A new proof of the Jawerth-Franke embedding

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Abstract

We present an alternative proof of the Jawerth embedding

$$F^{s_0}_{p_0q}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1p_0}(\mathbb{R}^n),$$

where

$$-\infty < s_1 < s_0 < \infty, \quad 0 < p_0 < p_1 \le \infty, \quad 0 < q \le \infty$$

and

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

The original proof given in [3] uses interpolation theory. Our proof relies on wavelet decompositions and transfers the problem from function spaces to sequence spaces. Using similar techniques, we also recover the embedding of Franke, [2].

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1 Introduction

Let $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ denote the Besov and Triebel-Lizorkin function spaces, respectively. The classical Sobolev embedding theorem may be rewritten also for these two scales.

Theorem 1.1. *Let* $-\infty < s_1 < s_0 < \infty$ *and* $0 < p_0 < p_1 \le \infty$ *with*

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}. \tag{1.1}$$

(i) If $0 < q_0 \le q_1 \le \infty$*, then*

$$B^{s_0}_{p_0q_0}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1q_1}(\mathbb{R}^n). \tag{1.2}$$

(ii) If $0 < q_0, q_1 \le \infty$ *and* $p_1 < \infty$*, then*

$$F^{s_0}_{p_0q_0}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1q_1}(\mathbb{R}^n). \tag{1.3}$$

We observe, that there is no condition on the fine paramters q_0, q_1 in (1.3). This surprising effect was first observed in full generality by Jawerth, [3]. Using (1.3), we may prove

$$F^{s_0}_{p_0q}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1p_1}(\mathbb{R}^n) = B^{s_1}_{p_1p_1}(\mathbb{R}^n) \quad \text{and} \quad B^{s_0}_{p_0p_0}(\mathbb{R}^n) = F^{s_0}_{p_0p_0}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1q}(\mathbb{R}^n)$$

for every $0 < q \le \infty$. But Jawerth ([3]) and Franke ([2]) showed, that these embeddings are not optimal and may be improved.

Theorem 1.2. *Let* $-\infty < s_1 < s_0 < \infty$*,* $0 < p_0 < p_1 \le \infty$ *and* $0 < q \le \infty$ *with* (1.1).
(i) Then

$$F^{s_0}_{p_0q}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1p_0}(\mathbb{R}^n). \tag{1.4}$$

(ii) If $p_1 < \infty$*, then*

$$B^{s_0}_{p_0p_1}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1q}(\mathbb{R}^n). \tag{1.5}$$

The original proofs (see [3] and [2]) use interpolation techniques. We use a different method. First, we observe that using (for example) the wavelet decomposition method, (1.4) and (1.5) is equivalent to

$$f^{s_0}_{p_0q} \hookrightarrow b^{s_1}_{p_1p_0} \quad \text{and} \quad b^{s_0}_{p_0p_1} \hookrightarrow f^{s_1}_{p_1q} \tag{1.6}$$

under the same restrisctions on parameters s_0, s_1, p_0, p_1, q as in Theorem 1.2. Here, b^s_{pq} and f^s_{pq} stands for the sequence spaces of Besov and Triebel-Lizorkin type. We prove (1.6) directly using the technique of non-increasing rearrangement on a rather elementary level.

All the unimportant constants are denoted by the letter c, whose meaning may differ from one occurrence to another. If $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are two sequences of positive real numbers, we write $a_n \lesssim b_n$ if, and only if, there is a positive real number $c > 0$ such that $a_n \le c\, b_n, n \in \mathbb{N}$. Furthermore, $a_n \approx b_n$ means that $a_n \lesssim b_n$ and simultaneously $b_n \lesssim a_n$.

2 Notation and definitions

We introduce the sequence spaces associated with the Besov and Triebel-Lizrokin spaces. Let $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$. Then $Q_{\nu m}$ denotes the closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, centred at $2^{-\nu}m$, and with side length $2^{-\nu}$. By $\chi_{\nu m} = \chi_{Q_{\nu m}}$ we denote the characteristic function of $Q_{\nu m}$. If

$$\lambda = \{\lambda_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\},$$

$-\infty < s < \infty$ and $0 < p, q \le \infty$, we set

$$||\lambda|b^s_{pq}|| = \left(\sum_{\nu=0}^{\infty} 2^{\nu(s-\frac{n}{p})q}\Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu\, m}|^p\Big)^{\frac{q}{p}}\right)^{\frac{1}{q}} \tag{2.1}$$

appropriately modified if $p = \infty$ and/or $q = \infty$. If $p < \infty$, we define also

$$||\lambda|f^s_{pq}|| = \left|\left|\Big(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} |2^{\nu s}\lambda_{\nu\, m}\chi_{\nu\, m}(\cdot)|^q\Big)^{1/q}|L_p(\mathbb{R}^n)\right|\right|. \tag{2.2}$$

The connection between the function spaces $B^s_{pq}(\mathbb{R}^n)$, $F^s_{pq}(\mathbb{R}^n)$ and the sequence spaces b^s_{pq}, f^s_{pq} may be given by various decomposition techniques, we refer to [7, Chapters 2 and 3] for details and further references.

As a result of these characterisations, (1.4) is equivalent to (1.6).

We use the technique of non-increasing rearrangement. We refer to [1, Chapter 2] for details.

Definition 2.1. Let μ be the Lebesgue measure in $\mathbb{R}^n$. If h is a measurable function on $\mathbb{R}^n$, we define the non-increasing rearrangement of h through

$$h^*(t) = \sup\{\lambda > 0 : \mu\{x \in \mathbb{R}^n : |h(x)| > \lambda\} > t\}, \qquad t \in (0, \infty). \tag{2.3}$$

We denote its averages by

$$h^{**}(t) = \frac{1}{t}\int_0^t h^*(s)ds, \quad t > 0.$$

We shall use the following properties. The first two are very well known and their proofs may be found in [1], Proposition 1.8 in Chapter 2, and Theorem 3.10 in Chapter 3.

Lemma 2.2. *If $0 < p \le \infty$, then*

$$||h|L_p(\mathbb{R}^n)|| = ||h^*|L_p(0,\infty)||$$

for every measurable function h.

Lemma 2.3. *If $1 < p \le \infty$, then there is a constant c_p such that*

$$||h^{**}|L_p(0,\infty)|| \le c_p||h^*|L_p(0,\infty)||$$

for every measurable function h.

Lemma 2.4. *Let h_1 and h_2 be two non-negative measurable functions on $\mathbb{R}^n$. If $1 \le p \le \infty$, then*

$$||h_1 + h_2|L_p(\mathbb{R}^n)|| \le ||h_1^* + h_2^*|L_p(0,\infty)||.$$

Proof. The proof follows from Theorems 3.4 and 4.6 in [1, Chapter 2] □

3 Main results

In this part, we present a direct proof of the discrete versions of Jawerth and Franke embedding. We start with the Jawerth embedding.

Theorem 3.1. *Let $-\infty < s_1 < s_0 < \infty$, $0 < p_0 < p_1 \le \infty$ and $0 < q \le \infty$. Then*

$$f^{s_0}_{p_0q} \hookrightarrow b^{s_1}_{p_1p_0} \quad \textit{if} \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}. \tag{3.1}$$

Proof. Using the elementary embedding

$$f^s_{pq_0} \hookrightarrow f^s_{pq_1} \quad \text{if} \quad 0 < q_0 \le q_1 \le \infty \tag{3.2}$$

and the lifting property of Besov and Triebel-Lizorkin spaces (which is even more simple in the language of sequence spaces), we may restrict ourselves to the proof of

$$f^s_{p_0\infty} \hookrightarrow b^0_{p_1p_0}, \quad \text{where} \quad s = n\Big(\frac{1}{p_0} - \frac{1}{p_1}\Big). \tag{3.3}$$

Let $\lambda \in f^s_{p_0\infty}$ and set

$$h(x) = \sup_{\nu\in\mathbb{N}_0} 2^{\nu s} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|\chi_{\nu m}(x).$$

Hence

$$|\lambda_{\nu m}| \le 2^{-\nu s} \inf_{x\in Q_{\nu m}} h(x), \quad \nu\in\mathbb{N}_0, \quad m\in\mathbb{Z}^n.$$

Using this notation,

$$||\lambda|f^s_{p_0\infty}|| = ||h|L_{p_0}(\mathbb{R}^n)||$$

and

$$||\lambda|b^0_{p_1p_0}||^{p_0} \le \sum_{\nu=0}^\infty 2^{-\nu n}\Big(\sum_{m\in\mathbb{Z}^n} \inf_{x\in Q_{\nu m}} h(x)^{p_1}\Big)^{p_0/p_1} \le \sum_{\nu=0}^\infty 2^{-\nu n}\Big(\sum_{k=1}^\infty h^*(2^{-\nu n}k)^{p_1}\Big)^{p_0/p_1},$$

Using the monotonicity of h^* and the inequality $p_0 < p_1$ we get

$$\begin{aligned}
||\lambda|b^0_{p_1p_0}||^{p_0} &\lesssim \sum_{\nu=0}^\infty 2^{-\nu n}\Big(\sum_{l=0}^\infty 2^{nl}\cdot(2^n-1)\cdot h^*(2^{-\nu n}2^{nl})^{p_1}\Big)^{p_0/p_1}\\
&\lesssim \sum_{\nu=0}^\infty 2^{-\nu n}\sum_{l=0}^\infty 2^{nl\frac{p_0}{p_1}} h^*(2^{-\nu n}2^{nl})^{p_0}.
\end{aligned}$$

We substitute $j = l-\nu$ and obtain

$$\begin{aligned}
||\lambda|b^0_{p_1p_0}||^{p_0} &\lesssim \sum_{j=-\infty}^\infty \sum_{\nu=-j}^\infty 2^{-\nu n}2^{n(\nu+j)\frac{p_0}{p_1}} h^*(2^{jn})^{p_0}\\
&= \sum_{j=-\infty}^\infty 2^{nj\frac{p_0}{p_1}} h^*(2^{jn})^{p_0} \sum_{\nu=-j}^\infty 2^{n\nu\left(\frac{p_0}{p_1}-1\right)}\\
&\approx \sum_{j=-\infty}^\infty 2^{nj} h^*(2^{nj})^{p_0} \approx ||h^*|L_{p_0}(0,\infty)||^{p_0} = ||h|L_{p_0}(\mathbb{R}^n)||^{p_0}.
\end{aligned}$$

If $p_1 = \infty$, only notational changes are necessary. □

Theorem 3.2. *Let $-\infty < s_1 < s_0 < \infty, 0 < p_0 < p_1 < \infty$ and $0 < q \le \infty$. Then*

$$b^{s_0}_{p_0p_1} \hookrightarrow f^{s_1}_{p_1q} \quad \textit{if} \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}. \tag{3.4}$$

Proof. Using the lifting property and (3.2), we may suppose that $s_1 = 0$ and $0 < q < p_0$.
Using Lemma 2.4, we observe that

$$||\lambda|f^0_{p_1q}|| = \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^q \chi_{\nu m}(x) \right)^{1/q} |L_{p_1}(\mathbb{R}^n) \right\|$$

may be estimated from above by

$$\left\| \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} \tilde{\lambda}^q_{\nu m} \tilde{\chi}_{\nu m}(\cdot) |L_{\frac{p_1}{q}}(0,\infty) \right\|^{1/q}, \tag{3.5}$$

where $\tilde{\lambda}_\nu = \{\tilde{\lambda}_{\nu m}\}_{m=0}^{\infty}$ is a non-increasing rearrangement of $\lambda_\nu = \{\lambda_{\nu m}\}_{m\in\mathbb{Z}^n}$ and $\tilde{\chi}_{\nu m}$ is a characteristic function of the interval $(2^{-\nu n}m, 2^{-\nu n}(m+1))$.
Using duality, (3.5) may be rewritten as

$$\sup_g \left(\int_0^\infty g(x) \left(\sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} \tilde{\lambda}^q_{\nu m} \tilde{\chi}_{\nu m}(x) \right) dx \right)^{1/q} = \sup_g \left(\sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} 2^{-\nu n} \tilde{\lambda}^q_{\nu m} g_{\nu m} \right)^{1/q}, \tag{3.6}$$

where the supremum is taken over all non-increasing non-negative measurable functions g with $||g|L_\beta(0,\infty)|| \le 1$ and $g_{\nu m} = 2^{\nu n} \int g(x) \tilde{\chi}_{\nu m}(x) dx$. Here, β is the conjugated index to $\frac{p_1}{q}$. Similarly, α stands for the conjugated index to $\frac{p_0}{q}$.
We use twice Hölder's inequality and estimate (3.6) from above by

$$\left(\sum_{\nu=0}^{\infty} 2^{-\nu n} \left(\sum_{m=0}^{\infty} \tilde{\lambda}^{p_0}_{\nu m} \right)^{\frac{p_1}{p_0}} \right)^{1/p_1} \cdot \sup_g \left(\sum_{\nu=0}^{\infty} 2^{-\nu n} \left(\sum_{m=0}^{\infty} g^\alpha_{\nu m} \right)^{\frac{\beta}{\alpha}} \right)^{\frac{1}{\beta q}} \tag{3.7}$$

The first factor in (3.7) is equal to $||\lambda|b^{s_0}_{p_0p_1}||$. To finish the proof, we have to show that

$$\left(\sum_{\nu=0}^{\infty} 2^{-\nu n} \left(\sum_{m=0}^{\infty} g^\alpha_{\nu m} \right)^{\frac{\beta}{\alpha}} \right)^{\frac{1}{\beta q}} \le c \tag{3.8}$$

for every non-increasing non-negative measurable functions g with $||g|L_\beta(0,\infty)|| \le 1$. We fix such a function g and write

$$\begin{aligned}
\left(\sum_{\nu=0}^{\infty} 2^{-\nu n} \left(\sum_{m=0}^{\infty} g^\alpha_{\nu m} \right)^{\frac{\beta}{\alpha}} \right)^{1/\beta} &\le \left(\sum_{\nu=0}^{\infty} 2^{-\nu n} \left(\sum_{l=0}^{\infty} 2^{ln} (g^{**})^\alpha (2^{(-\nu+l)n}) \right)^{\frac{\beta}{\alpha}} \right)^{1/\beta} \\
&\le \left(\sum_{\nu=0}^{\infty} 2^{-\nu n} \sum_{l=0}^{\infty} 2^{ln} (g^{**})^\beta (2^{(-\nu+l)n}) \right)^{1/\beta} \\
&\le \left(\sum_{k=-\infty}^{\infty} 2^{kn\frac{\beta}{\alpha}} \sum_{\nu=-k}^{\infty} 2^{\nu n(\frac{\beta}{\alpha}-1)} (g^{**})^\beta (2^{kn}) \right)^{1/\beta} \\
&\lesssim \left(\sum_{k=-\infty}^{\infty} 2^{kn} (g^{**})^\beta (2^{kn}) \right)^{1/\beta} \\
&\lesssim ||g^{**}|L_\beta(0,\infty)|| \le c\,||g|L_\beta(0,\infty)|| \le c\,.
\end{aligned}$$

Taking the $\frac{1}{q}$−power of this estimate, we finish the proof of (3.8). $\square$

The Theorems 3.1 and 3.2 are sharp in the following sense.

Theorem 3.3. *Let* $-\infty < s_1 < s_0 < \infty$, $0 < p_0 < p_1 \leq \infty$ *and* $0 < q_0, q_1 \leq \infty$ *with*

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

(i) If

$$f^{s_0}_{p_0 q_0} \hookrightarrow b^{s_1}_{p_1 q_1}, \tag{3.9}$$

then $q_1 \geq p_0$.
(ii) If $p_1 < \infty$ *and*

$$b^{s_0}_{p_0 q_0} \hookrightarrow f^{s_1}_{p_1 q_1}, \tag{3.10}$$

then $q_0 \leq p_1$.

Remark 3.4. Using (any of) the usual decomposition techniques, the same statements hold true also for the function spaces. These results were first proved in [4].

Proof. (i) Suppose that $0 < q_1 < p_0 < \infty$ and set

$$\lambda_{\nu m} = \begin{cases} \nu^{-\frac{1}{q_1}} 2^{\nu(\frac{n}{p_1} - s_1)} & \text{if} \quad \nu \in \mathbb{N}_0 \quad \text{and} \quad m = 0, \\ 0, & \text{otherwise.} \end{cases}$$

A simple calculation shows, that $||\lambda|f^{s_0}_{p_0 q_0}|| < \infty$ and $||\lambda|b^{s_1}_{p_1 q_1}|| = \infty$. Hence, (3.9) does not hold.
(ii) Suppose that $0 < p_1 < q_0 \leq \infty$ and set

$$\lambda_{\nu m} = \begin{cases} \nu^{-\frac{1}{p_1}} 2^{\nu(\frac{n}{p_1} - s_1)} & \text{if} \quad \nu \in \mathbb{N}_0 \quad \text{and} \quad m = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Again, it is a matter of simple calculation to show, that $||\lambda|b^{s_0}_{p_0 q_0}|| < \infty$ and $||\lambda|f^{s_1}_{p_1 q_1}|| = \infty$. Hence, (3.10) is not true. ∎

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