The Jawerth-Franke embedding of spaces with dominating mixed smoothness

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August 12, 2008

Abstract

We give a proof of the Jawerth and Franke embedding for function spaces with dominating mixed smoothness of Besov and Triebel-Lizorkin type

$$S^{\overline{r}^0}_{p_0,q_0}F(\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_N})\hookrightarrow S^{\overline{r}^1}_{p_1,p_0}B(\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_N})$$

and

$$S^{\overline{r}^0}_{p_0,p_1}B(\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_N})\hookrightarrow S^{\overline{r}^1}_{p_1,q_1}F(\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_N})$$

where

$$0<p_0<p_1\le\infty \text{ and } 0<q_0,q_1\le\infty$$

and

$$\overline{d}=(d_1,\dots,d_N)\in\mathbb{N}^N, \overline{r}^i=(r^i_1,\dots,r^i_N)\in\mathbb{R}^N, i=0,1$$

with

$$r^0_i-\frac{d_i}{p_0}=r^1_i-\frac{d_i}{p_1},\quad i=1,\dots,N.$$

Our main tools are discretization by a wavelet isomorphism and multivariate rearrangements.

AMS Classification: 42B35, 46E30, 46E35

Keywords and phrases: Besov spaces, Triebel-Lizorkin spaces, Sobolev embedding, dominating mixed smoothness, Jawerth-Franke embedding

1 Introduction and main results

1.1 Introduction

Our aim is to study function spaces with dominating mixed smoothness properties. These spaces were first defined by S. M. Nikol'skij in [18] and [19]. He introduced the spaces of Sobolev type

$$S_p^{\overline{r}}W(\mathbb{R}^2) = \Big\{ f \in L_p(\mathbb{R}^2) : ||f|S_p^{\overline{r}}W(\mathbb{R}^2)|| = ||f|L_p(\mathbb{R}^2)|| + \Big|\Big| \frac{\partial^{r_1} f}{\partial x_1^{r_1}} \mid L_p(\mathbb{R}^2) \Big|\Big| + \\ + \Big|\Big| \frac{\partial^{r_2} f}{\partial x_2^{r_2}} \mid L_p(\mathbb{R}^2) \Big|\Big| + \Big|\Big| \frac{\partial^{r_1+r_2} f}{\partial x_1^{r_1} \partial x_2^{r_2}} \mid L_p(\mathbb{R}^2) \Big|\Big| < \infty \Big\},$$

where $1 < p < \infty, r = (r_1, r_2) \in \mathbb{N}_0^2$. The mixed derivative $\frac{\partial^{r_1+r_2} f}{\partial x_1^{r_1} \partial x_2^{r_2}}$ plays the dominant part here and gave the name to this class of spaces.

We prefer to work with the following more general version. Namely, let $N \geq 2$ be a natural number and let $d_1, \dots, d_N$ be natural numbers. We set $\overline{d} = (d_1, \dots, d_N)$ and $d = d_1 + \dots + d_N$. Let further $\overline{r} = (r_1, \dots, r_N) \in \mathbb{N}_0^N$ and $1 < p < \infty$. Then

$$S_p^{\overline{r}}W(\mathbb{R}^{\overline{d}}) = S_p^{\overline{r}}W(\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_N}) = \Big\{ f \in L_p(\mathbb{R}^d) : ||D^\alpha f|L_p(\mathbb{R}^d)|| < \infty \quad \text{for all} \\ \alpha = (\alpha_1, \dots, \alpha_N), \alpha_i \in \mathbb{N}_0^{d_i} \quad \text{and} \quad |\alpha_i| \leq r_i \quad \text{for} \quad i = 1, \dots, N \Big\}.$$

The spaces of this type found many applications in connection with partial differential equations ([18], [19], [37], [17], [38]), approximation theory ([28], [29], [33], [27]), information based complexity ([36], [20]) and other areas of mathematics. The reader may consult the survey [22] for more references.

The Fourier-analytic approach to these function spaces is based on the so-called *decomposition of unity.*

Let $\varphi \in S(\mathbb{R}^n)$ be from the Schwartz-space of smooth rapidly decreasing functions with

$$\varphi(x) = 1 \qquad \text{if } |x| \leq 4/3 \quad \text{and} \quad \varphi(x) = 0 \qquad \text{if } |x| \geq 3/2.$$

We put $\varphi_0 = \varphi$, $\varphi_1 = \varphi(\cdot/2) - \varphi$ and

$$\varphi_j(x) = \varphi_1(2^{-j+1}x), \qquad x \in \mathbb{R}^n, j \in \mathbb{N}.$$

We observe, that the system $\{\varphi_j\}_{j=0}^\infty$ satisfies

$$\sum_{j=0}^{\infty} \varphi_j(t) = 1 \quad \text{for all } t \in \mathbb{R}^n. \tag{1.1}$$

Let $N \geq 2$ be again a natural number and let $d_1, \dots d_N$ be natural numbers. We define d and $\overline{d}$ as above. For $i = 1, \dots, N$ we define $\{\varphi_j^i\}_{j=0}^\infty \subset S(\mathbb{R}^{d_i})$ as described above and put for $\overline{k} = (k_1, \dots, k_N) \in \mathbb{N}_0^N$ and $x = (x^1, \dots, x^N) \in \mathbb{R}^d$

$$\varphi_{\overline{k}}(x) := \varphi_{k_1}^1(x^1) \cdots \varphi_{k_N}^N(x^N). \tag{1.2}$$

As

$$\sum_{\overline{k} \in \mathbb{N}_0^N} \varphi_{\overline{k}}(x) = \left(\sum_{k_1=0}^{\infty} \varphi_{k_1}(x^1) \right) \cdots \left(\sum_{k_N=0}^{\infty} \varphi_{k_N}(x^N) \right) = 1$$

for all $x = (x^1, \dots, x^N) \in \mathbb{R}^d$, we see that $\{\varphi_{\overline{k}}\}_{\overline{k} \in \mathbb{N}_0^N}$ forms also a decomposition of unity on $\mathbb{R}^d$ with the tensor product structure.

We denote by $\widehat{f}$ the Fourier transform of a distribution $f \in S'(\mathbb{R}^d)$ and by $f^\vee$ its inverse transform.

Definition 1.1. Let $\overline{r} \in \mathbb{R}^N$, $0 < q \leq \infty$ and $\varphi = \{\varphi_{\overline{k}}\}_{\overline{k} \in \mathbb{N}_0^N}$ be as above.

1. Let $0 < p \leq \infty$. Then $S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N})$ is the set of all $f \in S'(\mathbb{R}^d)$, such that
$$\big\|f\big|S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N})\big\|_\varphi := \left(\sum_{\overline{k} \in \mathbb{N}_0^N} 2^{\overline{k}\cdot\overline{r}q} \big\|\big(\varphi_{\overline{k}}\widehat{f}\big)^\vee\big|L_p(\mathbb{R}^d)\big\|^q\right)^{1/q} \tag{1.3}$$
is finite.

2. Let $0 < p < \infty$. Then $S^{\overline{r}}_{p,q}F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N})$ is the set of all $f \in S'(\mathbb{R}^d)$, such that
$$\big\|f\big|S^{\overline{r}}_{p,q}F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N})\big\|_\varphi := \left\|\left(\sum_{\overline{k} \in \mathbb{N}_0^N} 2^{\overline{k}\cdot\overline{r}q} \big|\big(\varphi_{\overline{k}}\widehat{f}\big)^\vee(\cdot)\big|^q\right)^{1/q}\right|L_p(\mathbb{R}^d)\Bigg\| \tag{1.4}$$
is finite.

Let us mention, that (1.3) and (1.4) lead to equivalent quasi-norms for different choices of $\{\varphi_{\overline{k}}\}$. If $d_1 = d_2 = \cdots = d_N$, then this and other basic aspects of the theory of function spaces with dominating mixed smoothness may be found in [1], [24], [2] or [34]. We refer to [10] for the general case. To shorten the notation, we write sometimes $S^{\overline{r}}_{p,q}B(\mathbb{R}^{\overline{d}})$ instead of $S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N})$ and similar in the $F-$case.

One of the main features of the classes $S^{\overline{r}}_{p,q}B(\mathbb{R}^{\overline{d}})$ and $S^{\overline{r}}_{p,q}F(\mathbb{R}^{\overline{d}})$ consists in the fact, that their quasi-norms are cross-quasi-norms, i.e. if
$$f = (f_1 \otimes \cdots \otimes f_N)$$
where $f_i \in S'(\mathbb{R}^{d_i}), i = 1, \ldots, N$ and f is a tensor product of tempered distributions in the sense of [25, Chapters IV and VII] or [12, Chapter X], then
$$||f_1 \otimes \cdots \otimes f_N|S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N})|| = \prod_{i=1}^N ||f_i|B^{r_i}_{p,q}(\mathbb{R}^{d_i})|| \tag{1.5}$$
and
$$||f_1 \otimes \cdots \otimes f_N|S^{\overline{r}}_{p,q}F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N})|| = \prod_{i=1}^N ||f_i|F^{r_i}_{p,q}(\mathbb{R}^{d_i})|| \tag{1.6}$$
where $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ are the Fourier-analytic Besov spaces and Triebel-Lizorkin spaces, respectively.

1.2 Main result

Our main result is the following theorem:

Theorem 1.2. *Let $\overline{r}^0, \overline{r}^1 \in \mathbb{R}^N$, $0 < p_0 < p_1 \leq \infty$ and $0 < q_0, q_1 \leq \infty$ with*
$$r^0_j - \frac{d_j}{p_0} = r^1_j - \frac{d_j}{p_1}, \quad j = 1, \ldots, N. \tag{1.7}$$

1. *Then*
$$S^{\overline{r}^0}_{p_0,q_0}F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow S^{\overline{r}^1}_{p_1,q_1}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \tag{1.8}$$
if, and only if, $p_0 \leq q_1$.

2. If $p_1 < \infty$*, then*

$$S^{\overline{r}^0}_{p_0,q_0}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow S^{\overline{r}^1}_{p_1,q_1}F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \tag{1.9}$$

if, and only if, $q_0 \le p_1$.

Remark 1.3. (i) The original proofs in the isotropic case, cf. [13] and [9], use interpolation techniques. This approach was applied in [23] also to function spaces with dominating mixed smoothness but (although these authors succeeded to overcome numerous obstacles) led only to partial results. Here, we shall use a different method of proof, originally introduced in [35] to prove Theorem 1.2 in the isotropic situation.

(ii) Embeddings of Jawerth-Franke type have been proved already for several other scales of function spaces of Besov and Triebel-Lizorkin type. We refer to [8, Appendix C.3] for anisotropic case, to [6] and [11] for weighted function spaces and to [5] and [7] for spaces with generalised smoothness. In general, all these authors used the method of Jawerth and Franke and we believe that in all these cases one could apply our approach as well.

(iii) Embedding (1.8) was already obtained by Krbec and Schmeisser (cf. [15, Lemma 4.7]) in the special case $N = 2$ and $p_1 = \infty$. Furthermore, Schmeisser and Sickel (cf. [23, Theorem 3]) proved (1.9) in the Banach space setting, i.e. $1 \le p_0 < p_1 < \infty$ and $1 \le q_0, q_1 \le \infty$. The use of duality arguments allowed to prove also (1.8) but only for $1 < p_0 < \infty$. Our approach yields the proof of Theorem 1.2 without any further restrictions on the parameters.

1.3 Further consequences

Let $C(\mathbb{R}^d)$ be the space of all complex-valued bounded and uniformly continuous functions on $\mathbb{R}^d$. One of the well studied problems in the isotropic case is the embedding of Besov and Triebel-Lizorkin spaces into $C(\mathbb{R}^d)$ or $L_r(\mathbb{R}^d)$ with $1 \le r \le \infty$. This problem is connected with the works of Grisvard, Peetre, Golovkin, Stein, Zygmund, Besov or Iljin. We refer to [4] for details.

We use (1.8) to characterize those spaces $S^{\overline{r}}_{p,q}B(\mathbb{R}^{\overline{d}})$ and $S^{\overline{r}}_{p,q}F(\mathbb{R}^{\overline{d}})$ which are embedded in $C(\mathbb{R}^d)$ and $L_u(\mathbb{R}^d)$, $1 < u \le \infty$. This approach was applied already in [16], cf. also [22]. Unfortunately, there was a flaw in the arguments used in [16].

Theorem 1.4. *(i) Let* $\overline{r} \in \mathbb{R}^N$, $0 < p \le \infty$ *and* $0 < q \le \infty$. *Then the following three assertions are equivalent.*

(a) $\quad S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow C(\mathbb{R}^d)$,

(b) $\quad S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow L_\infty(\mathbb{R}^d)$,

$$(c) \quad \begin{cases} r_i - \dfrac{d_i}{p} > 0 & \text{for all} \quad i = 1, \dots, N \quad \text{or} \\ r_i - \dfrac{d_i}{p} \ge 0 & \text{for all} \quad i = 1, \dots, N \quad \text{and} \quad 0 < q \le 1. \end{cases}$$

(ii) Let $\overline{r} \in \mathbb{R}^N$, $0 < p < \infty$ *and* $0 < q \le \infty$. *Then the following three assertions are equivalent.*

(a') $\quad S^{\overline{r}}_{p,q}F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow C(\mathbb{R}^d)$,

(b') $\quad S^{\overline{r}}_{p,q}F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow L_\infty(\mathbb{R}^d)$,

$$(c') \quad \begin{cases} r_i - \dfrac{d_i}{p} > 0 & \text{for all} \quad i = 1, \dots, N \quad \text{or} \\ r_i - \dfrac{d_i}{p} \ge 0 & \text{for all} \quad i = 1, \dots, N \quad \text{and} \quad 0 < p \le 1. \end{cases}$$

We consider a similar problem also for L_u, $1 < u < \infty$. Due to the Littlewood-Paley theory the number 2 plays an exceptional role if $1 < u < \infty$.

Theorem 1.5. *(i) Let* $\overline{r}\in\mathbb{R}^N$, $1<u<\infty$, $0<p\le\infty$ *and* $0<q\le\infty$. *Then* $S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_N})\hookrightarrow L_u(\mathbb{R}^d)$ *if, and only if,* $p\le u$ *and*

$$\begin{cases} r_i>\dfrac{d_i}{p}-\dfrac{d_i}{u} & \text{for all}\quad i=1,\dots,N\quad \text{or}\\ r_i\ge\dfrac{d_i}{p}-\dfrac{d_i}{u} & \text{for all}\quad i=1,\dots,N,\quad 0<p<u\quad\text{and}\quad 0<q\le u\quad\text{or}\\ r_i\ge 0\quad \text{for all}\quad i=1,\dots,N, & p=u\quad\text{and}\quad 0<q\le\min(u,2).\end{cases}$$

(ii) Let $\overline{r}\in\mathbb{R}^N$, $1<u<\infty$, $0<p<\infty$ *and* $0<q\le\infty$. *Then* $S^{\overline{r}}_{p,q}F(\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_N})\hookrightarrow L_u(\mathbb{R}^d)$ *if, and only if,* $p\le u$ *and*

$$\begin{cases} r_i>\dfrac{d_i}{p}-\dfrac{d_i}{u} & \text{for all}\quad i=1,\dots,N\quad \text{or}\\ r_i\ge\dfrac{d_i}{p}-\dfrac{d_i}{u} & \text{for all}\quad i=1,\dots,N\quad\text{and}\quad 0<p<u\quad\text{or}\\ r_i\ge 0\quad \text{for all}\quad i=1,\dots,N, & p=u\quad\text{and}\quad 0<q\le 2.\end{cases}$$

Remark 1.6. Let $1<u\le\infty$. Direct comparison of Theorems 1.4-1.5 with similar assertions for isotropic Besov and Triebel-Lizorkin spaces (cf. [26]) shows, that $S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_N})\hookrightarrow L_u(\mathbb{R}^d)$ if, and only if, $B^{r_i}_{p,q}(\mathbb{R}^{d_i})\hookrightarrow L_u(\mathbb{R}^{d_i})$ for all $i=1,\dots,N$. The same statement holds true, if $L_\infty(\mathbb{R}^d)$ is replaced by $C(\mathbb{R}^d)$ and also for the Triebel-Lizorkin spaces.

Remark 1.7. Also the optimal embeddings into $L_1(\mathbb{R}^n)$ and $L_1^{\rm loc}(\mathbb{R}^n)$ - the space of locally integrable functions - are very well known in the isotropic case. To extend these results to function spaces with dominating mixed smoothness, it would be probably necessary to consider the analog of the Hardy space H^1 and of the space of bounded mean oscilation BMO in the framework of dominating mixed smoothness. But this goes beyond the scope of this work.

2 Proofs

2.1 Preliminaries

Our approach is based on two classical techniques - decomposition theorems and multivariate rearrangements.

First, we describe the sequence spaces associated to $S^{\overline{r}}_{p,q}B(\mathbb{R}^{\overline{d}})$ and $S^{\overline{r}}_{p,q}F(\mathbb{R}^{\overline{d}})$.

Let $m\in\mathbb{Z}^d$, $m=(m^1,\dots,m^N)$ with $m^i\in\mathbb{Z}^{d_i}$, and $\overline{\nu}\in\mathbb{N}_0^N$. Then $Q_{\overline{\nu},m}$ denotes the closed cube in $\mathbb{R}^d$ with sides parallel to the coordinate axes, centred at the point $2^{-\overline{\nu}}m=(2^{-\nu_1}m^1,\dots,2^{-\nu_N}m^N)$, and with sides of the lengths $2^{-\nu_1},\dots,2^{-\nu_N}$. Explicitly,

$$Q_{\overline{\nu},m}=\{x\in\mathbb{R}^d:|x^i-2^{-\nu_i}m^i|_\infty\le 2^{-\nu_i-1}, i=1,\dots,N\},\tag{2.1}$$

where $x=(x^1,\dots,x^N), x^i\in\mathbb{R}^{d_i}$, and $|t|_\infty=\max\limits_{i=1,\dots,n}|t_i|$, $t\in\mathbb{R}^n$. By $\chi_{\overline{\nu},m}=\chi_{Q_{\overline{\nu},m}}$ we denote the characteristic function of $Q_{\overline{\nu},m}$. If

$$\lambda=\{\lambda_{\overline{\nu},m}\in\mathbb{C}:\overline{\nu}\in\mathbb{N}_0^N, m\in\mathbb{Z}^d\},$$

$\overline{r}\in\mathbb{R}^N$ and $0<p,q\le\infty$, we set

$$\big\|\lambda|s^{\overline{r}}_{p,q}b\big\|=\left(\sum_{\overline{\nu}\in\mathbb{N}_0^N}2^{\overline{\nu}\cdot(\overline{r}-\overline{d}/p)q}\left(\sum_{m\in\mathbb{Z}^d}|\lambda_{\overline{\nu},m}|^p\right)^{q/p}\right)^{1/q},\tag{2.2}$$

appropriately modified if $p=\infty$ and/or $q=\infty$. If $p<\infty$, we define also

$$\big\|\lambda|s^{\overline{r}}_{p,q}f\big\|=\left\|\left(\sum_{\overline{\nu}\in\mathbb{N}_0^N}\sum_{m\in\mathbb{Z}^d}|2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu},m}|^q\chi_{\overline{\nu},m}(\cdot)\right)^{1/q}\right|L_p(\mathbb{R}^d)\Bigg\|.\tag{2.3}$$

Using the wavelet decomposition techniques, one may give linear isomorphisms between function spaces with dominating mixed smoothness properties and corresponding sequence spaces. We refer to [35] if $d_1 = d_2 = \cdots = d_N = 1$ and to [10] in the general case.

This allows to reduce the proof of Theorem 1.2 to the embeddings of sequence spaces. Hence, it is enough to prove that under condition (1.7)

$$s^{\overline{r}_0}_{p_0,q_0}f \hookrightarrow s^{\overline{r}_1}_{p_1,q_1}b \tag{2.4}$$

if, and only if, $p_0 \le q_1$ and

$$s^{\overline{r}_0}_{p_0,q_0}b \hookrightarrow s^{\overline{r}_1}_{p_1,q_1}f, \tag{2.5}$$

if, and only if, $q_0 \le p_1$.

Now, we present briefly the concept of non-increasing rearrangement. We refer to [3, Chapter 2] for details.

Definition 2.1. Let μ be the Lebesgue measure in $\mathbb{R}^n$. If h is a measurable function on $\mathbb{R}^n$, we define the *non-increasing rearrangement* of h through

$$h^*(t) = \sup\{\lambda > 0 : \mu\{x \in \mathbb{R}^n : |h(x)| > \lambda\} > t\}, \qquad t \in (0,\infty). \tag{2.6}$$

We shall need also the so-called multivariate rearrangements.

Let $f : (0,\infty)^{k-1} \times \mathbb{R}^{d_k} \times \cdots \times \mathbb{R}^{d_N} \to \mathbb{C}$, $k \le N$, be a measurable function. We set

$$(R_k f)(t_1, \dots, t_{k-1}, s, y^{k+1}, \dots, y^N) = [f(t_1, \dots, t_{k-1}, \cdot, y^{k+1}, \dots, y^N)]^*(s),$$
$$s > 0, \quad t_1, \dots, t_{k-1} \in (0,\infty), \quad y^i \in \mathbb{R}^{d_i}, i = k+1, \dots, N.$$

We define the *multivariate non-increasing rearrangement* of $f : \mathbb{R}^d \to \mathbb{C}$ by

$$(Rf)(\overline{s}) = (R_N \circ \cdots \circ R_1 f)(\overline{s}), \quad \overline{s} = (s_1, \dots, s_N) \in (0,\infty)^N.$$

The utility of multivariate rearrangements in connection with embeddings of Sobolev type has been discovered by Kolyada [14]. Later on, it was used by Krbec and Schmeisser [16] in connection with function spaces with dominating mixed smoothness.

We shall use the following two properties. They are well known in the scalar case $N = 1$ (cf. [3]) and may be easily generalised to $N > 1$.

Lemma 2.2. *If* $0 < p \le \infty$, *then*

$$||h|L_p(\mathbb{R}^d)|| = ||Rh|L_p((0,\infty)^N)||$$

for every measurable function h.

Lemma 2.3. *Let* h_1 *and* h_2 *be two non-negative measurable functions on* $\mathbb{R}^d$. *If* $1 \le p \le \infty$, *then*

$$||h_1 + h_2|L_p(\mathbb{R}^d)|| \le ||Rh_1 + Rh_2|L_p((0,\infty)^N)||.$$

If $g : (0,\infty)^N \to \mathbb{R}$ is measurable, we define the average operator $\mathcal{A}g$ by

$$(\mathcal{A}g)(\overline{t}) := \left(\prod_{i=1}^N t_i\right)^{-1} \int_{[0,\overline{t}]} |g(x)|dx \quad \text{for} \quad \overline{t} \in (0,\infty)^N.$$

The following property is also well known if $N = 1$, the generalisation to $N > 1$ follows by iteration.

Lemma 2.4. *If* $1 < p \le \infty$, *then there is a constant* c_p *such that*

$$||\mathcal{A}h|L_p((0,\infty)^N)|| \le c_p||h|L_p((0,\infty)^N)||$$

for every measurable function h *defined on* $(0,\infty)^N$.

2.2 Proof of Theorem 1.2

Step 1. Proof of (2.4)

We observe, that the operator

$$I_{\overline{r}} : \lambda_{\overline{\nu},m} \to \tilde{\lambda}_{\overline{\nu},m} = 2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu},m}, \quad \overline{\nu} \in \mathbb{N}_0^N, \quad m \in \mathbb{Z}^d$$

forms a linear isomorphism from $s^{\overline{r}^0}_{p,q}b$ onto $s^{\overline{r}_0-\overline{r}}_{p,q}b$, where $\overline{r}, \overline{r}^0 \in \mathbb{R}^N$ are arbitrary. The same statement holds for the f-spaces as well.

We combine this with the simple embedding

$$s^{\overline{r}}_{p,q_0}f \hookrightarrow s^{\overline{r}}_{p,q_1}f \quad \text{if} \quad 0 < q_0 \le q_1 \le \infty,$$

and hence it is enough to prove that

$$s^{\overline{r}}_{p_0,\infty}f \hookrightarrow s^0_{p_1,p_0}b, \tag{2.7}$$

where

$$r_i = d_i\left(\frac{1}{p_0} - \frac{1}{p_1}\right), \quad i = 1, \dots, N. \tag{2.8}$$

Let $\lambda \in s^{\overline{r}}_{p_0,\infty}f$. We set

$$h(x) = \sup_{\overline{\nu}\in\mathbb{N}_0^N} 2^{\overline{\nu}\cdot\overline{r}} \sum_{m\in\mathbb{Z}^d} |\lambda_{\overline{\nu},m}|\chi_{\overline{\nu},m}(x), \quad x \in \mathbb{R}^d. \tag{2.9}$$

Using this notation, we get

$$|\lambda_{\overline{\nu},m}| \le 2^{-\overline{\nu}\cdot\overline{r}} \inf_{x\in Q_{\overline{\nu},m}} h(x), \quad \overline{\nu} \in \mathbb{N}_0^N, \quad m \in \mathbb{Z}^d$$

and

$$\big\|h\big|L_{p_0}(\mathbb{R}^d)\big\| = \big\|\lambda\big|s^{\overline{r}}_{p_0,\infty}f\big\| < \infty. \tag{2.10}$$

The main step of our calculation is the following estimate:

$$\left(\sum_{m\in\mathbb{Z}^d} \inf_{x\in Q_{\overline{\nu},m}} h(x)^{p_1}\right)^{1/p_1} \le \left(\sum_{\overline{k}\in\mathbb{N}^N} (Rh)(2^{-(\nu_1+1)d_1}k_1, \dots, 2^{-(\nu_N+1)d_N}k_N)^{p_1}\right)^{1/p_1}, \quad \overline{\nu} \in \mathbb{N}_0^N, \tag{2.11}$$

for $0 < p_1 < \infty$ and

$$\sup_{m\in\mathbb{Z}^d} \inf_{x\in Q_{\overline{\nu},m}} h(x) \le (Rh)(2^{-(\nu_1+1)d_1}, \dots, 2^{-(\nu_N+1)d_N}) \tag{2.12}$$

for $p_1 = \infty$ and all $\overline{\nu} \in \mathbb{N}_0^N$.

We start with the case $p_1 = \infty$. To prove (2.12) we fix some $\overline{\nu} \in \mathbb{N}_0^N$. Then we make use of the fact, that the sets $Q_{\overline{\nu},m}$ have a product structure. Hence, they may be rewritten as

$$Q_{\overline{\nu},m} = Q_{\nu_1,m^1} \times \dots \times Q_{\nu_N,m^N}.$$

Let $\varepsilon > 0$, and fix $x^2 \in \mathbb{R}^{d_2}, \dots, x^N \in \mathbb{R}^{d_N}$. Then there is some $m^1_0 \in \mathbb{Z}^{d_1}$, such that

$$\sup_{m^1\in\mathbb{Z}^{d_1}} \inf_{y\in Q_{\nu_1,m^1}} h(y, x^2, \dots, x^N) < \inf_{y\in Q_{\nu_1,m^1_0}} h(y, x^2, \dots, x^N) + \varepsilon. \tag{2.13}$$

Let us point out, that (2.10) implies that the supremum on the left-hand side of (2.13) is finite for almost every $(x^2,\dots,x^N)\in\mathbb{R}^{d_2+\dots+d_N}$.

Obviously,

$$h(x^1,x^2,\dots,x^N) > \inf_{y\in Q_{\nu_1,m_0^1}} h(y,x^2,\dots,x^N)-\varepsilon$$

holds for all $x^1\in Q_{\nu_1,m_0^1}$. This is a set of Lebesgue-measure $2^{-\nu_1 d_1} > 2^{-(\nu_1+1)d_1}$. From this, it follows

$$\begin{aligned}(R_1h)(2^{-(\nu_1+1)d_1},x^2,\dots,x^N) &\ge \inf_{y\in Q_{\nu_1,m_0^1}} h(y,x^2,\dots,x^N)-\varepsilon\\ &\ge \sup_{m^1\in\mathbb{Z}^{d_1}}\inf_{y\in Q_{\nu_1,m^1}} h(y,x^2,\dots,x^N)-2\varepsilon.\end{aligned}$$

With $\varepsilon\to 0$ we get

$$\begin{aligned}\sup_{k_1\in\mathbb{N}}(R_1h)(2^{-(\nu_1+1)d_1}k_1,x^2,\dots,x^N) &= (R_1h)(2^{-(\nu_1+1)d_1},x^2,\dots,x^N)\\ &\ge \sup_{m^1\in\mathbb{Z}^{d_1}}\inf_{y\in Q_{\nu_1,m^1}} h(y,x^2,\dots,x^N).\end{aligned}$$

If we use the same argument for the function $(R_1h)(2^{-(\nu_1+1)d_1},\cdot,x^3,\dots,x^N)$, we get

$$\begin{aligned}&\sup_{k_1,k_2\in\mathbb{N}}(R_2\circ R_1h)(2^{-(\nu_1+1)d_1}k_1,2^{-(\nu_2+1)d_2}k_2,x^3,\dots,x^N)\\ &\quad\ge \sup_{m^2\in\mathbb{Z}^{d_2}}\inf_{y^2\in Q_{\nu_2,m^2}}(R_1h)(2^{-(\nu_1+1)d_1},y^2,x^3,\dots,x^N)\\ &\quad\ge \sup_{m^2\in\mathbb{Z}^{d_2}}\inf_{y^2\in Q_{\nu_2,m^2}}\sup_{m^1\in\mathbb{Z}^{d_1}}\inf_{y^1\in Q_{\nu_1,m^1}} h(y^1,y^2,x^3,\dots,x^N)\\ &\quad\ge \sup_{m^2\in\mathbb{Z}^{d_2},m^1\in\mathbb{Z}^{d_1}}\inf_{y^2\in Q_{\nu_2,m^2},y^1\in Q_{\nu_1,m^1}} h(y^1,y^2,x^3,\dots,x^N).\end{aligned}$$

Further iteration yields

$$\begin{aligned}&\sup_{\overline{k}\in\mathbb{N}^N}(Rh)(2^{-(\nu_1+1)d_1}k_1,\dots,2^{-(\nu_N+1)d_N}k_N)\\ &\quad\ge \sup_{m^N\in\mathbb{Z}^{d_N},\dots,m^1\in\mathbb{Z}^{d_1}}\inf_{y^N\in Q_{\nu_N,m^N},\dots,y^1\in Q_{\nu_1,m^1}} h(y^1,\dots,y^N)=\sup_{m\in\mathbb{Z}^d}\inf_{y\in Q_{\overline{\nu},m}} h(y),\end{aligned}$$

and (2.12) is proven.

If $0<p_1<\infty$ then (2.11) may be proved by similar arguments, but we prefer to present an alternative way. We shall use the abbreviation $\eta_m:=\inf_{x\in Q_{\overline{\nu},m}} h(x)$. By (2.9) and (2.10) we have

$$0\le\eta_m<\infty,\quad m\in\mathbb{Z}^d.$$

As the interiors of $Q_{\overline{\nu},m}$ are mutually disjoint, we may define a new function $\tilde{h}$ by

$$\tilde{h}(y)=\eta_m,\quad y\in interior(Q_{\overline{\nu},m})\quad\text{and}\quad \tilde{h}(y)=0\quad\text{if}\quad y\in boundary(Q_{\overline{\nu},m}).$$

One observes immediately, that $0\le\tilde{h}(x)\le h(x)$ for $x\in\mathbb{R}^{\overline{d}}$ and therefore $(R\tilde{h})(t)\le(Rh)(t)$ for $t\in(0,\infty)^N$.

It follows, that

$$\left(\sum_{m\in\mathbb{Z}^d}\inf_{x\in Q_{\overline{\nu},m}} h(x)^{p_1}\right)^{1/p_1} = 2^{\overline{\nu}\cdot\overline{d}/p_1}||\tilde{h}|L_{p_1}(\mathbb{R}^d)|| = 2^{\overline{\nu}\cdot\overline{d}/p_1}||R\tilde{h}|L_{p_1}((0,\infty)^N)||.$$

As $\tilde{h}$ is constant on the cubes $Q_{\overline{\nu},m}$, $R\tilde{h}$ is constant on the cubes $Q'_{\overline{\nu},\overline{m}}$ with vertices in $(2^{-\nu_1 d_1}m_1,\dots,2^{-\nu_N d_N}m_N)$ and $(2^{-\nu_1 d_1}(m_1+1),\dots,2^{-\nu_N d_N}(m_N+1))$ and sides parallel to the coordinate axes. Here, $\overline{m}=(m_1,\dots,m_N)\in\mathbb{N}_0^N$.
Hence

$$2^{\overline{\nu}\cdot\overline{d}/p_1}||R\tilde{h}|L_{p_1}((0,\infty)^N)|| \le \left(\sum_{\overline{k}\in\mathbb{N}^N}(R\tilde{h})(2^{-(\nu_1+1)d_1}k_1,\dots,2^{-(\nu_N+1)d_N}k_N)^{p_1}\right)^{1/p_1}$$
$$\le \left(\sum_{\overline{k}\in\mathbb{N}^N}(Rh)(2^{-(\nu_1+1)d_1}k_1,\dots,2^{-(\nu_N+1)d_N}k_N)^{p_1}\right)^{1/p_1}$$

and (2.11) follows.
Now we are ready to give the proof of (2.4). Using condition (1.7) we obtain $\overline{r}p_0+\overline{d}p_0/p_1=\overline{d}$ and hence

$$\big\|\lambda|s^0_{p_1,p_0}b\big\|^{p_0} \le \sum_{\overline{\nu}\in\mathbb{N}_0^N}2^{-\overline{\nu}\cdot\overline{d}}\left(\sum_{\overline{m}\in\mathbb{Z}^d}\inf_{x\in Q_{\overline{\nu}\,\overline{m}}}h(x)^{p_1}\right)^{p_0/p_1}$$
$$\le \sum_{\overline{\nu}\in\mathbb{N}_0^N}2^{-\overline{\nu}\cdot\overline{d}}\left(\sum_{\overline{k}\in\mathbb{N}^N}(Rh)(2^{-(\nu_1+1)d_1}k_1,\dots,2^{-(\nu_N+1)d_N}k_N)^{p_1}\right)^{p_0/p_1}$$
$$\le \sum_{\overline{\nu}\in\mathbb{N}_0^N}2^{-\overline{\nu}\cdot\overline{d}}\left(\sum_{\overline{l}\in\mathbb{N}_0^N}\sum_{\substack{\overline{k}\in\mathbb{N}^N:\\ \forall i:2^{l_id_i}\le k_i<2^{(l_i+1)d_i}}}(Rh)(2^{-(\nu_1+1)d_1}k_1,\dots,2^{-(\nu_N+1)d_N}k_N)^{p_1}\right)^{p_0/p_1}$$
$$\lesssim \sum_{\overline{\nu}\in\mathbb{N}_0^N}2^{-\overline{\nu}\cdot\overline{d}}\left(\sum_{\overline{l}\in\mathbb{N}_0^N}2^{\overline{l}\cdot\overline{d}}(Rh)(2^{(l_1-\nu_1-1)d_1},\dots,2^{(l_N-\nu_N-1)d_N})^{p_1}\right)^{p_0/p_1}$$
$$\le \sum_{\overline{\nu}\in\mathbb{N}_0^N}2^{-\overline{\nu}\cdot\overline{d}}\sum_{\overline{l}\in\mathbb{N}_0^N}2^{\overline{l}\cdot\overline{d}\frac{p_0}{p_1}}(Rh)(2^{(l_1-\nu_1-1)d_1},\dots,2^{(l_N-\nu_N-1)d_N})^{p_0}.$$

We substitute $\overline{n}=\overline{l}-\overline{\nu}-1$ and find

$$\big\|\lambda|s^0_{p_1,p_0}b\big\|^{p_0} \le \sum_{\overline{\nu}\in\mathbb{N}_0^N}2^{-\overline{\nu}\cdot\overline{d}}\sum_{\overline{n}\in\mathbb{Z}^N:\overline{n}+\overline{\nu}+1\in\mathbb{N}_0^N}2^{(\overline{n}+\overline{\nu}+1)\cdot\overline{d}\frac{p_0}{p_1}}(Rh)(2^{n_1d_1},\dots,2^{n_Nd_N})^{p_0}$$
$$\le 2^{d\frac{p_0}{p_1}}\sum_{\overline{n}\in\mathbb{Z}^N}2^{\overline{n}\cdot\overline{d}\frac{p_0}{p_1}}(Rh)(2^{n_1d_1},\dots,2^{n_Nd_N})^{p_0}\sum_{\overline{\nu}\in\mathbb{Z}^N:\overline{\nu}+1\ge-\overline{n}}2^{\overline{\nu}\cdot\overline{d}(\frac{p_0}{p_1}-1)}$$
$$\lesssim \sum_{\overline{n}\in\mathbb{Z}^N}2^{\overline{n}\cdot\overline{d}\frac{p_0}{p_1}}(Rh)(2^{n_1d_1},\dots,2^{n_Nd_N})^{p_0}2^{-\overline{n}\cdot\overline{d}(\frac{p_0}{p_1}-1)}$$
$$= \sum_{\overline{n}\in\mathbb{Z}^N}2^{\overline{n}\cdot\overline{d}}(Rh)(2^{n_1d_1},\dots,2^{n_Nd_N})^{p_0}\sim\big\|Rh\big|L_{p_0}((0,\infty)^N)\big\|^{p_0}=\big\|h\big|L_{p_0}(\mathbb{R}^d)\big\|^{p_0}.$$

This finishes the proof of (2.4) under the condition (1.7) and $p_1<\infty$. In case $p_1=\infty$ one can estimate more directly

$$
\begin{aligned}
\|\lambda|s^0_{\infty,p_0}b\|^{p_0} &\le \sum_{\overline{\nu}\in\mathbb{N}_0^N} 2^{-\overline{\nu}\cdot\overline{d}} \sup_{\overline{m}\in\mathbb{Z}^d} \inf_{x\in Q_{\overline{\nu}\,\overline{m}}} h(x)^{p_0}\\
&\le \sum_{\overline{\nu}\in\mathbb{N}_0^N} 2^{-\overline{\nu}\cdot\overline{d}} (Rh)(2^{-(\nu_1+1)d_1},\dots,2^{-(\nu_N+1)d_N})^{p_0}\\
&\le 2^d \sum_{\overline{\nu}\in\mathbb{Z}^N} 2^{-(\overline{\nu}+1)\cdot\overline{d}} (Rh)(2^{-(\nu_1+1)d_1},\dots,2^{-(\nu_N+1)d_N})^{p_0}\\
&\sim \|Rh|L_{p_0}((0,\infty)^N)\|^{p_0} = \|h|L_{p_0}(\mathbb{R}^d)\|^{p_0}.
\end{aligned}
$$

Step 2. Proof of (2.5)

We use similar arguments as in Step 1, this time combined with duality.

Using lifting properties and trivial embeddings, we may again restrict the proof to

$$s^{\overline{r}}_{p_0,p_1}b \hookrightarrow s^0_{p_1,q}f,$$

where

$$r_i = d_i\left(\frac{1}{p_0}-\frac{1}{p_1}\right),\quad i=1,\dots,N$$

and $0<q<p_0$.

Let $\lambda=\{\lambda_{\overline{\nu}}\}_{\overline{\nu}\in\mathbb{N}_0^N}=\{\lambda_{\overline{\nu},m}\}_{\overline{\nu}\in\mathbb{N}_0^N,m\in\mathbb{Z}^d}$ be in $s^{\overline{r}}_{p_0,p_1}b$. The multivariate non-increasing rearrangement of $\lambda_{\overline{\nu}}=\{\lambda_{\overline{\nu},m}\}_{m\in\mathbb{Z}^d}$ is defined similar to Definition 2.1 and denoted by $\widetilde{\lambda}_{\overline{\nu}}=\{\widetilde{\lambda}_{\overline{\nu},\overline{m}}\}_{\overline{m}\in\mathbb{N}_0^N}$. As $\lambda_{\overline{\nu}}\in\ell_{p_0}(\mathbb{Z}^d)$, this rearrangement is also a rearrangement of a sequence in the classical sense. Furthermore, we write $\widetilde{\chi}_{\overline{\nu}\,\overline{m}}$ for characteristic functions of cubes $Q'_{\overline{\nu},\overline{m}}\subset(0,\infty)^N$, which were used already in the Step 1.

Then, using $q<p_1$ and Lemma 2.3,

$$
\begin{aligned}
\|\lambda|s^0_{p_1q}f\| &= \left\|\left(\sum_{\overline{\nu}\in\mathbb{N}_0^N}\sum_{m\in\mathbb{Z}^d}|\lambda_{\overline{\nu}\,m}|^q\chi_{\overline{\nu},m}(x)\right)^{1/q}\Bigg|L_{p_1}(\mathbb{R}^d)\right\|\\
&\le \left\|\sum_{\overline{\nu}\in\mathbb{N}_0^N}\sum_{\overline{m}\in\mathbb{N}_0^N}\widetilde{\lambda}^q_{\overline{\nu}\,\overline{m}}\widetilde{\chi}_{\overline{\nu}\,\overline{m}}(x)\Bigg|L_{\frac{p_1}{q}}((0,\infty)^N)\right\|^{1/q}.
\end{aligned}
\tag{2.14}
$$

Let α and β be the conjugate exponents of $\frac{p_0}{q}$ and of $\frac{p_1}{q}$, respectively. Using duality, (2.14) may be rewritten as

$$
\begin{aligned}
\|\lambda|s^0_{p_1q}f\| &\le \sup_g\left(\int_{(0,\infty)^N} g(x)\left(\sum_{\overline{\nu}\in\mathbb{N}_0^N}\sum_{\overline{m}\in\mathbb{N}_0^N}\widetilde{\lambda}^q_{\overline{\nu},\overline{m}}\widetilde{\chi}_{\overline{\nu},\overline{m}}(x)\right)dx\right)^{1/q}\\
&= \sup_g\left(\sum_{\nu\in\mathbb{N}_0^N}\sum_{\overline{m}\in\mathbb{N}_0^N}2^{-\overline{\nu}\cdot\overline{d}}\widetilde{\lambda}^q_{\overline{\nu},\overline{m}}g_{\overline{\nu},\overline{m}}\right)^{1/q},
\end{aligned}
\tag{2.15}
$$

where the supremum is taken over all non-negative functions $g:(0,\infty)^N\to[0,\infty]$, which are non-increasing in each variable, $\|g|L_\beta((0,\infty)^N)\|\le 1$ and $g_{\overline{\nu},\overline{m}}=2^{\overline{\nu}\cdot\overline{d}}\int g(x)\widetilde{\chi}_{\overline{\nu},\overline{m}}(x)dx$.

We use twice Hölder's inequality and estimate (2.15) from above by

$$\left(\sum_{\nu\in\mathbb{N}_0^N} 2^{-\overline{\nu}\cdot\overline{d}}\left(\sum_{\overline{m}\in\mathbb{N}_0^N}\widetilde{\lambda}^{p_0}_{\overline{\nu},\overline{m}}\right)^{\frac{p_1}{p_0}}\right)^{1/p_1}\cdot\sup_g\left(\sum_{\overline{\nu}\in\mathbb{N}_0^N} 2^{-\overline{\nu}\cdot\overline{d}}\left(\sum_{\overline{m}\in\mathbb{N}_0^N} g^{\alpha}_{\overline{\nu}\,\overline{m}}\right)^{\beta/\alpha}\right)^{\frac{1}{\beta q}}. \tag{2.16}$$

The first factor in (2.16) is equal to $\|\lambda|s^{\overline{r}}_{p_0,p_1}b\|$ due to condition (1.7). Hence it is enough to prove that there is a constant $c>0$, such that

$$\left(\sum_{\overline{\nu}\in\mathbb{N}_0^N} 2^{-\overline{\nu}\cdot\overline{d}}\left(\sum_{\overline{m}\in\mathbb{N}_0^N} g^{\alpha}_{\overline{\nu},\overline{m}}\right)^{\beta/\alpha}\right)^{\frac{1}{\beta q}}\le c$$

for every non-negative measurable function g, which is non-increasing in each component and with $\|g|L_\beta((0,\infty)^N)\|\le 1$.
First, we use the monotonicity of g and obtain

$$\begin{aligned}\sum_{\overline{m}\in\mathbb{N}_0^N} g^{\alpha}_{\overline{\nu}\,\overline{m}} &= \sum_{\overline{l}\in\mathbb{N}_0^N}\ \sum_{\substack{\overline{m}\in\mathbb{N}_0^N:\\ \forall i:2^{l_i d_i}-1\le m_i<2^{(l_i+1)d_i}-1}} g^{\alpha}_{\overline{\nu},\overline{m}} \lesssim \sum_{\overline{l}\in\mathbb{N}_0^N} 2^{\overline{l}\cdot\overline{d}}\left(2^{\overline{\nu}\cdot\overline{d}}\int_{W_{\overline{\nu},(2^{l_1d_1},\dots,2^{l_Nd_N})}} g(x)dx\right)^{\alpha}\\ &\lesssim \sum_{\overline{l}\in\mathbb{N}_0^N} 2^{\overline{l}\cdot\overline{d}}(\mathcal{A}g)(2^{(l_1-\nu_1)d_1},\dots,2^{(l_N-\nu_N)d_N})^{\alpha},\end{aligned}$$

where $W_{\overline{\nu},\overline{k}}=[2^{-\nu_1d_1}(k_1-1),2^{-\nu_1d_1}k_1]\times\cdots\times[2^{-\nu_Nd_N}(k_N-1),2^{-\nu_Nd_N}k_N]$.
Using $1<\beta<\alpha<\infty$, this leads to

$$\begin{aligned}\left(\sum_{\overline{\nu}\in\mathbb{N}_0^N} 2^{-\overline{\nu}\cdot\overline{d}}\left(\sum_{\overline{m}\in\mathbb{N}_0^N} g^{\alpha}_{\overline{\nu}\,\overline{m}}\right)^{\beta/\alpha}\right)^{\frac{1}{\beta}} &\le \left(\sum_{\overline{\nu}\in\mathbb{N}_0^N} 2^{-\overline{\nu}\cdot\overline{d}}\left(\sum_{\overline{l}\in\mathbb{N}_0^N} 2^{\overline{l}\cdot\overline{d}}(\mathcal{A}g)(2^{(l_1-\nu_1)d_1},\dots,2^{(l_N-\nu_N)d_N})^{\alpha}\right)^{\frac{\beta}{\alpha}}\right)^{1/\beta}\\ &\le \left(\sum_{\overline{\nu}\in\mathbb{N}_0^N} 2^{-\overline{\nu}\cdot\overline{d}}\sum_{\overline{l}\in\mathbb{N}_0^N} 2^{\overline{l}\cdot\overline{d}\frac{\beta}{\alpha}}(\mathcal{A}g)(2^{(l_1-\nu_1)d_1},\dots,2^{(l_N-\nu_N)d_N})^{\beta}\right)^{1/\beta}\\ &= \left(\sum_{\overline{k}\in\mathbb{Z}^N} 2^{\overline{k}\cdot\overline{d}\frac{\beta}{\alpha}}\sum_{\overline{\nu}\in\mathbb{N}_0^N:\overline{\nu}\ge-\overline{k}} 2^{-\overline{\nu}\cdot\overline{d}}2^{\overline{\nu}\cdot\overline{d}\frac{\beta}{\alpha}}(\mathcal{A}g)(2^{k_1d_1},\dots,2^{k_Nd_N})^{\beta}\right)^{1/\beta}\\ &\le \left(\sum_{\overline{k}\in\mathbb{Z}^N} 2^{\overline{k}\cdot\overline{d}\frac{\beta}{\alpha}}(\mathcal{A}g)(2^{k_1d_1},\dots,2^{k_Nd_N})^{\beta}\sum_{\overline{\nu}\in\mathbb{Z}^N:\overline{\nu}\ge-\overline{k}} 2^{\overline{\nu}\cdot\overline{d}(\frac{\beta}{\alpha}-1)}\right)^{1/\beta}\\ &\lesssim \left(\sum_{\overline{k}\in\mathbb{Z}^N} 2^{\overline{k}\cdot\overline{d}\frac{\beta}{\alpha}}(\mathcal{A}g)(2^{k_1d_1},\dots,2^{k_Nd_N})^{\beta}2^{-\overline{k}\cdot\overline{d}(\frac{\beta}{\alpha}-1)}\right)^{1/\beta}\\ &\sim \big\|\mathcal{A}g\big|L_\beta((0,\infty)^N)\big\|\sim\big\|g\big|L_\beta((0,\infty)^N)\big\|\le 1.\end{aligned}$$

This finishes the proof of (2.5).
Step 3.

We show, that if (1.7) and (2.4) hold, then $p_0 \le q_1$. Suppose, that $0 < q_1 < p_0 < \infty$ and set

$$\lambda_{\overline{\nu},m} = \begin{cases} \nu_1^{-1/q_1} 2^{\nu_1(d_1/p_1 - r_1^1)} & \text{if} \quad \overline{\nu} = (\nu_1, 0, \dots, 0), \ \nu_1 \in \mathbb{N} \quad \text{and} \quad m = (0, \dots, 0) \in \mathbb{Z}^d, \\ 0, & \text{otherwise}. \end{cases}$$

Direct calculations show that $||\lambda|s^{\overline{r}_1}_{p_1q_1}b|| = \infty$ and $||\lambda|s^{\overline{r}_0}_{p_0q_0}f|| < \infty$. Hence (2.4) does not hold.
Step 4.
We show, that (2.5) implies $q_0 \le p_1$. To this end we assume that $0 < p_1 < q_0 \le \infty$ and set

$$\lambda_{\overline{\nu},m} = \begin{cases} \nu_1^{-1/p_1} 2^{\nu_1(d_1/p_1 - r_1^1)} & \text{if} \quad \overline{\nu} = (\nu_1, 0, \dots, 0), \ \nu_1 \in \mathbb{N} \quad \text{and} \quad m = (0, \dots, 0) \in \mathbb{Z}^d, \\ 0, & \text{otherwise}. \end{cases}$$

This leads to $||\lambda|s^{\overline{r}_1}_{p_1q_1}f|| = \infty$ and $||\lambda|s^{\overline{r}_0}_{p_0q_0}b|| < \infty$. Hence (2.5) does not hold.

2.3 Proof of Theorem 1.4

If (c) is satisfied, then we use the embedding

$$S^0_{\infty,1}B(\mathbb{R}^{\overline{d}}) \hookrightarrow C(\mathbb{R}^d), \tag{2.17}$$

which follows directly from Definition 1.1, and the Sobolev embedding (cf. [24, Theorem 2.4.1])

$$S^{\overline{r}^0}_{p_0,q_0}B(\mathbb{R}^{\overline{d}}) \hookrightarrow S^{\overline{r}^1}_{p_1,q_1}B(\mathbb{R}^{\overline{d}})$$

if

$$r_j^0 - \frac{d_j}{p_0} = r_j^1 - \frac{d_j}{p_1}, \quad j = 1, \dots, N, \quad 0 < p_0 < p_1 \le \infty \quad \text{and} \quad 0 < q_0 \le q_1 \le \infty.$$

Hence, $S^{\overline{r}}_{p,q}B(\mathbb{R}^{\overline{d}}) \hookrightarrow C(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d)$. This proves $(c) \implies (a) \implies (b)$.
If (c) is not satisfied, we look for a distribution $f \in S^{\overline{r}}_{pq}B(\mathbb{R}^{\overline{d}})$, which may *not* be represented by a bounded measurable function in the usual sense. The counterexamples may be given directly using the wavelet expansions as presented in [10]. But one may proceed also indirectly:
Let us assume that $r_j - \dfrac{d_j}{p} < 0$ for some $1 \le j \le N$ or $r_j - \dfrac{d_j}{p} \le 0$ for some $1 \le j \le N$ and $q > 1$. In both cases, it is known that there is a distribution $\psi_j \in B^{r_j}_{pq}(\mathbb{R}^{d_j})$, such that $\psi_j \notin L_\infty(\mathbb{R}^{d_j})$, cf. [26, Theorem 3.3.1]. Now it is enough to consider

$$f = \psi_1 \otimes \psi_2 \otimes \dots \otimes \psi_N,$$

where $\psi_i \in S(\mathbb{R}^{d_i}), i \ne j$, are suitably chosen smooth functions. The proof of (ii) uses similar arguments, this time combined with (1.8).

2.4 Proof of Theorem 1.5

The proof of Theorem 1.5 follows by similarly with (2.17) replaced by

$$S^0_{u,2}F(\mathbb{R}^{\overline{d}}) = L_u(\mathbb{R}^d), \quad 1 < u < \infty.$$

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