have used Lemma 2.7 with $d=2$ and $A=\emptyset$.

As for the last term in the decomposition (2.78), we may assume that $\overline{M}$ is large enough to apply Step 3. So we may assume that we have a decomposition (2.77) for g with, let's say, $\lambda^4_{\overline{\nu}\,\overline{m}}$ and $a^4_{\overline{\nu}\,\overline{m}}(x)$ instead of $\lambda_{\overline{\nu}\,\overline{m}}$ and $a_{\overline{\nu}\,\overline{m}}(x)$ and $||\lambda^4_{\overline{\nu}\,\overline{m}}|s_{p,q}f|| \le c\,||g|S^{\overline{r}+2\overline{M}}_{p,q}F(\mathbb{R}^2)||$. As $a^4_{\overline{\nu}\,\overline{m}}(x)$ are $(\overline{r}+2\overline{M},p)_{\overline{K}+2\overline{M},-1}$-atoms, the functions $D^{2(M_1,M_2)}a^4_{\overline{\nu}\,\overline{m}}(x)$ are $(\overline{r},p)_{\overline{K},2\overline{M}-1}$-atoms.

In the case of the second term we use the decomposition

$$g(x) = \sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2\\ \nu_2=0}} \sum_{\overline{m}\in\mathbb{Z}^d} \psi_{\overline{\nu}\,\overline{m}}(x)k_{\overline{\nu},A}(g)(x) = \sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2\\ \nu_2=0}} \sum_{\overline{m}\in\mathbb{Z}^d} \lambda^2_{\overline{\nu}\,\overline{m}} a^2_{\overline{\nu}\,\overline{m}}(x), \tag{2.79}$$

where $A=\{1\}$, $k_{\overline{\nu},A}(g)(x)$ are defined by (2.40),

$$\lambda^2_{\overline{\nu}\,\overline{m}} = c_1 2^{2\nu_1 M_1} 2^{\nu_1(r_1-\frac{1}{p})} \sum_{\beta\le\overline{K}+(2M_1,0)} \sup_{y\in c_2 Q_{\overline{\nu}\,\overline{m}}} |D^\beta(k_{\overline{\nu},A}(g))(y)| \tag{2.80}$$

and

$$a^2_{\overline{\nu}\,\overline{m}}(x) = \frac{1}{\lambda^2_{\overline{\nu}\,\overline{m}}}\psi_{\overline{\nu}\,\overline{m}}(x)k_{\overline{\nu},A}(g)(x). \tag{2.81}$$

If c_1 and c_2 are large enough, then $D^{(2M_1,0)}a^2_{\overline{\nu}\,\overline{m}}(x)$ are $(\overline{r},p)_{\overline{K},\overline{L}}$-atoms for $L_1 \le 2M_1-1$. Finally, we use Lemma 2.7 to estimate $||\lambda^2|s_{pq}f||$

$$\begin{aligned}||\lambda^2|s_{pq}f|| &\le c_1 \sum_{\beta\le\overline{K}+(2M_1,0)} \left|\left| \left(\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2\\ \nu_2=0}} 2^{q\nu_1(2M_1+r_1)} \sup_{\cdot-y\in c_2 Q_{\overline{\nu}0}} |D^\beta(k_{\overline{\nu},A}(g))(y)|^q \right)^{1/q} |L_p \right|\right| \\ &\le c\,||g|S^{\overline{r}+2\overline{M}}_{p,q}F|| \le c\,||f|S^{\overline{r}}_{p,q}F||\end{aligned} \tag{2.82}$$

if $\overline{M}$ is chosen sufficiently large. We have used Lemma 2.7 with $D^{\beta_1}k_1$ and $D^{\beta_2}g$ instead of k_1 and f. The third term can be estimated in a similar way. The sum of these four decompositions then gives the decomposition for f.

In general dimension d one has to use the full generality of Lemma 2.7 but the proof is the same. □

3. Subatomic decomposition

In this section we describe the subatomic decomposition for spaces $S^{\overline{r}}_{p,q}F$. We follow closely [**Tr3**] and [**T03**].

First of all, we shall introduce some special building blocks called quarks.

DEFINITION 2.12. Let $\psi \in S(\mathbb{R})$ be a non-negative function with

$$\operatorname{supp}\psi \subset \{t \in \mathbb{R} : |t| < 2^{\phi}\} \tag{2.83}$$

for some $\phi \geq 0$ and

$$\sum_{n\in\mathbb{Z}} \psi(t-n) = 1, \quad t \in \mathbb{R}. \tag{2.84}$$

We define $\Psi(x) = \psi_1(x_1) \cdot \ldots \cdot \psi_d(x_d)$ and $\Psi^{\beta}(x) = x^{\beta}\Psi(x)$ for $x = (x_1, \ldots, x_d)$ and $\beta \in \mathbb{N}_0^d$. Further let $\overline{r} \in \mathbb{R}^d$ and $0 < p \leq \infty$. Then

$$(\beta qu)_{\overline{\nu}\,\overline{m}}(x) = 2^{-\overline{\nu}\cdot(\overline{r}-\frac{1}{p})}\Psi^{\beta}(2^{\overline{\nu}}x - \overline{m}), \quad \overline{\nu} \in \mathbb{N}_0^d,\ \overline{m} \in \mathbb{Z}^d \tag{2.85}$$

is called an $(\overline{r}, p)$-β-quark related to $Q_{\overline{\nu}\,\overline{m}}$.

Recall that the spaces $s_{pq}f$ were defined by (2.47).

THEOREM 2.13. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $\overline{r} \in \mathbb{R}^d$ be such that*

$$\overline{r} > \sigma_{pq}.$$

(i) *Let*

$$\lambda = \{\lambda^{\beta} : \beta \in \mathbb{N}_0^d\} \quad \textit{with} \quad \lambda^{\beta} = \{\lambda^{\beta}_{\overline{\nu}\,\overline{m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d\}$$

and let $\varrho > \phi$, where ϕ is the number from (2.83). *If*

$$\sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^{\beta}|s_{pq}f|| < \infty$$

then the series

$$\sum_{\beta\in\mathbb{N}_0^d}\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} \lambda^{\beta}_{\overline{\nu}\,\overline{m}}(\beta qu)_{\overline{\nu}\,\overline{m}}(x) \tag{2.86}$$

converges in $S'(\mathbb{R}^d)$, its limit f belongs to $S^{\overline{r}}_{p,q}F$ and

$$||f|S^{\overline{r}}_{p,q}F|| \leq c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^{\beta}|s_{pq}f||. \tag{2.87}$$

$(\beta qu)_{\overline{\nu}\,\overline{m}}$ has the same meaning as in (2.85).

(ii) *Every $f \in S^{\overline{r}}_{p,q}F$ can be represented by* (2.86) *with convergence in $S'(\mathbb{R}^d)$ and*

$$\sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^{\beta}|s_{pq}f|| \leq c||f|S^{\overline{r}}_{p,q}F||. \tag{2.88}$$

PROOF. Step 1.

First of all, we shall discuss convergence of (2.86). It turns out that this series converges not only in $S'(\mathbb{R}^d)$ but also in some $L_u(\mathbb{R}^d), u \geq 1$.

Let $1 \leq p < \infty$. Then $\overline{r} > 0$ and we get

$$|f(x)| \leq c \sum_{\beta\in\mathbb{N}_0^d}\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} 2^{\phi|\beta|}|\lambda^{\beta}_{\overline{\nu}\,\overline{m}}|2^{-\overline{\nu}\cdot(\overline{r}-\frac{1}{p})}\tilde{\chi}_{\overline{\nu}\,\overline{m}}(x), \tag{2.89}$$

where $\tilde{\chi}_{\overline{\nu}\,\overline{m}}$ is a characteristic function of $2^{\phi+1}Q_{\overline{\nu}\,\overline{m}}$. Using two times the Hölder inequality we get for every $\epsilon > 0$

$$|f(x)| \le c \sup_{\beta\in\mathbb{N}_0^d} 2^{(\phi+\epsilon)|\beta|} \sup_{\overline{\nu}\in\mathbb{N}_0^d} 2^{-\overline{\nu}\cdot(\overline{r}-\frac{1}{p}-\epsilon)} \sup_{\overline{m}\in\mathbb{Z}^d} |\lambda^\beta_{\overline{\nu}\,\overline{m}}|\tilde{\chi}_{\overline{\nu}\,\overline{m}}(x).$$

Taking the p-power and replacing the suprema with sums we get

$$|f(x)|^p \le c \sum_{\beta\in\mathbb{N}_0^d}\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} 2^{(\phi+\epsilon)|\beta|p} 2^{-\overline{\nu}\cdot(\overline{r}-\frac{1}{p}-\epsilon)p} |\lambda^\beta_{\overline{\nu}\,\overline{m}}|^p \tilde{\chi}_{\overline{\nu}\,\overline{m}}(x). \tag{2.90}$$

Let us denote $\tilde{q} = \max(p,q)$ and choose ϵ such that $0 < 2\epsilon < \varrho - \phi$ and $\epsilon < \overline{r}$. Integration of (2.90) and the Hölder inequality result in

$$\begin{aligned} ||f|L_p|| &\le c \sup_{\beta\in\mathbb{N}_0^d} 2^{(\phi+2\epsilon)|\beta|} \left(\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} 2^{-\overline{\nu}\cdot(\overline{r}-\epsilon)p} |\lambda^\beta_{\overline{\nu}\,\overline{m}}|^p \right)^{1/p} \\ &\le c \sup_{\beta\in\mathbb{N}_0^d} 2^{(\phi+2\epsilon)|\beta|} \left(\sum_{\overline{\nu}\in\mathbb{N}_0^d} \left(\sum_{\overline{m}\in\mathbb{Z}^d} |\lambda^\beta_{\overline{\nu}\,\overline{m}}|^p \right)^{\tilde{q}/p} \right)^{1/\tilde{q}} \\ &\le c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|} ||\lambda^\beta|s_{p,\max(p,q)}b|| \le c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|} ||\lambda^\beta|s_{p,q}f||. \end{aligned} \tag{2.91}$$

Therefore, for $1 \le p < \infty$, (2.86) converges in $L_p(\mathbb{R}^d)$.
Let $0 < p < 1$. Then $\overline{r} > \frac{1}{p} - 1$ and we get again (2.89). Integrating this estimate and using Hölder inequality, we get for every $\epsilon > 0$

$$\begin{aligned} ||f|L_1|| &\le c \sum_{\beta\in\mathbb{N}_0^d}\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} 2^{\phi|\beta|} 2^{-\overline{\nu}\cdot(\overline{r}-\frac{1}{p}+1)} |\lambda^\beta_{\overline{\nu}\,\overline{m}}| \\ &\le c \sup_{\beta\in\mathbb{N}_0^d} 2^{(\phi+\epsilon)|\beta|} \sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} 2^{-\overline{\nu}\cdot(\overline{r}-\frac{1}{p}+1)} |\lambda^\beta_{\overline{\nu}\,\overline{m}}|. \end{aligned}$$

By similar arguments as in (2.91) we get

$$||f|L_1|| \le c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|} ||\lambda^\beta|s_{p,q}f||$$

and (2.86) converges in $L_1(\mathbb{R}^d)$.
Step 2.
We now prove that the function f defined as a limit of (2.86) belongs to $S^{\overline{r}}_{p,q}F$ and the estimate (2.87).
We decompose (2.86) into

$$f = \sum_{\beta\in\mathbb{N}_0^d} f^\beta \tag{2.92}$$

with

$$f^\beta = \sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} \lambda^\beta_{\overline{\nu}\,\overline{m}} (\beta q u)_{\overline{\nu}\,\overline{m}}(x). \tag{2.93}$$

We show that $(\beta q u)_{\overline{\nu}\,\overline{m}}$ are (up to some normalising constants) $(\overline{r},p)_{\overline{K},-1}$-atoms centred at $Q_{\overline{\nu}\,\overline{m}}$ for every $K \in \mathbb{N}_0$. The conditions (2.48) and (2.50) are satisfied trivially. To prove

(2.49) we chose $0 \le \alpha \le \overline{K}$ and estimate

$$D^\alpha(\beta qu)_{\overline{\nu}\,\overline{m}}(x) = 2^{-\overline{\nu}\cdot(\overline{r}-\frac{1}{p})}\prod_{i=1}^d 2^{\nu_i\alpha_i}D^{\alpha_i}(\psi^{\beta_i})(2^{\nu_i}x_i - m_i)$$

where $\psi^{\beta_i}(t) = t^{\beta_i}\psi(t)$. But for $0 \le \alpha_i \le K_i$ and any $t \in \operatorname{supp}\psi$ we get by Leibnitz rule

$$|D^{\alpha_i}(\psi^{\beta_i})(t)| \le c_{K_i}\sup_{\gamma_1\le K_i}\sup_{\gamma_2\le K_i}|D^{\gamma_1}t^{\beta_i}|\cdot|(D^{\gamma_2}\psi)(t)| \le c_{K_i,\psi}\sup_{\gamma_1\le K_i}|D^{\gamma_1}t^{\beta_i}|.$$

The last absolute value may be estimated from above by $(1+\beta_i)^{K_i}2^{\phi\beta_i}$. Hence we obtain

$$|D^{\alpha_i}(\psi^{\beta_i})(t)| \le c_{K_i,\psi}(1+\beta_i)^{K_i}2^{\phi\beta_i}$$

and

$$|D^\alpha(\beta qu)_{\overline{\nu}\,\overline{m}}(x)| \le c_1\,2^{-\overline{\nu}\cdot(\overline{r}-\frac{1}{p})+\alpha\cdot\overline{\nu}}(1+\beta)^{\overline{K}}2^{\phi|\beta|} \le c_2\,2^{-\overline{\nu}\cdot(\overline{r}-\frac{1}{p})+\alpha\cdot\overline{\nu}}2^{(\phi+\epsilon)|\beta|}$$

for every $\epsilon > 0$. The constant c_2 is independent of β but may depend on $\overline{K}$, ψ and ϵ.
It follows that the functions $c_2^{-1}2^{-(\phi+\epsilon)|\beta|}(\beta qu)_{\overline{\nu}\,\overline{m}}(x)$ are $(\overline{r},p)_{K,-1}$-atoms and (2.93) may be understood as atomic decomposition of f^β. By Theorem 2.11 it follows that

$$||f^\beta|S^{\overline{r}}_{p,q}F|| \le c2^{(\phi+\epsilon)|\beta|}||\lambda^\beta|s_{pq}f||$$

and for $\eta = \min(1,p,q)$ get by triangle inequality for $S^{\overline{r}}_{pq}F$-quasi-norms

$$||f|S^{\overline{r}}_{p,q}F||^\eta \le \sum_{\beta\in\mathbb{N}_0^d}||f^\beta|S^{\overline{r}}_{p,q}F||^\eta \le c\sum_{\beta\in\mathbb{N}_0^d}2^{(\phi+\epsilon)\eta|\beta|}||\lambda^\beta|s_{pq}f||^\eta \le c\sup_{\beta\in\mathbb{N}_0^d}2^{(\phi+2\epsilon)\eta|\beta|}||\lambda^\beta|s_{pq}f||^\eta.$$

If we choose $\epsilon > 0$ so small that $\phi + 2\epsilon < \varrho$ we obtain (2.87). This finishes the proof of part (i).
Step 3.
By Remark 1.5 we have

$$\hat{f}(\xi) = \sum_{\overline{\nu}\in\mathbb{N}_0^d}\varphi_{\overline{\nu}}(\xi)\hat{f}(\xi)$$

with convergence in $S'(\mathbb{R}^d)$. Let $Q_{\overline{\nu}}$ be a cube in $\mathbb{R}^d$ centred at the origin with side lengths $2\pi2^{\nu_1},\ldots,2\pi2^{\nu_d}$. Hence $\operatorname{supp}\varphi_{\overline{\nu}} \subset Q_{\overline{\nu}}$ and we may interpret $\varphi_{\overline{\nu}}\hat{f}$ as a periodic distribution. Using its expansion into a Fourier series we get

$$(\varphi_{\overline{\nu}}\hat{f})(\xi) = \sum_{\overline{m}\in\mathbb{Z}^d}b_{\overline{\nu}\,\overline{m}}e^{-i(2^{-\overline{\nu}}\overline{m})\cdot\xi}, \quad \xi \in Q_{\overline{\nu}} \tag{2.94}$$

with

$$b_{\overline{\nu}\,\overline{m}} = c2^{-|\overline{\nu}|}\int_{Q_{\overline{\nu}}}e^{i(2^{-\overline{\nu}}\overline{m})\cdot\xi}(\varphi_{\overline{\nu}}\hat{f})(\xi)\mathrm{d}\xi = c'2^{-|\overline{\nu}|}(\varphi_{\overline{\nu}}\hat{f})^\vee(2^{-\overline{\nu}}\overline{m}).$$

Here we used again the notation $2^{-\overline{\nu}}\overline{m} = (2^{-\nu_1}m_1,\ldots,2^{-\nu_d}m_d)$ for $\overline{\nu}\in\mathbb{N}_0^d$ and $\overline{m}\in\mathbb{Z}^d$.
Let now $\omega \in S(\mathbb{R}^d)$ with $\operatorname{supp}\omega \subset Q_0$ and $\omega(\xi) = 1$ if $|\xi_i| \le 2$ for all $i = 1,\ldots,d$. Then the functions $\omega_{\overline{\nu}}(\xi) = \omega(2^{-\overline{\nu}}\xi)$ satisfy

$$\operatorname{supp}\omega_{\overline{\nu}} \subset Q_{\overline{\nu}}, \quad \omega_{\overline{\nu}}(\xi) = 1 \quad \text{if} \quad \xi \in \operatorname{supp}\varphi_{\overline{\nu}}$$

for all $\overline{\nu}\in\mathbb{N}_0^d$. We multiply (2.94) with $\omega_{\overline{\nu}}$, extend it by zero outside $Q_{\overline{\nu}}$ and take the inverse Fourier transform

$$(\varphi_{\overline{\nu}}\hat{f})^\vee(x) = \sum_{\overline{m}\in\mathbb{Z}^d}b_{\overline{\nu}\,\overline{m}}\omega^\vee_{\overline{\nu}}(x - 2^{-\overline{\nu}}\overline{m}) = \sum_{\overline{m}\in\mathbb{Z}^d}2^{|\overline{\nu}|}b_{\overline{\nu}\,\overline{m}}\omega^\vee(2^{\overline{\nu}}x - \overline{m}), \quad x \in \mathbb{R}^d.$$

Using (2.84) and the definition of Ψ, we get

$$(\varphi_{\overline{\nu}}\hat{f})^\vee(x) = \sum_{\overline{m}\in\mathbb{Z}^d} 2^{|\overline{\nu}|} b_{\overline{\nu}\,\overline{m}} \sum_{\overline{l}\in\mathbb{Z}^d} \Psi(2^{\overline{\nu}}x - \overline{l})\omega^\vee(2^{\overline{\nu}}x - \overline{m}).$$

Expanding the entire analytic function $\omega^\vee(2^{\overline{\nu}}\cdot - \overline{m})$ with respect to the off-point $2^{-\overline{\nu}}\overline{l}$ we arrive at

$$\begin{aligned}(\varphi_{\overline{\nu}}\hat{f})^\vee(x) &= \sum_{\overline{m}\in\mathbb{Z}^d} 2^{|\overline{\nu}|} b_{\overline{\nu}\,\overline{m}} \sum_{\overline{l}\in\mathbb{Z}^d} \Psi(2^{\overline{\nu}}x - \overline{l}) \sum_{\beta\in\mathbb{N}_0^d} 2^{\overline{\nu}\cdot\beta}\frac{(D^\beta\omega^\vee)(\overline{l}-\overline{m})}{\beta!}(x - 2^{-\overline{\nu}}\overline{l})^\beta \\ &= \sum_{\overline{m}\in\mathbb{Z}^d} 2^{|\overline{\nu}|} b_{\overline{\nu}\,\overline{m}} \sum_{\overline{l}\in\mathbb{Z}^d}\sum_{\beta\in\mathbb{N}_0^d} \Psi^\beta(2^{\overline{\nu}}x - \overline{l})\frac{(D^\beta\omega^\vee)(\overline{l}-\overline{m})}{\beta!}.\end{aligned}$$

Hence

$$f = \sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\beta\in\mathbb{N}_0^d}\sum_{\overline{l}\in\mathbb{Z}^d} \lambda^\beta_{\overline{\nu}\,\overline{l}} 2^{-\overline{\nu}\cdot(\overline{r}-\frac{1}{p})}\Psi^\beta(2^{\overline{\nu}}x - \overline{l}) = \sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\beta\in\mathbb{N}_0^d}\sum_{\overline{l}\in\mathbb{Z}^d} \lambda^\beta_{\overline{\nu}\,\overline{l}}(\beta qu)_{\overline{\nu}\,\overline{l}}(x),$$

where

$$\lambda^\beta_{\overline{\nu}\,\overline{l}} = 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p}+1)}\sum_{\overline{m}\in\mathbb{Z}^d} b_{\overline{\nu}\,\overline{m}}\frac{(D^\beta\omega^\vee)(\overline{l}-\overline{m})}{\beta!} = c\,2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})}\sum_{\overline{m}\in\mathbb{Z}^d}(\varphi_{\overline{\nu}}\hat{f})^\vee(2^{-\overline{\nu}}\overline{m})\frac{(D^\beta\omega^\vee)(\overline{l}-\overline{m})}{\beta!}.$$

It remains to prove (2.88). For this reason we define

$$\Lambda_{\overline{\nu}\,\overline{m}} = 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})}(\varphi_{\overline{\nu}}\hat{f})^\vee(2^{-\overline{\nu}}\overline{m})$$

and prove that

$$\sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^\beta|s_{pq}f|| \le c||\Lambda|s_{pq}f|| \le c'||f|S^{\overline{r}}_{pq}F||. \tag{2.95}$$

We start with the second inequality in (2.95).

Let $x\in Q_{\overline{\nu}\,\overline{m}}$ be fixed. Then

$$|(\varphi_{\overline{\nu}}\hat{f})^\vee(2^{-\overline{\nu}}\overline{m})| \le \sup_{x-y\in Q_{\overline{\nu},0}}|(\varphi_{\overline{\nu}}\hat{f})^\vee(y)| \le c(\varphi^*_{\overline{\nu}}f)_{\overline{a}}(x) \tag{2.96}$$

for every $\overline{a}\in\mathbb{R}^d_+$. We multiply (2.96) by $2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})}$, take the q-power and sum over $\overline{m}\in\mathbb{Z}^d$ to get

$$\sum_{\overline{m}\in\mathbb{Z}^d}|\Lambda_{\overline{\nu}\,\overline{m}}|^q|\chi^{(p)}_{\overline{\nu}\,\overline{m}}(x)|^q \le c\,2^{\overline{\nu}\cdot\frac{q}{p}}2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}(\varphi^*_{\overline{\nu}}f)^q_{\overline{a}}(x) = c\,2^{\overline{\nu}\cdot\overline{r}q}(\varphi^*_{\overline{\nu}}f)^q_{\overline{a}}(x),\quad x\in\mathbb{R}^d, \overline{\nu}\in\mathbb{N}_0^d.$$

Taking $\overline{a} > \frac{n}{\min(p,q)}$, we get finally with help of Theorem 1.15

$$\begin{aligned}||\Lambda|s_{pq}f|| &= \left|\left|\left(\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d}|\Lambda_{\overline{\nu}\,\overline{m}}\chi^{(p)}_{\overline{\nu}\,\overline{m}}(x)|^q\right)^{1/q}|L_p(\mathbb{R}^d)\right|\right| \\ &\le c\left|\left|\left(\sum_{\overline{\nu}\in\mathbb{N}_0^d}2^{\overline{\nu}\cdot\overline{r}q}(\varphi^*_{\overline{\nu}}f)^q_{\overline{a}}(x)\right)^{1/q}|L_p(\mathbb{R}^d)\right|\right| \\ &\le c||f|S^{\overline{r}}_{p,q}F||.\end{aligned}$$

To prove the first inequality in (2.95), we mention that

$$\lambda^\beta_{\overline{\nu}\,\overline{l}} = \frac{1}{\beta!}\sum_{\overline{m}\in\mathbb{Z}^d}\Lambda_{\overline{\nu}\,\overline{m}}(D^\beta\omega^\vee)(\overline{l}-\overline{m}) \tag{2.97}$$

and recall a result proven in [**T02**], namely that for any given $a > 0$ there are constants $c_a > 0$ and $C > 0$ such that

$$|D^\beta \omega^\vee(x)| \le c_a 2^{C|\beta|}(1+|x|^2)^{-a}, \quad x \in \mathbb{R}^d, \beta \in \mathbb{N}_0^d. \tag{2.98}$$

Furthermore, we define

$$h^\beta_{\overline{\nu}}(x) = \sum_{\overline{l}\in\mathbb{Z}^d} \lambda^\beta_{\overline{\nu}\,\overline{l}} \chi^{(p)}_{\overline{\nu}\,\overline{l}}(x), \tag{2.99}$$

$$H_{\overline{\nu}}(x) = \sum_{\overline{l}\in\mathbb{Z}^d} \Lambda_{\overline{\nu}\,\overline{l}} \chi^{(p)}_{\overline{\nu}\,\overline{l}}(x) \tag{2.100}$$

and let $0 < \kappa < \min(1, p, q)$. We prove (2.95) by the following chain of inequalities

$$\begin{aligned} 2^{\varrho|\beta|}||\lambda^\beta|s_{pq}f|| &= 2^{\varrho|\beta|}||h^\beta_{\overline{\nu}}|L_p(l_q)|| = 2^{\varrho|\beta|}||\ |h^\beta_{\overline{\nu}}|^\kappa\ |L_{\frac{p}{\kappa}}(l_{\frac{q}{\kappa}})||^{\frac{1}{\kappa}} \\ &\le c 2^{\varrho|\beta|}\left(\frac{2^{C|\beta|}}{\beta!}\right)^\kappa ||\overline{M}(|H_{\overline{\nu}}|^\kappa)|L_{\frac{p}{\kappa}}(l_{\frac{q}{\kappa}})||^{\frac{1}{\kappa}} \\ &\le c'||\ |H_{\overline{\nu}}|^\kappa\ |L_{\frac{p}{\kappa}}(l_{\frac{q}{\kappa}})||^{\frac{1}{\kappa}} = ||\Lambda|s_{pq}f||. \end{aligned} \tag{2.101}$$

The equalities in (2.101) involve only definitions of corresponding spaces. The second inequality follows from Theorem 1.10, choice of κ and the growth of $\beta!$ for $|\beta| \to \infty$. Hence only the first inequality in (2.101) needs to be proven.

To prove it, put (2.98) into (2.97) to obtain

$$|\lambda^\beta_{\overline{\nu}\,\overline{m}}| \le \frac{c_a 2^{C|\beta|}}{\beta!} \sum_{\overline{m}\in\mathbb{Z}^d} \frac{|\Lambda_{\overline{\nu}\,\overline{m}}|}{(1+|\overline{l}-\overline{m}|^2)^a}. \tag{2.102}$$

Let us take $x \in Q_{\overline{\nu}\,\overline{l}}$. Using the definition of $h^\beta_{\overline{\nu}}$ from (2.99), (2.102) and the property $\kappa < 1$ we get

$$|h^\beta_{\overline{\nu}}(x)|^\kappa = 2^{\overline{\nu}\cdot\frac{\kappa}{p}}|\lambda^\beta_{\overline{\nu}\,\overline{l}}|^\kappa \le \frac{c_a^\kappa 2^{C|\beta|\kappa}}{(\beta!)^\kappa} 2^{\overline{\nu}\cdot\frac{\kappa}{p}} \sum_{\overline{m}\in\mathbb{Z}^d} \frac{|\Lambda_{\overline{\nu}\,\overline{m}}|^\kappa}{(1+|\overline{l}-\overline{m}|^2)^{a\kappa}}. \tag{2.103}$$

We split the summation over $\overline{m} \in \mathbb{Z}^d$ into two sums according to the size of $|\overline{l}-\overline{m}|$

$$\sum_{\overline{m}\in\mathbb{Z}^d} \frac{|\Lambda_{\overline{\nu}\,\overline{m}}|^\kappa}{(1+|\overline{l}-\overline{m}|^2)^{a\kappa}} = \sum_{k=0}^\infty \frac{1}{(1+k^2)^{a\kappa}} \sum_{\overline{m}:|\overline{l}-\overline{m}|=k} |\Lambda_{\overline{\nu}\,\overline{m}}|^\kappa. \tag{2.104}$$

Finally we estimate the last sum using the iterated maximal operator $\overline{M}$

$$\begin{aligned} \sum_{\overline{m}:|\overline{l}-\overline{m}|=k} |\Lambda_{\overline{\nu}\,\overline{m}}|^\kappa &\le 2^{-\overline{\nu}\frac{\kappa}{p}} 2^{|\overline{\nu}|} \int_{y:y-x\in(k+2)Q_{\overline{\nu},0}} |H_{\overline{\nu}}(y)|^\kappa \mathrm{d}y \\ &\le 2^{-\overline{\nu}\frac{\kappa}{p}}(k+2)^d \overline{M}(|H_{\overline{\nu}}|^\kappa)(x). \end{aligned} \tag{2.105}$$

We combine (2.103), (2.104) and (2.105) and arrive at

$$|h^\beta_{\overline{\nu}}(x)|^\kappa \le c_a' \frac{2^{C|\beta|\kappa}}{(\beta!)^\kappa} \overline{M}(|H_{\overline{\nu}}|^\kappa)(x)$$

for every $a > \frac{d}{\kappa}$. This finishes the proof of (2.101) and, consequently, also the proof of (2.95) and hence also of the part (ii) of the Theorem. □

Next we shall deal with subatomic decompositions in the general case. Namely, we would like to prove an analogy of Theorem 2.13 without the restriction $\overline{r} > \sigma_{pq}$.

We start with the definition of quarks satisfying certain moment conditions.

DEFINITION 2.14. Let ψ, ϕ and Ψ be as in the Definition 2.12. Let

$$\overline{r} \in \mathbb{R}^d, \quad 0 < p \le \infty, \quad \beta \in \mathbb{N}_0^d \quad \text{and} \quad \overline{L} + 1 \in \mathbb{N}_0^d.$$

Then

$$(\beta qu)_{\overline{\nu}\,\overline{m}}^{\overline{L}}(x) = 2^{-\overline{\nu}\cdot(\overline{r}-\frac{1}{p})}(D^{\overline{L}+1}\Psi^\beta)(2^{\overline{\nu}}x - \overline{m}), \quad x \in \mathbb{R}^d,\ \overline{\nu} \in \mathbb{N}_0^d,\ \overline{m} \in \mathbb{Z}^d$$

is called an $(\overline{r}, p)^{\overline{L}}$-$\beta$-quark related to $Q_{\overline{\nu}\,\overline{m}}$.

REMARK 2.15. If $\overline{L} = -1$ then this definition coincides with Definition 2.12.

REMARK 2.16. For the need of this section we introduce temporarily following notation. Let $A \subset \{1, \dots, d\}$ and $\overline{N} = (N_1, \dots, N_d) \in \mathbb{R}^d$. Then we define the vector $\overline{N}^A = (N_1^A, \dots, N_d^A)$ by

$$N_i^A = \begin{cases} N_i & \text{if} \quad i \in A, \\ 0 & \text{if} \quad i \notin A. \end{cases}$$

THEOREM 2.17. *Let* $0 < p < \infty$, $0 < q \le \infty$ *and* $\overline{r} \in \mathbb{R}^d$. *Further let* $\overline{L} + 1 \in \mathbb{N}_0^d$ *be such that* $\overline{L} \ge \max(-1, [\sigma_{pq} - \overline{r}])$ *and* $\overline{\sigma} \in \mathbb{R}^d$ *with* $\overline{\sigma} > \max(\sigma_{pq}, \overline{r})$.

(i) *Let for every set* $A \subset \{1, \dots, d\}$

$$\lambda^A = \{\lambda^{A,\beta} : \beta \in \mathbb{N}_0^d\} \quad \textit{with} \quad \lambda^{A,\beta} = \{\lambda^{A,\beta}_{\overline{\nu}\,\overline{m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d\}$$

and let $\varrho > \phi$, *where* ϕ *is the number from* (2.83). *If*

$$\sup_{A \subset \{1,\dots,d\}} \sup_{\beta \in \mathbb{N}_0^d} 2^{\varrho|\beta|} ||\lambda^{A,\beta}|s_{pq}f|| < \infty$$

then the series

$$\sum_{A \subset \{1,\dots,d\}} \sum_{\beta \in \mathbb{N}_0^d} \sum_{\overline{\nu} \in \mathbb{N}_0^d} \sum_{\overline{m} \in \mathbb{Z}^d} \lambda^{A,\beta}_{\overline{\nu}\,\overline{m}} (\beta qu)^{\overline{L}^A}_{\overline{\nu}\,\overline{m}}(x) \tag{2.106}$$

converges in $S'(\mathbb{R}^d)$, *its limit* f *belongs to* $S^{\overline{r}}_{p,q}F$ *and*

$$||f|S^{\overline{r}}_{p,q}F|| \le c \sup_{A \subset \{1,\dots,d\}} \sup_{\beta \in \mathbb{N}_0^d} 2^{\varrho|\beta|} ||\lambda^{A,\beta}|s_{pq}f||. \tag{2.107}$$

Here $(\beta qu)^{\overline{L}^A}_{\overline{\nu}\,\overline{m}}$ *are* $(\overline{\sigma}^{A^c} + \overline{r}^A, p)^{\overline{L}^A - 1^{A^c}}$-$\beta$-*quarks.*

(ii) *Every* $f \in S^{\overline{r}}_{p,q}F$ *can be represented by* (2.106) *with convergence in* $S'(\mathbb{R}^d)$ *and*

$$\sup_{A \subset \{1,\dots,d\}} \sup_{\beta \in \mathbb{N}_0^d} 2^{\varrho|\beta|} ||\lambda^{A,\beta}|s_{pq}f|| \le c||f|S^{\overline{r}}_{p,q}F||. \tag{2.108}$$

REMARK 2.18. 1. Note that $\overline{\sigma}^{A^c} + \overline{r}^A$ is a vector whose i-th component is equal to σ_i if $i \notin A$ and equal to r_i if $i \in A$. Similarly, the i-th component of vector $\overline{L}^A - 1^{A^c}$ is equal to L_i for $i \in A$ and equal to -1 otherwise. So, for example, if $d = 2$ and $A = \{1\}$ then the quarks in (2.106) are $((r_1, \sigma_2), p)^{(L_1,-1)}$-$\beta$-quarks.

2. Because of the difficulties with notation we shall give the proof only for $d = 2$.

PROOF OF THEOREM 2.17 FOR $d = 2$.

Step 1.

First we discuss the convergence of (2.106). As the first sum is only finite, we may discuss the convergence of the triple sum over $\beta, \overline{\nu}$ and $\overline{m}$ separately for each $A \subset \{1,2\}$. Let us do this for example for $A = \{1\}$. Then we may rewrite the $((r_1,\sigma_2),p)^{(L_1,-1)}$-$\beta$-quarks from (2.106) as

$$\begin{aligned}(\beta qu)^{\overline{L}^A}_{\overline{\nu}\,\overline{m}}(x) &= 2^{-\nu_1(r_1-\frac{1}{p})-\nu_2(\sigma_2-\frac{1}{p})}(D^{(L_1+1,0)}\Psi^\beta)(2^{\overline{\nu}}x-\overline{m}) = \\ &= D^{(L_1+1,0)}\{2^{-\nu_1(r_1-\frac{1}{p}+L_1+1)-\nu_2(\sigma_2-\frac{1}{p})}\Psi^\beta(2^{\overline{\nu}}x-\overline{m})\} = \\ &= D^{(L_1+1,0)}(\beta qu)^{+}_{\overline{\nu}\,\overline{m}}(x)\end{aligned} \tag{2.109}$$

where $(\beta qu)^{+}_{\overline{\nu}\,\overline{m}}(x)$ are $((r_1+L_1+1,\sigma_2),p)$-β-quarks according to the Definition 2.12. As $r_1+L_1+1 > \sigma_{pq}$ and $\sigma_2 > \sigma_{pq}$ we may use the same arguments as in the proof of Theorem 2.13 and obtain the same kind of convergence as there.

Step 2.

Let us assume that the function f is given by (2.106). Then we may understand this decomposition as

$$f = \sum_{A \subset \{1,2\}} f^A. \tag{2.110}$$

We shall prove for every admissible set A that

$$||f^A|S^{\overline{r}}_{pq}f|| \le c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^{A,\beta}|s_{pq}f||. \tag{2.111}$$

If $A = \emptyset$ then the decomposition of $f^\emptyset$ in the triple sum according to (2.106) can be understood as a subatomic decomposition of $f^\emptyset$ in the space $S^{\overline{\sigma}}_{pq}F$ and from Theorem (2.13) follows that

$$f \in S^{\overline{\sigma}}_{pq}F \subset S^{\overline{r}}_{pq}F \quad \text{and} \quad ||f^\emptyset|S^{\overline{r}}_{pq}F|| \le c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^{\emptyset,\beta}|s_{pq}f||.$$

If $A = \{1\}$ then we use (2.109) and obtain that $f^{\{1\}} = D^{(L_1+1,0)}g$, where

$$g \in S^{(r_1+L_1+1,\sigma_2)}_{pq}F \quad \text{and} \quad ||g|S^{(r_1+L_1+1,\sigma_2)}_{pq}F|| \le c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^{\{1\},\beta}|s_{pq}f||.$$

Hence

$$\begin{aligned}||f^{\{1\}}|S^{(r_1,r_2)}_{pq}F|| &\le ||f^{\{1\}}|S^{(r_1,\sigma_2)}_{pq}F|| = ||D^{(L_1+1,0)}g|S^{(r_1,\sigma_2)}_{pq}F|| \\ &\le ||g|S^{(r_1+L_1+1,\sigma_2)}_{pq}F|| \le c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^{\{1\},\beta}|s_{pq}f||.\end{aligned} \tag{2.112}$$

Using similar technique we prove (2.111) also for $A = \{2\}$ and $A = \{1,2\}$. Now (2.110) together with (2.111) shows that (2.107) holds.

Step 3.

We prove the part (ii) of the Theorem. By similar arguments as in the Step 4. of the proof of Theorem 2.11 we prove in analogy with (2.78) that for every $\overline{M} \in \mathbb{N}_0^d$ such that

$$\overline{r} + \overline{M} + 1 \ge \overline{\sigma} \quad \text{and} \quad \overline{M} \ge \overline{L}$$

there is a function $g \in S^{\overline{r}+\overline{M}+1}_{pq}F$ with

$$f = g + \frac{\partial^{M_1+1} g}{\partial x_1^{M_1+1}} + \frac{\partial^{M_2+1} g}{\partial x_2^{M_2+1}} + \frac{\partial^{M_1+1+M_2+1} g}{\partial x_1^{M_1+1} x_2^{M_2+1}}. \tag{2.113}$$

Furthermore

$$||g|S^{\overline{r}+\overline{M}+1}_{pq}F|| \approx ||f|S^{\overline{r}}_{pq}F||. \tag{2.114}$$

Let us define

$$g_1 = g, \quad g_2 = D^{(M_1-L_1,0)}g, \quad g_3 = D^{(0,M_2-L_2)}g \quad \text{and} \quad g_4 = D^{(M_1-L_1,M_2-L_2)}g.$$

Then we can rewrite (2.113) and (2.114) as

$$f = g_1 + \frac{\partial^{L_1+1} g_2}{\partial x_1^{L_1+1}} + \frac{\partial^{L_2+1} g_3}{\partial x_2^{L_2+1}} + \frac{\partial^{L_1+1+L_2+1} g_4}{\partial x_1^{L_1+1} x_2^{L_2+1}} \tag{2.115}$$

with

$$\begin{cases} g_1 \in S^{\overline{r}+\overline{M}+1}_{pq}F \subset S^{\overline{\sigma}}_{pq}F, \\ g_2 \in S^{(r_1+L_1+1,r_2+M_2+1)}_{pq}F \subset S^{(r_1+L_1+1,\sigma_2)}_{pq}F, \\ g_3 \in S^{(r_1+M_1+1,r_2+L_2+1)}_{pq}F \subset S^{(\sigma_1,r_2+L_2+1)}_{pq}F, \\ g_4 \in S^{\overline{r}+\overline{L}+1}_{pq}F. \end{cases} \tag{2.116}$$

Furthermore, the norm of g_i in the corresponding space may be estimated from above by $||f|S^{\overline{r}}_{pq}F||$ for all $i = 1, \dots, 4$. We may use Theorem 2.13 for each function g_i to get four optimal decompositions and corresponding analogs of (2.88). Putting these estimates into (2.116) and using (2.109) we get (2.108). $\square$

CHAPTER 3

Remarks

In the previous chapter, we have followed the way of [**SchT**], [**Tr1**], [**Tr2**], [**Tr3**] and [**T03**] to carry over some important aspects from theory of $F^s_{p,q}(\mathbb{R}^d)$ spaces to $S^{\overline{r}}_{pq}F(\mathbb{R}^d)$ spaces. In this last chapter we add some useful comments and show another possible approach to this topic. Namely, we generalise the proof of Theorem of Bui, Paluszyński and Taibleson given in [**Rych**] and we give an alternative proof of the existence of optimal atomic decomposition following the ideas of Netrusov and Hedberg as they are described in [**HN**]. We also formulate our results for spaces $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$.

1. Improved version of Theorem 2.1.

The main disadvantage of Theorem 2.1 lies in the fact that (2.3) is an equivalent quasi-norm only in the set $S^{\overline{r}}_{p,q}F$. To be more concrete, we don't know if (2.3) is finite if and only if $f \in S^{\overline{r}}_{p,q}F$. The crucial point lies in the beginning of Part 2 of the proof of this Theorem. By estimating of $||f|S^{\overline{r}}_{p,q}F||$ from above by (2.3) we have to use the fact that $||f|S^{\overline{r}}_{p,q}F|| < \infty$. Closer look on this part of proof of Theorem 2.1 shows that in the case $p, q > 1$ the Nikol'skij inequality in (2.26) is not needed and, consequently, it is not necessary to use the fact $f \in S^{\overline{r}}_{p,q}F$. But this answer is only partial. For all indices $0 < p < \infty$ and $0 < q \leq \infty$ the answer may be given by following generalisation of Theorem of Bui, Paluszyński and Taibleson.

Theorem 3.1. *Let $p, q, \overline{r}$ and $\overline{N}$ be as in Theorem 2.1. Also the functions $\psi_{\overline{k}}, \overline{k} \in \mathbb{N}_0^d$ have the same meaning and satisfy* (2.1) *and* (2.2). *Then*

$$S^{\overline{r}}_{p,q}F = \left\{ f \in S' : \left|\left| \Big(\sum_{\overline{k} \in \mathbb{N}_0^d} |2^{\overline{k}\cdot\overline{r}}(\psi_{\overline{k}}\hat{f})^\vee(\cdot)|^q \Big)^{1/q} |L_p \right|\right| < \infty \right\} \tag{3.1}$$

and (2.3) *is an equivalent quasi-norm in S'.*

Proof. First of all we construct for every $1 \leq i \leq d$ a sequence of functions $\{\lambda^i_j(t)\}_{j=0}^\infty \subset S(\mathbb{R})$ such that

$$\sum_{j=0}^\infty \lambda^i_j(t)\psi^i_j(t) = 1, \quad t \in \mathbb{R}, \tag{3.2}$$

$$\lambda^i_j(t) = \lambda^i_1(2^{j-1}t), \quad t \in \mathbb{R}, \quad j > 0, \tag{3.3}$$

$$\operatorname{supp} \lambda^i_0 \subset \{t \in \mathbb{R} : |t| \leq 2\} \quad \text{and} \quad \operatorname{supp} \lambda^i_j \subset \{t \in \mathbb{R} : 2^{j-1} \leq |t| \leq 2^{j+1}\}. \tag{3.4}$$

We may construct such functions for example by the choice $\lambda^i_j(t) = \frac{\varphi_j(t)}{\psi^i_j(t)}$ for $t \in \operatorname{supp} \varphi_j$ and $\lambda^i_j(t) = 0$ elsewhere. Now we define, as usually, $\lambda_{\overline{k}}(x) = \lambda^1_{k_1}(x_1) \cdot \ldots \cdot \lambda^d_{k_d}(x_d)$ for every $\overline{k} \in \mathbb{N}_0^d$. From (3.2) we obtain

$$\sum_{\overline{k} \in \mathbb{N}_0^d} \lambda_{\overline{k}}(x)\psi_{\overline{k}}(x) = 1, \quad x \in \mathbb{R}^d.$$

It follows that the expansions

$$f = \sum_{\overline{k}\in\mathbb{N}_0^d} (\lambda_{\overline{k}}\psi_{\overline{k}}\hat{f})^\vee, \quad (\varphi_{\overline{\nu}}\hat{f})^\vee = \sum_{\overline{k}\in\mathbb{N}_0^d} (\varphi_{\overline{\nu}}\lambda_{\overline{k}}\psi_{\overline{k}}\hat{f})^\vee \tag{3.5}$$

converge in S' for every $f \in S'$ and every $\overline{\nu} \in \mathbb{N}_0^d$.

We have

$$\begin{aligned} |(\varphi_{\overline{\nu}}\lambda_{\overline{k}}\psi_{\overline{k}}\hat{f})^\vee(y)| &\le c \int_{\mathbb{R}^d} |(\varphi_{\overline{\nu}}\lambda_{\overline{k}})^\vee(z)|\ |(\psi_{\overline{k}}\hat{f})^\vee(y-z)|\mathrm{d}z \\ &\le c(\psi^*_{\overline{k}}f)_{\overline{a}}(y) \int_{\mathbb{R}^d} |(\varphi_{\overline{\nu}}\lambda_{\overline{k}})^\vee(z)| \prod_{i=1}^d (1+2^{k_i}|z_i|)^{a_i}\mathrm{d}z \end{aligned} \tag{3.6}$$

for every $\overline{a} > 0$. We denote the last integral by $I_{\overline{\nu}\,\overline{k}}$ and point out that

$$I_{\overline{\nu}\,\overline{k}} = \prod_{i=1}^d I_{\nu_i\,k_i} = \prod_{i=1}^d \int_{\mathbb{R}} |(\varphi_{\nu_i}\lambda^i_{k_i})^{\vee_1}(z_i)|(1+2^{k_i}|z_i|)^{a_i}\mathrm{d}z_i.$$

To estimate these integrals from above we use the following version of Lemma 1 from [**Rych**].

Lemma 3.2. *Let $g, h \in S(\mathbb{R})$, $M \ge -1$ be integer,*

$$D^\alpha g(0) = 0 \quad \textit{for all } \alpha \le M.$$

Then for any $N > 0$, there is a constant C_N such that

$$\sup_{t\in\mathbb{R}} |(g(b\cdot)h(\cdot))^{\vee_1}(t)|(1+|t|)^N \le C_N b^{M+1}, \quad b > 0.$$

With the change of variables $2^{k_i}z_i \to z_i$ we get

$$\int_{\mathbb{R}} |(\varphi_{\nu_i}\lambda^i_{k_i})^{\vee_1}(z_i)|(1+2^{k_i}|z_i|)^{a_i}\mathrm{d}z_i = \int_{\mathbb{R}} |(\varphi_{\nu_i}(2^{k_i}\cdot)\lambda^i_{k_i}(2^{k_i}\cdot))^{\vee_1}(z_i)|(1+|z_i|)^{a_i}\mathrm{d}z_i.$$

We assume that $k_i \ge 1$ and write $\lambda^i(t) = \lambda^i_{k_i}(2^{k_i}t)$. By (3.3), this definition doesn't depend on the choice of k_i.

Using Lemma 3.2 we may estimate for $k_i < \nu_i$

$$\int_{\mathbb{R}} |(\varphi_{\nu_i}\lambda^i_{k_i})^{\vee_1}(z_i)|(1+2^{k_i}|z_i|)^{a_i}\mathrm{d}z_i \le c \sup_{z_i\in\mathbb{R}} |(\varphi_{\nu_i-k_i}\lambda^i)^{\vee_1}(z_i)|(1+|z_i|)^{a_i+2} \le c\,2^{(k_i-\nu_i)(M+1)},$$

where M is at our disposal, as we may suppose that $\varphi_{\nu_i-k_i}(\cdot) = \varphi(2^{\nu_i-k_i}\cdot)$ and φ vanishes around zero. The exceptional terms with $0 = k_i < \nu_i$ may be incorporated in the same way (only some changes in notation are necessary).

As for $0 \le \nu_i \le k_i$, the situation is very similar. We get

$$\begin{aligned} \int_{\mathbb{R}} |(\varphi_{\nu_i}\lambda^i_{k_i})^{\vee_1}(z_i)|(1+2^{k_i}|z_i|)^{a_i}\mathrm{d}z_i &\le 2^{(k_i-\nu_i)a_i} \int_{\mathbb{R}} |(\varphi_{\nu_i}(2^{\nu_i}\cdot)\lambda^i_{k_i}(2^{\nu_i}\cdot))^{\vee_1}(z_i)|(1+|z_i|)^{a_i}\mathrm{d}z_i \\ &\le c2^{(\nu_i-k_i)(M-a_i+1)}, \end{aligned}$$

where M is again at our disposal. Now we use that λ^i vanishes in the neighbourhood of zero. The exceptional terms may be incorporated in the same way (only some changes in notation are necessary).

Hence choosing M large enough, we see that there is a number $\delta > 0$ such that

$$2^{\overline{\nu}\cdot\overline{r}} I_{\overline{\nu}\,\overline{k}} \le c2^{\overline{k}\cdot\overline{r}}2^{-|\overline{\nu}-\overline{k}|\delta}, \quad \overline{\nu} \in \mathbb{N}_0^d, \overline{k} \in \mathbb{N}_0^d. \tag{3.7}$$

We put (3.7) into (3.6) and (3.5) and get

$$2^{\overline{\nu}\cdot\overline{r}}|(\varphi_{\overline{\nu}}\hat{f})^\vee(y)| \le 2^{\overline{\nu}\cdot\overline{r}}\sum_{\overline{k}\in\mathbb{N}_0^d}|(\varphi_{\overline{\nu}}\lambda_{\overline{k}}\psi_{\overline{k}}\hat{f})^\vee(y)| \le \sum_{\overline{k}\in\mathbb{N}_0^d}2^{\overline{k}\cdot\overline{r}}|(\psi_{\overline{k}}^*f)_{\overline{a}}(y)|2^{-\delta|\overline{k}-\overline{\nu}|}. \tag{3.8}$$

The pointwise convergence of (3.5), which was used in the first inequality in (3.8), may be proved like in the Step 4 in the proof of Theorem 2.1. Namely, the second decomposition (3.5) converges in S'. And for each term from this sum we have estimates (3.6) and (3.7). These estimates allows to prove that partial sums of (3.5) form a fundamental sequence in some $L_r, r \ge 1$. Its L_r-limit must coincide with its S'-limit and (3.5) converges also in L_r. The pointwise convergence than follows in the same way.

For $q \ge 1$ we use Young's convolution inequality and for $q < 1$ we use the concavity of the function $t \to t^q$ to get

$$\left(\sum_{\overline{\nu}\in\mathbb{N}_0^d}2^{\overline{\nu}\cdot\overline{r}q}|(\varphi_{\overline{\nu}}\hat{f})^\vee(y)|^q\right)^{1/q} \le c\left(\sum_{\overline{k}\in\mathbb{N}_0^d}2^{\overline{k}\cdot\overline{r}q}|(\psi_{\overline{k}}^*f)_{\overline{a}}(y)|^q\right)^{1/q}.$$

We apply the L_p-quasi-norm and choose $\overline{a} > \frac{1}{\min(p,q)}$ and use Theorem 1.15 to obtained the desired result. $\square$

2. Alternative proof optimal atomic decomposition

In this section we give the alternative to the proof of part (ii) of Theorem 2.11 under the condition (2.67), namely the existence of optimal atomic decomposition. We use the ideas expressed in [**HN**].

First of all, we quote Theorem 1.4.2. from [**SchT**].

THEOREM 3.3. *Let $0 < \omega < \infty$. Let Ω be a compact subset of $\mathbb{R}^d$. Then there exists a positive constant $c > 0$ such that*

$$\sup_{z\in\mathbb{R}^d}\frac{|\nabla\psi(x-z)|}{1+|z|^{d/\omega}} \le c\sup_{z\in\mathbb{R}^d}\frac{|\psi(x-z)|}{1+|z|^{d/\omega}} \le c[M|\psi|^\omega(x)]^{1/\omega}$$

holds for all $\psi \in S^\Omega$ and all $x \in \mathbb{R}^d$.

The dependence of c on the size of Ω may be studied by the classical dilation arguments. If $\phi \in S$ and $\operatorname{supp}\hat{\phi} \subset Q_{\overline{R}} = [-R_1, R_1]\times\cdots\times[-R_d, R_d]$, we set $\hat{\psi}(x) = \hat{\phi}(R_1x_1, \ldots, R_dx_d)$ and use Theorem 3.3 for $\Omega = [-1,1]^d$. It follows that

$$\sup_{z\in\mathbb{R}^d}\frac{\overline{R}^{-\alpha}|D^\alpha\phi(x-z)|}{\prod_{i=1}^d(1+|R_iz_i|)^{1/\omega}} \le c\sup_{z\in\mathbb{R}^d}\frac{|\phi(x-z)|}{\prod_{i=1}^d(1+|R_iz_i|)^{1/\omega}} \le c[\overline{M}|\phi|^\omega(x)]^{1/\omega} \tag{3.9}$$

for every function $\phi \in S^{Q_{\overline{R}}}$ and every $x \in \mathbb{R}^d$ with c independent of $\overline{R}$ but dependent on the multiindex α.

The optimal atomic decomposition may now be obtained as

$$f = \sum_{\overline{k}\in\mathbb{N}_0^d}(\varphi_{\overline{k}}\hat{f})^\vee = \sum_{\overline{k}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d}(\varphi_{\overline{k}}\hat{f})^\vee(x)\psi(2^{\overline{k}}x-\overline{m}) = \sum_{\overline{k}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d}\lambda_{\overline{k}\,\overline{m}}a_{\overline{k}\,\overline{m}}(x),$$

where

$$\lambda_{\overline{k}\,\overline{m}} = c_1 2^{\overline{k}\cdot(\overline{r}-\frac{1}{p})-\alpha\cdot\overline{k}}\sup_{0\le\alpha\le\overline{K}}\ \sup_{x:|x-2^{-\overline{k}}m|\le c_2}|D^\alpha[(\varphi_{\overline{k}}\hat{f})^\vee(x)\psi(2^{\overline{k}}x-\overline{m})]| \tag{3.10}$$

and

$$a_{\overline{k}\,\overline{m}}(x)=\lambda^{-1}_{\overline{k}\,\overline{m}}(\varphi_{\overline{k}}\hat{f})^{\vee}(x)\psi(2^{\overline{k}}x-\overline{m}).$$

The function ψ is the function from (2.75). When c_1 and c_2 are chosen large enough, we see immediately that $a_{\overline{k}\,\overline{m}}(x)$ are $(\overline{r},p)_{\overline{K},-1}$-atoms.

Let us now take $x,y\in\mathbb{R}^d$ such that $|x_i-y_i|\le\gamma 2^{-k_i}$ where $\gamma>0$ is a fixed number. Then we use (3.9) and obtain

$$\begin{aligned}
|D^{\alpha}[(\varphi_{\overline{k}}\hat{f})^{\vee}(x)\psi(2^{\overline{k}}x-\overline{m})]| &\le c\sup_{0\le\beta\le\alpha}2^{\overline{k}\cdot\beta}|D^{\alpha-\beta}(\varphi_{\overline{k}}\hat{f})^{\vee}(x)|\\
&\le c\sup_{0\le\beta\le\alpha}2^{\overline{k}\cdot\beta}\sup_{z\in\mathbb{R}^d}\frac{|D^{\alpha-\beta}(\varphi_{\overline{k}}\hat{f})^{\vee}(x-z)|}{\prod_{i=1}^d(1+|R_iz_i|)^{1/\omega}}\\
&\le c\sup_{0\le\beta\le\alpha}2^{\overline{k}\cdot\beta}\sup_{z\in\mathbb{R}^d}\frac{|D^{\alpha-\beta}(\varphi_{\overline{k}}\hat{f})^{\vee}(y-z)|}{\prod_{i=1}^d(1+|R_iz_i|)^{1/\omega}}\\
&\le c\,2^{\overline{k}\cdot\beta}2^{\overline{k}\cdot(\alpha-\beta)}(\overline{M}|(\varphi_{\overline{k}}\hat{f})^{\vee}(y)|^{\omega})^{1/\omega},
\end{aligned}$$

where $0<\omega<\min(1,p,q)$. If we define

$$g_{\overline{k}}(x)=\sum_{\overline{m}\in\mathbb{Z}^d}|\lambda_{\overline{k}\,\overline{m}}\chi^{(p)}_{\overline{k}\,\overline{m}}(x)|$$

and choose $x\in Q_{\overline{k}\,\overline{m}}$ we get

$$g_{\overline{k}}(x)=2^{\overline{k}\cdot\frac{1}{p}}|\lambda_{\overline{k}\,\overline{m}}|\le c2^{\overline{k}\cdot\overline{r}}(\overline{M}|(\varphi_{\overline{k}}\hat{f})^{\vee}(x)|^{\omega})^{1/\omega}$$

and

$$\begin{aligned}
||\lambda|s_{pq}f||=||g_{\overline{k}}(\cdot)|L_p(l_q)||=||(g_{\overline{k}}(\cdot))^{\omega}|L_{\frac{p}{\omega}}(l_{\frac{q}{\omega}})||^{1/\omega}&\le c||2^{\overline{k}\cdot\overline{r}\omega}\overline{M}|(\varphi_{\overline{k}}\hat{f})^{\vee}(\cdot)|^{\omega}|L_{\frac{p}{\omega}}(l_{\frac{q}{\omega}})||^{1/\omega}\\
\le ||2^{\overline{k}\cdot\overline{r}\omega}|(\varphi_{\overline{k}}\hat{f})^{\vee}(\cdot)|^{\omega}|L_{\frac{p}{\omega}}(l_{\frac{q}{\omega}})||^{1/\omega}=||2^{\overline{k}\cdot\overline{r}}(\varphi_{\overline{k}}\hat{f})^{\vee}(\cdot)|L_p(l_q)||&=||f|S^{\overline{r}}_{p,q}F||,
\end{aligned}$$

which is (2.54).

The advantage of this proof is that it doesn't use Theorem 2.1 or any of its versions. But to count the coefficients (3.10) one has to know $\hat{f}$ and hence the global information about the function f is needed. In some sense, one may say, that coefficients (3.10) are *not local*.

3. Another version of atomic and subatomic decomposition

Recall that $(\overline{r},p)_{\overline{K},\overline{L}}$-atoms and $(\overline{r},p)$-β-quarks were defined in Definitions 2.9 and 2.12. Both these definitions include the dependence on $\overline{r}$ and p. But this fact represents a very uncomfortable complication for further applications. Hence we are going to reformulate these Definitions and following Theorems in such a way, that new atoms and quarks don't depend on $\overline{r}$ neither p.

Definition 3.4. Let $\overline{K}\in\mathbb{N}_0^d,\overline{L}+1\in\mathbb{N}_0^d$, and $\gamma>1$. A $\overline{K}$-times differentiable complex-valued function $a(x)$ is called $[\overline{K},\overline{L}]$-atom centred at $Q_{\overline{\nu}\,\overline{m}}$ if

$$\operatorname{supp}a\subset\gamma Q_{\overline{\nu}\,\overline{m}}, \tag{3.11}$$

$$|D^{\alpha}a(x)|\le 2^{\alpha\cdot\overline{\nu}}\qquad\text{for}\quad 0\le\alpha\le\overline{K} \tag{3.12}$$

and

$$\int_{\mathbb{R}}x_i^ja(x)\mathrm{d}x_i=0\qquad\text{if}\quad i=1,\dots,d;\ j=0,\dots,L_i\quad\text{and}\quad\nu_i\ge1. \tag{3.13}$$

DEFINITION 3.5. Let $\psi \in S(\mathbb{R})$ be a non-negative function with

$$\operatorname{supp}\psi \subset \{t \in \mathbb{R} : |t| < 2^{\phi}\} \tag{3.14}$$

for some $\phi \geq 0$ and

$$\sum_{n\in\mathbb{Z}} \psi(t-n) = 1, \quad t \in \mathbb{R}. \tag{3.15}$$

We define $\Psi(x) = \psi_1(x_1) \cdot \ldots \cdot \psi_d(x_d)$ and $\Psi^{\beta}(x) = x^{\beta}\Psi(x)$ for $x = (x_1, \ldots, x_d)$ and $\beta \in \mathbb{N}_0^d$. Further let $\overline{r} \in \mathbb{R}^d$ and $0 < p \leq \infty$. Then

$$(\beta qu)_{\overline{\nu}\,\overline{m}}(x) = \Psi^{\beta}(2^{\overline{\nu}}x - \overline{m}), \quad \overline{\nu} \in \mathbb{N}_0^d,\ \overline{m} \in \mathbb{Z}^d \tag{3.16}$$

is called an β-quark related to $Q_{\overline{\nu}\,\overline{m}}$.

This change is reflected also in the definition of sequence spaces $s_{pq}b$ and $s_{pq}f$.

DEFINITION 3.6. If $0 < p \leq \infty$, $0 < q \leq \infty$, $\overline{r} \in \mathbb{R}^d$ and

$$\lambda = \{\lambda_{\overline{\nu}\,\overline{m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d\} \tag{3.17}$$

then we define

$$s_{pq}^{\overline{r}}b = \left\{\lambda : ||\lambda|s_{pq}^{\overline{r}}b|| = \left(\sum_{\overline{\nu}\in\mathbb{N}_0^d} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\Big(\sum_{\overline{m}\in\mathbb{Z}^d} |\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\right)^{1/q} < \infty\right\} \tag{3.18}$$

and

$$s_{pq}^{\overline{r}}f = \left\{\lambda : ||\lambda|s_{pq}^{\overline{r}}f|| = \left|\left|\left(\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} |2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}}(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R}^d)\right|\right| < \infty\right\} \tag{3.19}$$

with the usual modification for p and/or q equal to ∞.

REMARK 3.7. We point out that with λ given by (3.17) and $g_{\overline{\nu}}(x) = \sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}}(x)$, we obtain that

$$||\lambda|s_{pq}^{\overline{r}}b|| = ||2^{\overline{\nu}\cdot\overline{r}}g_{\overline{\nu}}|l_q(L_p)||, \qquad ||\lambda|s_{pq}^{\overline{r}}f|| = ||2^{\overline{\nu}\cdot\overline{r}}g_{\overline{\nu}}|L_p(l_q)||.$$

Sequence spaces of this kind were denoted by E_{dis} in [**HN**] and may be understood as a discrete version of $S_{p,q}^{\overline{r}}F$ and $S_{p,q}^{\overline{r}}B$. Let us also mention, that $s_{pq}^{1/p}f = s_{pq}f$ and $s_{pq}^{1/p}b = s_{pq}b$, where $s_{pq}f$ and $s_{pq}b$ were defined by (2.47) and (2.46) respectively.

Using this notation we may reformulate Theorems 2.11 and 2.13.

THEOREM 3.8. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $\overline{r} \in \mathbb{R}^d$. Fix $\overline{K} \in \mathbb{N}_0^d$ and $\overline{L} + 1 \in \mathbb{N}_0^d$ with*

$$K_i \geq (1 + [r_i])_+ \quad \textit{and} \quad L_i \geq \max(-1, [\sigma_{pq} - r_i]), \qquad i = 1, \ldots, d. \tag{3.20}$$

(i) *If $\lambda \in s_{pq}^{\overline{r}}f$ and $\{a_{\overline{\nu}\,\overline{m}}(x)\}_{\overline{\nu}\in\mathbb{N}_0^d,\overline{m}\in\mathbb{Z}^d}$ are $[\overline{K}, \overline{L}]$-atoms centred at $Q_{\overline{\nu}\,\overline{m}}$, then the sum*

$$\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}}a_{\overline{\nu}\,\overline{m}}(x) \tag{3.21}$$

converges in $S'(\mathbb{R}^d)$, its limit f belongs to the space $S_{p,q}^{\overline{r}}F$ and

$$||f|S_{p,q}^{\overline{r}}F|| \leq c\,||\lambda|s_{pq}^{\overline{r}}f||, \tag{3.22}$$

where the constant c is universal for all admissible λ and $a_{\overline{\nu}\,\overline{m}}$.

(ii) *For every $f \in S^{\overline{r}}_{p,q}F$ there is a $\lambda \in s^{\overline{r}}_{pq}f$ and $[\overline{K},\overline{L}]$-atoms centred at $Q_{\overline{\nu}\,\overline{m}}$ (denoted again by $\{a_{\overline{\nu}\,\overline{m}}(x)\}_{\overline{\nu}\in\mathbb{N}_0^d,\overline{m}\in\mathbb{Z}^d}$) such that the sum (3.21) converges in $S'(\mathbb{R}^d)$ to f and*

$$||\lambda|s^{\overline{r}}_{pq}f|| \le c\,||f|S^{\overline{r}}_{p,q}F||. \tag{3.23}$$

The constant c is again universal for every $f \in S^{\overline{r}}_{p,q}F$.

Theorem 3.9. *Let $0<p<\infty$, $0<q\le\infty$ and $\overline{r}\in\mathbb{R}^d$ be such that*

$$\overline{r} > \sigma_{pq}.$$

(i) *Let*

$$\lambda = \{\lambda^\beta : \beta\in\mathbb{N}_0^d\} \quad \textit{with} \quad \lambda^\beta = \{\lambda^\beta_{\overline{\nu}\,\overline{m}} \in \mathbb{C} : \overline{\nu}\in\mathbb{N}_0^d, \overline{m}\in\mathbb{Z}^d\}$$

and let $\varrho > \phi$, where ϕ is the number from (3.14). If

$$\sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^\beta|s^{\overline{r}}_{pq}f|| < \infty$$

then the series

$$\sum_{\beta\in\mathbb{N}_0^d}\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} \lambda^\beta_{\overline{\nu}\,\overline{m}}(\beta qu)_{\overline{\nu}\,\overline{m}}(x) \tag{3.24}$$

converges in $S'(\mathbb{R}^d)$, its limit f belongs to $S^{\overline{r}}_{p,q}F$ and

$$||f|S^{\overline{r}}_{p,q}F|| \le c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^\beta|s_{pq}f||. \tag{3.25}$$

$(\beta qu)_{\overline{\nu}\,\overline{m}}$ are now β-quarks defined by (3.16).
(ii) *Every $f\in S^{\overline{r}}_{p,q}F$ can be represented by (3.24) with convergence in $S'(\mathbb{R}^d)$ and*

$$\sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^\beta|s^{\overline{r}}_{pq}f|| \le c||f|S^{\overline{r}}_{p,q}F||. \tag{3.26}$$

Remark 3.10. Proofs of this two Theorems are obvious. We have only moved the factor $2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})}$ from (2.49) and (2.85) into the coefficients $\lambda_{\overline{\nu}\,\overline{m}}$ or $\lambda^\beta_{\overline{\nu}\,\overline{m}}$ respectively.

4. Spaces of Besov Type

All our results may be reformulated for $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ spaces. The proofs are usually very similar, only some technical details are easier. We state these formulation to provide some reference to these results.

Theorem 3.11. *Let $0<p<\infty$, $0<q\le\infty$, $\overline{r}=(r_1,\dots,r_d)\in\mathbb{R}^d$. Let $\overline{N}=(N_1,\dots,N_d)\in\mathbb{N}_0^d$ be d even numbers with $\overline{r}<\overline{N}$.*
Let ψ_0 and $\psi^1,\dots,\psi^d$ be $d+1$ complex-valued functions from $S(\mathbb{R})$, which satisfy the Tauberian conditions

$$|\psi_0(t)|>0 \quad \textit{if} \quad |t|\le 2, \qquad \textit{and} \qquad |\psi^i(t)|>0 \quad \textit{if} \quad \frac{1}{2}\le|t|\le 2, \quad i=1,\dots,d. \tag{3.27}$$

Let us also suppose that

$$D^\alpha\psi^i(0)=0, \qquad 0\le\alpha\le N_i-1, \qquad i=1,\dots,d. \tag{3.28}$$

Let $\psi_0^i = \psi_0$ *and* $\psi_j^i(t) = \psi^i(2^{-j}t)$ *if* $t \in \mathbb{R}$, $j \in \mathbb{N}$ *and* $i = 1, \ldots, d$. *Further let* $\psi_{\overline{k}}(x) = \psi_{k_1}^1(x_1) \cdot \ldots \cdot \psi_{k_d}^d(x_d)$ *whenever* $\overline{k} = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$ *and* $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. *Then*

$$S_{p,q}^{\overline{r}}B = \left\{ f \in S' : \left(\sum_{\overline{k} \in \mathbb{N}_0^d} 2^{\overline{k}\cdot\overline{r}q} ||(\psi_{\overline{k}}\hat{f})^\vee | L_p(\mathbb{R}^d)||^q \right)^{1/q} < \infty \right\} \tag{3.29}$$

and

$$\left(\sum_{\overline{k} \in \mathbb{N}_0^d} 2^{\overline{k}\cdot\overline{r}q} ||(\psi_{\overline{k}}\hat{f})^\vee | L_p(\mathbb{R}^d)||^q \right)^{1/q} \tag{3.30}$$

is an equivalent quasi-norm in $S'(\mathbb{R}^d)$.

THEOREM 3.12. *Let* $0 < p < \infty$, $0 < q \le \infty$ *and* $\overline{r} \in \mathbb{R}^d$. *Fix* $\overline{K} \in \mathbb{N}_0^d$ *and* $\overline{L} + 1 \in \mathbb{N}_0^d$ *with*

$$K_i \ge (1 + [r_i])_+ \quad \text{and} \quad L_i \ge \max(-1, [\sigma_p - r_i]), \qquad i = 1, \ldots, d. \tag{3.31}$$

(i) *If* $\lambda \in s_{pq}^{\overline{r}}b$ *and* $\{a_{\overline{\nu}\,\overline{m}}(x)\}_{\overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d}$ *are* $[\overline{K}, \overline{L}]$*-atoms centred at* $Q_{\overline{\nu}\,\overline{m}}$, *then the sum*

$$\sum_{\overline{\nu} \in \mathbb{N}_0^d} \sum_{\overline{m} \in \mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}} a_{\overline{\nu}\,\overline{m}}(x) \tag{3.32}$$

converges in $S'(\mathbb{R}^d)$, *its limit* f *belongs to the space* $S_{p,q}^{\overline{r}}B$ *and*

$$||f | S_{p,q}^{\overline{r}}B|| \le c \, ||\lambda | s_{pq}^{\overline{r}}b||, \tag{3.33}$$

where the constant c *is universal for all admissible* λ *and* $a_{\overline{\nu}\,\overline{m}}$.

(ii) *For every* $f \in S_{p,q}^{\overline{r}}B$ *there is a* $\lambda \in s_{pq}^{\overline{r}}b$ *and* $[\overline{K}, \overline{L}]$*-atoms centred at* $Q_{\overline{\nu}\,\overline{m}}$ (*denoted again by* $\{a_{\overline{\nu}\,\overline{m}}(x)\}_{\overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d}$) *such that the sum* (3.32) *converges in* $S'(\mathbb{R}^d)$ *to* f *and*

$$||\lambda | s_{pq}^{\overline{r}}b|| \le c \, ||f | S_{p,q}^{\overline{r}}B||. \tag{3.34}$$

The constant c *is again universal for every* $f \in S_{p,q}^{\overline{r}}B$.

THEOREM 3.13. *Let* $0 < p < \infty$, $0 < q \le \infty$ *and* $\overline{r} \in \mathbb{R}^d$ *be such that*

$$\overline{r} > \sigma_p.$$

(i) *Let*

$$\lambda = \{\lambda^\beta : \beta \in \mathbb{N}_0^d\} \quad \text{with} \quad \lambda^\beta = \{\lambda_{\overline{\nu}\,\overline{m}}^\beta \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d\}$$

and let $\varrho > \phi$, *where* ϕ *is the number from* (3.14). *If*

$$\sup_{\beta \in \mathbb{N}_0^d} 2^{\varrho|\beta|} ||\lambda^\beta | s_{pq}^{\overline{r}}b|| < \infty$$

then the series

$$\sum_{\beta \in \mathbb{N}_0^d} \sum_{\overline{\nu} \in \mathbb{N}_0^d} \sum_{\overline{m} \in \mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}}^\beta (\beta qu)_{\overline{\nu}\,\overline{m}}(x) \tag{3.35}$$

converges in $S'(\mathbb{R}^d)$, *its limit* f *belongs to* $S_{p,q}^{\overline{r}}B$ *and*

$$||f | S_{p,q}^{\overline{r}}F|| \le c \sup_{\beta \in \mathbb{N}_0^d} 2^{\varrho|\beta|} ||\lambda^\beta | s_{pq}b||. \tag{3.36}$$

$(\beta qu)_{\overline{\nu}\,\overline{m}}$ *are now* β*-quarks defined by* (3.16).

(ii) *Every* $f \in S_{p,q}^{\overline{r}}B$ *can be represented by* (3.24) *with convergence in* $S'(\mathbb{R}^d)$ *and*

$$\sup_{\beta \in \mathbb{N}_0^d} 2^{\varrho|\beta|} ||\lambda^\beta | s_{pq}^{\overline{r}}f|| \le c||f | S_{p,q}^{\overline{r}}F||. \tag{3.37}$$

Bibliography

[Am] Amanov, T. I., Spaces of Differentiable Functions with Dominating Mixed Derivatives. (Russian), Alma-Ata: Nauka Kaz. SSR 1976

[Ba] Bagby, R. J., An extended inequality for the maximal function, Proc. Amer. Math. Soc. 48 (1975), 419-422.

[HN] Hedberg, L. I. and Netrusov, Y. V., An axiomatic approach to function spaces and spectral synthesis, Proceedings of the Spring School *Function Spaces and Applications*, Paseky nad Jizerou, 2003

[Ho] Hochmuth, R., A φ-Transform Result for Spaces with Dominating Mixed Smoothness Properties, Result. Math. 33 (1998), 106-119

[FS] Fefferman, C. and Stein, E. M., Some maximal inequalities, Amer. J. Math. 93 (1971), 107-115.

[FrJ1] Frazier, M. and Jawerth, B., Decomposition of Besov spaces, Indiana Univ. Math. J. 34 (1985), 777-799.

[FrJ2] Frazier, M. and Jawerth, B., A discrete transform and decomposition of distribution spaces, J. Funct. Anal. 93 (1990), 34-170.

[N1] Nikol'skij, S. M., On boundary properties of differentiable functions of several variables, (Russian) Dokl. Akad. Nauk SSSR 146 (1962), 542-545.

[N2] Nikol'skij, S. M., On stable boundary values of differentiable functions of several variables, (Russian) Mat. Sb. 61 (1963), 224-252.

[Rod] Rodríguez Fernández, M. C., Über die Spur von Funktionen mit dominierenden gemischten Glattheitseigenschaften auf der Diagonale. Ph.D-thesis, Jena, 1997.

[Rych] Rychkov, V. S., On a Theorem of Bui, Paluszyński, and Taibleson, Proceedings of Steklov Institute of Mathematics 227 (1999), 280-292.

[SchT] Schmeisser, H.-J. and Triebel, H., Topics in Fourier analysis and function spaces. Chister, Wiley, 1987.

[Tr1] Triebel, H., Theory of function spaces II. Basel, Birkhäuser, 1992.

[Tr2] Triebel, H., Fractals and Spectra, Basel, Birkhäuser, 1997.

[Tr3] Triebel, H., The Structure of Functions, Basel, Birkhäuser, 2001.

[T02] Triebel, H., Towards a Gausslet Analysis: Gaussian representations of functions, in: Function Spaces. Interpolation Theory and Related Topics (Lund, 2002) de Gruyter, Berlin, 2002, 425-449

[T03] Triebel, H., Wavelet frames for distributions; local and pointwise regularity, Stud. Math. 154 (2003), 59-88.