Thesis Proposal: Impact of delay on pattern formation and stability in the attraction-repulsion model

The attraction-repulsion model considers a system of $N \in \mathbb{N}$ self-propelled agents with positions $x_i = x_i(t)$ and velocities $v_i = v_i(t)$, $i = 1, \ldots, N$, interacting according to

$$\dot{x}_i(t) = v_i(t), \tag{1}$$
$$\dot{v}_i(t) = \left(\vartheta^2 - |v_i(t)|^2\right) v_i(t) - \nabla_{x_i} \sum_{j \neq i} U(|x_i(t) - x_j(t)|), \tag{2}$$

where $\vartheta > 0$ is the constant target velocity and $U = U(r)$ is the interaction potential. Generically, one chooses the Morse potential

$$U(r) = -C_a e^{-r/R_a} + C_r e^{-r/R_r}$$

with attraction constant $C_a > 0$, attraction radius $R_a > 0$, and repulsion constant $C_r > 0$, repulsion radius $R_r > 0$. The model is known to generate a variety of stable patterns, most notably the flocking (crystalic), milling and double milling (Fig. 1) patterns.

The goal of the thesis is to numerically investigate the impact of introduction of three types of delay to the model (1)–(2):

- Information propagation-type delay, where agent i at time t receives information about the phase-space vector of agent j with a delay $\tau > 0$, which we for simplicity assume to be globally constant and taking the same value for all pairs i, j. I.e., system (1)–(2) transforms into

$$\dot{x}_i(t) = v_i(t), \tag{3}$$
$$\dot{v}_i(t) = \left(\vartheta^2 - |v_i(t)|^2\right) v_i(t) - \nabla_{x_i} \sum_{j \neq i} U(|x_i(t) - x_j(t-\tau)|). \tag{4}$$

- Reaction-type delay, where agent i receives the information about all other agents immediately, however, its reaction takes place with a certain latency $\tau > 0$, which we again assume to be globally constant and identical for all i, j. Then, system (1)–(2) reads

$$\dot{x}_i(t) = v_i(t), \tag{5}$$
$$\dot{v}_i(t) = \left(\vartheta^2 - |v_i(t-\tau)|^2\right) v_i(t-\tau) - \nabla_{x_i} \sum_{j \neq i} U(|x_i(t-\tau) - x_j(t-\tau)|). \tag{6}$$

- Alternatively, we may focus on the effect of reaction-type delay in the atttraction-repulsion mechanism only, assuming that self-propulsion reacts immediately. Then, the underlying system reads

$$\dot{x}_i(t) = v_i(t), \tag{7}$$
$$\dot{v}_i(t) = \left(\vartheta^2 - |v_i(t)|^2\right) v_i(t) - \nabla_{x_i} \sum_{j \neq i} U(|x_i(t-\tau) - x_j(t-\tau)|). \tag{8}$$

The goals of this thesis are as follows:

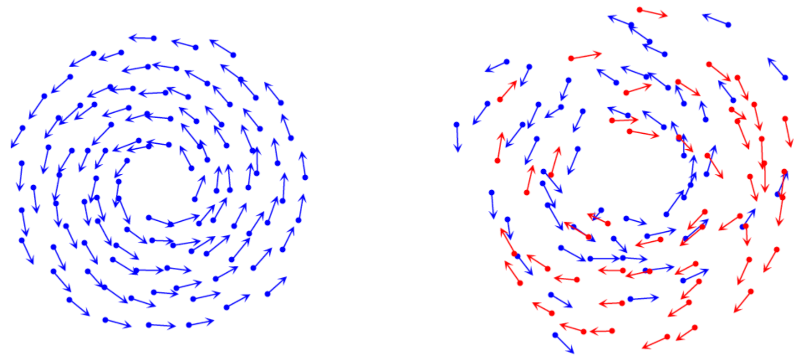

Figure 1: Milling and double milling patterns in the attraction-repulsion models [1].

1. Implement a numerical code (matlab, python) to resolve the systems (3)–(4), (5)–(6) and (7)–(8), choosing appropriate discretization methods.

2. Perform systematic numerical experiments to gain understanding of the impact of the different types of delay on the pattern formation dynamics of the attraction-repulsion system, in various parameter settings: Patterns are expected to lose their stability as the length of the delay $\tau > 0$ increases. The time a pattern needs to form, starting from random initial data, shall be investigated as a function of $\tau > 0$. Critical values of $\tau > 0$ (bifurcations) shall be estimated. Moreover, one expects that the effect of reaction-type delay is more destabilizing than the effect of propagation type delay. This hypotheses shall be investigated numerically.

3. Publication of the results of the systematic numerical simulations in appropriate scientific journal will be welcome, though not necessary.

References

[1] J. A. Carrillo, M. R. D'Orsogna and V. Panferov: *Double milling in self-propelled swarms from kinetic theory*. Kinetic and Related Models **2** (2009), 363–378.