On positive positive-definite functions and Bochner's Theorem

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Abstract

We recall an open problem on the error of quadrature formulas for the integration of functions from some finite dimensional spaces of trigonometric functions posed by Erich Novak in [8] ten years ago and summarised recently in [9]. It is relatively easy to prove an error formula for the best quadrature rules with positive weights which shows intractability of the tensor product problem for such rules. In contrast to that, the conjecture that also quadrature formulas with arbitrary weights can not decrease the error is still open.

We generalise Novak's conjecture to a statement about positive positive-definite functions and provide several equivalent reformulations, which show the connections to Bochner's Theorem and Toeplitz matrices.

1 Integration of trigonometric polynomials

In his work [8], E. Novak used quadrature formulas

$$Q_n(f)=\sum_{i=1}^n c_i f(x_i),\qquad c_i\in\mathbb{R},\quad x_i\in[0,1]^d \tag{1}$$

to approximate the integral

$$\mathrm{INT}_d(f)=\int_{[0,1]^d} f(x)\,\mathrm{d}x.$$

Here f belongs to a unit ball of a Hilbert space F_d, which is defined inductively as follows.

The space F_1 is linear and generated by three functions:

$$e_1(x) = 1, \quad e_2(x) = \cos(2\pi x), \quad e_3(x) = \sin(2\pi x)$$

for $x \in [0,1]$. The scalar product $\langle \cdot, \cdot \rangle_{F_1}$ on F_1 is *defined* by the statement, that $\{e_i\}_{i=1}^3$ is an orthonormal basis of F_1.

For $d > 1$, F_d is defined as the d-fold tensor product of F_1 with the tensor scalar product

$$\langle f_1 \otimes f_2 \otimes \cdots \otimes f_d, g_1 \otimes g_2 \otimes \cdots \otimes g_d \rangle_{F_d} = \prod_{j=1}^d \langle f_j, g_j \rangle_{F_1},$$

where $f_j, g_j \in F_1$ and

$$(f_1 \otimes f_2 \otimes \cdots \otimes f_d)(x) = f_1(x_1) f_2(x_2) \dots f_d(x_d), \quad x = (x_1, x_2, \dots, x_d) \in [0,1]^d.$$

Then F_d is a 3^d-dimensional Hilbert space with a reproducing kernel $K_d(x,y)$ given by

$$K_d(x,y) = \langle \delta_x, \delta_y \rangle_{F_d},$$

where δ_x is the function

$$\delta_x(z) = \prod_{j=1}^d [1 + \cos(2\pi(x_j - z_j))]$$

so that the point evaluation of $f \in F_d$ at x is given as

$$f(x) = \langle f, \delta_x \rangle_{F_d}.$$

Altogether, we obtain

$$K_d(x,y) = \prod_{j=1}^d [1 + \cos(2\pi(x_j - y_j))], \quad x, y \in [0,1]^d.$$

The worst-case error of Q_n as introduced by (1) is then given by

$$e^{\mathrm{wor}}(Q_n)^2 = \sup_{||f||_{F_d} \le 1} |\mathrm{INT}_d(f) - Q_n(f)|^2 = \left\| 1 - \sum_{j=1}^n c_j \delta_{x_j} \right\|_{F_d}^2$$
$$= 1 - 2 \sum_{j=1}^n c_j + \sum_{j,k=1}^n c_j c_k K_d(x_j, x_k).$$

It turned out, that the analysis of this error is much simpler, if we assume, that all the coefficients are positive. In that case, we have the estimate

$$e^{\mathrm{wor}}(Q_n)^2 \ge 1 - 2 \sum_{j=1}^n c_j + \sum_{j=1}^n c_j^2 2^d$$

and a simple calculation shows that the right hand side is minimal for $c_j = 2^{-d}$ which gives

$$e^{\mathrm{wor}}(Q_n)^2 \geq \max(1 - n2^{-d}, 0). \tag{2}$$

It was also observed in [8] that this estimate is optimal in the case positive coefficients, i.e. there exists a quadrature rule Q_n as defined in (1) with positive coefficients c_i such that equality holds in (2). In particular, this estimate shows that the problem is intractable with quadrature formulas with positive weights since for fixed error the number n of sample points needs to grow exponentially with the dimension d.

If the coefficients are allowed to change the signs, we use the simple fact, that the projection of any $y \in F_d$ onto the ray generated by $x \in F_d$ is given by $\dfrac{\langle y, x\rangle x}{\langle x, x\rangle}$ and obtain

$$\inf_{c_j, x_j} \left\| 1 - \sum_{j=1}^n c_j \delta_{x_j} \right\|_{F_d}^2 = 1 - \sup_{c_j, x_j} \frac{\left\langle 1, \sum_{j=1}^n c_j \delta_{x_j} \right\rangle_{F_d}^2}{\left\langle \sum_{j=1}^n c_j \delta_{x_j}, \sum_{j=1}^n c_j \delta_{x_j} \right\rangle_{F_d}} = 1 - \sup_{c_j, x_j} \frac{\left(\sum_{j=1}^n c_j \right)^2}{\sum_{j,k=1}^n c_j c_k K_d(x_j, x_k)}, \tag{3}$$

see [8].

Erich Novak conjectured, that the estimate (2) applies also for quadrature formulas (1) with (possibly) negative coefficients. In view of (3), this is equivalent to

Conjecture 1. *(E. Novak) Let $n, d \geq 2$ be natural numbers and let $x_1, \dots, x_n \in \mathbb{R}^d$. Then the $n \times n$ matrix*

$$\left\{ \prod_{i=1}^d \frac{1 + \cos(x_{j,i} - x_{k,i})}{2} - \frac{1}{n} \right\}_{j,k=1}^n$$

is positive semi-definite.

Choosing $x_1, \dots, x_n$ such that for each pair $j, k = 1, \dots, n$ with $j \neq k$ there is some $i = 1, \dots, d$ with $\cos(x_{j,i} - x_{k,i}) = -1$ produces the $n \times n$-matrix with diagonal entries $1 - 1/n$ and offdiagonal entries $-1/n$ and shows that the constant $1/n$ is optimal in Conjecture 1, i.e. it obviously does not hold, if $1/n$ is replaced by any bigger quantity independent of d. This choice of $x_1, \dots, x_n$ is only possible if $n \leq 2^d$, but this is also the only interesting case in Conjecture 1.

Although the conjecture may be easily formulated and tested by computer (at least for small dimensions n and d), it is not clear, which property (or properties) of the function $\frac{1+\cos x}{2}$ are the most important in this context. Hence, it is natural to try to generalise the conjecture to a wider class of functions, where only really significant properties would play a role. For example, it was conjectured already in [8], that this problem may be connected to the Hadamard product of matrices (cf. [6] or [7]) or to positive-definite functions (cf. [11]).

1.1 Hadamard product

We first introduce some notation. If $A, B \in \mathbb{R}^{n\times n}$ are two symmetric $n \times n$ matrices, we write $A \succeq B$ if $A - B$ is positive semi-definite, i.e. if $x^T(A-B)x \ge 0$ for all $x \in \mathbb{R}^n$. It is easy to see, that this relation represents a partial ordering on the set of all $n \times n$ matrices. Furthermore, we denote by E_n the $n \times n$ matrix with all components equal to one.

Using this notation, we may rewrite the Conjecture 1 as

$$\left\{\prod_{i=1}^d \frac{1+\cos(x_{j,i}-x_{k,i})}{2}\right\}_{j,k=1}^n \succeq \frac{E_n}{n} \quad \text{for all} \quad x_1,\dots,x_n \in \mathbb{R}^d.$$

Definition 1. *If $A = (a_{i,j})_{i,j=1}^n, B = (b_{i,j})_{i,j=1}^n \in \mathbb{R}^{n\times n}$ are two $n \times n$ matrices, we define their Hadamard product as their component-wise product, i.e.*

$$A \circ B = \{a_{i,j} b_{i,j}\}_{i,j=1}^n.$$

We remark, that the Hadamard product is sometimes also referred to as the *Schur product*, cf. [6, Chapter 7.5]. Obviously, the matrix E_n is a unit element with respect to the Hadamard multiplication. The celebrated *Schur product theorem* states that the Hadamard product of two positive semi-definite matrices is also positive semi-definite, see [6, Theorem 7.5.3].

It is easy to see that the following Statement holds for $n = 2$ and that it would provide a direct proof of Conjecture 1.

Statement. *Let $A = (a_{i,j})_{i,j=1}^n$ and $B = (b_{i,j})_{i,j=1}^n$ be two symmetric $n \times n$ matrices with*

$$0 \le a_{i,j}, b_{i,j} \le 1 \quad \text{for} \quad 1 \le i,j \le n \quad \text{and} \quad a_{i,i} = b_{i,i} = 1, \quad \text{for all} \quad i = 1,2,\dots,n.$$

Furthermore, let

$$A \succeq \frac{E_n}{n} \quad \text{and} \quad B \succeq \frac{E_n}{n}.$$

Then $A \circ B \succeq \frac{E_n}{n}$.

Unfortunately, in dimensions $n \ge 4$, the Statement fails. A counterexample for $n = 4$, which we obtained by computer calculations, may be found after the next statement.

One might think that the reason for this failure is that we considered only the complete $n \times n$ matrix from Conjecture 1. Obviously, any square submatrix obtained from it by deleting the rows and columns in a certain subset of the indices $1, 2, \dots, n$ is again a matrix of the same type. For a given $n \times n$ matrix A and a subset $I \subseteq \{1, 2, \dots, n\}$, the matrix $A(I)$ is the $|I| \times |I|$-matrix obtained from A by deleting all columns and rows with indices not in I. Again, the following Statement would easily give a proof of Conjecture 1.

Statement. *Let $A = (a_{i,j})_{i,j=1}^n$ and $B = (b_{i,j})_{i,j=1}^n$ be two symmetric $n \times n$ matrices with*

$$0 \le a_{i,j}, b_{i,j} \le 1 \quad \text{for} \quad 1 \le i, j \le n \quad \text{and} \quad a_{i,i} = b_{i,i} = 1, \quad \text{for all} \quad i = 1, 2, \ldots, n. \tag{4}$$

Furthermore, let

$$A(I) \succeq \frac{E_{|I|}}{|I|} \quad \text{and} \quad B(I) \succeq \frac{E_{|I|}}{|I|} \quad \text{for all} \quad I \subseteq \{1, 2, \ldots, n\}. \tag{5}$$

Then $A \circ B \succeq \frac{E_n}{n}$.

Again, there is a counterexample for $n = 4$, namely

$$A = \begin{pmatrix} 1 & 0.88 & 0.05 & 0.82 \\ 0.88 & 1 & 0.04 & 0.89 \\ 0.05 & 0.04 & 1 & 0.41 \\ 0.82 & 0.89 & 0.41 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0.4 & 0.5 \\ 0 & 1 & 0.87 & 0.8 \\ 0.4 & 0.87 & 1 & 0.97 \\ 0.5 & 0.8 & 0.97 & 1 \end{pmatrix}.$$

Let us mention, that we needed to generate about 10^8 random matrices A and B with (4) and (5) to find a counterexample and that the smallest eigenvalue of $A \circ B - \frac{E_4}{4}$ is just -0.00169. We did compute the eigenvalues of the computer generated matrices to an accuracy which shows that the signs of the eigenvalues are indeed as claimed in the statements here. So these are statements are mathematical facts.

2 Positive positive-definite functions

2.1 Euclidean spaces

Conjecture 1 may be interpreted as a search for a class of functions $f : \mathbb{R}^d \to \mathbb{C}$, such that $f(0) = 1$ and for every $n \in \mathbb{N}$ and every $x_1, \ldots, x_n \in \mathbb{R}^d$,

$$\{f(x_j - x_k)\}_{j,k=1}^n \succeq \frac{E_n}{n}. \tag{6}$$

In Section 1 we only considered real valued functions f. Now it is more convenient to treat complex valued functions. Since our main focus is on functions which are positive (and, therefore, real), this difference is merely cosmetic. If E_n is replaced by the zero matrix in (6), then the question is the subject of the celebrated Bochner Theorem, see [10].

Theorem 1. (Bochner) *Let f be a bounded continuous function on $\mathbb{R}^d$. Then*

$$\{f(x_j - x_k)\}_{j,k=1}^n \succeq 0 \tag{7}$$

for any choice of $n \in \mathbb{N}$ and any $x_1, \ldots, x_n \in \mathbb{R}^d$ if, and only if, there is a positive finite Borel measure μ on $\mathbb{R}^d$, such that

$$f(x) = (\mathcal{F}\mu)(x) = \int_{\mathbb{R}^d} e^{2\pi i \langle x, \xi \rangle} \, \mathrm{d}\mu(\xi), \quad x \in \mathbb{R}^d. \tag{8}$$

Obviously, (6) is stronger than (7). When trying to characterise functions, which satisfy (6), we are actually seeking for a subclass of functions described in Bochner's Theorem. Based on several numerical experiments, we were lead to the class of positive positive-definite functions.

Definition 2. *Let $f : \mathbb{R}^d \to \mathbb{C}$ be a bounded continuous function. We say, that f is positive positive-definite, if f is real-valued, $f(x) \geq 0$ for all $x \in \mathbb{R}^d$ and there is a positive Radon measure μ, such that* (8) *holds for all $x \in \mathbb{R}^d$. We denote by $\mathcal{P}_+(\mathbb{R}^d) \subset C(\mathbb{R}^d)$ the set of all positive positive-definite functions.*

Some of the properties of $\mathcal{P}_+(\mathbb{R}^d)$ are easy to see. They include

- $\mathcal{P}_+(\mathbb{R}^d)$ is a convex cone,
- if $f, g \in \mathcal{P}_+(\mathbb{R}^d)$, then $f \cdot g \in \mathcal{P}_+(\mathbb{R}^d)$,
- if $f, g \in \mathcal{P}_+(\mathbb{R}^d)$ and the convolution $f * g$ is a well-defined bounded continuous function, then also $f * g \in \mathcal{P}_+(\mathbb{R}^d)$,
- if $f \in \mathcal{P}_+(\mathbb{R}^{d_1})$ and $g \in \mathcal{P}_+(\mathbb{R}^{d_2})$, then $f \otimes g \in \mathcal{P}_+(\mathbb{R}^{d_1+d_2})$.

The properties of $\mathcal{P}_+(\mathbb{R}^d)$ were already studied in literature (cf. [1],[3] and [4]) but, as stated in [1], "the structure of the cone of such functions is not clear". A similar comment made in [4] reads: "...a full classification of extremals is probably impossible". Here the term extremals refers to the extremal rays of the convex cone $\mathcal{P}_+(\mathbb{R}^d)$.

Based on our numerical experiments, we formulate the following

Conjecture 2. *If $f : \mathbb{R}^d \to \mathbb{R}$ is a bounded positive positive-definite continuous function with $f(0) = 1$, then* (6) *holds for every $n \in \mathbb{N}$ and every $x_1, \ldots, x_n \in \mathbb{R}^d$.*

Of course, Conjecture 2 generalises Conjecture 1, as the function $\frac{1+\cos x}{2}$ is easily seen to be positive positive-definite. Moreover, by convexity it is enough to verify the conjecture for functions f on extremal rays of the convex cone $\mathcal{P}_+(\mathbb{R}^d)$. Also the converse of Conjecture 2 is of interest (and would actually lead to an interesting variant of Bochner's Theorem). Namely, is it true, that if a bounded continuous function $f : \mathbb{R}^d \to \mathbb{C}$ satisfies (6) with $f(0) = 1$ for every $n \in \mathbb{N}$ and every $x_1, \ldots, x_n \in \mathbb{R}^d$, then f is necessarily positive positive-definite?

If we assume in advance that f is real-valued, then the answer is positive.

Proposition 1. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous bounded function with $f(0) = 1$ which satisfies (6) for every $n \in \mathbb{N}$ and every $x_1, \ldots, x_n \in \mathbb{R}^d$, then f is positive positive-definite.*

The proof follows immediately from Bochner's Theorem and by considering the 2×2 matrices

$$\begin{pmatrix} f(0) & f(x-y) \\ f(y-x) & f(0) \end{pmatrix}$$

together with the observation that (8) and $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d$ implies $f(-x) = f(x)$ for all $x \in \mathbb{R}^d$.

2.2 The torus

Since the function $\frac{1+\cos x}{2}$ is 2π-periodic, the considerations of the previous section can also be formulated in terms of functions on the d-dimensional torus $\mathbb{T}^d = [-\pi, \pi]^d$ were we, as usual, identify the points $-\pi$ and π so that a function on $\mathbb{T}$ may be also interpreted as a 2π-periodic function on $\mathbb{R}$.

For a continuous (or just integrable) function f on $\mathbb{T}^d$, let $\hat{f}$ be the Fourier series of f. Now the analogue of Bochner's Theorem tells us that, for a continuous function f on $\mathbb{T}^d$, the matrix

$$\{f(x_j - x_k)\}_{j,k=1}^n \succeq 0$$

for any choice of $n \in \mathbb{N}$ and any $x_1, \ldots, x_n \in \mathbb{R}^d$ if, and only if, the Fourier series of f is nonnegative, i.e. $\hat{f} \geq 0$. So the class of continuous nonnegative positive-definite functions on $\mathbb{T}^d$ is just the class of all continuous nonnegative functions with nonnegative Fourier series. We are lead to the following conjecture.

Conjecture 3. *If $f : \mathbb{T}^d \to \mathbb{C}$ is a continuous positive function with nonnegative Fourier series and $f(0) = 1$, then (6) holds for every $n \in \mathbb{N}$ and every $x_1, \ldots, x_n \in \mathbb{T}^d$.*

2.3 Locally compact abelian groups

Conjectures 1, 2 and 3 may be seen as special cases of a general conjecture for functions on locally compact abelian groups. We use the notation of [2]. We denote by G a locally compact abelian group and by $\hat{G}$ its dual group. Let $\mathrm{d}x$ denote a Haar measure on G. Such a Haar measure is unique up to a positive multiplicative constant. If $f \in L_1(G)$, we define its Fourier transform by

$$\hat{f}(\xi) = \int_G f(x)\overline{\xi(x)}\,\mathrm{d}x, \qquad \xi \in \hat{G}. \tag{9}$$

The inverse Fourier transform formula takes the form

$$f(x) = \int_{\hat{G}} \hat{f}(\xi)\xi(x)\,\mathrm{d}\xi, \qquad x \in G. \tag{10}$$

Here we have to normalize the Haar measure on $\hat{G}$ by $\mathrm{d}\xi$, otherwise a multiplicative constant occurs in (10).

In this context, Bochner's Theorem tells us that, for a bounded continuous function f on G, the matrix

$$\{f(x_j - x_k)\}_{j,k=1}^n \succeq 0$$

for any choice of $n \in \mathbb{N}$ and any $x_1, \ldots, x_n \in G$ if, and only if, the Fourier transform of f is a positive Radon measure on $\hat{G}$, cf. [2, p. 95]. In analogy to Definition 2 we define the convex cone $\mathcal{P}_+(G)$ of positive positive definite functions on G. The general conjecture may be stated as

Conjecture 4. *Let G be a locally compact abelian group. If $f : G \to \mathbb{R}$ is a bounded positive positive-definite continuous function with $f(0) = 1$, then (6) holds for every $n \in \mathbb{N}$ and every $x_1, \ldots, x_n \in G$.*

The condition (6) can be rewritten as

$$\sum_{j,k=1}^n f(x_j - x_k) c_j \overline{c_k} \geq f(0) \cdot \frac{|\sum_{j=1}^n c_j|^2}{n}, \qquad c_1, \ldots, c_n \in \mathbb{C},$$

or, equivalently,

$$\int_{G\times G} f(x-y)\,\mathrm{d}\alpha(x)\,\overline{\mathrm{d}\alpha(y)} \geq f(0) \cdot \frac{|\int_G 1\,\mathrm{d}\alpha(x)|^2}{|\mathrm{supp}\ \alpha|} = f(0) \cdot \frac{|\hat{\alpha}(1)|^2}{|\mathrm{supp}\ \alpha|}$$

where α is a complex atomic measure on G with finite support and $|\mathrm{supp}\ \alpha|$ denotes the cardinality of the support of α.

Since f and $\hat{f}$ are real valued, we can use

$$f(y-x) = \int_{\hat{G}} \hat{f}(\xi)\xi(y-x)\,\mathrm{d}\xi = \int_{\hat{G}} \hat{f}(\xi)\xi(x-y)\,\mathrm{d}\xi = f(x-y)$$

to get the equivalent inequality

$$\int_{\hat{G}} \hat{f}(\xi)|\hat{\alpha}(\xi)|^2\,\mathrm{d}\xi \geq f(0) \cdot \frac{|\hat{\alpha}(1)|^2}{|\mathrm{supp}\ \alpha|}.$$

If G is a compact group (e.g. $G = \mathbb{T}^d$), then $\hat{G}$ is a discrete group ($\hat{G} = \mathbb{Z}^d$ if $G = \mathbb{T}^d$), and after appropriate normalisation of the Haar measures on G and $\hat{G}$ this inequality may be reformulated using sums as

$$\sum_{\xi\in\hat{G}} \hat{f}(\xi)|\hat{\alpha}(\xi)|^2 \geq f(0) \cdot \frac{|\hat{\alpha}(1)|^2}{|\mathrm{supp}\ \alpha|}. \tag{11}$$

Since any matrix as in (6) involves only finitely many elements in G, the structure theorem for compactly generated abelian groups, see [5], and approximation of functions on the torus by functions on finite cyclic groups can be used to reduce Conjecture 4 to the case of finite groups G. We finally reformulate the conjecture in these terms using (11). We assume that the Haar measure on G is chosen as the counting measure. Then the proper normalisation of the Haar measure on $\hat{G}$ is the normalised counting measure and Conjecture 4 is equivalent to

Conjecture 5. *Let G be any finite abelian group. If f is a nonnegative function on G whose Fourier transform is nonnegative, then*

$$\frac{1}{|G|}\sum_{\xi\in\hat{G}}\hat{f}(\xi)|\hat{\alpha}(\xi)|^2 \geq \frac{1}{|\text{supp }\alpha|}f(0)|\hat{\alpha}(1)|^2$$

for every complex valued function α on G.

One appealing feature of this formulation is that for any given finite abelian group G the truth of the conjecture can, in principle, be checked with a finite algorithm as follows. We identify the real valued functions on G with $\mathbb{R}^d$ where $d = |G|$. Then the conditions $f(x) \geq 0$ for $x \in G$ and $\hat{f}(\xi) \geq 0$ for $\xi \in \hat{G}$ give $2d$ linear conditions which determine the cone of positive positive-definite functions. We may assume also that $f(0) = 1$. Then we obtain a convex $(d-1)$-dimensional polytope P with at most $2d$ faces. We need to check the conjecture only for functions f corresponding to the vertices of P. For each of these finitely many functions we can check the conjecture by verifying for any subset of I of G that

$$\{f(x-y)\}_{x,y\in I} \succeq \frac{E_{|I|}}{|I|}.$$

Conjecture 5 is easily seen to be true if α is restricted to satisfy $|\text{supp }\alpha| \leq 2$. In the case $|\text{supp }\alpha| = 1$ it is just the equation

$$f(0) = \frac{1}{|G|}\sum_{\xi\in\hat{G}}\hat{f}(\xi)|, \tag{12}$$

in the case $|\text{supp }\alpha| = 2$ it can be translated back to the positive semi-definiteness of the matrix

$$\begin{pmatrix} f(0)-1/2 & f(x-y)-1/2 \\ f(y-x)-1/2 & f(0)-1/2 \end{pmatrix}.$$

The following two results provide further special cases.

Proposition 2. *Let G be any finite abelian group. If f is a nonnegative function on G whose Fourier transform is nonnegative, then*

$$\sum_{\xi\in\hat{G}}\hat{f}(\xi)|\hat{\alpha}(\xi)|^2 \geq f(0)|\hat{\alpha}(1)|^2$$

for every complex valued function α on G.

Proof. Since $\hat{f}(\xi) \geq 0$ for all $\xi \in \hat{G}$ it is enough to check that $\hat{f}(1) \geq f(0)$. Moreover, $\hat{f}(\xi) \leq \hat{f}(0)$ for all $\xi \in \hat{G}$ follows from the positive definiteness of $\hat{f}$. Now $\hat{f}(1) \geq f(0)$ is immediate from (12). □

In particular, Conjecture 5 also holds if α has full support. Analogously, the original Conjecture 1 is true for $n = 2^d$.

Theorem 2. *Let $G = \mathbb{Z}_2^d$ be the d-th power of the cyclic group of order 2. If $f : G \to \{0,1\}$ is a function which has tensor product structure $f(x) = f_1(x_1) \dots f_d(x_d)$ where $f_i : \mathbb{Z}_2 \to \{0,1\}$ is positive definite, then*

$$\frac{1}{|G|} \sum_{\xi \in \hat{G}} \hat{f}(\xi) |\hat{\alpha}(\xi)|^2 \geq \frac{1}{|\text{supp } \alpha|} f(0) |\hat{\alpha}(1)|^2$$

for every complex valued function α on G.

This implies that Conjecture 4 is true for the Cantor group $G = \mathbb{Z}_2^\infty$ at least for functions with tensor product structure taking values in $\{0,1\}$ and that Conjecture 1 is true if the angles $x_{i,j}$ are restricted to the set $\{0, \pm\pi\}$. Of course, this implies that it is also true for any convex combination of such functions.

Proof. We prove the equivalent statement from Conjecture 4, i.e

$$\sum_{j,k=1}^{n} f(x_j - x_k) c_j c_k \geq \frac{f(0)}{n} \Big(\sum_{j=1}^{n} c_j \Big)^2 \tag{13}$$

whenever $x_1, \dots, x_n \in G$ and $c_1, \dots, c_n \in \mathbb{R}$. Observe that each f_i in the tensor product decomposition of f is either the constant function 0, the constant function 1 or the function given by $g(0) = 1$ and $g(1) = 0$. If one of the functions is the zero function the inequality (13) is trivial. Any factor f_i which is the constant 1 function can be omitted, so we may as well assume that $f_i = g$ for $i = 1, \dots, d$.

Then $f(x) = 1$ for $x = (x^1, \cdots, x^d) \in G$ if and only if $x^1 = \cdots = x^d = 0$. Let $A = \{x_1, \dots, x_n\} \subset G$ and define for $x \in G$ the set $A_x = \{j = 1, \dots, n : x_j = x\}$. We conclude that

$$\sum_{j,k=1}^{n} f(x_j - x_k) c_j c_k = \sum_{x \in G} \sum_{j,k \in A_x} c_j c_k = \sum_{x \in A} \Big(\sum_{j \in A_x} c_j \Big)^2 \geq \frac{1}{|A|} \Big(\sum_{x \in A} \sum_{j \in A_x} c_j \Big)^2 \geq \frac{1}{n} \Big(\sum_{j=1}^{n} c_j \Big)^2.$$

□

Finally, we reformulate (11) using Plancherel's identity for f of type $f = g * g$, where g is a non-negative even function. Then (11) reads

$$\sum_{x \in G} |(g * \alpha)(x)|^2 \geq \sum_{x \in G} g(x)^2 \cdot \frac{|\sum_{x \in G} \alpha(x)|^2}{|\text{supp } \alpha|}.$$

For $G = \mathbb{Z}$, this leads to

Conjecture 6. *Let* $\{g_m\}_{m\in\mathbb{Z}}$ *be a nonnegative sequence and let* $\{\alpha_n\}_{n\in\mathbb{Z}}$ *be an arbitrary sequence with finite support. Then*

$$\sum_{m\in\mathbb{Z}}\left|\sum_{n\in\mathbb{Z}} g_{m-n}\alpha_n\right|^2 \ge \frac{(\sum_{m\in\mathbb{Z}} g_m^2)|\sum_{n\in\mathbb{Z}}\alpha_n|^2}{|\text{supp }\alpha|}. \tag{14}$$

For a nonnegative sequence $g=\{g_m\}_{m\in\mathbb{Z}}$ let $\mathcal{G}$ be the infinite-dimensional Toeplitz matrix generated by g, i.e the entry in row m and column n of $\mathcal{G}$ is g_{m-n} for all $m,n\in\mathbb{Z}$. Furthermore, let $\alpha=\{\alpha_n\}_{n\in\mathbb{Z}}$ be an arbitrary sequence with finite support. Then (14) means

$$||\mathcal{G}\alpha||_2^2 \ge \frac{||g||_2^2|\sum_{n\in\mathbb{Z}}\alpha_n|^2}{|\text{supp }\alpha|}$$

so this conjecture would provide a lower bound of the norm of the image of such a Toeplitz matrix with positive entries on sequences with finite support.

2.4 Computer experiments

Let us briefly describe the computer experiments, which motivated the conjectures above.

We tested Conjecture 2 for tensor products of the following functions

$$\frac{1+\cos t}{2},\quad \left(\frac{\sin t}{t}\right)^2,\quad e^{-t^2/2},\quad \frac{t^4+6}{6}\,e^{-t^2/2},\quad \frac{1}{3}+\frac{4}{9}\cos t+\frac{2}{9}\cos 2t,\quad \max(1-|t|,0)$$

and the function $e^{-|x|}$ for $x\in\mathbb{R}^d$ with

$$(n,d)\in\{(3,2),(4,3),(4,4),(6,4)\}.$$

The nonnegativity of these functions is easily checked, positive definiteness follows from Bochner's Theorem by computing the Fourier transform (or the Fourier series of the periodic functions) and checking its nonnegativity. For each case, we made about 10^8 numerical experiments - of course, without finding a single counterexample. Also Conjecture 6 was tested extensively.

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