On dilation operators in Triebel-Lizorkin spaces

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Abstract

We consider dilation operators $T_k : f \to f(2^k \cdot)$ in the framework of Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$. If $s > n\max\left(\frac{1}{p}-1,0\right)$, T_k is a bounded linear operator from $F^s_{p,q}(\mathbb{R}^n)$ into itself and there are optimal bounds for its norm. We study the situation on the line $s = n\max\left(\frac{1}{p}-1,0\right)$, an open problem mentioned in [ET96, 2.3.1]. It turns out that the results shed new light upon the diversity of different approaches to Triebel-Lizorkin spaces on this line, associated to definitions by differences, Fourier-analytical methods and subatomic decompositions.

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Introduction

In this article dilation operators acting on Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ are investigated. The idea for this paper originates from its forerunners [Vyb08] and [SchXX], where the authors studied corresponding problems for Besov spaces. Since the substantial theory of the Triebel-Lizorkin spaces is strongly linked with the theory of Besov spaces – in the sequel briefly denoted as F-spaces and B-spaces, respectively – the question came up whether those previous results could be carried over to the F-space setting. This paper aims at providing a rather final answer to this question.
We consider dilation operators of the form

$$T_k f(x) = f(2^k x), \quad x \in \mathbb{R}^n, \quad k \in \mathbb{N}, \tag{0.1}$$

which represent bounded operators from $F^s_{p,q}(\mathbb{R}^n)$ into itself. Their behaviour is well known when $s > \sigma_p = n\max\left(\frac{1}{p}-1,0\right)$. Then we have for $0 < p < \infty$, $0 < q \le \infty$,

$$\|T_k|\mathcal{L}(F^s_{p,q}(\mathbb{R}^n))\| \sim 2^{k(s-\frac{n}{p})}, \qquad s > \sigma_p,$$

cf. [ET96, 2.3.1, 2.3.2]. Here we investigate the situation on the line $s = \sigma_p$. For $1 < p < \infty$ and $0 < p \le 1$ with $p \le q$ we obtain sharp estimates for the norms of the operators T_k, i.e.,

$$\|T_k|\mathcal{L}(F^{\sigma_p}_{p,q}(\mathbb{R}^n))\| \sim 2^{k(\sigma_p-\frac{n}{p})} \cdot \begin{cases} k^{\frac{1}{q}-\frac{1}{\max(q,2)}} & \text{if } 1 < p < \infty, \\ k^{1/p} & \text{if } 0 < p \le 1, \quad p \le q, \end{cases}$$

whereas, for $0 < q < p < 1$, we only have

$$2^{k(\sigma_p-\frac{n}{p})}k^{1/p} \lesssim \|T_k|\mathcal{L}(F^{\sigma_p}_{p,q}(\mathbb{R}^n))\| \lesssim 2^{k(\sigma_p-\frac{n}{p})}k^{1/q}$$

or, when $0 < q < p = 1$,

$$2^{-kn}k^{\max(1,1/q-1/2)} \lesssim \|T_k|\mathcal{L}(F^0_{1,q}(\mathbb{R}^n))\| \lesssim 2^{-kn}k^{1/q}.$$

As a by-product, the results for the dilation operators lead to new insights concerning the nature of the different approaches to F-spaces with positive smoothness – namely the classical ($\mathbf{F}^s_{p,q}$), the Fourier-analytical ($F^s_{p,q}$) and the subatomic approach ($\mathfrak{F}^s_{p,q}$) – on the line $s = \sigma_p$. Recent results by Hedberg, Netrusov [HN07] on atomic decompositions and by Triebel [Tri06, Sect. 9.2] on the reproducing formula prove coincidences

$$\mathbf{F}^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n), \qquad s > n\left(\frac{1}{\min(p,q)}-\frac{1}{p}\right), \quad 0 < p < \infty, \quad 0 < q \le \infty,$$

and

$$F^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n), \qquad s > n\left(\frac{1}{\min(p,q,1)} - 1\right), \quad 0 < p < \infty, \quad 0 < q \leq \infty,$$

resulting in

$$F^s_{p,q}(\mathbb{R}^n) = \mathbf{F}^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n),$$

whenever

$$0 < p < \infty, \quad 0 < q \leq \infty, \quad s > n\left(\frac{1}{\min(p,q)} - \frac{1}{\max(1,p)}\right)$$

(in terms of equivalent quasi-norms).
Furthermore, since for $s < n(\frac{1}{p} - 1)$ the δ-distribution belongs to $F^s_{p,q}(\mathbb{R}^n)$ – which is a singular distribution and cannot be interpreted as a function – the spaces

$$F^s_{p,q}(\mathbb{R}^n) \qquad \text{and} \qquad \mathfrak{F}^s_{p,q}(\mathbb{R}^n), \qquad 0 < s < \sigma_p,$$

cannot be compared. The situation on the line $s = \sigma_p$, $0 < p < 1$, so far remained an open problem. In this case $F^s_{p,q}(\mathbb{R}^n)$ is a subspace of $L_1^{loc}(\mathbb{R}^n)$ and the two spaces $F^{\sigma_p}_{p,q}(\mathbb{R}^n)$ and $\mathfrak{F}^{\sigma_p}_{p,q}(\mathbb{R}^n)$ can be compared. But our results yield, that they do not coincide, i.e.,

$$F^{\sigma_p}_{p,q}(\mathbb{R}^n) \neq \mathfrak{F}^{\sigma_p}_{p,q}(\mathbb{R}^n), \qquad 0 < q \leq \infty.$$

1 Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$

We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be euclidean n-space, $n \in \mathbb{N}$, $\mathbb{C}$ the complex plane. The set of multi-indices $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_i \in \mathbb{N}_0$, $i = 1, \ldots, n$, is denoted by $\mathbb{N}_0^n$, with $|\beta| = \beta_1 + \cdots + \beta_n$, as usual. Moreover, if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ we put $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$.
We use the equivalence '$\sim$' in

$$a_k \sim b_k \quad \text{or} \quad \varphi(x) \sim \psi(x)$$

always to mean that there are two positive numbers c_1 and c_2 such that

$$c_1\, a_k \leq b_k \leq c_2\, a_k \quad \text{or} \quad c_1\, \varphi(x) \leq \psi(x) \leq c_2\, \varphi(x)$$

for all admitted values of the discrete variable k or the continuous variable x, where $\{a_k\}_k$, $\{b_k\}_k$ are non-negative sequences and φ, ψ are non-negative functions. If $a \in \mathbb{R}$, then $a_+ := \max(a, 0)$ and $[a]$ denotes the integer part of a.
All unimportant positive constants will be denoted by c, occasionally with subscripts. For convenience, let both $\mathrm{d}x$ and $|\cdot|$ stand for the (n-dimensional) Lebesgue measure in the sequel. As we shall always deal with function spaces on $\mathbb{R}^n$, we may usually omit the '$\mathbb{R}^n$' from their notation for convenience.

Let for $0 < p, q \leq \infty$ the numbers σ_p and σ_{pq} be given by

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ \qquad \text{and} \qquad \sigma_{pq} = n\left(\frac{1}{\min(p,q)} - 1\right)_+ . \tag{1.1}$$

Furthermore, let $Q_{\nu,m}$ with $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ denote a cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centered at $2^{-\nu}m$, and with side length $2^{-\nu}$. For a cube Q in $\mathbb{R}^n$ and $r > 0$, we denote by rQ the cube in $\mathbb{R}^n$ concentric with Q and with side length r times the side length of Q. Moreover, $\chi^{(p)}_{\nu,m}$ stands for the p-normalized characteristic function of $Q_{\nu,m}$, i.e.,

$$\chi^{(p)}_{\nu,m}(x) = 2^{\frac{\nu n}{p}} \quad \text{if} \quad x \in Q_{\nu,m} \qquad \text{and} \qquad \chi^{(p)}_{\nu,m}(x) = 0 \quad \text{if} \quad x \notin Q_{\nu,m}.$$

Of course

$$\|\chi^{(p)}_{\nu,m}|L_p(\mathbb{R}^n)\| = 1.$$

The Fourier-analytical approach

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and its dual $\mathcal{S}'(\mathbb{R}^n)$ of all complex-valued tempered distributions have their usual meaning here. Let $\varphi_0 = \varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp}\varphi \subset \{y \in \mathbb{R}^n : |y| < 2\} \quad \text{and} \quad \varphi(x) = 1 \quad \text{if} \quad |x| \leq 1\,, \tag{1.2}$$

and for each $j \in \mathbb{N}$ let $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$. Then $\{\varphi_j\}_{j=0}^{\infty}$ forms a *smooth dyadic resolution of unity*. Given any $f \in \mathcal{S}'(\mathbb{R}^n)$, we denote by $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ its Fourier transform and its inverse Fourier transform, respectively. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, then the compact support of $\varphi_j \mathcal{F}f$ implies by the Paley-Wiener-Schwartz theorem that $\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)$ is an entire analytic function on $\mathbb{R}^n$.

Definition 1.1 *Let* $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, *and* $\{\varphi_j\}_j$ *a smooth dyadic resolution of unity. The space* $F^s_{p,q}(\mathbb{R}^n)$ *is the set of all distributions* $f \in \mathcal{S}'(\mathbb{R}^n)$ *such that*

$$\left\|f \,|F^s_{p,q}(\mathbb{R}^n)\right\| = \left\|\left\| \left\{2^{js}\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)(\cdot)\right\}_{j\in\mathbb{N}_0} |\ell_q\right\| |L_p(\mathbb{R}^n)\right\| \tag{1.3}$$

is finite.

Remark 1.2 The spaces $F^s_{p,q}(\mathbb{R}^n)$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}_j$ appearing in their definition. They are quasi-Banach spaces (Banach spaces for $p, q \geq 1$), and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, where the first embedding is dense if $q < \infty$. An extension of Definition 1.1 to $p = \infty$ does not make sense if $0 < q < \infty$ (in particular, a corresponding space is not independent of the choice $\{\varphi_j\}_j$). The case $p = q = \infty$ yields the Besov spaces $B^s_{\infty,\infty}(\mathbb{R}^n)$.
In general, the Fourier-analytical Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ are defined correspondingly to the spaces $F^s_{p,q}(\mathbb{R}^n)$ by interchanging the order in which the quasi-norms are taken, i.e., first using the L_p-norm and afterwards applying the ℓ_q-norm – in view of (1.3). These B-spaces are closely linked with the Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ via

$$B^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^n). \tag{1.4}$$

The theory of the spaces $F^s_{p,q}(\mathbb{R}^n)$ (and $B^s_{p,q}(\mathbb{R}^n)$) has been developed in detail in [Tri83] and [Tri92] (and continued and extended in the more recent monographs [Tri01], [Tri06]), but has a longer history already including many contributors; we do not further want to discuss this here.
Note that the spaces $F^s_{p,q}(\mathbb{R}^n)$ contain tempered distributions which can only be interpreted as regular distributions (functions) for sufficiently high smoothness. More precisely, we have

$$F^s_{p,q}(\mathbb{R}^n) \subset L_1^{\mathrm{loc}}(\mathbb{R}^n) \quad \text{if, and only if,} \quad \begin{cases} s \geq \sigma_p, & \text{for } 0 < p < 1,\ 0 < q \leq \infty, \\ s > \sigma_p, & \text{for } 1 \leq p < \infty,\ 0 < q \leq \infty, \\ s = \sigma_p, & \text{for } 1 \leq p < \infty,\ 0 < q \leq 2, \end{cases} \tag{1.5}$$

cf. [ST95, Thm. 3.3.2]. In particular, for $s < \sigma_p$ one cannot interpret $f \in F^s_{p,q}(\mathbb{R}^n)$ as a regular distribution in general.
The scale $F^s_{p,q}(\mathbb{R}^n)$ contains many well-known function spaces. We list a few special cases.
Let $1 < p < \infty$, then

$$F^s_{p,2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n), \qquad s \in \mathbb{R},$$

are the (*fractional*) *Sobolev spaces* containing all $f \in S'(\mathbb{R}^n)$ with

$$\mathcal{F}^{-1}(1 + |\xi|^2)^{s/2}\mathcal{F}f \in L_p(\mathbb{R}^n).$$

In particular, for $k \in \mathbb{N}_0$, we obtain the *classical Sobolev spaces*

$$F^k_{p,2}(\mathbb{R}^n) = W^k_p(\mathbb{R}^n), \qquad \text{i.e.,} \qquad F^0_{p,2}(\mathbb{R}^n) = L_p(\mathbb{R}^n),$$

usually normed by

$$\|f|W^k_p(\mathbb{R}^n)\| = \left(\sum_{|\alpha|\leq k} \|\mathrm{D}^\alpha f|L_p(\mathbb{R}^n)\|^p\right)^{1/p}.$$

Furthermore,

$$F^0_{p,2}(\mathbb{R}^n) = h_p(\mathbb{R}^n), \qquad 0 < p < \infty,$$

the latter being the *inhomogenoues Hardy spaces.*

Local means and atomic decompositions

There are equivalent characterizations for the F-spaces $F^s_{p,q}(\mathbb{R}^n)$ in terms of *local means* and *atomic decompositions*. We first sketch the approach via local means. For further details we refer to [BPT96], [BPT97], and [Tri06] with forerunners in [Tri92, Sect. 2.5.3].
Let $B = \{y \in \mathbb{R}^n : |y| < 1\}$ be the unit ball in $\mathbb{R}^n$ and let κ be a C^∞ function in $\mathbb{R}^n$ with supp $\kappa \subset B$. Then

$$k(t,f)(x) = \int_{\mathbb{R}^n} \kappa(y) f(x+ty)\mathrm{d}y = t^{-n} \int_{\mathbb{R}^n} \kappa\Big(\frac{y-x}{t}\Big) f(y)\mathrm{d}y \tag{1.6}$$

with $x \in \mathbb{R}^n$, and $t > 0$ are *local means* (appropriately interpreted for $f \in S'(\mathbb{R}^n)$). For given $s \in \mathbb{R}$ it is assumed that the kernel κ satisfies in addition for some $\varepsilon > 0$,

$$\kappa^\vee(\xi) \neq 0 \text{ if } 0 < |\xi| < \varepsilon \qquad \text{and} \qquad (\mathrm{D}^\alpha \kappa^\vee)(0) = 0 \text{ if } |\alpha| \leq s. \tag{1.7}$$

The second condition is empty if $s < 0$. Furthermore, let κ_0 be a second C^∞ function in $\mathbb{R}^n$ with supp $\kappa_0 \subset B$ and $\kappa_0^\vee(0) \neq 0$. The meaning of $k_0(f,t)$ is defined in the same way as (1.6) with κ_0 instead of κ.
We have the following characterization in terms of local means, cf. [Tri06, Th. 1.10] and [Ryc99].

Theorem 1.3 *Let* $0 < p < \infty$, $0 < q \leq \infty$ *and* $s \in \mathbb{R}$. *Let* κ_0 *and* κ *be the above kernels of local means. Then for* $f \in S'(\mathbb{R}^n)$,

$$\|k_0(1,f)|L_p(\mathbb{R}^n)\| + \left\| \left(\sum_{j=1}^\infty 2^{jsq} |k(2^{-j},f)(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \tag{1.8}$$

is an equivalent quasi-norm in $F^s_{p,q}(\mathbb{R}^n)$.

Remark 1.4 We shall only need one part of Theorem 1.3, namely that $\|f|F^s_{p,q}(\mathbb{R}^n)\|$ can be estimated from below by (1.8). In that case some of the asumptions in (1.7) may be omitted. The inspection of the proof, cf. [Ryc99, Rem. 3], shows that if κ is a C^∞ function in $\mathbb{R}^n$ with

$$\text{supp } \kappa \subset B \qquad \text{and} \qquad \mathrm{D}^\alpha \kappa^\vee(0) = 0, \qquad |\alpha| \leq N,$$

where $N > s-1$, then

$$\|f|F^s_{p,q}(\mathbb{R}^n)\| \geq c \left\| \left(\sum_{j=1}^\infty 2^{jsq} |k(2^{-j},f)(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\|$$

for some $c > 0$.

The following atomic characterization of function spaces of type $F^s_{p,q}(\mathbb{R}^n)$ is sometimes preferred (compared with the above Fourier-analytical approach), e.g. when establishing the lower bound for the dilation operators later on; we closely follow the presentation in [Tri97, Sect. 13].

Definition 1.5 *Let* $0 < p < \infty$, $0 < q \leq \infty$, *and* $\lambda = \{\lambda_{\nu,m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$. *Then*

$$f_{p,q} = \left\{ \lambda : \|\lambda|f_{p,q}\| = \left\| \left(\sum_{\nu=0}^\infty \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m} \chi^{(p)}_{\nu,m}(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| < \infty \right\}$$

(*with the usual modification if* $p = \infty$ *and/or* $q = \infty$).

Definition 1.6

(i) *Let* $K \in \mathbb{N}_0$ *and* $d > 1$. *A* K*-times differentiable complex-valued function* a *on* $\mathbb{R}^n$ (*continuous if* $K = 0$) *is called a* 1_K*-atom if*

$$\text{supp}\, a \subset dQ_{0,m} \quad \textit{for some} \quad m \in \mathbb{Z}^n, \tag{1.9}$$

and

$$|\mathrm{D}^\alpha a(x)| \leq 1 \quad \textit{for} \quad |\alpha| \leq K.$$

(ii) *Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $K \in \mathbb{N}_0$, $L+1 \in \mathbb{N}_0$, and $d > 1$. A K-times differentiable complex-valued function a on $\mathbb{R}^n$ (continuous if $K = 0$) is called an $(s,p)_{K,L}$-atom if for some $\nu \in \mathbb{N}_0$*

$$\operatorname{supp} a \subset dQ_{\nu,m} \quad \textit{for some} \quad m \in \mathbb{Z}^n, \tag{1.10}$$

$$|\mathrm{D}^\alpha a(x)| \leq 2^{-\nu(s-\frac{n}{p})+|\alpha|\nu} \quad \textit{for } |\alpha| \leq K, \tag{1.11}$$

and

$$\int_{\mathbb{R}^n} x^\beta a(x)\mathrm{d}x = 0 \quad \textit{if } |\beta| \leq L. \tag{1.12}$$

It is convenient to write $a_{\nu,m}(x)$ instead of $a(x)$ if this atom is located at $Q_{\nu,m}$ according to (1.9) and (1.10). Assumption (1.12) is called a *moment condition*, where $L = -1$ means that there are no moment conditions. Furthermore, K denotes the smoothness of the atom, cf. (1.11). The atomic characterization of function spaces of type $F^s_{p,q}(\mathbb{R}^n)$ is given by the following result, cf. [Tri97, Thm. 13.8].

Theorem 1.7 *Let $0 < p < \infty$, $0 < q \leq \infty$, and $s \in \mathbb{R}$. Let $K \in \mathbb{N}_0$ and $L+1 \in \mathbb{N}_0$ with*

$$K \geq (1+[s])_+ \qquad \textit{and} \qquad L \geq \max(-1, [\sigma_{pq} - s])$$

be fixed. Then $f \in S'(\mathbb{R}^n)$ belongs to $F^s_{p,q}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}(x), \quad \textit{convergence being in} \quad S'(\mathbb{R}^n), \tag{1.13}$$

where the $a_{\nu,m}$ are 1_K-atoms ($\nu = 0$) or $(s,p)_{K,L}$-atoms ($\nu \in \mathbb{N}$) with

$$\operatorname{supp} a_{\nu,m} \subset dQ_{\nu,m}, \qquad \nu \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \quad d > 1,$$

and $\lambda \in f_{p,q}$. Furthermore,

$$\inf \|\lambda | f_{p,q}\|,$$

where the infimum is taken over all admissible representations (1.13), is an equivalent quasi-norm in $F^s_{p,q}(\mathbb{R}^n)$.

2 Dilation Operators

In this section we present our main results concerning dilation operators T_k in F-spaces when $s = \sigma_p$. We distinguish between the cases $1 < p < \infty$ and $0 < p \leq 1$, when $\sigma_p = 0$ and $\sigma_p = n(1/p - 1)$, respectively.

Theorem 2.1 *Let $1 < p < \infty$ and $0 < q \leq \infty$. Then*

$$\|T_k | \mathcal{L}(F^0_{p,q}(\mathbb{R}^n))\| \sim 2^{-k\frac{n}{p}} \cdot k^{\frac{1}{q} - \frac{1}{\max(q,2)}}, \quad k \in \mathbb{N}.$$

Proof : Step 1. Recall Definition 1.1, where in particular the dyadic resolution of unity was constructed such that

$$\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x), \qquad j \in \mathbb{N}.$$

Elementary calculation yields

$$(\varphi_j(\xi)\widehat{f(2^k\cdot)}(\xi))^\vee(x) = 2^{-kn}(\varphi_j(\xi)\widehat{f}(2^{-k}\xi))^\vee(x) = (\varphi_j(2^k\xi)\widehat{f}(\xi))^\vee(2^kx). \tag{2.1}$$

For convenience we assume $q < \infty$ in the sequel, but the counterpart for $q = \infty$ is obvious. From the definition of F-spaces with $f(2^k x)$ in place of $f(x)$ we obtain

$$
\begin{aligned}
&\|f(2^k\cdot)|F^{\sigma_p}_{p,q}(\mathbb{R}^n)\| \\
&= \left\| \left(\sum_{j=0}^{\infty} 2^{j\sigma_p q} |(\varphi_j(2^k\cdot)\widehat{f})^\vee(2^k\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \\
&= 2^{-k\frac{n}{p}} \left\| \left(\sum_{j=0}^{\infty} 2^{j\sigma_p q} |(\varphi_j(2^k\cdot)\widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \\
&\sim 2^{-k\frac{n}{p}} \left\{ \left\| (\varphi_0(2^k\cdot)\widehat{f}(\cdot))^\vee(\cdot)|L_p(\mathbb{R}^n) \right\| + \left\| \left(\sum_{j=1}^{k} 2^{j\sigma_p q} |(\varphi_j(2^k\cdot)\widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \right. \\
&\qquad \left. + \left\| \left(\sum_{j=k+1}^{\infty} 2^{j\sigma_p q} |(\varphi_j(2^k\cdot)\widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \right\}
\end{aligned}
\tag{2.2}
$$

If $j \geq k+1$, then $\varphi_j(2^k x) = \varphi_{j-k}(x)$. This yields for the last term

$$
\begin{aligned}
&2^{-k\frac{n}{p}} \left\| \left(\sum_{j=k+1}^{\infty} 2^{j\sigma_p q} |(\varphi_j(2^k\cdot)\widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \\
=&2^{-k\frac{n}{p}} \left\| \left(\sum_{j=k+1}^{\infty} 2^{(j-k)\sigma_p q} 2^{k\sigma_p q} |(\varphi_{j-k}(\cdot)\widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \\
=&2^{k(\sigma_p - \frac{n}{p})} \left\| \left(\sum_{l=1}^{\infty} 2^{l\sigma_p q} |(\varphi_l(\cdot)\widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \\
\leq&2^{-\frac{kn}{p}} \|f|F^{\sigma_p}_{p,q}(\mathbb{R}^n)\|.
\end{aligned}
\tag{2.3}
$$

If $j = 0$, we use the Hausdorff-Young inequality and obtain

$$
\begin{aligned}
\|(\varphi_0(2^k\cdot)\widehat{f})^\vee|L_p\| &= \|(\varphi_0(2^k\cdot)\varphi_0\widehat{f})^\vee|L_p\| \\
&= \|(\varphi_0(2^k\cdot)^\vee * (\varphi_0\widehat{f})^\vee|L_p\| \\
&\leq \|(\varphi_0(2^k\cdot)^\vee|L_1\| \cdot \|(\varphi_0\widehat{f})^\vee|L_p\| \\
&\leq c\|f|F^0_{p,q}\|.
\end{aligned}
$$

Step 2. In view of Step 1 it remains to consider $j = 1, \ldots, k$. Using Hölder's inequality with

$$\frac{1}{u} = \frac{q}{2} \qquad \text{and} \qquad \frac{1}{u'} = 1 - \frac{q}{2} \qquad \text{if} \quad q < 2$$

or

$$||id : \ell_2^k \hookrightarrow \ell_q^k|| = 1 \qquad \text{when} \qquad q \geq 2 \quad \text{and} \quad k \in \mathbb{N}$$

together with the Littlewood-Paley theorem, we see that

$$
\begin{aligned}
&\left\| \left(\sum_{j=1}^{k} |(\varphi_j(2^k\cdot)\widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p \right\| \\
&\leq k^{\frac{1}{q} - \frac{1}{\max(q,2)}} \left\| \left(\sum_{j=1}^{k} |(\varphi_j(2^k\cdot)\widehat{f})^\vee(\cdot)|^2 \right)^{1/2} |L_p \right\| \\
&\leq k^{\frac{1}{q} - \frac{1}{\max(q,2)}} \|(\varphi_0\widehat{f})^\vee|L_p\| \\
&\leq k^{\frac{1}{q} - \frac{1}{\max(q,2)}} \|f|F^0_{p,q}\|,
\end{aligned}
$$

giving the desired upper bound.

Step 3. In order to establish the lower bound we take $\psi \in S(\mathbb{R}^n)$ with

$$\operatorname{supp}\psi \subset \{x \in \mathbb{R}^n : |x| \le 1/8\}.$$

We define the functions f_k through their Fourier transforms

$$\widehat{f_k}(\xi) = \sum_{j=1}^{k} \psi(2^k(\xi - \xi_j)), \qquad \xi \in \mathbb{R}^n, \quad k \in \mathbb{N},$$

where $\xi_j = (2^{-j}, 0, \dots, 0)$. We shall show that

$$\|f_k|F^0_{p,q}\| \lesssim k^{1/2} 2^{kn(1/p-1)}, \tag{2.4}$$

and

$$\|f_k(2^k\cdot)|F^0_{p,q}\| \gtrsim k^{1/q} 2^{-kn}. \tag{2.5}$$

We deal with (2.4) first. As the support of $\widehat{f_k}$ lies in the unit ball of $\mathbb{R}^n$, we may omit the terms with $j \ge 1$ in (1.3). Furthermore, since $1 < p < \infty$ we may use the Littlewood-Paley decomposition theorem to estimate

$$\begin{aligned} \|f_k|F^0_{p,q}\| &= \|(\varphi_0 \widehat{f_k})^\vee|L_p\| \\ &\lesssim \left\| \left(\sum_{j=1}^{\infty} |(\varphi_1(2^j\cdot)\cdot\varphi_0\widehat{f_k})^\vee(x)|^2 \right)^{1/2} |L_p \right\| \qquad (2.6)\\ &= \left\| \left(\sum_{j=1}^{k} |\psi(2^k(\xi-\xi_j))^\vee(x)|^2 \right)^{1/2} |L_p \right\| \\ &= \left\| \left(\sum_{j=1}^{k} |2^{-kn}\psi^\vee(2^{-k}x)e^{ix\xi_j}|^2 \right)^{1/2} |L_p \right\| \\ &= k^{1/2}2^{-kn}\left\|\psi^\vee(2^{-k}x)|L_p\right\| = k^{1/2}2^{kn(1/p-1)}\left\|\psi^\vee|L_p\right\|. \end{aligned}$$

Let us mention, that (2.4) holds also for $p = 1$. In this case, the inequality on the second line of (2.6) follows (roughly speaking) by the embedding

$$F^0_{1,2}(\mathbb{R}^n) \hookrightarrow L_1(\mathbb{R}^n),$$

cf. [ST95, Th. 3.1.1]. To prove (2.5), observe that

$$\widehat{f_k(2^k\cdot)}(\xi) = 2^{-kn}\widehat{f_k}(2^{-k}\xi) = 2^{-kn}\sum_{j=1}^{k}\psi(\xi - 2^k\xi_j), \qquad \xi \in \mathbb{R}^n.$$

Using again the support properties of ψ and φ_j, we arrive at

$$\begin{aligned} \|f_k(2^k\cdot)|F^0_{p,q}\| &= \left\| \left(\sum_{j=1}^{k} |(\varphi_j \widehat{f_k(2^k\cdot)}(\xi))^\vee(x)|^q \right)^{1/q} |L_p \right\| \\ &= 2^{-kn}\left\| \left(\sum_{j=1}^{k} |(\psi(\xi - 2^k\xi_j))^\vee(x)|^q \right)^{1/q} |L_p \right\| \\ &= 2^{-kn}\left\| \left(\sum_{j=1}^{k} |(\psi^\vee(x)e^{ix2^k\xi_j}|^q \right)^{1/q} |L_p \right\| \\ &= 2^{-kn}k^{1/q}\|\psi^\vee|L_p\|. \end{aligned}$$

Observe, that also (2.5) holds even for $p=1$.
This finally leads to

$$\|T_k|\mathcal{L}(F^0_{p,q})\| \geq \frac{\|f_k(2^k\cdot)|F^0_{p,q}\|}{\|f_k|F^0_{p,q}\|} \geq k^{1/q-1/2}2^{-\frac{kn}{p}}.$$

Step 4. Let $1<p<\infty$ and $q\geq 2$. Chose an arbitrary non-vanishing $\psi\in S(\mathbb{R}^n)$. Using the trivial embedding $F^0_{p,2}\hookrightarrow F^0_{p,q}$, we obtain

$$\|T_k|\mathcal{L}(F^0_{p,q})\| \geq \frac{\|\psi|F^0_{p,q}\|}{\|\psi(2^{-k}\cdot)|F^0_{p,q}\|} \geq \frac{\|\psi|F^0_{p,q}\|}{\|\psi(2^{-k}\cdot)|F^0_{p,2}\|} \sim 2^{-k\frac{n}{p}}.$$

□

Theorem 2.2 *Let* $0<p\leq 1$, $0<p\leq q\leq\infty$. *Then*

$$\|T_k|\mathcal{L}(F^{\sigma_p}_{p,q}(\mathbb{R}^n))\| \sim 2^{k(\sigma_p-\frac{n}{p})}k^{1/p}, \quad k\in\mathbb{N}.$$

Proof :
Step 1. We give an estimate for the upper bounds of the dilation operators T_k similar to Theorem 2.1. We need to find suitable substitutes when $0<p\leq 1$.

For the further calculations we make use of the following Fourier multiplier theorem, cf. [Tri83, Prop. 1.5.1],

$$\|(M\widehat{h})^\vee|L_p\| \leq c\|M^\vee|L_p\|\cdot\|h|L_p\|, \qquad \text{if } 0<p\leq 1, \tag{2.7}$$

with $M^\vee\in S'\cap L_p$, and $\operatorname{supp}\widehat{h}\subset\Omega$, $\operatorname{supp}M\subset\Gamma$, where Ω and Γ are compact subsets of $\mathbb{R}^n$ (c does not depend on M and h, but may depend on Ω and Γ). Of course for $p=1$ this is just the Hausdorff-Young inequality (which was also used in Theorem 2.1). We put $h=(\varphi_0\widehat{f})^\vee$, where $\operatorname{supp}\widehat{h}\subset\operatorname{supp}\varphi_0=\Omega$.
If $j=0$, we take $M_0=\varphi_0(2^k\cdot)$ where $\operatorname{supp}M_0\subset\operatorname{supp}\varphi_0=\Gamma$ and calculate

$$\begin{aligned} 2^{-k\frac{n}{p}}\|(\varphi_0(2^k\cdot)\widehat{f})^\vee|L_p\| &\leq c2^{-k\frac{n}{p}}\|\varphi_0(2^k\cdot)^\vee|L_p\|\cdot\|(\varphi_0\widehat{f})^\vee|L_p\|,\\ &= c2^{-k\frac{n}{p}}2^{k\sigma_p}\|\varphi_0{}^\vee|L_p\|\cdot\|(\varphi_0\widehat{f})^\vee|L_p\|\\ &= c'2^{k(\sigma_p-\frac{n}{p})}\|(\varphi_0\widehat{f})^\vee|L_p\|\\ &= c2^{-kn}\|f|F^{\sigma_p}_{p,q}\|. \end{aligned} \tag{2.8}$$

According to the observations in Step 1 of Theorem 2.1 it remains to consider $1\leq j\leq k$. This is the crucial step, leading to $k^{1/p}$. In this case $\varphi_j(x)=\bar{\varphi}(2^{-j}x)$, where $\bar{\varphi}=\varphi_0(x)-\varphi_0(2x)$. Hence

$$\begin{aligned} &\left\|\left(\sum_{j=1}^k 2^{j\sigma_p q}|(\varphi_j(2^k\cdot)\widehat{f})^\vee(\cdot)|^q\right)^{1/q}|L_p(\mathbb{R}^n)\right\|\\ &= \left(\int_{\mathbb{R}^n}\left(\sum_{j=1}^k 2^{j\sigma_p q}|(\varphi_j(2^k\cdot)\widehat{f})^\vee(x)|^q\right)^{p/q}\mathrm{d}x\right)^{1/p}\\ &\leq \left(\sum_{j=1}^k\int_{\mathbb{R}^n}2^{j\sigma_p p}|(\varphi_j(2^k\cdot)\widehat{f})^\vee(x)|^p\mathrm{d}x\right)^{1/p}\\ &= \left(\sum_{j=1}^k 2^{j\sigma_p p}\|(\varphi_j(2^k\cdot)\widehat{f})^\vee|L_p\|^p\right)^{1/p}\\ &= \left(\sum_{j=1}^{k-1}2^{j\sigma_p p}\|(\bar{\varphi}(2^{k-j}\cdot)\widehat{f})^\vee|L_p\|^p+2^{k\sigma_p p}\|(\bar{\varphi}\widehat{f})^\vee|L_p\|^p\right)^{1/p} \end{aligned} \tag{2.9}$$

where the inequality follows from $\ell_{\frac{p}{q}} \hookrightarrow \ell_1$ since $p < q$.
The term for $j = k$ in (2.9) needs some extra care. Using (2.7) where we set $M_k = \varphi_0(2\cdot)$, $\operatorname{supp} M_k \subset \operatorname{supp} \varphi_0 = \Gamma$ we obtain

$$\begin{aligned} 2^{k\sigma_p p}\|(\bar{\varphi}\widehat{f})^\vee|L_p\|^p &= 2^{k\sigma_p p}\|(\varphi_0\widehat{f})^\vee - (\varphi_0(2\cdot)\widehat{f})^\vee|L_p\|^p \\ &\leq c2^{k\sigma_p p}\left(\|(\varphi_0\widehat{f})^\vee|L_p\| + \|(\varphi_0(2\cdot)\varphi_0\widehat{f})^\vee|L_p\|\right)^p \\ &\leq c' 2^{k\sigma_p p}\|(\varphi_0\widehat{f})^\vee|L_p\|^p\,(1 + \|\varphi_0^\vee(2\cdot)|L_p\|)^p \\ &= c_1 2^{k\sigma_p p}\|(\varphi_0\widehat{f})^\vee|L_p\|^p. \end{aligned} \tag{2.10}$$

This estimate can be incorporated into our further calculations. Now for $1 \leq j \leq k-1$ we use the multiplier theorem with $M_j = \bar{\varphi}(2^{k-j}\cdot)$, and observe that

$$\operatorname{supp} M_j \subset \{x : |2^{k-j}x| \leq 2\} \subset \{x : |x| \leq 2\} = \Gamma.$$

Now inserting (2.10) into (2.9) yields

$$\begin{aligned} &\left(\sum_{j=1}^{k-1} 2^{j\sigma_p p}\|(\bar{\varphi}(2^{k-j}\cdot)\varphi_0\widehat{f})^\vee|L_p\|^p + c_1 2^{k\sigma_p p}\|(\varphi_0\widehat{f})^\vee|L_p\|^p\right)^{1/p} \\ \leq& c\left(\sum_{j=1}^{k-1} 2^{j\sigma_p p}\|(\bar{\varphi}(2^{k-j}\cdot))^\vee(\cdot)|L_p\|^p\|(\varphi_0\widehat{f})^\vee|L_p\|^p + 2^{k\sigma_p p}\|(\varphi_0\widehat{f})^\vee|L_p\|^p\right)^{1/p} \\ \leq& c\|(\varphi_0\widehat{f})^\vee|L_p\|\left(\sum_{j=1}^{k-1} 2^{j\sigma_p p}\|2^{(j-k)n}\bar{\varphi}^\vee(2^{j-k}\cdot)|L_p\|^p + 2^{k\sigma_p p}\right)^{1/p} \\ =& c\|(\varphi_0\widehat{f})^\vee|L_p\|\left(\sum_{j=1}^{k-1} 2^{j\sigma_p p}2^{(j-k)np}2^{-(j-k)\frac{n}{p}p}\|\bar{\varphi}^\vee|L_p\|^p + 2^{k\sigma_p p}\right)^{1/p} \\ \leq& c2^{k\sigma_p p}k^{1/p}\|F^{\sigma_p}_{p,q}(\mathbb{R}^n)\|. \end{aligned} \tag{2.11}$$

Now (2.2) together with (2.3), (2.8), and (2.11) give the upper estimate.

Step 2. We construct a function that gives the lower bound. Let $\psi \in S(\mathbb{R})$ be a non-negative function with $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^n : |x| \leq 1/8\}$ and $\int_{\mathbb{R}^n}\psi(x)\mathrm{d}x = 1$. We show that

$$\|\psi(2^k\cdot)|F^{\sigma_p}_{p,q}(\mathbb{R}^n)\| \geq c2^{-kn}k^{1/p}, \qquad k \in \mathbb{N}, \quad 0 < q \leq \infty.$$

Let us take a function $\kappa \in S(\mathbb{R}^n)$ with

$$(\mathrm{D}^\alpha\kappa^\vee)(0) = 0, \qquad |\alpha| \leq r, \tag{2.12}$$

where $r > \sigma_p - 1$, according to [Ryc99, Th. BPT]. In particular, by [Ryc99, Rem. 3] these conditions on κ are sufficient for our purposes. Furthermore, we require

$$\kappa(x) = 1 \qquad \text{if} \qquad x \in M = \{z \in \mathbb{R}^n : |z - (\frac{1}{2}, 0\ldots, 0)| < 1/4\}. \tag{2.13}$$

Such a function κ was constructed in [SchXX, Th. 2.1].
Simple calculation shows that if $j = 1, 2, \ldots, k$ and $|x - (-\frac{1}{2}\cdot\frac{1}{2^j}, 0\ldots, 0)| < \frac{1}{2^j}\frac{1}{8}$, which is equivalent to writing

$$x \in B_{2^{-(j+3)}}(x_j), \qquad x_j = (-2^{-(j+1)}, 0, \ldots, 0),$$

then

$$\operatorname{supp}_y\psi(2^k x + 2^{k-j}y) \subset M.$$

For these x we get

$$\mathcal{K}(2^{-j}, \psi(2^k\cdot))(x) = \int_{\mathbb{R}^n}\kappa(y)\psi(2^k x + 2^{k-j}y)\mathrm{d}y = \int_{\mathbb{R}^n}\psi(2^k x + 2^{k-j}y)\mathrm{d}y = 2^{(j-k)n}.$$

Note that the for different values of j, the balls $B_{2^{-(j+3)}}(x_j)$ are pairwise disjoint. Hence we calculate

$$\begin{aligned}
\|\psi(2^k\cdot)|F^{\sigma_p}_{p,q}\| &\geq \left\| \left(\sum_{j=1}^{k} 2^{j\sigma_p q}|\mathcal{K}(2^{-j},\psi(2^k\cdot))(\cdot)|^q\right)^{1/q} |L_p\right\| \\
&= \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^{k} 2^{j\sigma_p q}|\mathcal{K}(2^{-j},\psi(2^k\cdot))(x)|^q\right)^{p/q} \mathrm{d}x\right)^{1/p} \\
&\geq \left(\sum_{l=1}^{k} \int_{B_{2^{-(l+3)}}(x_l)} \left(\sum_{j=1}^{k} \delta_{lj} 2^{j\sigma_p q}|\mathcal{K}(2^{-j},\psi(2^k\cdot))(x)|^q\right)^{p/q} \mathrm{d}x\right)^{1/p} \\
&\geq \left(\sum_{j=1}^{k} 2^{j\sigma_p p} 2^{(j-k)np} 2^{-jn}\right)^{1/p} \\
&= 2^{-kn}\left(\sum_{j=1}^{k} 2^{jn(\frac{1}{p}-1)p} 2^{jnp} 2^{-jn}\right)^{1/p} \\
&= 2^{-kn} k^{1/p},
\end{aligned}$$

which gives the desired result. Our estimate holds as well in the case $p = 1$. □

Refining the methods used in Theorem 2.2 we obtain the following generalization. However, our estimates are not sharp and might still be improved.

Theorem 2.3 *Let* $0 < q < p < 1$*. Then*

$$2^{k(\sigma_p - \frac{n}{p})} k^{1/p} \lesssim \|T_k|\mathcal{L}(F^{\sigma_p}_{p,q}(\mathbb{R}^n))\| \lesssim 2^{k(\sigma_p - \frac{n}{p})} k^{1/q}.$$

Furthermore, if $0 < q < p = 1$ *we have*

$$2^{-kn} k^{\max(1,1/q-1/2)} \lesssim \|T_k|\mathcal{L}(F^{0}_{1,q}(\mathbb{R}^n))\| \lesssim 2^{-kn} k^{1/q}.$$

Proof :
Step 1. Refining the estimates for the upper bound used in Step 1 of Theorem 2.2 we see that we only need to consider the 'critical terms' when $j = 1, \ldots, k$. In this case we now calculate

$$\begin{aligned}
&\left\| \left(\sum_{j=1}^{k} 2^{j\sigma_p q}|(\varphi_j(2^k\cdot)\widehat{f})^\vee(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R}^n)\right\| \\
&= \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^{k} 2^{j\sigma_p q}|(\varphi_j(2^k\cdot)\widehat{f})^\vee(x)|^q\right)^{p/q} \mathrm{d}x\right)^{1/p} \\
&= \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^{k} 2^{j\sigma_p q}|(\varphi_j(2^k\cdot)\widehat{f})^\vee(x)|^q\right)^{p/q} \mathrm{d}x\right)^{\frac{q}{p}\cdot\frac{1}{q}} \\
&\leq \left(\sum_{j=1}^{k} \left(\int_{\mathbb{R}^n} 2^{j\sigma_p p}|(\varphi_j(2^k\cdot)\widehat{f})^\vee(x)|^p \mathrm{d}x\right)^{q/p}\right)^{1/q} \\
&= \left(\sum_{j=1}^{k} 2^{j\sigma_p q}\|(\varphi_j(2^k\cdot)\widehat{f})^\vee|L_p\|^q\right)^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&\leq c\left(\sum_{j=1}^{k} 2^{j\sigma_p q}\|\bar{\varphi}(2^{k-j}\cdot)^\vee|L_p\|^q \cdot \|(\varphi_0\widehat{f})^\vee|L_p\|^q\right)^{1/q}\\
&\leq c\|(\varphi_0\widehat{f})^\vee|L_p\|\left(\sum_{j=1}^{k} 2^{j\sigma_p q}\|\bar{\varphi}(2^{k-j}\cdot)^\vee|L_p\|^q\right)^{1/q}\\
&\leq c\|(\varphi_0\widehat{f})^\vee|L_p\|\left(\sum_{j=1}^{k} 2^{j\sigma_p q}2^{(j-k)nq}2^{-(j-k)\frac{n}{p}q}\|\bar{\varphi}(\cdot)^\vee|L_p\|^q\cdot\right)^{1/q}\\
&\leq c'\|(\varphi_0\widehat{f})^\vee|L_p\|2^{k\sigma_p n}k^{1/q}\\
&\leq c''2^{k\sigma_p n}k^{1/q}\|f|F^{\sigma_p}_{p,q}\|,
\end{aligned}
$$

where in the third step we used the generalized triangle inequality, cf. [HLP52, p. 148], since $\frac{p}{q} > 1$, before applying the Fourier Multiplier theorem (2.7).

Step 2. The proof of the lower bound

$$\|T_k|\mathcal{L}(F^{\sigma_p}_{p,q}(\mathbb{R}^n))\| \gtrsim k^{1/p}2^{k(\sigma_p-\frac{n}{p})}, \quad k \in \mathbb{N}$$

is the same as in Step 2 of Theorem 2.2.
Step 3. Finally, the estimate

$$\|T_k|\mathcal{L}(F^{0}_{1,q}(\mathbb{R}^n))\| \gtrsim k^{1/q-1/2}2^{-kn}, \quad k \in \mathbb{N}$$

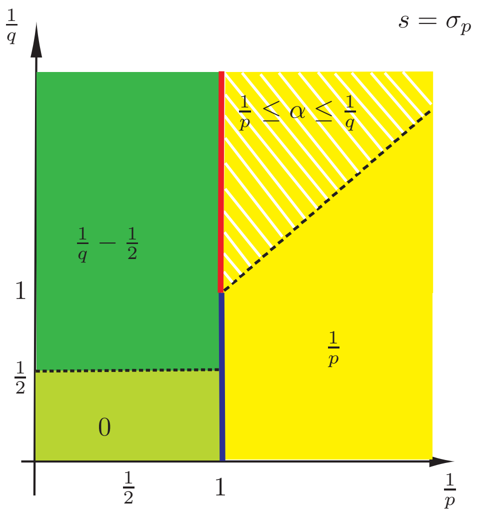
for $0 < q < p = 1$ follows from the Step 3 of Theorem 2.1.

□

Remark 2.4 The picture aside summarizes our results and illustrates the dependency of the additional factors k^α on p and q that were obtained for upper bounds of the dilation operators when $s = \sigma_p$, i.e.

$$T_k \sim 2^{k(\sigma_p-n/p)}\cdot k^\alpha.$$

There is a jump at $p = 1$ in the exponent of k caused by the absence of the Littlewood-Paley assertion in this case. Furthermore, our estimates when $0 < q < p < 1$ and $0 < q < p = 1$ are not sharp and might be improved.



3 Applications

3.1 F-spaces with positive smoothness on $\mathbb{R}^n$

In this section we want to discuss the connection and diversity of three different approaches to F-spaces with positive smoothness, using the previous results on dilation operators.
In addition to the Fourier-analytical approach, cf. Definition 1.1, we now present two further characterizations – associated to definitions by differences and subatomic decompositions – before we come to some comparisions.

The classical approach: Triebel-Lizorkin spaces $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$

If f is an arbitrary function on $\mathbb{R}^n$, $h \in \mathbb{R}^n$ and $r \in \mathbb{N}$, then

$$(\Delta^1_h f)(x) = f(x+h) - f(x) \quad \text{and} \quad (\Delta^{r+1}_h f)(x) = \Delta^1_h(\Delta^r_h f)(x), \quad x \in \mathbb{R}^n.$$

For convenience we may write Δ_h instead of Δ^1_h. Furthermore, for a function $f \in L_p(\mathbb{R}^n)$, $0 < p < \infty$, $r \in \mathbb{N}$, the *ball means* are denoted by

$$d^r_{t,p} f(x) = \left(t^{-n} \int_{|h|\le t} |(\Delta^r_h f)(x)|^p \mathrm{d}h \right)^{1/p}, \quad x \in \mathbb{R}^n, \quad t > 0. \tag{3.1}$$

Definition 3.1 *Let $0 < p < \infty$, $0 < q \le \infty$, $s > 0$, and $r \in \mathbb{N}$ such that $r > s$. Then $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in L_p(\mathbb{R}^n)$ such that*

$$\|f|\mathbf{F}^s_{p,q}(\mathbb{R}^n)\|_r = \|f|L_p(\mathbb{R}^n)\| + \left\| \left(\int_0^1 t^{-sq} d^r_{t,p} f(\cdot)^q \frac{\mathrm{d}t}{t} \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \tag{3.2}$$

(*with the usual modification if $q = \infty$*) *is finite.*

Remark 3.2 The approach by differences for the spaces $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$ has been described in detail in [Tri83, 2.5.10] for those spaces which can also be considered as subspaces of $S'(\mathbb{R}^n)$. Otherwise one finds in [Tri06, 9.2.2] the necessary explanations and references to the relevant literature. In particular, the spaces in Definition 3.1 are independent of r, meaning that different values of $r > s$ result in quasi-norms which are equivalent. Furthermore, the spaces are quasi-Banach spaces (Banach spaces, if $1 \le p < \infty$, $1 \le q \le \infty$). Recall that we deal with subspaces of $L_p(\mathbb{R}^n)$, in particular, we have the embedding

$$\mathbf{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n), \qquad s > 0, \quad 0 < q \le \infty, \quad 0 < p < \infty.$$

Further information on the classical approach to F-spaces – treated in a more general context – may be found in [HN07].

We add the following homogeneity estimate, which will serve us later on. Let $s > 0$, $0 < p < \infty$, $0 < q \le \infty$, and $k \in \mathbb{N}_0$. Then for all $f \in \mathbf{F}^s_{p,q}(\mathbb{R}^n)$

$$\|f(2^k\cdot)|\mathbf{F}^s_{p,q}(\mathbb{R}^n)\| \le 2^{k(s-\frac{n}{p})} \|f|\mathbf{F}^s_{p,q}(\mathbb{R}^n)\|. \tag{3.3}$$

Let $f \in \mathbf{F}^s_{p,q}(\mathbb{R}^n)$. For the proof we observe that

$$\|f|\mathbf{F}^s_{p,q}(\mathbb{R}^n)\| = \|f|L_p(\mathbb{R}^n)\| + \left(\int_{\mathbb{R}^n} \left(\int_0^1 t^{-(s+\frac{n}{p})q} \left(\int_{|h|\le t} |\Delta^r_h f(x)|^p \mathrm{d}h \right)^{q/p} \frac{\mathrm{d}t}{t} \right)^{p/q} \mathrm{d}x \right)^{1/p},$$

where $\int_0^1 \ldots \frac{\mathrm{d}t}{t}$ can be replaced by $\int_0^\lambda \ldots \frac{\mathrm{d}t}{t}$ with arbitrary $0 < \lambda \le \infty$ in the sense of equivalent quasi-norms.

Now straightforward calculation yields

$$\begin{aligned}\|f(2^k\cdot)|\mathbf{F}^s_{p,q}(\mathbb{R}^n)\| &= \|f(2^k\cdot)|L_p(\mathbb{R}^n)\| + \left(\int_{\mathbb{R}^n}\left(\int_0^1 t^{-(s+\frac{n}{p})q}\left(\int_{|h|\le t}|\Delta_h^r f(2^k x)|^p \mathrm{d}h\right)^{q/p}\frac{\mathrm{d}t}{t}\right)^{p/q}\mathrm{d}x\right)^{1/p}\\
&\le 2^{-k\frac{n}{p}}\|f|L_p(\mathbb{R}^n)\| + 2^{k(s-\frac{n}{p})}\left(\int_{\mathbb{R}^n}\left(\int_0^\infty t'^{-(s+\frac{n}{p})q}\left(\int_{|h'|\le t'}|\Delta_{h'}^r f(x')|^p \mathrm{d}h'\right)^{q/p}\frac{\mathrm{d}t'}{t'}\right)^{p/q}\mathrm{d}x\right)^{1/p}\\
&\le \max\left(2^{-k\frac{n}{p}}, 2^{k(s-\frac{n}{p})}\right)\|f|\mathbf{F}^s_{p,q}(\mathbb{R}^n)\|\\
&= 2^{k(s-\frac{n}{p})}\|f|\mathbf{F}^s_{p,q}(\mathbb{R}^n)\|,\end{aligned}$$

where we used in the second step that

$$\Delta_h^r f(2^k x) = \sum_{l=0}^r \binom{r}{l}(-1)^{r-l} f(2^k x + l2^k h) =: \Delta_{h'}^r f(x'),$$

by substituting $x' = 2^k x$, $h' = 2^k h$, and $t' = 2^k t$.

The subatomic approach: Triebel-Lizorkin spaces $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$

We complement our notation. Let

$$\mathbb{R}^n_{++} := \{y \in \mathbb{R}^n : y = (y_1, \dots, y_n),\ y_j > 0\}.$$

Moreover, $\chi_{\nu,m}$ denotes the characteristic function of the cube $Q_{\nu,m}$. The subatomic approach provides a constructive definition for Triebel-Lizorkin spaces, expanding functions f via building blocks and suitable coefficients, where the latter belong to certain sequence spaces $f^{s,\varrho}_{p,q}$.

Definition 3.3 *Let k be a non-negative C^∞-function in $\mathbb{R}^n$ with*

$$\operatorname{supp} k \subset \left\{y \in \mathbb{R}^n : |y| < 2^{J-\varepsilon}\right\} \cap \mathbb{R}^n_{++} \tag{3.4}$$

for some fixed $\varepsilon > 0$ and some fixed $J \in \mathbb{N}$, satisfying

$$\sum_{m\in\mathbb{Z}^n} k(x-m) = 1, \;\; x \in \mathbb{R}^n. \tag{3.5}$$

Let $\beta \in \mathbb{N}_0^n$, $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, and set $k^\beta(x) = (2^{-J}x)^\beta k(x)$. Then

$$k^\beta_{\nu,m}(x) = k^\beta(2^\nu x - m) \tag{3.6}$$

denote the building blocks related to $Q_{\nu,m}$.

Remark 3.4 The above definition implies that the building blocks are bounded by

$$0 \le k^\beta_{\nu,m}(x) \le 2^{-\varepsilon|\beta|}, \quad x \in \mathbb{R}^n, \tag{3.7}$$

uniformly in $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, and for their supports we observe that

$$\operatorname{supp} k^\beta_{\nu,m} \subset 2^{J-\varepsilon}Q_{\nu,m} \tag{3.8}$$

uniformly in $\beta \in \mathbb{N}_0^n$.

Definition 3.5 *Let $\varrho \ge 0$, $s \in \mathbb{R}$, $0 < p,q \le \infty$ and*

$$\lambda = \left\{\lambda^\beta_{\nu,m} \in \mathbb{C} : \beta \in \mathbb{N}_0^n, m \in \mathbb{Z}^n, \nu \in \mathbb{N}_0\right\}.$$

Then the sequence space $f^{s,\varrho}_{p,q}$ is defined as

$$f^{s,\varrho}_{p,q} := \left\{\lambda : \|\lambda|f^{s,\varrho}_{p,q}\| < \infty\right\}, \tag{3.9}$$

where

$$\|\lambda|f^{s,\varrho}_{p,q}\| = \sup_{\beta\in\mathbb{N}_0^n} 2^{\varrho|\beta|}\left\|\left(\sum_{\nu=0}^\infty\sum_{m\in\mathbb{Z}^n} 2^{\nu sq}|\lambda^\beta_{\nu,m}|^q\chi_{\nu,m}(\cdot)\right)^{1/q}|L_p(\mathbb{R}^n)\right\| \tag{3.10}$$

(*with the usual modification if $p = \infty$ and/or $q = \infty$*).

We now define the related function spaces.

Definition 3.6 *Let* $s > 0$, $0 < p < \infty$, $0 < q \le \infty$, *and* $\varrho \ge 0$. *Then* $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ *contains all* $f \in L_p(\mathbb{R}^n)$ *which can be represented as*

$$f(x) = \sum_{\beta\in\mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda^\beta_{\nu,m} k^\beta_{\nu,m}(x), \quad x \in \mathbb{R}^n, \tag{3.11}$$

with coefficients $\lambda = \left\{\lambda^\beta_{\nu,m}\right\}_{\beta\in\mathbb{N}_0^n,\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n} \in f^{s,\varrho}_{p,q}$. *Then*

$$\left\|f|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\right\| = \inf \left\|\lambda|f^{s,\varrho}_{p,q}\right\|, \tag{3.12}$$

where the infimum is taken over all possible representations (3.11).

Remark 3.7 The definitions given above follow closely [Tri06, Sect. 9.2]. The spaces $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ are quasi-Banach spaces (Banach spaces for $p, q \ge 1$) and independent of k and ϱ (in terms of equivalent quasi-norms). Furthermore, for all admitted parameters p, q, s, we have

$$\mathfrak{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n) \qquad .$$

see [Tri06, Th. 9.8]. Concerning the convergence of (3.11) one obtains as a consequence of $\lambda \in f^{s,\varrho}_{p,q}$, that the series on the right-hand sides converge absolutely in $L_p(\mathbb{R}^n)$ if $p < \infty$. Since this implies unconditional convergence we may simplify (3.11) and write in the sequel

$$f = \sum_{\beta,\nu,m} \lambda^\beta_{\nu,m} k^\beta_{\nu,m}.$$

Remark 3.8 Considering the spaces $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ we obtain the following upper bounds for the dilation operators T_k. Let $s > 0$, $0 < p < \infty$, $0 < q \le \infty$, and $k \in \mathbb{N}_0$. Then for all $f \in \mathfrak{F}^s_{p,q}(\mathbb{R}^n)$

$$\|f(2^k\cdot)|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\| \le 2^{k(s-\frac{n}{p})}\|f|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\|. \tag{3.13}$$

The proof is fairly simple. We take $f \in \mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ with optimal representation

$$f(x) = \sum_{\beta,\nu,m} \lambda^\beta_{\nu,m} k^\beta_{\nu,m}(x),$$

i.e.,

$$\|f|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\| \sim \|\lambda|f^{s,\varrho}_{p,q}\| = \sup_\beta 2^{\varrho|\beta|} \left\| \left(\sum_\nu \sum_m 2^{\nu s q} |\lambda^\beta_{\nu,m}|^q \chi_{\nu,m}(\cdot) \right)^{1/q} |L_p \right\|,$$

where $\chi_{\nu,m}(\cdot)$ is the characteristic function of $Q_{\nu,m}$. Put

$$g(x) := f(2^k\cdot) = \sum_{\beta,\nu,m} \lambda^\beta_{\nu,m} k^\beta_{\nu,m}(2^k x) = \sum_{\beta,m} \sum_{l=k}^{\infty} \lambda^\beta_{l-k,m} k^\beta_{l,m}(x),$$

where $l := \nu + k$, since $k^\beta_{\nu,m}(2^k x) = (2^{\nu+k}x - m)^\beta k(2^{\nu+k}x - m) = k^\beta_{l,m}(x)$. This yields

$$\begin{aligned}
\|f(2^k\cdot)|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\| &\le \sup_\beta 2^{\varrho|\beta|} \left\| \left(\sum_{l=k}^{\infty} \sum_m 2^{lsq} |\lambda^\beta_{l-k,m}|^q \chi_{l,m}(\cdot) \right)^{1/q} |L_p \right\| \\
&= \sup_\beta 2^{\varrho|\beta|} \left\| \left(\sum_{l=k}^{\infty} \sum_m 2^{ksq} 2^{(l-k)sq} |\lambda^\beta_{l-k,m}|^q \chi_{l-k,m}(2^k\cdot) \right)^{1/q} |L_p \right\| \\
&= 2^{k(s-\frac{n}{p})} \sup_\beta 2^{\varrho|\beta|} \left\| \left(\sum_\nu \sum_m 2^{\nu s q} |\lambda^\beta_{\nu,m}|^q \chi_{\nu,m}(\cdot) \right)^{1/q} |L_p \right\| \\
&= 2^{k(s-\frac{n}{p})} \|f|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\|.
\end{aligned}$$

Connections and diversity

We now discuss the coincidence and diversity of the above presented concepts of F-spaces and may restrict ourselves to positive smoothness $s > 0$. In view of our Remarks 1.2, 3.2 and 3.7 concerning the different nature of these spaces, it is obvious that there cannot be established a complete coincidence of all approaches when $s < \sigma_p$.
In particular, no equivalent quasi-norms of type (3.2) can be expected for the spaces $F^s_{p,q}(\mathbb{R}^n)$ if $s < \sigma_p$. It seems to be clear that such a characterization is also impossible if $\sigma_p < s < \sigma_{pq}$ (in particular, when $0 < q < p$), i.e.

$$\mathbf{F}^s_{p,q}(\mathbb{R}^n) \neq F^s_{p,q}(\mathbb{R}^n), \qquad 0 < p < \infty, \quad 0 < q \leq \infty, \quad 0 < s < \sigma_{pq},$$

cf. [Tri06, Rem. 9.15], based on [CS06] – so the situation is even more complicated. Nevertheless, under certain restrictions on the smoothness parameter s, the above approaches result in the same F-space.

Theorem 3.9 *Let* $s > 0$, $0 < p < \infty$, $0 < q \leq \infty$.

(i) *Then*

$$\mathbf{F}^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n), \qquad s > n\left(\frac{1}{\min(p,q)} - \frac{1}{p}\right), \tag{3.14}$$

and

$$F^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n), \qquad s > \sigma_{pq} \tag{3.15}$$

(*in the sense of equivalent quasi-norms*).

(ii) *Furthermore,*

$$F^s_{p,q}(\mathbb{R}^n) = \mathbf{F}^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n), \qquad s > n\left(\frac{1}{\min(p,q)} - \frac{1}{\max(1,p)}\right) \tag{3.16}$$

(*in the sense of equivalent quasi-norms*).

Remark 3.10 The first equality in (3.16) is longer known, see [Tri83, Section 2.5.11], [Tri92, Thm. 3.5.3], whereas the second equality in (3.16) is a consequence of the recently proved coincidence (3.14), see [Tri06, Prop. 9.14] (with forerunners in [Tri97, Sect. 13.8], [Tri01, Thm. 2.9]). In the figures aside and below we have indicated the situation in the usual $(\frac{1}{p}, s)$-diagram for different values of q.

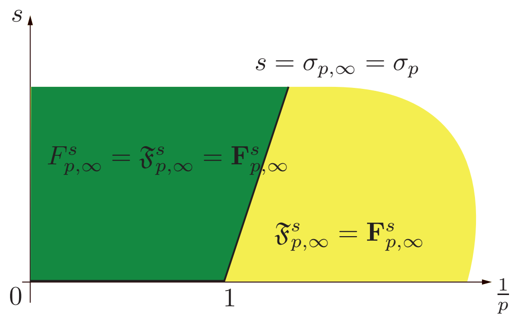


Figure 1: Parameter $q = \infty$

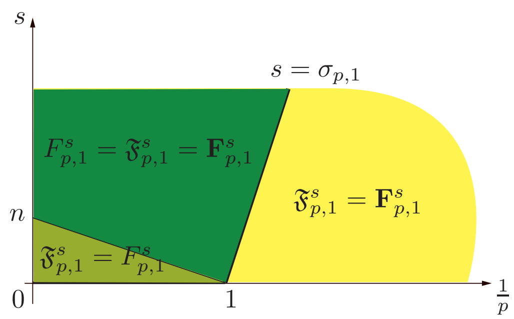


Figure 2: Parameter $q = 1$

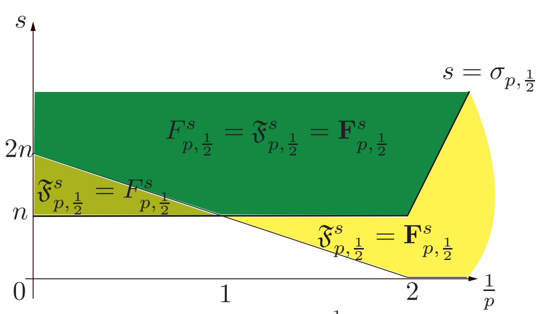


Figure 3: Parameter $q = \frac{1}{2}$

Our new results concerning the norms of the dilation operators T_k established in Section 2 now lead to new insights when dealing with different approaches for F-spaces in the limiting case $s = \sigma_p$. We obtain the following assertions which are especially interesting when $p < q$.

Corollary 3.11 *Let $0 < p < 1$ and $0 < q \leq \infty$. Then*

$$F^{\sigma_p}_{p,q}(\mathbb{R}^n) \neq \mathbf{F}^{\sigma_p}_{p,q}(\mathbb{R}^n)$$

and

$$F^{\sigma_p}_{p,q}(\mathbb{R}^n) \neq \mathfrak{F}^{\sigma_p}_{p,q}(\mathbb{R}^n)$$

(*in terms of equivalent quasi-norms*) *as sets of measurable functions.*

Proof : We use the homogeneity estimate (3.3),

$$\left\|f(2^k\cdot)|\mathbf{F}^s_{p,q}\right\| \leq 2^{k(s-\frac{n}{p})}\left\|f|\mathbf{F}^s_{p,q}\right\|,$$

where $s > 0$, $0 < p < \infty$, and $0 < q \leq \infty$. We proceed indirectly, assuming that $F^{\sigma_p}_{p,q}(\mathbb{R}^n) = \mathbf{F}^{\sigma_p}_{p,q}(\mathbb{R}^n)$ for $0 < q \leq \infty$. But then using Theorem 2.2 when $p \leq q$ or Theorem 2.3 for $q < p$, together with (3.3) we could find a function $\psi \in F^{\sigma_p}_{p,q}$ such that

$$2^{k(\sigma_p-\frac{n}{p})}k^{1/p}\|\psi|F^{\sigma_p}_{p,q}\| \leq c\|\psi(2^k\cdot)|F^{\sigma_p}_{p,q}\| \sim \|\psi(2^k\cdot)|\mathbf{F}^{\sigma_p}_{p,q}\| \leq 2^{k(\sigma_p-\frac{n}{p})}\|\psi|\mathbf{F}^{\sigma_p}_{p,q}\| \sim 2^{k(\sigma_p-\frac{n}{p})}\|\psi|F^{\sigma_p}_{p,q}\|,$$

which leads to

$$k^{1/p} \leq c, \qquad k \in \mathbb{N}.$$

This gives the desired contradiction.
The proof for the spaces $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ is the same; we only need to use the estimate (3.13) instead of (3.3). We give an alternative proof of this result in the next subsection. □

Remark 3.12 We know that $F^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ if $s > \sigma_{pq}$. Corollary 3.11 yields $F^{\sigma_{pq}}_{p,q}(\mathbb{R}^n) \neq \mathfrak{F}^{\sigma_{pq}}_{p,q}(\mathbb{R}^n)$ if $p \leq q$ since in this case $\sigma_{pq} = \sigma_p$. If $p > q$, then $\sigma_{pq} > \sigma_p$ and the sharp estimates for the norms of the dilation operators T_k in $F^{\sigma_{pq}}_{p,q}(\mathbb{R}^n)$, cf. [ET96, 2.3.1], coincide with the bounds for spaces $\mathfrak{F}^{\sigma_{pq}}_{p,q}(\mathbb{R}^n)$ as given in (3.13). So in this case studying dilation operators will not help solving the problem. It does not seem unlikely that the approaches coincide in this case.

3.2 A comment on atomic expansion

It might not be obvious immediately, but the building blocks $k^{\beta}_{\nu,m}$ in our subatomic approach differ from the atoms $a_{\nu,m}$ – used to characterize the spaces $F^s_{p,q}(\mathbb{R}^n)$ in Theorem 1.7 – mainly by the imposed moment conditions on the latter and some unimportant technicalities. In particular, the normalizing factors $2^{\nu(s-\frac{n}{p})}$ are incorporated in the sequence spaces $f^{s,\varrho}_{p,q}$ in the subatomic approach; recall Definition 1.5. We refer as well to [Tri01, Th. 3.6]. Now following [SchXX, Sect. 3.2] one can show that first moment conditions on the line $s = \sigma_{pq}$ are necessary. This immediately leads to

$$F^{\sigma_p}_{p,q}(\mathbb{R}^n) \neq \mathfrak{F}^{\sigma_p}_{p,q}(\mathbb{R}^n),$$

yielding an alternative proof of Corollary 3.11. We present the main ideas.
Every $f \in F^{\sigma_p}_{p,q}(\mathbb{R}^n)$ may be represented by optimal atomic decompositions

$$f(x) = \sum_{\nu,m} \lambda_{\nu,m} a_{\nu,m}(x), \qquad x \in \mathbb{R}^n,$$

with

$$\|\lambda|f_{p,q}\| \leq c\|f|F^{\sigma_p}_{p,q}\|, \qquad f \in F^{\sigma_p}_{p,q}(\mathbb{R}^n),$$

see [Tri06, Ch. 1.5] for details. If no moment conditions were required here, then

$$g_k(x) = f(2^k x) = \sum_{\nu,m} \lambda_{\nu,m} a_{\nu,m}(2^k x), \qquad x \in \mathbb{R}^n$$

would represent an atomic decomposition of $f(2^k x)$. This can be seen by setting

$$g_k(x) = \sum_{\nu,m} \lambda_{\nu,m} 2^{k(\sigma_p - \frac{n}{p})} 2^{-k(\sigma_p - \frac{n}{p})} a_{\nu,m}(2^k x) = \sum_{\nu,m} \lambda^k_{\nu,m} a^k_{\nu,m}(x),$$

where $a^k_{\nu,m}(x) = 2^{-k(\sigma_p - \frac{n}{p})} a_{\nu,m}(2^k x) \sim \tilde{a}_{\nu+k,m}(x)$, since

$$\operatorname{supp} a^k_{\nu,m} \subset Q_{\nu+k,m},$$

$$|\mathrm{D}^\alpha a^k_{\nu,m}(x)| = 2^{-k(\sigma_p - \frac{n}{p}) + k|\alpha|} |\mathrm{D}^\alpha a_{\nu,m}(x)| \le 2^{-(\nu+k)(\sigma_p - \frac{n}{p}) + (\nu+k)|\alpha|}.$$

Therefore we obtain

$$\|g_k | F^{\sigma_p}_{p,q}\| \le \|\lambda^k | f_{p,q}\| = 2^{k(\sigma_p - \frac{n}{p})} \|\lambda | f_{p,q}\| = 2^{-nk} \|\lambda | f_{p,q}\|,$$

resulting in

$$\|f(2^k \cdot) | F^{\sigma_p}_{p,q}\| \le c 2^{-nk} \|f | F^{\sigma_p}_{p,q}\|.$$

But we know by Theorem 2.2 and Theorem 2.3 that this is *not* true in general when $0 < p < \infty$.

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