Widths of embeddings in function spaces

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Abstract

We study the approximation, Gelfand and Kolmogorov numbers of embeddings in function spaces of Besov and Triebel-Lizorkin type. Our aim here is to provide sharp estimates in several cases left open in the literature and give a complete overview of the known results. We also add some historical remarks.

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1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $1 \le p \le \infty$ and let k be a natural number. We denote by $W_p^k(\Omega)$ the Sobolev spaces of functions from $L_p(\Omega)$ with all distributive derivatives of order smaller or equal to k in $L_p(\Omega)$. If

$$k_1 - k_2 \ge d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+, \tag{1.1}$$

and the boundary of Ω is Lipschitz then $W_{p_1}^{k_1}(\Omega)$ is continuously embedded into $W_{p_2}^{k_2}(\Omega)$. This theorem goes back to Sobolev [55].

If the inequality in (1.1) is strict, the embedding is even compact, cf. [48] and [31]. During the second half of the last century, this fact (and its numerous generalisations) found its applications in many areas of modern analysis, especially in connection with partial differential (and pseudo-differential) equations.

Later on, mathematicians started to be interested in measuring the *quality of compactness* of the embedding

$$I : W_{p_1}^{k_1}(\Omega) \hookrightarrow W_{p_2}^{k_2}(\Omega).$$

The very first question is, of course, how to measure compactness. During the years, several methods were developed. The most popular one assigns to I a non-increasing sequence of non-negative real numbers, say $\{s_n(I)\}_{n\in\mathbb{N}}$, often based on specific approximation quantities, and measures the decay of s_n as n tends to infinity.

Let us present this approach on the following example. Let X and Y be Banach spaces and let $T : X \to Y$ be a bounded linear operator between them. Then the nth approximation number of T is defined by

$$a_n(T) = \inf\{||T - L|| : L \in \mathcal{L}(X,Y),\ \text{rank } (L) < n\}, \quad n \in \mathbb{N}, \tag{1.2}$$

where $\mathcal{L}(X,Y)$ is the space of all bounded linear operators mapping X into Y endowed with the classical operator norm and rank L denotes the dimension of $L(X)$. Hence, we measure how well the operator T may be approximated by finite rank operators. If $\lim_{n\to\infty} a_n(T) = 0$, then T is compact. And in some sense, the faster the sequence $\{a_n(T)\}_{n\in\mathbb{N}}$ tends to zero, the more compact T is.

There are many other ways, how to define a sequence $\{s_n(T)\}_{n\in\mathbb{N}}$ for an operator $T \in \mathcal{L}(X,Y)$ such that the decay of $\{s_n\}$ describes in some sense the compactness of T; we refer to [43, 44, 6], where the axiomatic theory of the so-called s-numbers can be found.

It was observed by many authors, that even in the most simple case

$$id : \ell_{p_1}^m \to \ell_{p_2}^m, \quad m \in \mathbb{N}$$

it is surprisingly difficult to calculate (or at least estimate) the approximation numbers, as well as the other s-numbers, corresponding to id. The complexity of the problem may be demonstrated by the fact, that in several cases the proofs are based on probabilistic arguments and no optimal constructive approximation procedure is known up to now.

As a part of the good news is that these results may be combined with the discretization technique of Maĭorov [37] to get direct counterparts for embeddings between function spaces. Nowadays, there are many discretization techniques well known and studied in the literature.

Let us mention at least spline and wavelet decompositions and the φ-transform, cf. [8, 7, 49, 64, 23, 11, 16, 17].

The research in this area was complicated also by another regretful phenomena, namely communication problems between several groups working on the field. This effect was already pointed out by Caetano [4] and Pietsch [45, Section 6.2.6]. Also the separation of the Russian mathematical school causes some obstacles. Many breakthroughs achieved by Kashin, Gluskin and others were published in Russian. The nicely written dissertation of Lubitz [36] was written in German, never translated into English and never published.

The aim of this paper is rather extensive. We wish to

- give an overview of known results in this area,
- collect some historical references,
- close several minor gaps left open until now,
- present the power of the discretization method, but also its limits,
- provide an easy reference to the results about function spaces.

Several overviews may already be found in the literature, cf. [46, 34, 35, 45]. Unfortunately, they sometimes restrict themselves to $d = 1$, state the results only implicitly, or deal only with integer smoothness parameters $s_1, s_2 \in \mathbb{N}$. Here, leaded by the needs of possible applications, we shall study three types of s-numbers, namely approximation, Kolmogorov and Gelfand numbers, with respect to embeddings of function spaces defined on Lipschitz domains. This generalisation is not particularly interesting from the standpoint of functional analysis, but is of course crucial as far as the applications are concerned.

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2 Function and sequence spaces

2.1 Notation

We use standard notation: $\mathbb{N}$ denotes the collection of all natural numbers, $\mathbb{Z}$ the collection of all integers, $\mathbb{R}^d$ is the Euclidean d-dimensional space, where $d \in \mathbb{N}$, and $\mathbb{C}$ stands for the complex plane. Let $S(\mathbb{R}^d)$ be the Schwartz space of all complex-valued rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^d$ and let $S'(\mathbb{R}^d)$ be its dual, the space of all tempered distributions.

Furthermore, $L_p(\mathbb{R}^d)$ with $0 < p \le \infty$, are the classical Lebesgue spaces endowed with the (quasi-)norm

$$||f|L_p(\mathbb{R}^d)|| = \begin{cases} \left(\int_{\mathbb{R}^d} |f(x)|^p dx\right)^{1/p}, & 0 < p < \infty, \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} |f(x)|, & p = \infty. \end{cases}$$

For $\psi \in S(\mathbb{R}^d)$ we denote by

$$\widehat{\psi}(\xi) = (F\psi)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i<x,\xi>} \psi(x)dx, \quad x \in \mathbb{R}^d,$$

its Fourier transform and by $\psi^\vee$ or $F^{-1}\psi$ its inverse Fourier transform. Through duality, F and F^{-1} are extended to $S'(\mathbb{R}^d)$.

If $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are two sequences of non-negative real numbers, we write $a_n \lesssim b_n$ if there is a constant $c > 0$, such that $a_n \le c\, b_n$ for all natural numbers n. The symbols $a_n \gtrsim b_n$ and $a_n \approx b_n$ are defined similarly.

2.2 Function spaces

We give a Fourier-analytic definition of Besov and Triebel-Lizorkin spaces, which relies on the so-called *smooth dyadic resolution of unity.* Let $\varphi \in S(\mathbb{R}^d)$ with

$$\varphi(x) = 1 \quad \text{if} \quad |x| \le 1 \quad \text{and} \quad \varphi(x) = 0 \quad \text{if} \quad |x| \ge \frac{3}{2}. \tag{2.1}$$

We put $\varphi_0 = \varphi$ and $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$ for $j \in \mathbb{N}$ and $x \in \mathbb{R}^d$. This leads to the identity

$$\sum_{j=0}^\infty \varphi_j(x) = 1, \qquad x \in \mathbb{R}^d.$$

Definition 2.1. (i) Let $s \in \mathbb{R}, 0 < p, q \le \infty$. Then $B^s_{pq}(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|B^s_{pq}(\mathbb{R}^d)|| = \left(\sum_{j=0}^\infty 2^{jsq}||(\varphi_j\widehat{f})^\vee|L_p(\mathbb{R}^d)||^q\right)^{1/q} < \infty \tag{2.2}$$

(with the usual modification for $q = \infty$).

(ii) Let $s \in \mathbb{R}, 0 < p < \infty, 0 < q \le \infty$. Then $F^s_{pq}(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|F^s_{pq}(\mathbb{R}^d)|| = \left|\left|\left(\sum_{j=0}^\infty 2^{jsq}|(\varphi_j\widehat{f})^\vee(\cdot)|^q\right)^{1/q}|L_p(\mathbb{R}^d)\right|\right| < \infty \tag{2.3}$$

(with the usual modification for $q = \infty$).

Remark 2.2. We recommend [40, 59, 60, 51, 61] as standard references with respect to these classes of distributions. Extensive historical overviews, remarks and comments may be found in [60, Chapter 1], [61, Chapter 1] and [45, Chapter 6.7]. Let us mention that the spaces $B^s_{pq}(\mathbb{R}^d)$ and $F^s_{pq}(\mathbb{R}^d)$ do not depend on the choice of φ in the sense of equivalent (quasi-)norms. Many classical function spaces are included in these two scales.

1. If $1 < p < \infty$, then the Littlewood-Paley theorem states that

$$F^0_{p2}(\mathbb{R}^d) = L_p(\mathbb{R}^d).$$

2. Let $1 < p < \infty$ and $s \in \mathbb{N}$. Then

$$F^s_{p2}(\mathbb{R}^d) = W^s_p(\mathbb{R}^d)$$

are the classical Sobolev spaces.

3. Let $s > 0, s \notin \mathbb{N}$. Then
$$B^s_{\infty\infty}(\mathbb{R}^d) = \mathcal{C}^s(\mathbb{R}^d)$$
are the Hölder-Zygmund spaces.

On the other hand, many important function spaces (especially $L_1(\mathbb{R}^d), L_\infty(\mathbb{R}^d)$, $BV(\mathbb{R})$ - the space of functions with bounded variation and $C^k(\mathbb{R}^d)$ - the space of functions with all partial derivatives of order smaller or equal to k uniformly continuous and bounded) are *not* included.

If X and Y are two topological vector spaces, we write $X \hookrightarrow Y$ if X is continuously embedded in Y. The following embeddings describe the interplay between these function spaces and the Besov scale.

$$\begin{gathered} B^0_{11}(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d) \hookrightarrow B^0_{1\infty}(\mathbb{R}^d), \\ B^0_{\infty 1}(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d) \hookrightarrow B^0_{\infty\infty}(\mathbb{R}^d), \\ B^k_{\infty 1}(\mathbb{R}^d) \hookrightarrow C^k(\mathbb{R}^d) \hookrightarrow B^k_{\infty\infty}(\mathbb{R}^d). \end{gathered} \tag{2.4}$$

In many cases it will be possible to use the Fourier-analytical methods in the framework of Besov spaces and afterwards, simply by applying these simple continuous embeddings, to derive the same results also for the "bad" spaces $L_1(\mathbb{R}^d), L_\infty(\mathbb{R}^d)$ and $C^k(\mathbb{R}^d)$. The same procedure may be used also for the Triebel-Lizorkin scale because of

$$B^s_{p,\min(p,q)}(\mathbb{R}^d) \hookrightarrow F^s_{pq}(\mathbb{R}^d) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^d). \tag{2.5}$$

Remark 2.3. If $0 < p_1 \le p_2 \le \infty$, $0 < q_1, q_2 \le \infty$ and $s_2 \le s_1$, then the following version of the Sobolev embedding is true, see [2], [40, Chapters 3 and 11] and [58, Section 2.8.1].

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^d), \quad \text{if} \quad s_1 - \frac{d}{p_1} > s_2 - \frac{d}{p_2}.$$

There are several modifications of this embedding, which result in compact mappings. The first possibility is to restrict to function spaces on smooth bounded domains, the second involves *weighted spaces* and another one considers the so-called *radial spaces*, i.e. spaces of radial symmetric functions. We concentrate on the first possibility and refer to [61, Chapter 6] and [54] for the second and third approach.

Let Ω be a bounded domain. Let $D(\Omega) = C^\infty_0(\Omega)$ be the collection of all complex-valued infinitely-differentiable functions with compact support in Ω and let $D'(\Omega)$ be its dual - the space of all complex-valued distributions on Ω.
Let $g \in S'(\mathbb{R}^d)$. Then we denote by $g|\Omega$ its restriction to Ω:

$$(g|\Omega) \in D'(\Omega), \qquad (g|\Omega)(\psi) = g(\psi) \quad \text{for} \quad \psi \in D(\Omega).$$

Definition 2.4. Let Ω be a bounded domain in $\mathbb{R}^d$. Let $s \in \mathbb{R}$, $0 < p, q \le \infty$ with $p < \infty$ in the F-case. Let A^s_{pq} stand either for B^s_{pq} or F^s_{pq}. Then

$$A^s_{pq}(\Omega) = \{f \in D'(\Omega) : \exists g \in A^s_{pq}(\mathbb{R}^d) : g|\Omega = f\}$$

and

$$||f|A^s_{pq}(\Omega)|| = \inf ||g|A^s_{pq}(\mathbb{R}^d)||,$$

where the infimum is taken over all $g \in A^s_{pq}(\mathbb{R}^d)$ such that $g|\Omega = f$.

Intrinsic characterization of $B^s_{p,q}(\Omega), s > \sigma_p = d\left(\frac{1}{p} - 1\right)_+ = \max\left(\frac{1}{p} - 1, 0\right)$ are known to exist in case of Lipschitz domains, see [12, 13, 14] and [61, Section 1.11.9].

2.3 Sequence spaces

In this section we comment on the discretization techniques mentioned in the Introduction. First, we describe the situation on $\mathbb{R}^d$. Therefore, we introduce the sequence spaces b^s_{pq} and give a wavelet decomposition theorem for Besov spaces on $\mathbb{R}^d$. Good references in our context are [8, 11, 23, 38, 39, 63, 64].

Second, we deal with bounded domains $\Omega \subset \mathbb{R}^d$. The wavelet decomposition techniques may be adapted also to these function spaces, cf. [9, 61], but unfortunately, there are still open problems in this setting. To avoid these gaps, we use the theory on $\mathbb{R}^d$ and combine it with suitable extension and restriction operators.

Theorem 2.5. *For any $k \in \mathbb{N}$ there are real-valued compactly supported functions*

$$\psi_0, \psi_1 \in C^k(\mathbb{R})$$

satisfying

$$\int_{\mathbb{R}} t^\alpha \psi_1(t)dt = 0, \quad \alpha = 0, 1, \ldots, k-1,$$

such that

$$\{2^{\nu/2}\psi_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}\}$$

with

$$\psi_{\nu m}(t) = \begin{cases} \psi_0(t-m) & \text{if } \nu = 0, m \in \mathbb{Z}, \\ 2^{-\frac{1}{2}}\psi_1(2^{\nu-1}t - m) & \text{if } \nu \in \mathbb{N}, m \in \mathbb{Z} \end{cases}$$

is an orthonormal basis in $L_2(\mathbb{R})$.

Remark 2.6. This theorem was first proven by Daubechies in [10]. The functions ψ_0 and ψ_1 are therefore usually called Daubechies wavelets. We refer to [63, Theorem 19] for the proof of the next theorem.

Theorem 2.7. *Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and $k \in \mathbb{N}$ with $k > \max(s, \sigma_p - s)$. Let ψ_0, ψ_1 be the Daubechies wavelets of smoothness k. Let $E = \{0,1\}^d \setminus (0, \ldots, 0)$. For $e = (e_1, \ldots, e_d) \in E$ let*

$$\Psi_e(x) = \prod_{j=1}^d \psi_{e_j}(x_j), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

(i) Then

$$\begin{cases} \Psi(x-m) = \prod_{j=1}^d \psi_0(x_j - m_j) & m = (m_1, \ldots, m_d) \in \mathbb{Z}^d, \\ 2^{\frac{\nu-1}{2}d}\Psi_e(2^{\nu-1}x - m) & e \in E, \nu \in \mathbb{N}, m \in \mathbb{Z}^d \end{cases}$$

is an orthonormal basis in $L_2(\mathbb{R}^d)$.

(ii) Let $f \in S'(\mathbb{R}^d)$. Then $f \in B^s_{pq}(\mathbb{R}^d)$ if, and only if, it can be represented as

$$f = \sum_{m \in \mathbb{Z}^d} \lambda_m \Psi(x-m) + \sum_{\nu \in \mathbb{N}} \sum_{e \in E} \sum_{m \in \mathbb{Z}^d} \lambda^e_{\nu m} 2^{-\nu d/2} \Psi_e(2^{\nu-1}x - m) \tag{2.6}$$

with

$$||\lambda|\mathsf{b}^s_{pq}|| = \Big(\sum_{m \in \mathbb{Z}^d} |\lambda_m|^p\Big)^{\frac{1}{p}} + \Big(\sum_{\nu=1}^\infty 2^{\nu(s-\frac{d}{p})q} \sum_{e \in E}\Big(\sum_{m \in \mathbb{Z}^d} |\lambda^e_{\nu m}|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} < \infty$$

appropriately modified if $p = \infty$ and/or $q = \infty$. The representation in (2.6) *is unique, the complex coefficients $\{\lambda_m\}_{m\in\mathbb{Z}^d}$ and $\{\lambda^e_{\nu m}\}_{e\in E,\nu\in\mathbb{N}_0,m\in\mathbb{Z}^d}$ depend linearly on f and the mapping, which associates to $f \in B^s_{pq}(\mathbb{R}^d)$ the sequence of coefficients, is an isomorphic map of $B^s_{pq}(\mathbb{R}^d)$ onto b^s_{pq}.*

2.4 s-numbers

Given $p \in (0, 1]$, we say, that the quasi-Banach space Y is a p-Banach space if the inequality

$$||x + y|Y||^p \leq ||x|Y||^p + ||y|Y||^p, \qquad x, y \in Y.$$

is satisfied.

We recall a few basic facts of the theory of s-numbers. We refer to [44, 6] for further details. In this theory, one associates to every linear operator $T : X \to Y$ (X and Y quasi-Banach spaces) a sequence of scalars

$$s_1(T) \geq s_2(T) \geq \cdots \geq 0.$$

Let W, X, Y, Z be (quasi-)Banach spaces and let Y be a p-Banach space, $0 < p \leq 1$. If the rule $s : T \to \{s_n(T)\}_{n\in\mathbb{N}}$ satisfies

(S1) $||T|| = s_1(T) \geq s_2(T) \geq \cdots \geq 0$,

(S2) $s^p_{m+n-1}(S + T) \leq s^p_m(T) + s^p_n(S)$ for all $S, T \in \mathcal{L}(X, Y)$ and $m, n \in \mathbb{N}$,

(S3) $s_n(STU) \leq ||S||s_n(T)||U||$ for all $U \in \mathcal{L}(W, X), T \in \mathcal{L}(X, Y), S \in \mathcal{L}(Y, Z)$ and $n \in \mathbb{N}$,

(S4) If rank $T < n$, then $s_n(T) = 0$,

(S5) $s_n(I : \ell_2(n) \to \ell_2(n)) = 1$.

then the $s_n(T)$ are called s-numbers of the operator T.

Let us point out, that we shall not use **(S4)** and **(S5)** in what follows. Hence, our approach applies also to rules $s : T \to \{s_n(T)\}_{n\in\mathbb{N}}$ which satisfy only **(S1)**-**(S3)**. Such rules are called *pseudo-s-numbers* in [43, Chapter 12] and cover also the concept of entropy numbers.

Let

$$\mathcal{I}d : B^{s_1}_{p_1q_1}(\Omega) \to B^{s_2}_{p_2q_2}(\Omega) \tag{2.7}$$

be compact, i.e.

$$s_1 - s_2 > d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+. \tag{2.8}$$

We denote by

$$\mathrm{ext} : B^{s_1}_{p_1q_1}(\Omega) \to B^{s_1}_{p_1q_1}(\mathbb{R}^d) \tag{2.9}$$

a bounded linear extension operator. A convenient reference for this is Rychkov, cf. [52], but see also the references given there. Here we use the Lipschitz smoothness of $\partial\Omega$. The natural restriction will be denoted by

$$\mathrm{re} : B^{s_2}_{p_2q_2}(\mathbb{R}^d) \to B^{s_2}_{p_2q_2}(\Omega).$$

Clearly, it also represents a bounded linear operator.

Let $k > \max(s_1, \sigma_{p_1} - s_1, s_2, \sigma_{p_2} - s_2)$ be a natural number and let $\mathcal{W}$ be the mapping which associates to each $f \in B^{s_1}_{p_1q_1}(\mathbb{R}^d)$ its wavelet coefficients with respect to the Daubechies wavelets of smoothness k, as described in Theorem 2.7. Our choice of k ensures, that Theorem 2.7 may be applied to both, $B^{s_1}_{p_1q_1}(\mathbb{R}^d)$ and $B^{s_2}_{p_2q_2}(\mathbb{R}^d)$, simultaneously and that $\mathcal{W}^{-1}$ is a bounded linear operator, which maps $\mathsf{b}^{s_2}_{p_2q_2}$ isomorphically onto $B^{s_2}_{p_2q_2}(\mathbb{R}^d)$.
Finally, we adapt the sequence spaces b^s_{pq} to the function spaces on domains.

Definition 2.8. (i) Let $M = \{M_\nu\}_{\nu=0}^\infty$ be a sequence of non-negative integers. We say, that M is admissible, if there is some $\nu_0 \in \mathbb{N}_0$ and two positive real constants c_1, c_2 such that

$$M_\nu = 0 \qquad \text{for all} \quad \nu < \nu_0$$

and

$$c_1 2^{\nu d} \le M_\nu \le c_2 2^{\nu d}, \qquad \nu \ge \nu_0.$$

(ii) If $0 < p, q \le \infty$, $s \in \mathbb{R}$, $E = \{0,1\}^d \setminus (0,\dots,0)$, $M = \{M_\nu\}_{\nu=0}^\infty$ is an admissible sequence and

$$\lambda = \{\lambda_k : k = 1,\dots,M_0\} \cup \{\lambda^e_{\nu k} : e \in E, \nu \in \mathbb{N}, k \in M_\nu\},$$

we set

$$||\lambda|\mathsf{b}^{s,M}_{pq}|| = \Big(\sum_{k=1}^{M_0} |\lambda_k|^p\Big)^{\frac{1}{p}} + \Big(\sum_{\nu=1}^{\infty} 2^{\nu(s-\frac{d}{p})q} \sum_{e\in E}\Big(\sum_{k=1}^{M_\nu} |\lambda^e_{\nu k}|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}, \tag{2.10}$$

again appropriately modified if $p = \infty$ and/or $q = \infty$.

Let now Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and let the number $k \in \mathbb{N}$ describing the smoothness of the wavelets be fixed. Then we collect those wavelets, whose support intersects $\bar{\Omega}$:

$$\mathcal{M}_\nu = \begin{cases} \{m \in \mathbb{Z}^d : \operatorname{supp} \Psi(\cdot - m) \cap \bar{\Omega} \neq \emptyset\} & \text{if } \nu = 0, \\ \{m \in \mathbb{Z}^d : \exists e \in E : \operatorname{supp} \Psi_e(2^{\nu-1} \cdot - m) \cap \bar{\Omega} \neq \emptyset\} & \text{if } \nu \ge 1. \end{cases}$$

We observe that the sequence $M = \{M_\nu\}_{\nu=0}^\infty$ with

$$M_\nu = \#(\mathcal{M}_\nu) = \text{number of elements of } \mathcal{M}_\nu, \quad \nu \in \mathbb{N}_0,$$

is an admissible sequence in the sense of Definition 2.8.
With a slight abuse of notation, there is a natural projection operator $P : \mathsf{b}^s_{pq} \to \mathsf{b}^{s,M}_{pq}$ and a natural embedding operator $Q : \mathsf{b}^{s,M}_{pq} \to \mathsf{b}^s_{pq}$.
Using the weak multiplicativity property **(S3)** of s-numbers and the commutative diagram

$$\begin{array}{ccccccc} B^{s_1}_{p_1q_1}(\Omega) & \xrightarrow{\text{ext}} & B^{s_1}_{p_1q_1}(\mathbb{R}^d) & \xrightarrow{\mathcal{W}} & \mathsf{b}^{s_1}_{p_1q_1} & \xrightarrow{P} & \mathsf{b}^{s_1,M}_{p_1q_1} \\ \mathcal{I}d \big\downarrow & & & & & & \big\downarrow id \\ B^{s_2}_{p_2q_2}(\Omega) & \xleftarrow{\text{re}} & B^{s_2}_{p_2q_2}(\mathbb{R}^d) & \xleftarrow{\mathcal{W}^{-1}} & \mathsf{b}^{s_2}_{p_2q_2} & \xleftarrow{Q} & \mathsf{b}^{s_2,M}_{p_2q_2} \end{array}$$

we conclude that

$$s_n(\mathcal{I}d) \lesssim s_n(id), \qquad n \in \mathbb{N}.$$

To obtain the reverse inequality, we first set

$$\mathcal{M}'_\nu = \begin{cases} \{m \in \mathbb{Z}^d : \operatorname{supp} \Psi(\cdot - m) \subset \Omega\} & \text{if } \nu = 0, \\ \{m \in \mathbb{Z}^d : \forall e \in E : \operatorname{supp} \Psi_e(2^{\nu-1} \cdot - m) \subset \Omega\} & \text{if } \nu \geq 1. \end{cases} \tag{2.11}$$

Again, we observe, that the sequence $M' = \{M'_\nu\}_{\nu=0}^\infty$ with

$$M'_\nu = \#(\mathcal{M}'_\nu) = \text{number of elements of } \mathcal{M}'_\nu, \quad \nu \in \mathbb{N}_0,$$

is an admissible sequence in the sense of Definition 2.8.
If we use **(S3)** and

$$\begin{array}{ccccccc} \mathsf{b}^{s_1,M'}_{p_1q_1} & \xrightarrow{Q'} & \mathsf{b}^{s_1}_{p_1q_1} & \xrightarrow{\mathcal{W}^{-1}} & B^{s_1}_{p_1q_1}(\mathbb{R}^d) & \xrightarrow{\text{re}} & B^{s_1}_{p_1q_1}(\Omega) \\ {\scriptstyle id'}\downarrow & & & & & & \downarrow{\scriptstyle \mathcal{I}d} \\ \mathsf{b}^{s_2,M'}_{p_2q_2} & \xleftarrow{P'} & \mathsf{b}^{s_2}_{p_2q_2} & \xleftarrow{\mathcal{W}} & B^{s_2}_{p_2q_2}(\mathbb{R}^d) & \xleftarrow{\text{ext}} & B^{s_2}_{p_2q_2}(\Omega), \end{array}$$

we get the inequality.

$$s_n(id') \lesssim s_n(\mathcal{I}d), \quad n \in \mathbb{N}.$$

Hence

$$s_n(id') \lesssim s_n(\mathcal{I}d) \lesssim s_n(id), \quad n \in \mathbb{N}. \tag{2.12}$$

This formula is the main result of this section. It tells us, roughly speaking, that we may restrict ourselves to sequence spaces and all the results translate also into the language of function spaces. Before we start with the study of $s_n(id)$ and $s_n(id')$, we make another simplification. The (finite) sum over $e \in E$ in (2.10) comes from the theory of multivariate wavelet decompositions, but has no influence on the s-numbers.
If $M = \{M_\nu\}_{\nu=0}^\infty$ is an admissible sequence, we set

$$||\lambda|b^{s,M}_{pq}|| = \left(\sum_{\nu=0}^\infty 2^{\nu(s-\frac{d}{p})q} \left(\sum_{k=1}^{M_\nu} |\lambda_{\nu k}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}.$$

It follows that

$$s_n(\mathcal{I}d : B^{s_1}_{p_1q_1}(\Omega) \to B^{s_2}_{p_2q_2}(\Omega)) \approx s_n(id : \mathsf{b}^{s,M}_{pq} \to \mathsf{b}^{s,M}_{pq}) \approx s_n(id : b^{s,M}_{pq} \to b^{s,M}_{pq}). \tag{2.13}$$

Remark 2.9. The formula 2.13 represents the main result of this section and is of a crucial importance for our study of s-numbers of (2.7). We have proved (2.13) under the assumption that Ω is a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary. Using more sophisticated tools from the theory of function spaces, it may be proven that (2.13) holds also for more general classes of domains, at least under some restrictions on the parameters $s_1, s_2, p_1, p_2, q_1, q_2$. A detailed inspection of our proof shows, that (2.13) is true anytime there is a bounded linear extension operator (2.9) and its counterpart for $B^{s_2}_{p_2q_2}(\Omega)$. We refer to [62, Section 4.3.4] for a detail treatment of these questions.

3 Approximation numbers

Definition 3.1. Let X, Y be two quasi-Banach spaces and let $T \in \mathcal{L}(X,Y)$. For $n \in \mathbb{N}$, we define the nth approximation number by

$$a_n(T) = \inf\{||T - L|| : L \in \mathcal{L}(X,Y), \operatorname{rank}(L) < n\}.$$

In the setting of Banach spaces, this definition goes back to Pietsch [41] and Tikhomirov [57]. The generalisation to quasi-Banach spaces may be found in [15, Section 1.3.1]. In this section, we characterize the approximation numbers of (2.7) with (2.8).

First, we recall some lemmas which we shall need on the sequence space level. Lemma 3.2 is taken from [22] and Lemma 3.3 in the case $1 \le p_2 \le p_1 \le \infty$ may be found in [43, Section 11.11.5]. The proof may be directly generalised to the quasi-Banach setting $0 < p_2 \le p_1 \le \infty$.

For $0 < p \le \infty$, we set

$$p' = \begin{cases} \frac{p}{p-1} & \text{if} \quad 1 < p < \infty, \\ 1 & \text{if} \quad p = \infty, \\ \infty & \text{if} \quad 0 < p \le 1. \end{cases}$$

Lemma 3.2. *For $1 \le n \le m < \infty$ and $1 \le p_1 < p_2 \le \infty$, we define*

$$\Phi(m,n,p_1,p_2) := \begin{cases} \left(\min\{1, m^{\frac{1}{p_2}} n^{-\frac{1}{2}}\}\right)^{\frac{\frac{1}{p_1} - \frac{1}{p_2}}{\frac{1}{2} - \frac{1}{p_2}}} & \textit{if} \quad 2 \le p_1 < p_2 \le \infty, \\ \max\{m^{\frac{1}{p_2} - \frac{1}{p_1}}, \min\{1, m^{\frac{1}{p_2}} n^{-\frac{1}{2}}\} \cdot \sqrt{1 - \frac{n}{m}}\} & \textit{if} \quad 1 \le p_1 < 2 \le p_2 \le \infty, \\ \max\{m^{\frac{1}{p_2} - \frac{1}{p_1}}, \sqrt{1 - \frac{n}{m}}^{\frac{\frac{1}{p_1} - \frac{1}{p_2}}{\frac{1}{p_1} - \frac{1}{2}}}\} & \textit{if} \quad 1 \le p_1 < p_2 \le 2 \end{cases}$$

and

$$\Psi(m,n,p_1,p_2) := \begin{cases} \Phi(m,n,p_1,p_2) & \textit{if} \quad 1 \le p_1 < p_2 \le p_1', \\ \Phi(m,n,p_2',p_1') & \textit{if} \quad \max(p_1,p_1') < p_2 \le \infty. \end{cases}$$

Then if $1 \le p_1 < p_2 \le \infty$ and $(p_1,p_2) \ne (1,\infty)$

$$a_n(id : \ell_{p_1}^m \to \ell_{p_2}^m) \approx \Phi(m,n,p_1,p_2), \qquad 1 \le n \le m < \infty.$$

The constants of equivalence may depend on p_1 and p_2 but are independent of m and n.

Lemma 3.3. *If $0 < p_2 \le p_1 \le \infty$, then*

$$a_n(id : \ell_{p_1}^m \to \ell_{p_2}^m) = (m - n + 1)^{\frac{1}{p_2} - \frac{1}{p_1}}.$$

Lemma 3.4. *Let $0 < p \le 1$.*

(i) Let $0 < \lambda < 1$. Then there is a number $c_\lambda > 0$ such that

$$a_n(id : \ell_p^m \to \ell_\infty^m) \le \frac{c_\lambda}{\sqrt{n}} \tag{3.1}$$

holds for all natural numbers n and m with $m^\lambda < n \le m$.

(ii) There is a number $c > 0$ such that

$$a_n(id : \ell_p^{2n} \to \ell_\infty^{2n}) \ge \frac{c}{\sqrt{n}}, \quad n \ge 1. \tag{3.2}$$

Proof. Let $A = (a_{i,j})_{i,j=1}^m$ be an $m \times m$ matrix. Then

$$||A|\mathcal{L}(\ell_1^m, \ell_\infty^m)|| = ||A|\mathcal{L}(\ell_p^m, \ell_\infty^m)|| = \max_{i,j=1,\dots,m} |a_{i,j}|$$

for every $0 < p \le 1$. Hence, the approximation numbers of $id : \ell_p^m \to \ell_\infty^m$ do not depend on $0 < p \le 1$ and it is enough, when we prove Lemma 3.4 only for $p = 1$.

The first part follows from a combinatorial result of Kashin, cf. [26, 27] and [43, Section 11.11.11]:

Let $0 < \lambda < 1$ and $m^\lambda \le n \le m$ be natural numbers. Then there are m ℓ_2^n-unit vectors $\{f_i\}_{i=1}^m \subset \mathbb{R}^n$, such that

$$|(f_i, f_j)| \le \frac{c_\lambda}{\sqrt{n}}, \quad \text{if } i \ne j.$$

We set $A = (a_{i,j})_{i,j=1}^m$ with $a_{i,j} = (f_i, f_j)$. Then A is a matrix with rank $A \le n$ and $||I - A|\mathcal{L}(\ell_1^m, \ell_\infty^m)|| \le \frac{c_\lambda}{\sqrt{n}}$.

The proof of the second part follows trivially from the result of Stechkin, cf. [56] and [43, Section 11.11.8]:

$$a_n(id : \ell_1^m \to \ell_2^m) = \left(\frac{m-n+1}{m}\right)^{1/2}$$

and

$$||id : \ell_\infty^m \to \ell_2^m|| = \sqrt{m}.$$

□

Theorem 3.5. *Let* $-\infty < s_2 < s_1 < \infty$ *and* $0 < p_1, p_2, q_1, q_2 \le \infty$ *with* (2.8). *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded Lipschitz domain. Then* (2.7) *is compact and for* $n \in \mathbb{N}$

$$a_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}+\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+} \quad \text{if} \quad \begin{cases} \text{either} & 0 < p_1 \le p_2 \le 2, \\ \text{or} & 2 \le p_1 \le p_2 \le \infty, \\ \text{or} & 0 < p_2 \le p_1 \le \infty, \end{cases} \tag{3.3}$$

$$a_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}+\frac{1}{p}-\frac{1}{2}} \quad \text{if} \quad 0 < p_1 < 2 < p_2 \le \infty \quad \text{and} \quad \frac{s_1-s_2}{d} > \frac{1}{p} = \max\Big(1-\frac{1}{p_2}, \frac{1}{p_1}\Big), \tag{3.4}$$

$$a_n(\mathcal{I}d) \approx n^{\left(-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{p_2}\right)\cdot\frac{\min(p_1',p_2)}{2}} \quad \text{if} \quad \frac{s_1-s_2}{d} < \frac{1}{p} = \max\Big(1-\frac{1}{p_2}, \frac{1}{p_1}\Big), \tag{3.5}$$

and either $1 < p_1 < 2 < p_2 = \infty$

or $0 < p_1 < 2 < p_2 < \infty$

$$a_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{2}} \quad \text{if} \quad 0 < p_1 \le 1 < p_2 = \infty. \tag{3.6}$$

Proof. Approximation numbers form an additive and multiplicative scale of s-numbers. This fact may be verified directly, or the reader may consult [43, Section 11.2] in the Banach space settings and [15, Section 1.3] for the extension to quasi-Banach spaces.

Hence (2.12) applies to approximation numbers and we may restrict ourselves to sequence spaces.

The estimates covered by (3.3)-(3.5) are known. We refer to [15, Section 3.3.4] and [4]. The proof given in [15] is rather complicated, but [4] uses an approach very similar to ours.

It remains to prove the only missing case (3.6). We use Lemma 3.4 to estimate the approximation numbers of

$$id : b^{s_1,M}_{p_1 q_1} = \ell_{q_1}(2^{\nu(s_1-\frac{d}{p_1})}\ell^{M_\nu}_{p_1}) \to \ell_{q_2}(2^{\nu s_2}\ell^{M_\nu}_{\infty}) = b^{s_2,M}_{\infty\, q_2},$$

where $M = \{M_\nu\}_{\nu=0}^{\infty}$ is an admissible sequence. Let

$$id_\nu : 2^{\nu(s_1-\frac{d}{p_1})}\ell^{M_\nu}_{p_1} \to 2^{\nu s_2}\ell^{M_\nu}_{\infty}, \quad \nu = 0, 1, 2, \ldots$$

denote the identity operator between the finite dimensional building blocks of the considered sequence spaces. With a slight abuse of notation, we get

$$id = \sum_{\nu=0}^{\infty} id_\nu, \tag{3.7}$$

which, combined with the additivity of approximation numbers, leads to

$$a^\omega_{n'}(id) \le \sum_{\nu=0}^{N_1} a^\omega_{n_\nu}(id_\nu) + \sum_{\nu=N_1+1}^{N_2} a^\omega_{n_\nu}(id_\nu) + \sum_{\nu=N_2+1}^{\infty} ||id_\nu||^\omega,$$

where $N_1 < N_2$ are natural numbers, $n' - 1 = \sum_{\nu=0}^{N_2}(n_\nu - 1)$ and $\omega = \min(1, q_2)$. We set

$$n_\nu = \begin{cases} M_\nu + 1 & \text{if } 0 \le \nu \le N_1, \\ n^{1+\alpha}2^{-\alpha\nu d} & \text{if } N_1 + 1 \le \nu \le N_2, \end{cases}$$

where

$$0 < \alpha < 2\Big(\frac{s}{d} - \frac{1}{p_1}\Big) \tag{3.8}$$

and

$$N_1 = \left[\frac{\log_2 n}{d}\right], \quad N_2 = \left[\frac{\frac{s}{d} - \frac{1}{p} + \frac{1}{2}}{\frac{s}{d} - \frac{1}{p}} \cdot \frac{\log_2 n}{d}\right] \ge N_1.$$

Here, $[a]$ denotes the integer part of a real number a.
For this choice we get

$$n' = \sum_{\nu=0}^{N_2}(n_\nu - 1) + 1 \approx 2^{\nu N_1 d} + N_1^{1+\alpha}2^{-\alpha\nu d} \approx n.$$

A simple calculation shows that there is a number $\lambda > 0$ such that $M_\nu^\lambda \le n_\nu \le M_\nu$. Hence

$$a_{n_\nu}(id_\nu) \le \begin{cases} 0 & \text{if } 0 \le \nu \le N_1, \\ \frac{c_\lambda}{\sqrt{n_\nu}}2^{-\nu(s-\frac{d}{p_1})} & \text{if } N_1 + 1 \le \nu \le N_2 \end{cases}$$

and

$$\sum_{\nu=0}^{N_1} a^\omega_{n_\nu}(id_\nu) = 0,$$

$$\sum_{\nu=N_1+1}^{N_2} a^\omega_{n_\nu}(id_\nu) \le \sum_{\nu=N_1+1}^{N_2} \frac{c^\omega_\lambda}{\sqrt{n^\omega_\nu}} \le cn^{-\frac{1+\alpha}{2}\omega} \sum_{\nu=N_1+1}^{N_2} 2^{-\nu d\omega(\frac{s}{d}-\frac{1}{p_1}-\frac{\alpha}{2})} \lesssim n^{-\omega\left(\frac{s}{d}-\frac{1}{p_1}+\frac{1}{2}\right)},$$

$$\sum_{\nu=N_2+1}^{\infty} ||id_\nu||^\omega \le \sum_{\nu=N_2+1}^{\infty} 2^{-\nu\omega(s-\frac{d}{p_1})} \lesssim n^{-\omega\left(\frac{s}{d}-\frac{1}{p_1}+\frac{1}{2}\right)}.$$

It follows, that there is a constant $c > 0$ such that

$$a_{cn}(id) \lesssim n^{-\left(\frac{s}{d}-\frac{1}{p_1}+\frac{1}{2}\right)}, \quad n \geq 1,$$

which is equivalent to

$$a_n(id) \lesssim n^{-\left(\frac{s}{d}-\frac{1}{p_1}+\frac{1}{2}\right)}, \quad n \geq 1. \tag{3.9}$$

The proof of the reverse inequality to (3.9) follows easily from the second part of Lemma 3.4.

Let $M' = \{M'_\nu\}_{\nu=0}^\infty$ be an admissible sequence. Then, for $\nu \geq \nu_0$

$$a_n(id) \geq a_n(id_\nu) \gtrsim 2^{-\nu(s-\frac{d}{p_1})} \cdot \frac{1}{\sqrt{n}}$$

if $n = \left[\frac{M_\nu}{2}\right]$. This leads to

$$a_n(id) \gtrsim n^{-\left(\frac{s}{d}-\frac{1}{p_1}+\frac{1}{2}\right)}, \quad n = \left[\frac{M_\nu}{2}\right], \quad \nu \geq \nu_0$$

and by means of the monotonicity of the approximation numbers the result follows. □

Remark 3.6. We have used the open case (3.6) to demonstrate the typical use of the wavelet decomposition method and (2.12). Also (3.3)–(3.5) could be proven exactly in the same manner. For example, the proof of (3.5) in [4] follows along this line.

Remark 3.7. Although the results were stated only for Besov spaces, with the aid of (2.4) and (2.5) we may extend them also to Triebel-Lizorkin spaces, Sobolev and Lebesgue spaces and $C(\Omega), L_1(\Omega)$ and $L_\infty(\Omega)$. We return to this point later on.

Remark 3.8. The first estimates on approximation numbers of Sobolev embeddings of function spaces were obtained by Kolmogorov [30], who dealt with the Hilbert space case $p_1 = q_1 = p_2 = q_2 = 2$. Later on, Birman and Solomyak [3] studied the embeddings of Sobolev spaces. Finally, Kashin [29] observed the effect of "small smoothness" expressed by (3.5). In the framework of Besov spaces the results are contained in [15, 4]. Nowadays, the proof of (3.3)–(3.5) could be done very similar to the proof of (3.6), only using Lemmas 3.2 and 3.3 instead of Lemma 3.4.

4 Kolmogorov and Gelfand numbers

In this chapter we deal with Kolmogorov and Gelfand numbers. To begin with we recall their definition and describe their decay in connection with Sobolev embeddings of Besov spaces. We use the symbol $A \subset\subset B$ if A is a closed subspace of a topological vector space B.

Definition 4.1. Let X, Y be two quasi-Banach spaces and let $T \in \mathcal{L}(X, Y)$.
(i) For $n \in \mathbb{N}$, we define the nth Kolmogorov number by

$$d_n(T) = \inf\{||Q_N^Y T|| : N \subset\subset Y, \dim(N) < n\}.$$

Here, Q_N^Y stands for the natural surjection of Y onto the quotient space Y/N.

(ii) For $n \in \mathbb{N}$, we define the nth Gelfand number by

$$c_n(T) = \inf\{||TJ_M^X|| : M \subset\subset X, \operatorname{codim}(M) < n\}.$$

Here, J_M^X stands for the natural injection of M into X.

Clearly, the notion *dimension of a subspace* is purely algebraic and may be freely used also in the setting of quasi-Banach spaces. We refer to [50, Section 1.40] for the definition of a quotient subspace in the framework of general topological vector spaces (including quasi-Banach spaces as a special case). Finally, the codimension of a subspace may be defined as the dimension of the quotient space.

Both, Gelfand and Kolmogorov numbers, are additive and multiplicative s-scales. This follows directly from Definition 4.1, but the reader may wish to consult [44, Sections 2.4, 2.5] for the proof in the Banach space case. The generalisation to p-Banach spaces is obvious and causes no complications. Also the following relations are trivial:

$$c_n(T) \le a_n(T), \qquad d_n(T) \le a_n(T), \qquad n \in \mathbb{N}. \tag{4.1}$$

The Gelfand and Kolmogorov numbers are dual to each other in the following sense, cf. [44, Section 11.7.6-7]: If X and Y are Banach spaces, then

$$c_n(T^*) = d_n(T) \tag{4.2}$$

for all compact operators $T \in \mathcal{L}(X,Y)$ and

$$d_n(T^*) = c_n(T) \tag{4.3}$$

for all $T \in \mathcal{L}(X,Y)$.

The following result is due to Gluskin, cf. [21, 22] with [56, 24, 26, 27] as forerunners. It gives a very precise information on the behaviour of $d_n(id : \ell_{p_1}^m \to \ell_{p_2}^m)$ in the Banach space setting.

Lemma 4.2. *For $1 \le n \le m < \infty$ and $1 \le p_1, p_2 \le \infty$, we define*

$$\Phi(m,n,p_1,p_2) := \begin{cases} (m-n+1)^{\frac{1}{p_2}-\frac{1}{p_1}} & \text{if } 1 \le p_2 \le p_1 \le \infty, \\ \left(\min\{1, m^{\frac{1}{p_2}} n^{-\frac{1}{2}}\}\right)^{\frac{\frac{1}{p_1}-\frac{1}{p_2}}{\frac{1}{2}-\frac{1}{p_2}}} & \text{if } 2 \le p_1 < p_2 \le \infty, \\ \max\{m^{\frac{1}{p_2}-\frac{1}{p_1}}, \sqrt{1-\frac{n}{m}}^{\frac{\frac{1}{p_1}-\frac{1}{p_2}}{\frac{1}{p_1}-\frac{1}{2}}}\} & \text{if } 1 \le p_1 < p_2 \le 2, \\ \max\{m^{\frac{1}{p_2}-\frac{1}{p_1}}, \min\{1, m^{\frac{1}{p_2}} n^{-\frac{1}{2}}\} \cdot \sqrt{1-\frac{n}{m}}\} & \text{if } 1 \le p_1 < 2 < p_2 \le \infty. \end{cases}$$

Then

$$d_n(id : \ell_{p_1}^m \to \ell_{p_2}^m) \approx \Phi(m,n,p_1,p_2), \qquad 1 \le n \le m < \infty,$$

if $p_2 < \infty$. The constants of equivalence may depend on p_1 and p_2 but are independent of m and n.

Furthermore, there are two constants c_{p_1} and C_{p_1} such that

$$c_{p_1}\Phi(m,n,p_1,\infty) \le d_n(id : \ell_{p_1}^m \to \ell_\infty^m) \le C_{p_2}\Phi(m,n,p_1,\infty)\left(\log\left(\frac{em}{n}\right)\right)^{3/2},$$

for $1 \le p_1 \le \infty$.

Again we shall add some estimates which apply to quasi-Banach spaces.

Lemma 4.3. *If $0 < p_2 \le p_1 \le \infty$, then there is a constant $c > 0$ such that*

$$d_{[cn]+1}(\ell_{p_1}^{2n}, \ell_{p_2}^{2n}) \gtrsim n^{\frac{1}{p_2}-\frac{1}{p_1}}, \quad n \in \mathbb{N},$$

where $[cn]$ denotes the upper integer part of cn.

Proof. If $p_2 \ge 1$, then the result is a special case of [43, Section 11.11.4], which states that

$$d_n(\ell_{p_1}^m, \ell_{p_2}^m) = (m-n+1)^{\frac{1}{p_2}-\frac{1}{p_1}}, \quad 1 \le n \le m.$$

Let us mention, that (in contrast to Lemma 3.3 and Lemma 4.8) the estimate

$$d_n(\ell_{p_1}^m, \ell_{p_2}^m) = (m-n+1)^{\frac{1}{p_2}-\frac{1}{p_1}}, \quad 1 \le n \le m \le \infty,$$

is *not* true for Kolmogorov numbers if $0 < p_2 \le p_1 \le \infty$ and $p_2 < 1$. Simple counterexamples can be constructed directly.

If $p_2 < 1$ the proof is based on an inequality between entropy numbers and Kolmogorov numbers. First, we recall the basic facts about entropy numbers. Let $T : X \to Y$ be a bounded linear operator between two quasi-Banach spaces X and Y and let U_X and U_Y be the unit ball of X and Y, respectively. If $k \in \mathbb{N}$, we define the kth entropy number $e_k(T)$ as the infimum of all $\epsilon > 0$ such that

$$T(U_X) \subset \bigcup_{j=1}^{2^{k-1}} (y_j + \epsilon U_Y) \quad \text{for some} \quad y_1, \ldots, y^{2^{k-1}} \in Y.$$

We refer to [43] and [15] for detailed discussions of this concept, its history and further references.

The following Lemma may be found in [1], cf. also [5] and [47, Section 5].

Lemma 4.4. *If $\alpha > 0$ and $0 < p < 1$, then there is a constant $c_{\alpha,p} > 0$ such that for all p-Banach spaces X and Y, all linear mappings $T : X \to Y$ and all $n \in \mathbb{N}$ we have*

$$\sup_{k \le n} k^\alpha e_k(T) \le c_{\alpha,p} \sup_{k \le n} k^\alpha d_k(T).$$

We apply this lemma to $T = id : \ell_{p_1}^{2n} \to \ell_{p_2}^{2n}$ and combine it with the estimate (cf. [53])

$$e_k(T) \gtrsim 2^{-\frac{k}{4n}} (2n)^{\frac{1}{p_2}-\frac{1}{p_1}}, \quad k, n \in \mathbb{N}.$$

This leads to

$$n^\alpha n^{\frac{1}{p_2}-\frac{1}{p_1}} \lesssim \sup_{k \le n} k^\alpha d_k(T).$$

Hence, for every $n \in \mathbb{N}$ there is a $k_n \le n$ such that

$$n^\alpha n^{\frac{1}{p_2}-\frac{1}{p_1}} \lesssim k_n^\alpha d_{k_n}(T) \le k_n^\alpha (2n)^{\frac{1}{p_2}-\frac{1}{p_1}}. \tag{4.4}$$

We conclude, that there is a constant $1 \ge c > 0$ such that $n \ge k_n \ge cn$ for all $n \in \mathbb{N}$. Finally, we insert this estimate into (4.4) and the result follows. □

It is an obvious fact that the convex hull of the unit ball of $\ell_p^m, 0 < p < 1$, is the unit ball of ℓ_1^m. This can be combined with the following simple observation, cf. [35, Section 13.1].

Lemma 4.5. *Let X be a Banach space and let $K \subset X$. We define by*

$$d_n(K, X) = \inf\{\sup_{x\in K} \inf_{y\in N} ||x - y|| : N \subset\subset Y, \dim(N) < n\}$$

the nth Kolmogorov number of the set K.
Then

$$d_n(K, X) = d_n(\mathrm{conv}K, X),$$

where conv$K$ *is the convex hull of K.*

Theorem 4.6. *Let $-\infty < s_2 < s_1 < \infty$ and $0 < p_1, p_2, q_1, q_2 \le \infty$ with (2.8). Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then (2.7) is compact and for $n \in \mathbb{N}$*

$$d_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}+\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+} \quad \text{if} \quad \begin{cases} \text{either} & 0 < p_1 \le p_2 \le 2, \\ \text{or} & 0 < p_2 \le p_1 \le \infty, \end{cases} \tag{4.5}$$

$$d_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}} \quad \text{if} \quad 2 < p_1 \le p_2 \le \infty \quad \text{and} \quad \frac{s_1 - s_2}{d} > \frac{1}{2}\frac{\frac{1}{p_1}-\frac{1}{p_2}}{\frac{1}{2}-\frac{1}{p_2}}, \tag{4.6}$$

$$d_n(\mathcal{I}d) \approx n^{\frac{p_2}{2}\left(-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{p_2}\right)} \quad \text{if} \quad 2 < p_1 \le p_2 \le \infty \quad \text{and} \quad \frac{s_1 - s_2}{d} < \frac{1}{2}\frac{\frac{1}{p_1}-\frac{1}{p_2}}{\frac{1}{2}-\frac{1}{p_2}}, \tag{4.7}$$

$$d_n(\mathcal{I}d) \approx n^{\left(-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{2}\right)} \quad \text{if} \quad 0 < p_1 < 2 < p_2 \le \infty \quad \text{and} \quad \frac{s_1 - s_2}{d} > \frac{1}{p_1}, \tag{4.8}$$

$$d_n(\mathcal{I}d) \approx n^{\frac{p_2}{2}\left(-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{p_2}\right)} \quad \text{if} \quad 0 < p_1 < 2 < p_2 < \infty \quad \text{and} \quad \frac{1}{p_1}-\frac{1}{p_2} < \frac{s_1 - s_2}{d} < \frac{1}{p_1}. \tag{4.9}$$

Proof. Lubitz [36] used the results of [21] and was able to prove (4.5)–(4.9) if $1 \le p_1, p_2 \le \infty$ up to a certain logarithmic gap. This gap originates from using only the weaker results of [21] instead of the sharp inequalities in [22]. Using [22] and the method of Lubitz (which is very similar to the discretization method presented above), the proof of (4.5)–(4.9) in the Banach space setting follows immediately.

Hence, we concentrate on the proof of

($\clubsuit$) (4.5) if $0 < p_2 \le p_1 \le \infty$ and $0 < p_2 < 1$,

($\heartsuit$) (4.5) if $0 < p_1 < p_2 \le 2$ and $0 < p_1 < 1$,

($\spadesuit$) (4.8) if $0 < p_1 < 1$, $2 < p_2 \le \infty$ and $\dfrac{s_1 - s_2}{d} > \dfrac{1}{p_1}$,

($\diamondsuit$) (4.8) if $0 < p_1 < 1$, $2 < p_2 < \infty$ and $\dfrac{1}{p_1} - \dfrac{1}{p_2} < \dfrac{s_1 - s_2}{d} < \dfrac{1}{p_1}$.

Let us mention that all the estimates from above follow from the estimates given in Theorem 3.5 and (4.1). We shall give the proof of the estimates from below in following three steps.

Step 1. - Proof of ($\clubsuit$)

The proof of (4.5) can be finished in the same manner as in the proof of Theorem 3.5. Namely, if $M' = \{M'_\nu\}_{\nu=0}^\infty$ is an admissible sequence, we get for $\nu \ge \nu_0$

$$d_n(id) \ge d_n(id_\nu) \gtrsim 2^{-\nu(s_1 - s_2 - \frac{d}{p_1} + \frac{d}{p_2})} \cdot M_\nu^{\frac{1}{p_2} - \frac{1}{p_1}}$$

for $n = \left[\frac{c}{2} \cdot M'_\nu\right]$, where c is the constant from Lemma 4.3. This leads to

$$d_n(id) \gtrsim n^{-\frac{s_1 - s_2}{d}}, \quad n = \left[\frac{c}{2} \cdot M'_\nu\right], \quad \nu \ge \nu_0$$

Again the monotonicity of the Kolmogorov numbers completes the proof.

Step 2. - Proof of ($\spadesuit$) *and* ($\diamondsuit$)

It follows from Lemma 4.5, that if $0 < p_1 < 1$ and $2 < p_2 \le \infty$

$$d_n(\ell_{p_1}^m, \ell_{p_2}^m) = d_n(\ell_1^m, \ell_{p_2}^m), \quad 1 \le n \le m < \infty. \tag{4.10}$$

The proof of ($\spadesuit$) follows from (4.10), (4.2), Lemma 4.2 and the choice $n = \left[\frac{M'_\nu}{2}\right]$.

The proof of ($\diamondsuit$) follows in the same way, but with $n = \left[(M'_\nu)^{\frac{2}{p_2}}\right]$.

Step 3. - Proof of ($\heartsuit$)

We generalise the idea of Lemma 4.5 to p-Banach spaces, namely we show that for $0 < p_1 < p_2 \le 2$

$$d_n(\ell_{p_1}^m, \ell_{p_2}^m) = d_n(\ell_{\min(1,p_2)}^m, \ell_{p_2}^m), \quad 1 \le n \le m < \infty. \tag{4.11}$$

If $p_2 \ge 1$, this follows immediately from Lemma 4.5. If $p_2 \le 1$, we show that

$$d_n(\ell_{p_1}^m, \ell_{p_2}^m) \ge d_n(E_m, \ell_{p_2}^m) \ge d_n(\ell_{p_2}^m, \ell_{p_2}^m). \tag{4.12}$$

Here, $E_m = \{e_i\}_{i=1}^m \subset \mathbb{R}^m$ and $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ are the canonical unit vectors having all but one components 0 and the i^{th} component 1.

Of course, (4.12) implies one half of (4.11), the second one being obvious. From (4.12), only the second inequality needs a proof. Let $N \subset\subset \ell_{p_2}^m = Y$ be such that

$$\sup_{i=1,\dots,n} \inf_{y \in N} ||e_i - y||_{p_2} \le (1 + \varepsilon) d_n(E_m, \ell_{p_2}^m)$$

with $\dim N < n$. Hence, to every $e_i \in E_m$ there is a $f_i \in N$ such that

$$||e_i - f_i||_Y \le (1 + \varepsilon)^2 d_n(E_m, \ell_{p_2}^m).$$

To every $x \in \ell_{p_2}^m, x = \sum_{i=1}^m x_i e_i$ with $\sum_{i=1}^m |x_i|^{p_2} \le 1$ we associate $\tilde{x}(x) = \sum_{i=1}^m x_i f_i \in N$. The estimate

$$\begin{aligned} d_n(id : \ell_{p_2}^m \to \ell_{p_2}^m)^{p_2} &\le \sup_{||x||_{p_2} \le 1} \inf_{y \in N} ||x - y||_{p_2}^{p_2} \\ &\le \sup_{||x||_{p_2} \le 1} ||x - \tilde{x}(x)||_{p_2}^{p_2} = \sup_{||x||_{p_2} \le 1} \left|\left| \sum_{i=1}^m x_i (e_i - f_i) \right|\right|_{p_2}^{p_2} \\ &\le \sup_{||x||_{p_2} \le 1} \sum_{i=1}^m ||x_i (e_i - f_i)||_{p_2}^{p_2} = \sup_{||x||_{p_2} \le 1} \sum_{i=1}^m |x_i|^{p_2} ||e_i - f_i||_{p_2}^{p_2} \\ &\le \sup_{||x||_{p_2} \le 1} \sum_{i=1}^m |x_i|^{p_2} (1+\varepsilon)^{2p_2} d_n(E_m, \ell_{p_2}^m)^{p_2} \\ &\le (1+\varepsilon)^{2p_2} d_n(E_m, \ell_{p_2}^m)^{p_2} \end{aligned}$$

finishes the proof of (4.12).
The proof of (♡) follows in the same way as in the first and the second step. □

Now, we turn our attention to Gelfand numbers. First, we collect some information about $c_n(id : \ell_{p_1}^m \to \ell_{p_2}^m)$, cf. [22], (4.2) and (4.3).

Lemma 4.7. *For $1 \le n \le m < \infty$ and $1 \le p_1, p_2 \le \infty$, we define*

$$\Phi(m,n,p_1,p_2) := \begin{cases} (m-n+1)^{\frac{1}{p_2} - \frac{1}{p_1}} & \text{if } 1 \le p_2 \le p_1 \le \infty, \\ \left(\min\{1, m^{1-\frac{1}{p_1}} n^{-\frac{1}{2}}\}\right)^{\frac{\frac{1}{p_1} - \frac{1}{p_2}}{\frac{1}{p_1} - \frac{1}{2}}} & \text{if } 1 < p_1 < p_2 \le 2, \\ \max\{m^{\frac{1}{p_2} - \frac{1}{p_1}}, \sqrt{1 - \frac{n}{m}}^{\frac{\frac{1}{p_1} - \frac{1}{p_2}}{\frac{1}{2} - \frac{1}{p_2}}}\} & \text{if } 2 \le p_1 < p_2 \le \infty, \\ \max\{m^{\frac{1}{p_2} - \frac{1}{p_1}}, \min\{1, m^{1-\frac{1}{p_1}} n^{-\frac{1}{2}}\} \cdot \sqrt{1 - \frac{n}{m}}\} & \text{if } 1 < p_1 \le 2 < p_2 \le \infty. \end{cases}$$

Then, if $p_1 > 1$,

$$c_n(id : \ell_{p_1}^m \to \ell_{p_2}^m) \approx \Phi(m,n,p_1,p_2), \qquad 1 \le n \le m < \infty.$$

Furthermore, there are two constants c_{p_2} and C_{p_2} such that

$$c_{p_2} \Psi(m,n,p_2) \le c_n(id : \ell_1^m \to \ell_{p_2}^m) \le C_{p_2} \Psi(m,n,p_2) \left(\log\left(\frac{em}{n}\right)\right)^{3/2},$$

where

$$\Psi(m,n,p_2) := \begin{cases} n^{1-\frac{1}{p_2}} & \text{if } 1 < p_2 \le 2, \\ \min\{1, \max\{m^{1-\frac{1}{p_2}}, m^{-\frac{1}{2}} \sqrt{\frac{m}{n} - 1}\}\} & \text{if } 2 \le p_2 \le \infty. \end{cases}$$

The proof of this lemma follows by (4.2) or (4.3) and Lemma 4.2.

Lemma 4.8. *If* $0 < p_2 \le p_1 \le \infty$*, then*

$$c_n(\ell_{p_1}^m, \ell_{p_2}^m) = (m - n + 1)^{\frac{1}{p_2} - \frac{1}{p_1}}.$$

The proof of this lemma follows literally [44, Section 11.11.4].

Lemma 4.9. *Let* $0 < p < 1$*. Then there is a real constant* $c > 0$ *such that*

$$c_n(id : \ell_p^m \to \ell_2^m) \le c \left[\frac{n}{\log\left(1 + \frac{m}{n}\right)} \right]^{\frac{1}{2} - \frac{1}{p}}, \quad 1 \le n \le m < \infty.$$

Proof. This lemma slightly generalises a result of Kashin [28], which was later improved by Gluskin [22] and Garnaev and Gluskin [20]. We closely follow the presentation given in [35, Chapter 14].

Let $\mathbf{y} = (y_1, \dots, y_n)$ be a multivector, with $y_1, \dots, y_n \in S^{m-1}$, the unit sphere of $\mathbb{R}^m$. We set

$$F_{m,n}(x, \mathbf{y}) = \frac{|(x, y_1)| + \cdots + |(x, y_n)|}{n}, \quad x \in \mathbb{R}^m.$$

We equip the space

$$\Sigma_{m,n} = \underbrace{S^{m-1} \times \cdots \times S^{m-1}}_{n \text{ times}}$$

with the natural rotation invariant probability measure P. Then (cf. [35, Lemma 4.1, Chapter 14]) we have the following

Lemma 4.10. *For any* $x \in S^{m-1}$ *and* $m, n \ge 2$

$$P\left\{ \mathbf{y} \in \Sigma_{m,n} : \frac{1}{8\sqrt{m}} \le F(x, \mathbf{y}) \le \frac{3}{\sqrt{m}} \right\} > \begin{cases} 1 - e^{-n}, & n > 2, \\ \frac{1}{2}, & n = 2. \end{cases}$$

Let l and m be natural numbers with $1 \le l \le m$. Let b_p^m denote the unit ball of ℓ_p^m. We denote by $b_p^{m,l}$ the subset of all vectors from b_p^m whose coordinates are of the form $\frac{k}{l}$, $k \in \mathbb{Z}$. Then there is a real constant $\tilde{c} > 0$ such that for any natural number $n \le m$ with

$$l = \left[\frac{1}{2\tilde{c}} \left(\frac{n}{\log\left(1 + \frac{m}{n}\right)} \right)^{1/p} \right] \ge 1$$

there exists a multivector $\mathbf{y} = (y_1, \dots, y_n)$ such that for all $x \in b_p^{m,l}$

$$\frac{1}{8\sqrt{m}} ||x||_2 \le F(x, \mathbf{y}) \le \frac{3}{\sqrt{m}} ||x||_2. \tag{4.13}$$

To prove it, we need to estimate the number of the elements of $b_p^{m,l}$ from above. It could be done directly, but we prefer to use known results. Observe that the mutual ℓ_∞^m distance of the points in $b_p^{m,l}$ is at least $\frac{1}{l}$. Hence, if $M_p^{m,l} = \# b_p^{m,l}$ (i.e. the number of elements of $b_p^{m,l}$) is greater than 2^n for some natural number n, then

$$e_n(id : \ell_p^m \to \ell_\infty^m) \ge \frac{1}{2l}. \tag{4.14}$$

But, according to [53] and [15, Section 3.2.2], there is a constant $\tilde{c}$ such that

$$e_n(id : \ell_p^m \to \ell_\infty^m) \le \tilde{c} \left(\frac{\log(1+\frac{m}{n})}{n} \right)^{1/p}, \quad 1 \le n \le m. \tag{4.15}$$

From (4.14) and (4.15), it follows that if

$$\frac{1}{2l} > \tilde{c} \left(\frac{\log(1+\frac{m}{n})}{n} \right)^{1/p},$$

then $M_p^{m,l} \le 2^n < e^n$. This, combined with Lemma 4.10 ensures the existence of the multivector $\mathbf{y}$.

Let $b_p^{m,l}$ be as above and let b_∞^m be a unit ball of ℓ_∞^m. Let $V_p^{m,l} = b_p^{m,l} \cap (\frac{1}{l} b_\infty^m)$ be the set of all vectors in $\mathbb{R}^m$ with the ℓ_p^m-quasinorm at most one and with components in $\{0, \pm\frac{1}{l}\}$. Then we claim that

$$b_p^m \cap \left(\frac{1}{l} b_\infty^m \right) = \mathrm{conv}_p(V_p^{m,l}) \subset \mathrm{conv}(V_p^{m,l}), \tag{4.16}$$

where $\mathrm{conv}_p(V_p^{m,l})$ is the so-called p-convex hull of $V_p^{m,l}$. We refer to [18, 19, 25] for the notion of p-convexity, p-extreme points and the quasi-convex variant of the Krein-Milman theorem, which gives the identity in (4.16). The inclusion is a simple consequence of the fact that $p < 1$.

To prove Lemma 4.9, we need to find $N \subset\subset \mathbb{R}^m$ of codimension at most n such that for each point $x \in N \cap b_p^m$ we have $||x||_2 \le \frac{c}{\sqrt{l}}$.

Let $\mathbf{y}$ be one multivector with (4.13). We set

$$N = \{x \in \mathbb{R}^m : F(x, \mathbf{y}) = 0\}.$$

Let $x \in N \cap b_p^m$ and let $x' \in b_p^{m,l}$ be the closest point to x, hence $||x - x'||_\infty \le \frac{1}{l}$. We set $x'' = x - x'$. Then

$$||x''||_2 \le ||x''||_p^{\frac{p}{2}} \cdot ||x''||_\infty^{1-\frac{p}{2}} \le l^{\frac{p}{2}-1}. \tag{4.17}$$

It remains to estimate $||x'||_2$. This will be done by estimating the value of $F(x', \mathbf{y})$. The estimate

$$F(x', \mathbf{y}) \ge \frac{1}{8\sqrt{m}} ||x'||_2 \tag{4.18}$$

follows from (4.13) and the fact that $x' \in b_p^{m,l}$. On the other hand, because of $x \in N$ and F is subadditive,

$$F(x', \mathbf{y}) \le F(x, \mathbf{y}) + F(x'', \mathbf{y}) = F(x'', \mathbf{y}). \tag{4.19}$$

For all $\tilde{x} \in V_p^{m,l} \subset b_p^{m,l}$, we have

$$F(\tilde{x}, \mathbf{y}) \le \frac{3}{\sqrt{m}} ||\tilde{x}||_2 \le 3m^{-\frac{1}{2}} l^{\frac{p}{2}-1} \tag{4.20}$$

and by subadditivity of F and (4.16), the same holds also for $x'' \in b_p^m \cap \left(\frac{1}{l} b_\infty^m\right)$.

We insert (4.20) into (4.19) and (4.18) and get $||x'||_2 \le 24 l^{\frac{p}{2}-1}$, and together with (4.17), $||x|| \le \frac{25}{\sqrt{l}}$. □

Lemma 4.11. *Let $0 < p_1 < 1$ and $p_1 < p_2 \le \infty$. Then there is a real constant $c > 0$ such that*

$$c_n(id : \ell_{p_1}^m \to \ell_{p_2}^m) \le c \left[\frac{n}{\log\left(1 + \frac{m}{n}\right)}\right]^{\frac{1}{\min(p_2,2)} - \frac{1}{p_1}}, \quad 1 \le n \le m < \infty.$$

Theorem 4.12. *Let $-\infty < s_2 < s_1 < \infty$ and $0 < p_1, p_2, q_1, q_2 \le \infty$ with (2.8). Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then (2.7) is compact and for $n \in \mathbb{N}$*

$$c_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d} + \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+} \quad \text{if} \quad \begin{cases} \text{either} & 2 \le p_1 < p_2 \le \infty, \\ \text{or} & 0 < p_2 \le p_1 \le \infty, \end{cases} \tag{4.21}$$

$$c_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}} \quad \text{if} \quad 0 < p_1 < p_2 \le 2 \quad \text{and} \quad \frac{s_1 - s_2}{d} > \frac{1}{2}\frac{\frac{1}{p_1} - \frac{1}{p_2}}{\frac{1}{p_1} - \frac{1}{2}}, \tag{4.22}$$

$$c_n(\mathcal{I}d) \approx n^{\frac{p_1'}{2}\left(-\frac{s_1-s_2}{d} + \frac{1}{p_1} - \frac{1}{p_2}\right)} \quad \text{if} \quad 1 < p_1 < p_2 \le 2 \quad \text{and} \quad \frac{s_1 - s_2}{d} < \frac{1}{2}\frac{\frac{1}{p_1} - \frac{1}{p_2}}{\frac{1}{p_1} - \frac{1}{2}}, \tag{4.23}$$

$$c_n(\mathcal{I}d) \approx n^{\left(-\frac{s_1-s_2}{d} + \frac{1}{2} - \frac{1}{p_2}\right)} \quad \text{if} \quad 0 < p_1 < 2 < p_2 \le \infty \quad \text{and} \quad \frac{s_1 - s_2}{d} > 1 - \frac{1}{p_2}, \tag{4.24}$$

$$c_n(\mathcal{I}d) \approx n^{\frac{p_1'}{2}\left(-\frac{s_1-s_2}{d} + \frac{1}{p_1} - \frac{1}{p_2}\right)} \quad \text{if} \quad 1 < p_1 < 2 < p_2 \le \infty \quad \text{and} \quad \frac{1}{p_1} - \frac{1}{p_2} < \frac{s_1 - s_2}{d} < 1 - \frac{1}{p_2}. \tag{4.25}$$

Proof. As Gelfand numbers are multiplicative and additive s-numbers, we may invoke (2.12) and restrict again to sequence spaces. Then, the method of the proof of Theorem 3.5 applies. The estimates on the sequence space side are given by Lemma 4.2 and (4.2). This approach finishes the proof in case $1 \le p_1, p_2 \le \infty$.

In the cases, when $p_1 < 1$ and/or $p_2 < 1$, (4.2) and (4.3) fail and Lemma 4.2 does not provide suitable estimates for $c_n(id : \ell_{p_1}^m \to \ell_{p_2}^m)$. Hence, we are forced to treat these cases separately.

- (♣) (4.21) if $0 < p_2 \le p_1 \le \infty$ and $0 < p_2 < 1$,
- (♡) (4.22) if $0 < p_1 < p_2 \le 2$ and $0 < p_1 < 1$,
- (♠) (4.24) if $0 < p_1 < 1$ and $2 < p_2 \le \infty$.

Step 1. - Proof of (♣)

The proof of the estimate from below in (♣) follows exactly as in the proof of Theorem 4.6 with Lemma 4.3 replaced by Lemma 4.8.

The estimate from above in (♣) is provided by the corresponding statement about approximation numbers, cf. Theorem 3.5 and (4.1).

Step 2. - Proof of the estimates from below in (♡) *and* (♠)

If $1 \le p_2 \le \infty$, we use the estimate

$$c_n(id : \ell_1^m \to \ell_{p_2}^m) \le ||id : \ell_1^m \to \ell_{p_1}^m|| \cdot c_n(id : \ell_{p_1}^m \to \ell_{p_2}^m) \tag{4.26}$$

and if $p_2 < 1$, we use the estimate

$$c_n(id : \ell_{p_2}^m \to \ell_{p_2}^m) \le ||id : \ell_{p_2}^m \to \ell_{p_1}^m|| \cdot c_n(id : \ell_{p_1}^m \to \ell_{p_2}^m). \tag{4.27}$$

This leads to

$$c_n(id : \ell_{p_1}^{2n} \to \ell_{p_2}^{2n}) \gtrsim \begin{cases} n^{\frac{1}{2}-\frac{1}{p_1}} & \text{if } \; 2 \le p_2 \le \infty, \\ n^{\frac{1}{p_2}-\frac{1}{p_1}} & \text{if } \; 0 < p_2 \le 2 \end{cases} \tag{4.28}$$

and the proof of the estimates from below included in (♡) and (♠) may be again finished as in the proof of Theorem 4.6.

Step 3. - Proof of the estimates from above in (♡) *and* (♠)

Again, the knowledge of the behaviour of $c_n(id : \ell_{p_1}^m \to \ell_{p_2}^m)$ is of a crucial importance. Lemma 4.11 contains already the necessary information and the proof can be finished using the standard discretization method. □

5 Conclusion

In Theorems 3.5, 4.6 and 4.12 we gave an overview of the behaviour of approximation, Kolmogorov and Gelfand numbers of

$$\mathcal{I}d : B^{s_1}_{p_1q_1}(\Omega) \to B^{s_2}_{p_2q_2}(\Omega),$$

where Ω is a bounded domain in $\mathbb{R}^d$ with smooth (i.e. Lipschitz) boundary and the parameters satisfy

$$s_1 - s_2 > d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+.$$

The reader has surely noticed, that all the obtained results about the asymptotic decay of $a_n(\mathcal{I}d)$, $d_n(\mathcal{I}d)$ and $c_n(\mathcal{I}d)$ do not depend on the fine parameters $0 < q_1, q_2 \le \infty$. This is of course no coincidence. The reason lies in the roots of the method we have used, namely in (3.7).

Nevertheless, the presented bounds from above and from below coincide in all "non-limiting" cases. Unfortunately, this method has also its natural bounds. For example, if $0 < p_1 < 2 < p_2 \le \infty$ and $s_1 - s_2 = d\max(1 - \frac{1}{p_2}, \frac{1}{p_1})$, then Theorem 3.5 fails to characterize the decay of $a_n(\mathcal{I}d)$. One observes, that in this case both (3.4) and (3.5) meet at $n^{-\frac{1}{2}}$, but (in general) this is not the exact speed of the decay of $a_n(\mathcal{I}d)$. It was shown by Kulanin [33], that additional logarithmic factors come into play. Their exact order seems to be unknown, but we believe that it depends on q_1 and q_2. So, for principle reasons, the decomposition method can not be extended to this "limiting" case.

Using the elementary embeddings (2.4), we conclude, that all the results hold for Triebel-Lizorkin spaces, Lebesgue spaces, Sobolev spaces, Bessel potential spaces and Hölder-Zygmund spaces as well.

For example, Theorem 3.5 may be stated in the framework of Bessel potential spaces and their embeddings into $C(\Omega)$ and $L_\infty(\Omega)$.

Theorem 5.1. *Let $1 \le p \le \infty$, $s > \frac{d}{p}$ and let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then the embeddings*

$$\mathcal{I}d_1 : H_p^s(\Omega) \to \mathcal{C}(\Omega) \tag{5.1}$$

$$\mathcal{I}d_2 : H_p^s(\Omega) \to L_\infty(\Omega) \tag{5.2}$$

are compact and

$$\begin{aligned}
&a_n(\mathcal{I}d_1) \approx a_n(\mathcal{I}d_2) \approx n^{-\frac{s}{d}+\frac{1}{p}} && \text{if} \quad 2 \le p \le \infty, \\
&a_n(\mathcal{I}d_1) \approx a_n(\mathcal{I}d_2) \approx n^{-\frac{s}{d}+\frac{1}{\tilde{p}}-\frac{1}{2}} && \text{if} \quad 0 < p < 2 \quad \text{and} \quad \frac{s}{d} > \frac{1}{\tilde{p}} = \max\Big(1, \frac{1}{p}\Big), \\
&a_n(\mathcal{I}d_1) \approx a_n(\mathcal{I}d_2) \approx n^{\left(-\frac{s}{d}+\frac{1}{p}\right)\cdot\frac{p'}{2}} && \text{if} \quad 1 < p < 2 \quad \text{and} \quad \frac{1}{p} < \frac{s}{d} < 1.
\end{aligned}$$

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