Corrigenda to the paper: "On approximation numbers of Sobolev embeddings of weighted function spaces, J. Appr. Theory 136 (2005), 91-107"

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This remark refers to the Proposition 11 in [3]:
Suppose $1 \le p_0 < 2 < p_1 \le \infty, (p_0, p_1) \neq (1, \infty)$ and $\delta \neq \alpha$. We put

$$\varkappa = \begin{cases} \frac{\min(\alpha,\delta)}{d} + \frac{1}{2} - \frac{1}{\min(p_0',p_1)} & \text{if } \min(\alpha,\delta) > \frac{d}{\min(p_0',p_1)}, \\ \frac{\min(\alpha,\delta)}{d} \cdot \frac{\min(p_0',p_1)}{2} & \text{if } \min(\alpha,\delta) \le \frac{d}{\min(p_0',p_1)}. \end{cases}$$

Then

$$a_k\Big(\mathrm{id} : \ell_{q_0}\left(2^{j\delta}\ell_{p_0}(w_\alpha)\right) \to \ell_{q_1}(\ell_{p_1})\Big) \sim k^{-\varkappa}.$$

In the first case, when $\min(\alpha,\delta) > \frac{d}{\min(p_0',p_1)}$, the statement of the Proposition (and the proof given in [3]) is correct. Unfortunately, in the case $\min(\alpha,\delta) \le \frac{d}{\min(p_0',p_1)}$, the proof contains a mistake. The crucial point is the different choice of k in Substep 2.1 and Substep 2.3.
Nevertheless, if $\min(\alpha,\delta) < \frac{d}{\min(p_0',p_1)}$, the Proposition still holds and we present an alternative proof (cf. [4] on details of this method). In the limiting

case $\min(\alpha,\delta) = \frac{d}{\min(p_0',p_1)}$ we follow the work of Kulanin, cf. [2], and give a counterexample.
To prove Proposition 11 of [3] in the case $\min(\alpha,\delta) < \frac{d}{\min(p_0',p_1)}$, we substitute (26) in [3] by

$$id = \sum_{m=0}^{M_1} \sum_{i+j=m} P_{j,i} + \sum_{m=M_1+1}^{M_2} \sum_{i+j=m} P_{j,i} + \sum_{m=M_2+1}^{\infty} \sum_{i+j=m} P_{j,i},$$

where $M_1 < M_2$ are natural numbers, which shall be chosen later on. Using the subadditivity of approximation numbers this leads to

$$a_{k'}(id) \leq \underbrace{\sum_{m=0}^{M_1} \sum_{i+j=m} a_{k_{j,i}}(P_{j,i})}_{I} + \underbrace{\sum_{m=M_1+1}^{M_2} \sum_{i+j=m} a_{k_{j,i}}(P_{j,i})}_{II} + \underbrace{\sum_{m=M_2+1}^{\infty} \sum_{i+j=m} ||P_{j,i}||}_{III},$$

where $k'-1 = \sum_{m=0}^{M_2} \sum_{i+j=m} (k_{j,i}-1)$.
Let $k \in \mathbb{N}$ be given. We set

$$M_1 = \left[\frac{\log_2 k}{d} - \frac{\log_2 \log_2 k}{d}\right] \quad \text{and} \quad M_2 = \left[\frac{\min(p_0',p_1)}{2} \cdot \frac{\log_2 k}{d}\right],$$

where $[a]$ denotes the whole part of $a \in \mathbb{R}$ and $\log_2 k$ is a dyadic logarithm of k. Then

$$\begin{aligned} III &= \sum_{m=M_2+1}^{\infty} \sum_{i+j=m} ||P_{j,i}|| \leq c_1 \sum_{m=M_2+1}^{\infty} \sum_{i+j=m} 2^{-j\delta} 2^{-i\alpha} \\ &\leq c_2 \sum_{m=M_2+1}^{\infty} 2^{-m\min(\alpha,\delta)} \leq c_3 2^{-M_2 \cdot \min(\alpha,\delta)} \leq c_3 k^{-\varkappa}. \end{aligned}$$

To estimate I and II, we have to choose $k_{j,i}$. If $i+j \leq M_1$, we set $k_{j,i} = M_{j,i}+1$, hence $a_{k_{j,i}}(P_{j,i}) = 0$ and $I = 0$. We also observe, that

$$\sum_{m=0}^{M_1} \sum_{i+j=m} M_{j,i} \leq c_1 \sum_{m=0}^{M_1} (m+1) 2^{md} \leq c_2 M_1 \cdot 2^{M_1 d} \leq c_3 k.$$

The second sum represents the crux of the matter. We set

$$k_{j,i} = [k^{1-\varepsilon} \cdot 2^{iu_1} \cdot 2^{ju_2}],$$

where the positive real parameters ε, u_1, u_2 satisfy

$$\alpha + \frac{u_1}{2} < \frac{d}{t}, \quad 0 < \frac{u_1 - u_2}{2} < \delta - \alpha \quad \text{and} \quad \frac{u_1 t}{2d} = \varepsilon \qquad \text{if} \quad \delta > \alpha,$$

or

$$\delta + \frac{u_2}{2} < \frac{d}{t}, \quad 0 < \frac{u_2 - u_1}{2} < \alpha - \delta \quad \text{and} \quad \frac{u_2 t}{2d} = \varepsilon \qquad \text{if} \quad \delta < \alpha.$$

We recall that $\min(\alpha, \delta) < \frac{d}{t}$, $t = \min(p_0', p_1)$. With this choice, we get

$$\sum_{m=M_1+1}^{M_2} \sum_{i+j=m} k_{j,i} \le c_1 \, k^{1-\varepsilon} \sum_{m=M_1+1}^{M_2} 2^{m \cdot \max(u_1,u_2)} \le c_2 \, k^{1-\varepsilon} \cdot k^{\frac{t}{2d} \max(u_1,u_2)} = c_2 \, k$$

and

$$\begin{aligned}
\sum_{m=M_1+1}^{M_2} \sum_{i+j=m} a_{k_{j,i}}(P_{j,i}) &\le c_1 \sum_{m=M_1+1}^{M_2} \sum_{i+j=m} 2^{-j\delta - i\alpha} 2^{(i+j)d/t} [k^{(1-\varepsilon)} 2^{iu_1} 2^{ju_2}]^{-1/2} \\
&\le c_2 k^{-\frac{1}{2}+\frac{\varepsilon}{2}} \sum_{m=M_1+1}^{M_2} 2^{md/t} 2^{-m \min(\delta + \frac{u_2}{2}, \alpha + \frac{u_1}{2})} \\
&\le c_3 k^{\frac{\varepsilon}{2} - \frac{t}{2d} \cdot \min(\delta + \frac{u_2}{2}, \alpha + \frac{u_1}{2})} = c_3 k^{-\varkappa}.
\end{aligned}$$

This finishes the proof.

Very similar comments could be done to Theorem 17, part (iv) and Corollary 19, part (iv), whose proof follows directly from Proposition 11.

Finally, we present a counterexample to Corollary 19, part (iv) in the limiting case $\delta = \frac{d}{\min(p_0', p_1)}$. The construction comes from [2]. As this work may be difficult to reach, we sketch briefly the details.

We shall need the following Lemma of Kashin, [1].

Lemma 1. *Let $2 < p < \infty$ and let $L \subset \mathbb{R}^m$ be an k-dimensional subspace with $4k \le m$. Then*

$$C(\min(1, m^{1/p} k^{-1/2}))^2 \le \frac{1}{m} \sum_{j=1}^{m} \mathrm{dist}_p^2(e_j, L),$$

where $(e_j)_{j=1}^m$ are the canonical unit-vectors and $\mathrm{dist}_p(e_j, L)$ denotes their ℓ_p^m-distance from L.

We will show that $(2 < p < \infty, 1 \le q \le \infty)$

$$a_k(\mathrm{id} : X = \ell_\infty(2^{j/p} \ell_1^{2^j}) \to Y = \ell_q(\ell_p^{2^j})) \ge c(p,q) \cdot k^{-1/2} (\log_2 k)^{1/q}. \qquad (1)$$

This corresponds to $d = 1$, $p_0 = 1$, $2 < p_1 = p < \infty$ and $\delta = \frac{1}{p} = \frac{d}{\min(p_0', p_1)}$.
Let $T : \ell_\infty(2^{j/p}\ell_1^{2^j}) \to \ell_q(\ell_p^{2^j})$ be a linear operator with rank $T < k$ and let L be its range. It is sufficient to consider the summation index j running only through the set $\mathcal{J} = \{[\log_2(4k)], \ldots, [\frac{p}{2}\log_2(4k)]\}$.
It follows from Lemma 1, that

$$\frac{1}{2^j}\sum_{i=1}^{2^j} \mathrm{dist}_p^2(2^{-j/p}e_i, P_jL) \geq Ck^{-1}, \quad j \in \mathcal{J}.$$

Hence, for every $j \in \mathcal{J}$, there is an $i(j)$ such that $\mathrm{dist}_p(2^{-j/p}e_{i(j)}, P_jL) \geq Ck^{-1/2}$. We denote by A the element of X defined by

$$P_j(A) = \begin{cases} 2^{-j/p}e_{i(j)}, & j \in \mathcal{J}, \\ 0, & j \notin \mathcal{J}. \end{cases}$$

Observe, that $||A||_X = 1$. Finally, to this A we consider $y_A \in L$ such that $\mathrm{dist}_Y(A, L) = ||A - y_A||_Y$. We obtain

$$\begin{aligned} ||\mathrm{id} - T||^q &\geq \mathrm{dist}_Y^q(A, L) = ||A - y_A||_Y^q \geq ||\sum_{j\in\mathcal{J}} P_j(A - y_A)||_Y^q \\ &\geq c\sum_{j\in\mathcal{J}} \mathrm{dist}_p^q(2^{-j/p}e_{i(j)}, P_jL) \geq c'\sum_{j\in\mathcal{J}} k^{-q/2} \geq c''k^{-q/2}\log_2 k. \end{aligned}$$

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