WEAK ESTIMATES CANNOT BE OBTAINED BY EXTRAPOLATION

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Abstract. We prove that weak-type estimates cannot be obtained via extrapolation.

1. Introduction

One of the consequences of the classical extrapolation theorem of Yano ([5]; for a comprehensive theory see [2] or [4]) asserts that if a sublinear operator T is bounded on $L^p(0,1)$ for every $p \in (1,2)$ and

$$\|T\|_{L^p(0,1)\to L^p(0,1)} := \sup_{f\not\equiv 0} \frac{\|Tf\|_{L^p(0,1)}}{\|f\|_{L^p(0,1)}} \le \frac{C}{p-1} \tag{1}$$

with C independent of p, then

$$T : L\log L(0,1) \to L^1(0,1),$$

where $L\log L(0,1)$ is the *logarithmic Zygmund class* defined as the set of all measurable functions f on $(0,1)$ such that

$$\int_0^1 |f(x)|(1+\log_+|f(x)|)\,dx < \infty.$$

This behaviour is typical for many important operators including the Hardy–Littlewood maximal operator, the Hilbert transform or singular integrals. Another typical property of operators satisfying (1) is their weak $(1,1)$ boundedness, that is,

$$T : L^1(0,1) \to L^{1,\infty}(0,1),$$

where $L^{1,\infty}(0,1)$ is the *weak Lebesgue space*, defined as the set of all measurable functions f on $(0,1)$ such that

$$\sup_{\lambda>0} \lambda\,|\{x\in(0,1),\ |f(x)|>\lambda\}| < \infty.$$

However, this property cannot be extrapolated from the behaviour of the L^p-norms even when their blow-up is arbitrarily slow. This fact was noted by several authors, we refer to a construction briefly described in [3, Section 5.9] involving convolution operators or to the recent paper [1, Remark 4.5].

In this note we give an elementary proof of this fact, based on an assertion of independent ineterest (Proposition below), which yields a construction of a function with a more or less prescribed behaviour of its L^p-norm in dependence on p.

2. The result and the proof

Theorem. *Let F be a function defined on $(1,2)$ with values in $(1,\infty)$, satisfying*

$$\lim_{p\to 1_+} F(p) = \infty.$$

Then there exists a sublinear operator T defined on $L^1(0,1)$ such that

$$\|T\|_{L^p(0,1)\to L^p(0,1)} \le F(p), \qquad p\in(1,2),$$

but T is not bounded from $L^1(0,1)$ to $L^{1,\infty}(0,1)$.

The key step in our proof of Theorem is the following proposition.

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Proposition. *Assume that G is a function defined on $(2,\infty)$ with values in $(1,\infty)$, such that*

$$\lim_{q\to\infty} G(q)=\infty. \tag{2}$$

Then there exists a nonnegative measurable function w on $(0,1)$ such that

$$\lim_{q\to\infty} \|w\|_{L^q(0,1)}=\infty \tag{3}$$

and

$$\|w\|_{L^q(0,1)}\le G(q), \qquad q\in(2,\infty). \tag{4}$$

Proof. We will first construct a decreasing sequence $\{a_k\}\subset(0,1)$. Let $a_1=1$. Fix $k\in\mathbb{N}$ and assume that $a_1,\dots,a_{k-1}$ have been already chosen. The set of all $q\in(2,\infty)$ such that $G(q)q^{-\frac{1}{q}}\le k$ is bounded by some q_k. This easily follows from (2). Therefore, the number

$$b_k:=\inf_{q\in(2,q_k]}\frac{G(q)^q}{2qk^q(k+1)}$$

is strictly positive. We set

$$a_k:=\min\left\{\frac{1}{2(k+1)},b_k,\frac{a_{k-1}}{2}\right\}.$$

Then,

$$qk^{q-1}a_k\le\begin{cases}\frac{G(q)^q}{2k(k+1)} & \text{when } G(q)q^{-\frac{1}{q}}\le k;\\ \frac{qk^{q-1}}{2(k+1)}\le\frac{G(q)^q}{2k(k+1)} & \text{when } G(q)q^{-\frac{1}{q}}> k.\end{cases}$$

Summing over all k, we get

$$\sum_{k\in\mathbb{N}} qk^{q-1}a_k\le G(q)^q. \tag{5}$$

We finally define

$$w(x):=k \qquad \text{when } x\in(a_{k+1},a_k).$$

Then (3) is obviously satisfied, since w is unbounded. Moreover, by (5)

$$\begin{aligned}\|w\|^q_{L^q(0,1)}&=\sum_{k\in\mathbb{N}}\int_{k-1}^{k} q\lambda^{q-1}\left|\{x\in(0,1);\ w(x)>\lambda\}\right|\,d\lambda\\ &\le\sum_{k\in\mathbb{N}} qk^{q-1}a_k\\ &\le G(q)^q,\end{aligned}$$

and (4) follows. The proof is complete. □

Proof of Theorem. For $p\in(1,\infty)$, define $p'=\frac{p}{p-1}$. Let $F(p)$ be a function satisfying the assumptions of the Theorem. Applying the Proposition to the function $G(p'):=F(p)$, we obtain a nonnegative measurable function w on $(0,1)$ such that, for $p\in(1,2)$, $\|w\|_{L^{p'}(0,1)}\le G(p')=F(p)$. We define the operator T by

$$Tf:=\left(\int_0^1|f(y)|w(y)\,dy\right)\cdot\chi_{(0,1)}.$$

Then

$$\|T\|_{L^p(0,1)\to L^p(0,1)}=\|w\|_{L^{p'}(0,1)}\le F(p), \qquad p\in(1,2),$$

but

$$\|Tf\|_{L^1(0,1)\to L^{1,\infty}(0,1)}=\sup_{f\not\equiv0}\frac{\sup_{\lambda>0}\lambda\left|\{x\in(0,1);\ |Tf(x)|>\lambda\}\right|}{\|f\|_{L^1(0,1)}}=\sup_{f\not\equiv0}\frac{\int_0^1|f(y)|w(y)\,dy}{\|f\|_{L^1(0,1)}}=\infty,$$

since w is unbounded. The proof is complete. □

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