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Note

Spherical codes and Borsuk's conjecture ☆

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Abstract

The approach of Kalai and Kahn towards counterexamples of Borsuk's conjecture is generalized to spherical codes. This allows the construction of a finite set in $\mathbb{R}^{323}$ which cannot be partitioned into 561 sets of smaller diameter, thus improving upon the previous known examples. The construction is based on the subset of vectors of minimal length in the Leech lattice.

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Borsuk's conjecture stated in [1] asks whether every bounded set $S \subset \mathbb{R}^d$ containing at least two points can be partitioned into at most $d+1$ sets of smaller diameter. This conjecture was confirmed only for $d \leqslant 3$. Kahn and Kalai [4] constructed sets which, for large enough d, cannot be partitioned into at most $1.1^{\sqrt{d}}$ subsets of smaller diameter. Improvements on the least dimension where Borsuk's conjecture is shown to be false were obtained by Nilli ($d=946$, [5]), Raigorodskii ($d=561$ [6]), and Weissbach ($d=560$, [8]). Raigorodskii also showed that Borsuk's conjecture is false in all dimensions $d \geqslant 561$, [7]. It is the purpose of this note to improve upon these bounds.

We prove the following theorem, which provides a counterexample to Borsuk's conjecture in dimension d with $323 \leqslant d < 561$.

Theorem 1. *There exists a finite set in the unit sphere in* $\mathbb{R}^{323}$ *which cannot be partitioned into* 561 *sets of smaller diameter. Hence Borsuk's conjecture fails in all dimensions exceeding* 322.

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Let us first recall some definitions from the theory of spherical codes. We use notations as can be found in [2]. Ω_d denotes the unit sphere in $\mathbb{R}^d$. A finite subset C of Ω_d is called a spherical code. If $S \subset [-1,1)$ and $\langle x, y\rangle \in S$ for all $x, y \in C$ with $x \neq y$, then C is said to be a spherical S-code. The largest cardinality of a spherical S-code in Ω_d is denoted by $A(d,S)$. We shall also consider spherical S-codes contained in a subset $M \subset \Omega_d$. The largest cardinality of such a code is denoted by $A(d,S,M)$.

Let us define the Borsuk number $b(d)$ to be the smallest positive integer m such that any finite subset of $\mathbb{R}^d$ with at least two points can be partitioned into m subsets of smaller diameter. The proof of Theorem 1 is based on the following general result.

Theorem 2. *Let S be a finite subset of $[-1,1)$, $M \subset \Omega_d$, $n = d(d+3)/2$, and define $\alpha = \max S \cap [-1,0)$ and $\beta = \min S \cap [0,1)$. If $\alpha + \beta < 0$, then*

$$b(n-1)A(d,S\backslash\{\alpha,\beta\}) \geqslant A(d,S)$$

and

$$b(n-1)A(d,S\backslash\{\alpha,\beta\},M) \geqslant A(d,S,M).$$

Proof of Theorem 2. Let $C \in \Omega_d$ be a maximal spherical S-code, i.e. $|C| = A(d,S)$. Fix an orthonormal basis $((e_i)_{i=1}^d, (f_i)_{i=1}^d, (g_{i,j})_{1\leqslant i<j\leqslant d})$ in $\mathbb{R}^n$. Consider the map $\Phi: \mathbb{R}^d \to \mathbb{R}^n$ given by

$$\Phi(x) = \frac{1}{\sqrt{1-\alpha-\beta}}\left(\sum_{i=1}^{d} x_i^2 e_i + \sqrt{-\alpha-\beta}\sum_{i=1}^{d} x_i f_i + \sqrt{2}\sum_{1\leqslant i<j\leqslant d} x_i x_j g_{i,j}\right),$$

for $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$. Since $\alpha+\beta < 0$, this map is well defined and injective. Moreover, it is easily computed that

$$\langle \Phi(x), \Phi(y)\rangle = \frac{(\langle x,y\rangle - \alpha)(\langle x,y\rangle - \beta) - \alpha\beta}{1-\alpha-\beta}.$$

Hence Φ maps Ω_d into Ω_n and $\langle \Phi(x), \Phi(y)\rangle$ is minimal for $x, y \in C$ if and only if $\langle x, y\rangle = \alpha$ or $\langle x, y\rangle = \beta$ implying

$$distance(\Phi(x), \Phi(y)) = diameter(\Phi(C)) \quad \Leftrightarrow \quad \langle x, y\rangle \in \{\alpha,\beta\}.$$

Observe also, that the image of Ω_d is contained in the $(n-1)$-dimensional affine subspace consisting of all vectors for which the coordinates of the e_i-basis vectors sum up to $1/\sqrt{1-\alpha-\beta}$.

Assume now that $\Phi(C)$ is partitioned into $b(n-1)$ subsets of smaller diameter. This corresponds to a partition of C into subsets which are spherical $S\backslash\{\alpha,\beta\}$-codes. This finally yields

$$b(n-1)A(d,S\backslash\{\alpha,\beta\}) \geqslant |C| = A(d,S).$$

The same argument provides the second inequality. □

We will now prove some cardinality estimates for spherical codes in dimension 24 which are to be used in the proof of Theorem 1.

Proposition 3. *Let* $M = \Omega_{24} \cap 2^{-5/2}\mathbb{Z}^{24}$. *Then*

(i) $A(24, \{-\frac{1}{2}, \frac{1}{2}\}, M) \leqslant 25$,
(ii) $A(24, \{-\frac{1}{2}, \frac{1}{4}, \frac{1}{2}\}, M) \leqslant 325$,
(iii) $A(24, \{-1, -\frac{1}{2}, \frac{1}{4}, \frac{1}{2}\}, M) \leqslant 350$.

Proof. (i) For a spherical $\{\pm\frac{1}{2}\}$-code $C \subset M$, we consider the linear polynomials L_c given for $c \in C$ by $L_c(x) = 2\langle x, c\rangle + 1$. These polynomials have coefficients in the field $F = \mathbb{Q}(\sqrt{2})$. Moreover, $L_c(c) = 3$, $L_c(x) = 0$ if $\langle x, c\rangle = -\frac{1}{2}$, and $L_c(x) = 2$ if $\langle x, c\rangle = \frac{1}{2}$. This implies that the polynomials $\{L_c\colon c \in C\}$ are linear independent over F. Indeed, assuming $\sum_{c\in C} \lambda_c L_c = 0$ for some nontrivial $\lambda_c = \alpha_c + \beta_c\sqrt{2}$ with $\alpha_c, \beta_c \in \mathbb{Q}$, we may as well assume that the α_c, β_c are integers which are not all even. But then evaluation at the points $c \in C$ shows that α_c and β_c have to be even for each $c \in C$, a contradiction. Thus, the cardinality of C cannot exceed the dimension of the linear space (over F) of all linear polynomials in 24 indeterminates, which is 25.

(ii) The proof is similar to the proof of (i). Instead of the linear functions L_c we now consider the quadratic polynomials P_c given by $P_c(x) = (2\langle x, c\rangle - 1)(4\langle x, c\rangle - 1)$. Then $P_c(c) = 3$ is odd, but $P_c(x)$ is even for different points c, x of a spherical $\{-\frac{1}{2}, \frac{1}{4}, \frac{1}{2}\}$-code. Thus, the cardinality of such a code in M cannot exceed the dimension of the linear space of polynomials of total degree at most 2 in 24 indeterminates, which is 325.

(iii) Now, let $C \subset M$ be a spherical $\{-1, -\frac{1}{2}, \frac{1}{4}, \frac{1}{2}\}$-code. Consider the subset $C_1 \subset C$ containing all points whose antipodal point is not in C. Choose a subset $C_2 \subset C \setminus C_1$ consisting of one point of each pair of antipodal points. Then $C_1 \cup C_2 \subset M$ is a spherical $\{-\frac{1}{2}, \frac{1}{4}, \frac{1}{2}\}$-code. Moreover, $\langle x, y\rangle \neq \frac{1}{4}$ for any $x, y \in C_2$, since otherwise $x, -y \in C$ would have inner product $-\frac{1}{4}$. So $C_2 \subset M$ is a spherical $\{-\frac{1}{2}, \frac{1}{2}\}$-code. Now parts (i) and (ii) imply

$$|C| = |C_1| + 2|C_2| \leqslant A(24, \{-\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{2}\}, M) + A(24, \{-\tfrac{1}{2}, \tfrac{1}{2}\}, M) \leqslant 350. \qquad \square$$

Proof of Theorem 1. Let C be the set of normalized vectors of minimal length in the Leech lattice. It is well known, see e.g. [3] or [2, Chapter 14] that C is a spherical $\{-1, 0, \pm\frac{1}{2}, \pm\frac{1}{4}\}$-code of cardinality 196560 in M, where M is as in Proposition 3. Hence $A(24, \{-1, 0, \pm\frac{1}{2}, \pm\frac{1}{4}\}, M\} \geqslant 196560$. Together with (iii) in Proposition 3 and Theorem 2 we obtain $b(323) \geqslant 562$. $\square$

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