Sobolev and Jawerth embeddings for spaces with variable smoothness and integrability

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Abstract

We consider the Triebel-Lizorkin spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ of variable smoothness and integrability as introduced recently by Diening, Hästö and Roudenko in [9]. Under certain regularity conditions on the function parameters involved we show that

$$F^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow F^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)$$

if

$$s_0(x) \geq s_1(x) \quad \text{and} \quad s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)} \quad \text{for all} \quad x \in \mathbb{R}^n$$

with embeddings of Sobolev and Bessel potential spaces included as special cases.

If $\inf\limits_{x\in\mathbb{R}^n}(s_0(x) - s_1(x)) > 0$ we recover also the analogue of the Jawerth embedding

$$F^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}(\mathbb{R}^n) \hookrightarrow F^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)$$

for any q_0, q_1.

The proofs are based on the decomposition techniques of [9] and work exclusively with the associated sequence spaces $f^{s(\cdot)}_{p(\cdot),q(\cdot)}$.

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1 Introduction

The interplay between smoothness and integrability constitutes one of the corner stones of the theory of function spaces. It can be traced back as far as to Hardy and Littlewood [13, 14], but the decisive breakthrough was achieved by Sobolev [33], who proved the famous embedding

$$W_p^m(\Omega) \hookrightarrow L_q(\Omega), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, $L_q(\Omega)$ stands for the usual Lebesgue space and $W_p^m(\Omega)$ denotes the Sobolev space of functions with all distributive derivatives of order smaller or equal to m bounded in the $L_p(\Omega)$ norm. The crucial relation between the involved parameters $m \in \mathbb{N}$, $1 < p < n/m$ and $1 < q < \infty$ is

$$\frac{1}{q} = \frac{1}{p} - \frac{m}{n}. \tag{1.2}$$

During the last seventy years, many scales of spaces of smooth functions were defined using various techniques (e.g. derivatives, differences, Fourier coefficients or Fourier transform) with the corresponding analogues of (1.1) and (1.2) playing usually an important role in most of the applications. Actually, it seems that any new scale of spaces of smooth functions needs to exhibit some kind of interaction between smoothness and integrability to be accepted by the mathematical audience.

In recent years there has been a growing interest in function spaces describing local regularity properties of functions. The first spaces of this type are the spaces of variable integrability, which were implicitly used by Orlicz [27] already in 1931 and studied in detail by Kováčik and Rákosník [24] in 1991 together with the corresponding Sobolev spaces of variable integrability. During 1990's these spaces found applications in the study of variational integrals with non-standard growth, but it was probably the work of Růžička [29, 30, 31] on electrorheological fluids what promoted an enormous interest in these spaces. Since then, more than one hundred papers on this topic appeared. We refer to [8] for a brief overview and an extensive collection of references.

Another way how to describe the local properties of a function was outlined already by Peetre in [28, page 266] in Chapter 12 named "Some strange new spaces" and resulted in the concept of 2-microlocal spaces, cf. [5] and [20]. Along a different line of study, Leopold [25] introduced spaces of Besov-type with variable smoothness, but constant integrability. This approach was further developed by Besov [3, 4].

The Sobolev embedding for the spaces with variable integrability was addressed already by Kováčik and Rákosník [24] and later on by Růžička [31]. But their results failed to cover the optimal exponent according to (1.1). Edmunds and Rákosník [10, 11] proved the optimal Sobolev embedding theorem under Lipschitz and Hölder continuity of the exponents, cf. also [16]. Finally, Diening [7] and Samko [32] showed, that log-Hölder continuity is sufficient.

The embeddings of Besov and Triebel-Lizorkin spaces of variable smoothness were obtained by Besov [4] in a fairly general form. It seems that Leopold [26] was the only one up to now who tried to connect the function spaces with variable smoothness with spaces of variable integrability. Unfortunately, he also failed to recover the optimal exponent.

The last step (up to now) was done by Diening, Hästö and Roudenko in [9]. These authors combined the concept of spaces with variable integrability of Orlicz, Kováčik and Rákosník with the concept of variable smoothness of Leopold and Besov (which is in some sense very similar to the ideas of Peetre, Bony and Jaffard) and proposed the function spaces of Triebel-Lizorkin type of variable smoothness *and* integrability, cf. Definition 2.5. They proved (under some restrictions on the function parameters involved), that these spaces include the Lebesgue and Sobolev spaces of variable integrability and the spaces of variable smoothness as special cases. They proved also a certain version of the atomic decomposition theorem, which is a well known tool in the theory of function spaces of Besov and Triebel-Lizorkin type. Finally, they proved an analogue of the usual trace theorem, which exhibits the interplay between smoothness and integrability. The reader may consult also [17], [15] and references given there for other versions of the trace embedding theorem for Sobolev spaces with varying integrability.

Although mentioned on several places in [9] (and even in the abstract), the authors have not presented any version of Sobolev embedding, which would not only result in a generalization of (1.1) with (1.2) holding pointwise, but would (in the sense described above) help to justify the existence of this scale of function spaces - at least until this promising line of research finds any applications.

Our aim is to fill this gap. In the frame of Triebel-Lizorkin spaces with constant parameters, the following analogue of Sobolev embedding is true.

Theorem 1.1. *(Jawerth, [21]). Let*

$$-\infty < s_1 < s_0 < \infty, \quad 0 < p_0 < p_1 < \infty, \quad 0 < q \le \infty \tag{1.3}$$

with

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}. \tag{1.4}$$

Then

$$F^{s_0}_{p_0,\infty}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1,q}(\mathbb{R}^n). \tag{1.5}$$

The remarkable effect, which was first observed by Jawerth and which is in some sense unique to the Triebel-Lizorkin spaces, is the improvement in the third fine parameter $q > 0$, which may be chosen arbitrarily small. Of course, (1.5) holds only for $q = \infty$ if $s_0 = s_1$ (or, equivalently, $p_0 = p_1$). If the smoothness and integrability parameters s and p become functions of $x \in \mathbb{R}^n$, then it seems to be appropriate to assume that (1.4) holds pointwise, i.e.

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbb{R}^n \tag{1.6}$$

and if the improvement in the fine parameter is to be achieved, that also

$$\inf_{x\in\mathbb{R}^n}(s_0(x) - s_1(x)) = n \inf_{x\in\mathbb{R}^n}\Big(\frac{1}{p_0(x)} - \frac{1}{p_1(x)}\Big) > 0. \tag{1.7}$$

We prove that these "natural" assumptions (combined with appropriate regularity conditions) are really sufficient. We show, that if $s_1(x) \le s_0(x)$ and $p_0(x) \le p_1(x)$ with (1.6) and $0 < q(x) \le \infty$ for all $x \in \mathbb{R}^n$, then

$$F^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow F^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n). \tag{1.8}$$

If also (1.7) is satisfied, then even

$$F^{s_0(\cdot)}_{p_0(\cdot),\infty}(\mathbb{R}^n) \hookrightarrow F^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)$$

holds.

2 Preliminaries

Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$ and let $S'(\mathbb{R}^n)$ be its dual - the space of all tempered distributions. For $f \in S'(\mathbb{R}^n)$ we denote by $\widehat{f} = Ff$ its Fourier transform and by $f^\vee$ or $F^{-1}f$ its inverse Fourier transform. We give a Fourier-analytic definition of Triebel-Lizorkin spaces, which relies on the so-called *dyadic resolution of unity.* Let $\varphi \in S(\mathbb{R}^n)$ with

$$\varphi(x) = 1 \quad \text{if} \quad |x| \le 1 \quad \text{and} \quad \varphi(x) = 0 \quad \text{if} \quad |x| \ge \frac{3}{2}. \tag{2.1}$$

We put $\varphi_0 = \varphi$ and $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$ for $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$. This leads to the identity

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1, \qquad x \in \mathbb{R}^n.$$

Definition 2.1. Let $s \in \mathbb{R}, 0 < p < \infty, 0 < q \le \infty$. Then $F^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$||f|F^s_{pq}(\mathbb{R}^n)|| = \left|\left| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right|\right| < \infty \tag{2.2}$$

(with the usual modification for $q = \infty$).

Remark 2.2. (i) These spaces have a long history. In this context we recommend [28, 34, 35, 37] as standard references. We point out that the spaces $F^s_{pq}(\mathbb{R}^n)$ are independent of the choice of φ in the sense of equivalent (quasi-)norms. Special cases of these two scales include Lebesgue spaces, Sobolev spaces and inhomogeneous Hardy spaces.

(ii) Interchanging the order of L_p and ℓ_q norm in (2.2) would lead to the Fourier-analytic definition of Besov spaces. Unfortunately, they seem to be less convenient for describing local regularity properties of distributions, because they lack the so-called *localization principle*, cf. [35, Theorem 2.4.7]. Hence (also in correspondence with [9]) we study only the F-scale.

Next, we introduce the Lebesgue spaces of variable integrability.

Definition 2.3. Let $p : \mathbb{R}^n \to (0,\infty)$ be a measurable function. Then the space $L_{p(\cdot)}(\mathbb{R}^n)$ consists of all measurable functions $f : \mathbb{R}^n \to [-\infty,\infty]$ such that $||f|L_{p(\cdot)}(\mathbb{R}^n)|| < \infty$, where

$$||f|L_{p(\cdot)}(\mathbb{R}^n)|| = \inf\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leq 1\}$$

is the Minkowski functional of the absolutely convex set $\{f : \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \leq 1\}$.

Remark 2.4. (i) One could also consider (and it was done so already by Kováčik and Rákosník in [24]) that $p(x) = \infty$ on a set of a positive measure. But Definition 2.3 is already sufficient for our purpose, cf. also Remark 2.6.

(ii) If $p(x) \geq 1$ for all $x \in \mathbb{R}^n$, then $L_{p(\cdot)}(\mathbb{R}^n)$ are Banach spaces. To ensure, that $L_{p(\cdot)}(\mathbb{R}^n)$ are at least quasi-Banach spaces, we assume that

$$p^- := \inf_{x\in\mathbb{R}^n} p(x) > 0.$$

The generalization of Definition 2.1 to the setting of variable smoothness and integrability as it was given by [9] is surprisingly simple.

Definition 2.5. Let $-\infty < s(x) < +\infty, 0 < p(x) < \infty, 0 < q(x) \leq \infty$. Then $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$||f|F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)|| = \left|\left|\left(\sum_{j=0}^{\infty} 2^{js(\cdot)q(\cdot)}|(\varphi_j\widehat{f})^\vee(\cdot)|^{q(\cdot)}\right)^{1/q(\cdot)} |L_{p(\cdot)}(\mathbb{R}^n)\right|\right| < \infty \tag{2.3}$$

(with the usual modification for $q(x) = \infty$).

Remark 2.6. This definition introduces the Triebel-Lizorkin spaces of variable smoothness, integrability and summability under almost no conditions on $s(\cdot), p(\cdot)$ and $q(\cdot)$. Unfortunately, these spaces may depend on the choice of the function φ as described in (2.1). This is the case already when s and $q < \infty$ are constant and $p = \infty$. We refer to [34, Chapter 2.3.4] for a detailed discussion of related aspects. So, a first natural restriction seems to be the condition

$$p^+ = \sup_{x\in\mathbb{R}^n} p(x) < \infty.$$

Together with Remark 2.4(ii) this leads to

$$0 < p^- := \inf_{z\in\mathbb{R}^n} p(z) \leq p(x) \leq \sup_{z\in\mathbb{R}^n} p(z) =: p^+ < \infty, \quad x \in \mathbb{R}^n. \tag{2.4}$$

Next we present the regularity assumptions of [9].

Definition 2.7. Let g be a continuous function on $\mathbb{R}^n$.

(i) We say, that g is *1-locally* log-*Hölder continuous*, abbreviated $g \in C^{\log}_{1-\mathrm{loc}}(\mathbb{R}^n)$, if there exists $c > 0$ such that

$$|g(x) - g(y)| \leq \frac{c}{\log(e + 1/||x-y||_\infty)} \quad \text{for all} \quad x, y \in \mathbb{R}^n \quad \text{with} \quad ||x-y||_\infty \leq 1.$$

Here, $||z||_\infty = \max\{|z_1|, \dots, |z_n|\}$ denotes the maximum norm of $z \in \mathbb{R}^n$.

(ii) We say, that g is *locally* log-*Hölder continuous*, abbreviated $g \in C^{\log}_{\rm loc}(\mathbb{R}^n)$, if there exists $c > 0$ such that

$$|g(x) - g(y)| \le \frac{c}{\log(e + 1/|x - y|)}, \quad x, y \in \mathbb{R}^n.$$

(iii) We say, that g is *globally* log-*Hölder continuous*, abbreviated $g \in C^{\log}(\mathbb{R}^n)$, if it is locally log-Hölder continuous and there exists $c > 0$ and $g_\infty \in \mathbb{R}$ such that

$$|g(x) - g_\infty| \le \frac{c}{\log(e + |x|)}, \quad x \in \mathbb{R}^n.$$

Remark 2.8. (i) The conditions (ii) and (iii) are overtaken literally from [9] and we shall need them only for the transference of our results from sequence spaces to function spaces. It is the less restrictive condition (i), which we shall involve in our proofs.

(ii) The condition (i) is very similar to the original condition of Diening used in [6] to show the boundedness of the maximal operator.

We shall use the property (i) in the form formulated in next Lemma. We leave out the trivial proof.

Lemma 2.9. *Let $g \in C^{\log}_{1-\rm loc}(\mathbb{R}^n)$. Then there exists a constant $c > 0$ such that for every $j \in \mathbb{N}_0$ and every $x, y \in \mathbb{R}^n$ with $||x - y||_\infty \le 2^{-j}$ the following inequalities hold:*

$$\frac{1}{c} \le 2^{-j|g(x)-g(y)|} \le 2^{j(g(x)-g(y))} \le 2^{j|g(x)-g(y)|} \le c.$$

Definition 2.10. (Standing assumptions of [9]). Let p and q be positive functions on $\mathbb{R}^n$ such that $\frac{1}{p}, \frac{1}{q} \in C^{\log}(\mathbb{R}^n)$ and let $s \in C^{\log}_{\rm loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $s(x) \ge 0$ and let $s(x)$ have a limit at infinity.

Remark 2.11. (i) Let us note, that the *Standing assumptions* imply in particular (2.4) and a similar chain of inequalities for $q(x)$.

We introduce the sequence spaces associated with the Triebel-Lizorkin spaces of variable smoothness and integrability. Let $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. Then Q_{jm} denotes the closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, centered at $2^{-j}m$, and with side length 2^{-j}. By $\chi_{jm} = \chi_{Q_{jm}}$ we denote the characteristic function of $Q_{j\,m}$. If

$$\gamma = \{\gamma_{jm} \in \mathbb{C} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\},$$

$-\infty < s(x) < \infty$, $0 < p(x) < \infty$ and $0 < q(x) \le \infty$ for all $x \in \mathbb{R}^n$, we define

$$\begin{aligned} ||\gamma|f^{s(\cdot)}_{p(\cdot),q(\cdot)}|| &= \Bigg|\Bigg|\Bigg(\sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} 2^{js(\cdot)q(\cdot)}|\gamma_{jm}|^{q(\cdot)}\chi_{jm}(\cdot)\Bigg)^{1/q(\cdot)}\Big|L_{p(\cdot)}(\mathbb{R}^n)\Bigg|\Bigg| \qquad (2.5)\\ &= \Bigg|\Bigg|\sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} 2^{js(\cdot)}|\gamma_{jm}|\chi_{jm}(\cdot)\Big|L_{p(\cdot)}(\ell_{q(\cdot)})\Bigg|\Bigg|. \end{aligned}$$

The connection between the function spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and the sequence spaces $f^{s(\cdot)}_{p(\cdot),q(\cdot)}$ was one of the main aim of [9]. Following [18] and [19], these authors investigated the properties of the so-called φ-transform (denoted by S_φ) and obtained the following result.

Theorem 2.12. *Under the Standing assumptions of [9]*

$$||f|F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)|| \approx ||S_\varphi f|f^{s(\cdot)}_{p(\cdot),q(\cdot)}||$$

with constants independent of $f \in F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Remark 2.13. (i) The assumptions on s in the Theorem 2.12 seem to be too restrictive. It seems, that several authors now try to prove similar results also for $s(x)$, which are not necessarily positive or convergent at infinity. We refer at least to [23] and [39].

From this reason we formulate the theorems of embeddings of sequence spaces under minimal assumptions, which shall really be needed in the proof. If later on any improved version of Theorem 2.12 should appear, the results may then be easily overtaken.

(ii) We shall use only a simple corollary of Theorem 2.12, namely that (under the *Standing assumptions*) the spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $f^{s(\cdot)}_{p(\cdot),q(\cdot)}$ are isomorphic.

3 Main results

First, we state the results in the form of embeddings of sequence spaces under those assumptions really needed in the proof. Later on, we combine those with the *Standing assumptions of [9]* and obtain similar results also for the embeddings of function spaces. Finally, we state separately the embeddings of Sobolev and Bessel potential spaces.

Theorem 3.1. *Let* $-\infty < s_1(x) \le s_0(x) < \infty$, $0 < p_0(x) \le p_1(x) < \infty$ *for all* $x \in \mathbb{R}^n$ *with* $0 < p_0^- \le p_1^- \le p_1^+ < \infty$ *and*

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbb{R}^n.$$

Let $q(x) = \infty$ *for all* $x \in \mathbb{R}^n$ *or* $0 < q^- \le q(x) < \infty$ *for all* $x \in \mathbb{R}^n$ *and* $s_0, \frac{1}{p_0} \in C^{\log}_{1-\text{loc}}(\mathbb{R}^n)$. *Then*

$$f^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)} \hookrightarrow f^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}.$$

Proof. Step 1. $q(x) = \infty$ *for all* $x \in \mathbb{R}^n$.

We set

$$h(x) = \sup_{j,m} 2^{js_0(x)}|\gamma_{jm}|\chi_{jm}(x), \quad x \in \mathbb{R}^n. \tag{3.1}$$

Here, and later on, the supremum is taken over all $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. Then by (2.5)

$$||\gamma|f^{s_0(\cdot)}_{p_0(\cdot),\infty}|| = ||h|L_{p_0(\cdot)}(\mathbb{R}^n)|| \tag{3.2}$$

and trivially

$$2^{js_0(x)}|\gamma_{jm}| \le h(x), \quad x \in Q_{jm}, \tag{3.3}$$

which leads to

$$|\gamma_{jm}| \le \inf_{x\in Q_{jm}} 2^{-js_0(x)}h(x), \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}_n. \tag{3.4}$$

Using consequently (2.5), (3.4) and Lemma 2.9 for s_0 we may estimate

$$
\begin{aligned}
||\gamma|f^{s_1(\cdot)}_{p_1(\cdot),\infty}|| &= \left\|\left|\sup_{j,m} 2^{js_1(x)}|\gamma_{jm}|\chi_{jm}(x)|L_{p_1(\cdot)}(\mathbb{R}^n)\right|\right\| \\
&\le \left\|\left|\sup_{j,m} 2^{js_1(x)}\Big(\inf_{y\in Q_{jm}} 2^{-js_0(y)}h(y)\Big)\chi_{jm}(x)|L_{p_1(\cdot)}(\mathbb{R}^n)\right|\right\| \\
&= \left\|\left|\sup_{j,m} 2^{j(s_1(x)-s_0(x))}\Big(\inf_{y\in Q_{jm}} 2^{j(s_0(x)-s_0(y))}h(y)\Big)\chi_{jm}(x)|L_{p_1(\cdot)}(\mathbb{R}^n)\right|\right\| \\
&\le c\left\|\left|\sup_{j,m} 2^{jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)}\Big(\inf_{y\in Q_{jm}} h(y)\Big)\chi_{jm}(x)|L_{p_1(\cdot)}(\mathbb{R}^n)\right|\right\|.
\end{aligned}
$$

Let $A_{-1}\subset\mathbb{R}^n$ stand for those x, where

$$\sup_{j,m} 2^{jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)}\Big(\inf_{y\in Q_{jm}} h(y)\Big)\chi_{jm}(x)=0. \tag{3.5}$$

For each $x\in\mathbb{R}^n$ we denote by $J=J_x\in\mathbb{N}_0$ the smallest non-negative integer, such that

$$(3.5)\le 2\cdot 2^{Jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)}\sum_{m\in\mathbb{Z}^n}\Big(\inf_{y\in Q_{Jm}} h(y)\Big)\chi_{Jm}(x). \tag{3.6}$$

We may assume, that for almost all $x\in\mathbb{R}^n$ (3.5) is finite. Otherwise $h(x)=\infty$ on a set of positive measure and there is nothing to prove. Furthermore, we denote by $A_J\subset\mathbb{R}^n$ those x with $J_x=J\in\mathbb{N}_0$.

Let $\lambda>0$ be a positive real number such that

$$
\begin{aligned}
1 &\ge \int_{\mathbb{R}^n}\left(\frac{h(x)}{\lambda}\right)^{p_0(x)}dx=\sum_{J=-1}^{\infty}\int_{A_J}\left(\frac{h(x)}{\lambda}\right)^{p_0(x)}dx \qquad (3.7)\\
&\ge \sum_{J=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\int_{A_J\cap Q_{Jm}}\left(\frac{h(x)}{\lambda}\right)^{p_0(x)}dx.
\end{aligned}
$$

We set

$$h_{jm}:=\frac{\inf_{y\in Q_{jm}} h(y)}{\lambda},\quad j\in\mathbb{N}_0,\quad m\in\mathbb{Z}^n$$

and show, that there is a constant $C>0$ such that

$$\int_{\mathbb{R}^n}\left(C^{-1}\sup_{j,m}2^{jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)}h_{jm}\chi_{jm}(x)\right)^{p_1(x)}dx\le 1.$$

We split the integration over $\mathbb{R}^n$ into integrals over A_J and use (3.6).

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\left(C^{-1}\sup_{j,m}2^{jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)}h_{jm}\chi_{jm}(x)\right)^{p_1(x)}dx\\
&\le\sum_{J=0}^{\infty}\int_{A_J}\left((C/2)^{-1}\sum_{m\in\mathbb{Z}^n}2^{Jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)}h_{Jm}\chi_{Jm}(x)\right)^{p_1(x)}dx\\
&=\sum_{J=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\int_{A_J}\left((C/2)^{-1}2^{Jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)}h_{Jm}\right)^{p_1(x)}\chi_{Jm}(x)dx\\
&=\sum_{J=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\int_{A_J\cap Q_{Jm}}(C/2)^{-p_1(x)}2^{Jn\left(1-\frac{p_1(x)}{p_0(x)}\right)}h_{Jm}^{p_1(x)}dx \qquad (3.8)
\end{aligned}
$$

I. Let us fix $(J,m) \in \mathbb{N}_0 \times \mathbb{Z}^n$ such that

$$h_{Jm} \leq 1.$$

Then (as $p_0(x) \leq p_1(x)$)

$$2^{Jn\left(1-\frac{p_1(x)}{p_0(x)}\right)} \leq 1$$

and

$$h_{Jm}^{p_1(x)} \leq h_{Jm}^{p_0(x)}.$$

Hence for $C \geq 2$ we obtain

$$\begin{aligned} \int_{A_J \cap Q_{Jm}} (C/2)^{-p_1(x)} 2^{Jn\left(1-\frac{p_1(x)}{p_0(x)}\right)} h_{Jm}^{p_1(x)} dx &\leq \int_{A_J \cap Q_{Jm}} h_{Jm}^{p_0(x)} dx \\ &\leq \int_{A_J \cap Q_{Jm}} \left(\frac{h(x)}{\lambda}\right)^{p_0(x)} dx. \end{aligned} \tag{3.9}$$

II. Let us consider $(J,m) \in \mathbb{N}_0 \times \mathbb{Z}^n$ such that

$$h_{Jm} > 1.$$

Then

$$1 \geq \int_{Q_{Jm}} \left(\frac{h(x)}{\lambda}\right)^{p_0(x)} dx \geq \int_{Q_{Jm}} h_{Jm}^{p_0(x)} dx \geq 2^{-Jn} h_{Jm}^{p_0^{Jm}},$$

where $p_0^{Jm} = \inf_{x \in Q_{Jm}} p_0(x) > 0$. Hence

$$1 < h_{Jm} \leq 2^{Jn/p_0^{Jm}}. \tag{3.10}$$

We rewrite the integrals in (3.8) as

$$\int_{A_J \cap Q_{Jm}} (C/2)^{-p_1(x)} 2^{Jn\left(1-\frac{p_1(x)}{p_0(x)}\right)} h_{Jm}^{p_1(x)} dx = \int_{A_J \cap Q_{Jm}} \underbrace{(C/2)^{-p_1(x)} 2^{Jn\left(1-\frac{p_1(x)}{p_0(x)}\right)} h_{Jm}^{p_1(x)-p_0(x)}}_{(\star)} h_{Jm}^{p_0(x)} dx \tag{3.11}$$

and show that the estimate $(\star) \leq 1$ for $C \geq 2$ large enough and $x \in Q_{Jm}$ finishes immediately the proof. By (3.9) and (3.11) combined with $(\star) \leq 1$ and (3.7)

$$\begin{aligned} \sum_{J=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \int_{A_J \cap Q_{Jm}} (C/2)^{-p_1(x)} 2^{Jn\left(1-\frac{p_1(x)}{p_0(x)}\right)} h_{Jm}^{p_1(x)} dx &= \sum_{(J,m): h_{Jm} \leq 1} \cdots + \sum_{(J,m): h_{Jm} > 1} \cdots \\ &\leq \sum_{(J,m): h_{Jm} \leq 1} \int_{A_J \cap Q_{Jm}} \left(\frac{h(x)}{\lambda}\right)^{p_0(x)} dx + \sum_{(J,m): h_{Jm} > 1} \int_{A_J \cap Q_{Jm}} h_{Jm}^{p_0(x)} dx \\ &\leq \sum_{J=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \int_{A_J \cap Q_{Jm}} \left(\frac{h(x)}{\lambda}\right)^{p_0(x)} dx \leq 1. \end{aligned}$$

Hence, it remains to prove that $(\star) \leq 1$ for all $x \in Q_{Jm}$. By (3.10), it is enough to show that

$$(C/2)^{-p_1(x)} 2^{Jn\left(1-\frac{p_1(x)}{p_0(x)}\right)} \cdot 2^{Jn \cdot \frac{p_1(x)-p_0(x)}{p_0^{Jm}}} \leq 1$$

or, equivalently,

$$2^{Jn[p_1(x)-p_0(x)]\cdot[\frac{1}{p_0^{Jm}}-\frac{1}{p_0(x)}]} \le (C/2)^{p_1(x)}.$$

Using Lemma 2.9 for $\frac{1}{p_0}$ (with constant $2^{c_{\log}}$), this follows from

$$2^{n[1-\frac{p_0(x)}{p_1(x)}]\cdot c_{\log}} \le C/2.$$

As $0 \le 1 - \frac{p_0(x)}{p_1(x)} \le 1$, we may choose $C = 2^{nc_{\log}+1} \ge 2$.

Step 2. $0 < q(x) < \infty$ *for all* $x \in \mathbb{R}^n$.

Let $\lambda > 0$ be a positive real number with

$$\int_{\mathbb{R}^n} \Big(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{js_0(x)q(x)}|\gamma_{jm}|^{q(x)}\lambda^{-q(x)}\chi_{jm}(x)\Big)^{p_0(x)/q(x)} dx \le 1. \tag{3.12}$$

We have to show that there is a constant $C > 0$ independent of $\{\gamma_{jm}\}$, such that

$$\int_{\mathbb{R}^n} \Big(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{js_1(x)q(x)}|\gamma_{jm}|^{q(x)}(C\lambda)^{-q(x)}\chi_{jm}(x)\Big)^{p_1(x)/q(x)} dx \le 1. \tag{3.13}$$

We show, that under (3.12) the following inequality holds for almost all $x \in \mathbb{R}^n$

$$\Big(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{js_1(x)q(x)}\frac{|\gamma_{jm}|^{q(x)}}{(C\lambda)^{q(x)}}\chi_{jm}(x)\Big)^{p_1(x)} \le \Big(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{js_0(x)q(x)}\frac{|\gamma_{jm}|^{q(x)}}{\lambda^{q(x)}}\chi_{jm}(x)\Big)^{p_0(x)}. \tag{3.14}$$

Obviously, (3.14) implies (3.13).

For almost every $x \in \mathbb{R}^n$ and every $j \in \mathbb{N}_0$, there is exactly one $m = m(j) \in \mathbb{Z}^n$ such that $x \in Q_{j,m(j)}$. We fix one such an x. Then (3.14) reads like

$$\sum_{j=0}^{\infty} 2^{js_1(x)q(x)}|\gamma_{j,m(j)}|^{q(x)}(C\lambda)^{-q(x)} \le \Big(\sum_{j=0}^{\infty} 2^{js_0(x)q(x)}|\gamma_{j,m(j)}|^{q(x)}\lambda^{-q(x)}\Big)^{p_0(x)/p_1(x)}. \tag{3.15}$$

We set

$$\alpha_j := 2^{js_0(x)}\frac{|\gamma_{j,m(j)}|}{\lambda}, \quad j \in \mathbb{N}_0$$

and rewrite (3.15) once again. It now becomes

$$\sum_{j=0}^{\infty} 2^{jn\big(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\big)q(x)}(\alpha_j/C)^{q(x)} \le \Big(\sum_{j=0}^{\infty} \alpha_j^{q(x)}\Big)^{p_0(x)/p_1(x)}. \tag{3.16}$$

Using (3.12) and Lemma 2.9 for s_0, we get

$$\begin{aligned}
1 &\ge \int_{Q_{j,m(j)}} \Big(2^{js_0(y)q(y)}|\gamma_{j,m(j)}|^{q(y)}\lambda^{-q(y)}\Big)^{p_0(y)/q(y)} dy = \int_{Q_{j,m(j)}} \Big(2^{js_0(y)}|\gamma_{j,m(j)}|\lambda^{-1}\Big)^{p_0(y)} dy \\
&= \int_{Q_{j,m(j)}} \Big(2^{j(s_0(y)-s_0(x))}2^{js_0(x)}|\gamma_{j,m(j)}|\lambda^{-1}\Big)^{p_0(y)} dy \ge \int_{Q_{j,m(j)}} \Big(c\,2^{js_0(x)}|\gamma_{j,m(j)}|\lambda^{-1}\Big)^{p_0(y)} dy \\
&= \int_{Q_{j,m(j)}} (c\,\alpha_j)^{p_0(y)} dy.
\end{aligned}$$

If $c\alpha_j > 1$, we may further estimate

$$1 \geq 2^{-jn} \big(c\,\alpha_j \big)^{\inf_{z\in Q_{j,m(j)}} p_0(z)},$$

or, equivalently,

$$c\,\alpha_j \leq 2^{\frac{jn}{\inf_{z\in Q_{j,m(j)}} p_0(z)}} = 2^{\frac{jn}{p_0(x)}} 2^{\frac{jn}{\inf_{z\in Q_{j,m(j)}} p_0(z)} - \frac{jn}{p_0(x)}} \leq c' 2^{\frac{jn}{p_0(x)}} \tag{3.17}$$

and this estimate holds true also if $c\,\alpha_j \leq 1$.

If $\sum_{j=0}^{\infty} \alpha_j^{q(x)} \leq 1$, then (3.16) follows by monotonicity and $p_0(x) \leq p_1(x)$ for any $C \geq 1$. If $\sum_{j=0}^{\infty} \alpha_j^{q(x)} = \infty$, then there is nothing to prove. In the remaining case $1 < \sum_{j=0}^{\infty} \alpha_j^{q(x)} < \infty$ we find a non-negative integer $J \in \mathbb{N}_0$ such that

$$2^{\frac{Jnq(x)}{p_0(x)}} < \sum_{j=0}^{\infty} \alpha_j^{q(x)} \leq 2^{\frac{(J+1)nq(x)}{p_0(x)}}. \tag{3.18}$$

We split the sum over $j \in \mathbb{N}_0$ into two parts, apply (3.17) in the first part and use the inequality $p_0(x) \leq p_1(x)$ together with (3.18) in the second part.

$$\begin{aligned}
\sum_{j=0}^{\infty} 2^{jn\left(\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right) q(x)} \alpha_j^{q(x)} &= \sum_{j=0}^{J} 2^{jn\left(\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right) q(x)} \alpha_j^{q(x)} + \sum_{j=J+1}^{\infty} 2^{jn\left(\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right) q(x)} \alpha_j^{q(x)} \\
&\leq c^{q(x)} \sum_{j=0}^{J} 2^{jn\left(\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right) q(x)} 2^{\frac{jnq(x)}{p_0(x)}} + 2^{(J+1)n\left(\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right) q(x)} \sum_{j=J+1}^{\infty} \alpha_j^{q(x)} \\
&\leq c^{q(x)} \sum_{j=0}^{J} 2^{\frac{jnq(x)}{p_1(x)}} + 2^{\frac{(J+1)nq(x)}{p_1(x)}} \leq c_1^{q(x)}\, 2^{\frac{(J+1)nq(x)}{p_1(x)}} \\
&\leq c_1^{q(x)}\, 2^{\frac{nq(x)}{p_1(x)}} \Big(\sum_{j=0}^{\infty} \alpha_j^{q(x)} \Big)^{\frac{p_0(x)}{p_1(x)}} \leq C^{q(x)} \Big(\sum_{j=0}^{\infty} \alpha_j^{q(x)} \Big)^{\frac{p_0(x)}{p_1(x)}}.
\end{aligned}$$

In the last line, we used $0 < p_1^- \leq p_1^+ < \infty$ and again (3.18). This finishes the proof of (3.16) and consequently of the whole Step 2. □

Theorem 3.2. *Let $-\infty < s_1(x) < s_0(x) < \infty$ and $0 < p_0(x) < p_1(x) < \infty$ for all $x \in \mathbb{R}^n$ with $0 < p_0^- < p_1^+ < \infty$,*

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbb{R}^n$$

and

$$\varepsilon := \inf_{x\in\mathbb{R}^n} (s_0(x) - s_1(x)) = n \inf_{x\in\mathbb{R}^n} \left(\frac{1}{p_0(x)} - \frac{1}{p_1(x)} \right) > 0. \tag{3.19}$$

Let $s_0, \frac{1}{p_0} \in C^{\log}_{1-\mathrm{loc}}(\mathbb{R}^n)$. Then, for every $0 < q \leq \infty$,

$$f^{s_0(\cdot)}_{p_0(\cdot),\infty} \hookrightarrow f^{s_1(\cdot)}_{p_1(\cdot),q}.$$

Proof. We use again the notation of (3.1)-(3.4).

$$\begin{aligned}
||\gamma|f^{s_1(\cdot)}_{p_1(\cdot),q}|| &= \Big\|\Big(\sum_{j=0}^\infty\sum_{m\in\mathbb{Z}^n} 2^{js_1(x)q}|\gamma_{j\,m}|^q\chi_{j\,m}(x)\Big)^{1/q}|L_{p_1(\cdot)}(\mathbb{R}^n)\Big\|\\
&\le \Big\|\Big(\sum_{j=0}^\infty\sum_{m\in\mathbb{Z}^n} 2^{js_1(x)q}\Big(\inf_{y\in Q_{jm}} 2^{-js_0(y)}h(y)\Big)^q\chi_{jm}(x)\Big)^{1/q}|L_{p_1(\cdot)}(\mathbb{R}^n)\Big\| \qquad (3.20)\\
&\le \Big\|\Big(\sum_{j=0}^\infty\sum_{m\in\mathbb{Z}^n} 2^{j(s_1(x)-s_0(x))q}\Big(\inf_{y\in Q_{jm}} 2^{j(s_0(x)-s_0(y))}h(y)\Big)^q\chi_{jm}(x)\Big)^{1/q}|L_{p_1(\cdot)}(\mathbb{R}^n)\Big\|\\
&\le c\Big\|\Big(\sum_{j=0}^\infty\sum_{m\in\mathbb{Z}^n} 2^{jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)q}\Big(\inf_{y\in Q_{jm}} h(y)\Big)^q\chi_{jm}(x)\Big)^{1/q}|L_{p_1(\cdot)}(\mathbb{R}^n)\Big\|.
\end{aligned}$$

Let again $\lambda>0$ be a positive real number, such that

$$\int_{\mathbb{R}^n}\Big(\frac{h(x)}{\lambda}\Big)^{p_0(x)}dx\le 1. \qquad (3.21)$$

For almost every $x\in\mathbb{R}^n$ and every $j\in\mathbb{N}_0$ there is exactly one $m=m(j)$ such that $x\in Q_{j,m(j)}$. Fix one such $x\in\mathbb{R}^n$ and set

$$\alpha_j:=\frac{\inf\limits_{y\in Q_{j,m(j)}} h(y)}{\lambda}.$$

Then $\{\alpha_j\}_{j=0}^\infty$ is a non-decreasing sequence of non-negative real numbers with $\alpha:=\lim\limits_{j\to\infty}\alpha_j\le\dfrac{h(x)}{\lambda}$.

Let first $\alpha\le 1$. Then we use the monotonicity of $\{\alpha_j\}$, (3.19) and obtain for $C^q\ge(1-2^{-n\epsilon q})^{-1}$

$$\begin{aligned}
\Big(\sum_{j=0}^\infty C^{-q}2^{jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)q}\alpha_j^q\Big)^{p_1(x)/q} &\le \Big(\sum_{j=0}^\infty C^{-q}2^{jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)q}\alpha^q\Big)^{p_1(x)/q}\\
=\Big(\sum_{j=0}^\infty C^{-q}2^{jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)q}\Big)^{p_1(x)/q} &\cdot\alpha^{p_1(x)}\le\alpha^{p_0(x)}\le\Big(\frac{h(x)}{\lambda}\Big)^{p_0(x)}. \qquad (3.22)
\end{aligned}$$

Let us now consider the case $\alpha>1$. By (3.21), we get

$$1\ge\int_{\mathbb{R}^n}\Big(\frac{h(x)}{\lambda}\Big)^{p_0(x)}dx\ge\int_{Q_{j,m(j)}}\alpha_j^{p_0(x)}dx.$$

If $\alpha_j>1$, we may further estimate

$$1\ge 2^{-jn}\alpha_j^{\inf_{y\in Q_{j,m(j)}}p_0(y)}.$$

We apply Lemma 2.9 for $\frac{1}{p_0}$ to obtain an analogue of (3.17)

$$\alpha_j\le 2^{\frac{jn}{\inf_{y\in Q_{j,m(j)}}p_0(y)}}=2^{\frac{jn}{p_0(x)}}\cdot 2^{\frac{jn}{\inf_{y\in Q_{j,m(j)}}}-\frac{jn}{p_0(x)}}\le c_{\log}\,2^{\frac{jn}{p_0(x)}} \qquad (3.23)$$

and this estimate holds true also for $\alpha_j\le 1$.

We show, that for $C>0$ large enough (cf. (3.16))

$$\sum_{j=0}^\infty C^{-q}2^{jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)q}\alpha_j^q\le\alpha^{\frac{qp_0(x)}{p_1(x)}}. \qquad (3.24)$$

As $\alpha = \infty$ implies $h(x) = \infty$ and this happens only for a set of $x \in \mathbb{R}^n$ with measure zero, we may choose for almost every $x \in \mathbb{R}^n$ a non-negative integer $J \in \mathbb{N}_0$ such that

$$2^{\frac{Jn}{p_0(x)}} < \alpha \leq 2^{\frac{(J+1)n}{p_0(x)}} \tag{3.25}$$

and split

$$\sum_{j=0}^{\infty} C^{-q} 2^{jn\left(\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right)q} \alpha_j^q = \underbrace{\sum_{j=0}^{J} \dots}_{I} + \underbrace{\sum_{j=J+1}^{\infty} \dots}_{II}.$$

By (3.23) and (3.25)

$$I = \sum_{j=0}^{J} C^{-q} 2^{\frac{jnq}{p_1(x)}} \cdot 2^{-\frac{jnq}{p_0(x)}} \cdot \alpha_j^q \leq \sum_{j=0}^{J} C^{-q} c_{\log} 2^{\frac{jnq}{p_1(x)}} \leq c^{-1} 2^{\frac{(J+1)nq}{p_1(x)}} \leq 2^{\frac{Jnq}{p_1(x)}} \leq \alpha^{\frac{qp_0(x)}{p_1(x)}}.$$

The monotonicity of $\{\alpha_j\}$ and (3.25) lead to

$$\begin{aligned} II &\leq \sum_{j=J+1}^{\infty} 2^{jn\left(\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right)q} \alpha_j^q C^{-q} \leq \alpha^q C^{-q} \sum_{j=J+1}^{\infty} 2^{jn\left(\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right)q} \leq \alpha^q C^{-q} 2^{Jn\left(\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right)q} \\ &\leq \alpha^q C^{-q} \left(\alpha^{p_0(x)} 2^{-n}\right)^{\left(\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right)q} = \alpha^{\frac{qp_0(x)}{p_1(x)}} C^{-q} 2^{n\left(\frac{1}{p_0(x)} - \frac{1}{p_1(x)}\right)q} \leq \alpha^{\frac{qp_0(x)}{p_1(x)}} \end{aligned}$$

This finishes the proof of (3.24). Now (3.20), (3.22), (3.24) with (3.21) gives

$$||\gamma | f^{s_1(\cdot)}_{p_1(\cdot),q}|| \leq C ||\gamma | f^{s_0(\cdot)}_{p_0(\cdot),\infty}||.$$

□

Remark 3.3. The original proof of Jawerth of Theorem 1.1 used the technique of a distribution function, which fails for $L_{p(\cdot)}(\mathbb{R}^n)$. Another proof was given by Johnsen and Sickel [22] and relied on an inequality of Plancherel-Pólya-Nikol'skij type. Its classical proof [34, Chapter 1.3] is based on dilation arguments and (at least to our knowledge) there is still no analogue of these inequalities for $L_{p(\cdot)}(\mathbb{R}^n)$ up to now.

Our proofs of Theorems 3.1 and 3.2 were motived by [38]. An essential technique used there was the concept of non-increasing rearrangement. Unfortunately, it fails completely in the case of variable integrability exponents $p_0(x)$ and $p_1(x)$. To avoid this obstacle, we had to employ the somehow artificial inequality (3.24) - or its analogue (3.16). To motivate this step, let us consider the interpolation inequality between Lorentz spaces

$$||f|L_{p_1,q}(0,1)|| \leq c\,||f|L_{p_0,\infty}(0,1)||^{\theta} \cdot ||f|L_{\infty}(0,1)||^{1-\theta} \tag{3.26}$$

with

$$0 < p_0 < p_1 < \infty, \quad \frac{1}{p_1} = \frac{\theta}{p_0} + \frac{1-\theta}{\infty}, \quad 0 < \theta < 1$$

and its discrete version

$$\left(\sum_{j=0}^{\infty} 2^{-jnq(\frac{1}{p_0} - \frac{1}{p_1})} f^*(2^{-jn})^q\right)^{1/q} \leq c \left(\sup_{j \in \mathbb{N}_0} 2^{-jn/p_0} f^*(2^{-jn})\right)^{1-\frac{p_0}{p_1}} \cdot \left(\sup_{j \in \mathbb{N}_0} f^*(2^{-jn})\right)^{\frac{p_0}{p_1}}.$$

We refer to [2, Chapter 2] as a standard reference for non-increasing rearrangements and to [2, Chapter 4.4] for the notation connected with Lorentz spaces. We leave the details to the reader. The reader may also observe some similarities between (3.26) and the inequality (4) of [22].

Using Theorem 2.12, we obtain immediately following

Theorem 3.4. *Let s_0, s_1, p_0, p_1 and q be continuous functions satisfying the Standing assumptions of [9]. Let $s_0(x) \geq s_1(x)$ and $p_0(x) \leq p_1(x)$ for all $x \in \mathbb{R}^n$ with*

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbb{R}^n.$$

Then

$$F^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow F^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n).$$

We denote by $W^k_{p(\cdot)}(\mathbb{R}^n)$ the Sobolev space of functions form $L_{p(\cdot)}(\mathbb{R}^n)$, such that all its distributional derivatives of order smaller or equal to k exist and belong to $L_{p(\cdot)}(\mathbb{R}^n)$. Furthermore, we introduce the Bessel potential spaces of variable integrability introduced by Almeida and Samko [1] and by Gurka, Harjulehto and Nekvinda [12]. Let $\sigma \in \mathbb{R}$ and let $B^\sigma = F^{-1}(1+|\xi|^2)^{-\sigma/2}F$ be the Bessel potential operator. We set

$$L^\sigma_{p(\cdot)}(\mathbb{R}^n) = \{B^\sigma f : f \in L_{p(\cdot)}(\mathbb{R}^n)\}$$

and equip this space with norm $||f|L^\sigma_{p(\cdot)}(\mathbb{R}^n)|| = ||B^{-\sigma} f|L_{p(\cdot)}(\mathbb{R}^n)||$.

Let $p \in C^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $\sigma \in [0, \infty)$. It was shown in [9, Theorem 4.5] that $F^\sigma_{p(\cdot),2}(\mathbb{R}^n) \cong L^\sigma_{p(\cdot)}(\mathbb{R}^n)$ in the sense of equivalent norms. If moreover $\sigma \in \mathbb{N}_0$, then $F^\sigma_{p(\cdot),2}(\mathbb{R}^n) \cong W^\sigma_{p(\cdot)}(\mathbb{R}^n)$.

Hence setting $q = 2$ implies embeddings of Bessel potential spaces.

Theorem 3.5. *Let $0 \leq s_1 \leq s_0 < \infty$ and $p_0, p_1 \in C^{\log}(\mathbb{R}^n)$ with $1 < p_0^- \leq p_0(x) \leq p_1(x) \leq p_1^+ < \infty$ for all $x \in \mathbb{R}^n$. If*

$$s_0 - \frac{n}{p_0(x)} = s_1 - \frac{n}{p_1(x)}, \quad x \in \mathbb{R}^n,$$

then

$$L^{s_0}_{p_0(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{s_1}_{p_1(\cdot)}(\mathbb{R}^n).$$

If $s_1 \in \mathbb{N}_0$, then $L^{s_1}_{p_1(\cdot)}(\mathbb{R}^n)$ may be replaced by $W^{s_1}_{p_1(\cdot)}(\mathbb{R}^n)$ and similarly for s_0.

Remark 3.6. Let us only mention, that if $1 < p^- \leq p^+ < \infty$, then $p \in C^{\log}(\mathbb{R}^n)$ if, and only if, $\frac{1}{p} \in C^{\log}(\mathbb{R}^n)$. So the *Standing assumptions* on p_0 and p_1 are satisfied and the proof becomes trivial.

Theorem 3.7. *Let $s_0, s_1, p_0, p_1, q_0, q_1$ be continuous functions satisfying the Standing assumptions of [9] with*

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbb{R}^n$$

and

$$\inf_{x\in\mathbb{R}^n}(s_0(x) - s_1(x)) = n \inf_{x\in\mathbb{R}^n}\Big(\frac{1}{p_0(x)} - \frac{1}{p_1(x)}\Big) > 0.$$

Then

$$F^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}(\mathbb{R}^n) \hookrightarrow F^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}(\mathbb{R}^n).$$

Proof. By monotonicity and using Theorem 3.2, we obtain

$$f^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)} \hookrightarrow f^{s_0(\cdot)}_{p_0(\cdot),\infty} \hookrightarrow f^{s_1(\cdot)}_{p_1(\cdot),q_1^-} \hookrightarrow f^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}$$

and Theorem 2.12 finishes the proof. □

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