Odvození jpdf pro vlastní čísla Gaussovských matic

• Máme jpdf podle elementů H (resp. horního trojúhelníku)

$$\rho(H) = \prod_{i=1}^{N} \frac{e^{-H_{ii}^2/2}}{\sqrt{2\pi}} \prod_{i<j} \frac{e^{-H_{ij}^2}}{\sqrt{\pi}} = \prod_{i,j=1}^{N} e^{-H_{ij}^2/2} \cdot (2\pi)^{-N/2} \cdot \pi^{-\frac{N(N-1)}{4}}$$

• Z toho chceme odvodit hustotu na množině spekter H:

$$\tau(x_1,\dots,x_N) = \frac{1}{Z_N} e^{-\frac{1}{2}\|x\|_2^2} \prod_{j<k} |x_j - x_k|,$$

$$Z_N = (2\pi)^{N/2} \prod_{j=1}^{N} \frac{\Gamma(1+j/2)}{\Gamma(1+1/2)}$$

Poznámky: • uspořádané $(x_1 < x_2 < \dots < x_N)$ vs. neuspořádané spektrum ... $(x_1,\dots,x_N)$ obecné, určené jednozn.

... $\tau(x_1,\dots,x_N) = \tau(x_{\sigma(1)},\dots,x_{\sigma(N)})$, σ permutace

$\to$ liší se jen o faktor $N!$!

• pro $A \subset \mathbb{R}^N$ inv. vůči permutacím $(x \in A \Leftrightarrow (x_{\sigma(1)},\dots,x_{\sigma(N)}) \in A)$ bude platit $\mathbb{P}(\sigma(H) \in A) = \int_A \tau(x)\,dx$

• Vztah mezi H a $\sigma(H) = x$ je dán jako $H = O X O^T$

kde O ... ortogonální ... $O O^T = O^T O = I$

$X = \mathrm{diag}(x_1,\dots,x_N)$ ... $x = \mathrm{diag}(X)$.

Nejprve zkusíme v jednodušší formě: polární souřadnice

$(x,y) \;....\; (r,\varphi)$ $\qquad \Psi(r,\varphi) = (r\cos\varphi, r\sin\varphi) = (x,y)$

Mějme náhodný vektor V s $\mathbb{P}(V\in B) = \int_B \rho(x,y)\, d(x,y)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

Chceme najít hustotu τ tak, aby: $\displaystyle\int_{\{(r,\varphi):\, \Psi(r,\varphi)\in B\}} \tau(r,\varphi)\, d(r,\varphi)$

$$\| $$

$$\int_{\Psi^{-1}(B)} \tau(r,\varphi)\, d(r,\varphi)$$

Substituce v hustotě na $\mathbb{R}^2$

$$\int_B \rho(x,y)\, d(x,y) = \int_{\Psi^{-1}(B)} \rho(r\cos\varphi, r\sin\varphi)\cdot r\cdot dr\, d\varphi$$

$$\Rightarrow \tau(r,\varphi) = \rho(r\cos\varphi, r\sin\varphi)\cdot r$$

- vyjádřit x,y pomocí r,φ
- Jakobián $\quad r = \left| J((x,y),(r,\varphi)) \right| = \left| \det\begin{pmatrix} \frac{\partial x}{\partial r}, & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r}, & \frac{\partial y}{\partial \varphi} \end{pmatrix} \right|$.
- nechť τ nezávisí na φ : $\rho(x,y) = \bar\rho(x^2+y^2)$

$$\tau(r,\varphi) = r\cdot\rho(r\cos\varphi, r\sin\varphi) = r\cdot\bar\rho(r^2)$$

- Pokud B je ~~k~~rot. invariantní ... $\Psi^{-1}(B) = C\times[0,2\pi]$

$$\int_{\Psi^{-1}(B)} \tau(r,\varphi)\, dr\, d\varphi = \int_C \int_0^{2\pi} r\cdot\bar\rho(r^2)\, d\varphi\, dr = \int_C r\cdot\bar\rho(r^2)\, dr\cdot 2\pi$$

$$\bar\tau(r) = 2\pi\cdot r\,\bar\rho(r^2)$$

Nyní $\Psi : (X, \mathcal{O}) \to H = \mathcal{O} X \mathcal{O}^T$

$H = H(X, \mathcal{O})$

- Vyjádříme $\rho(H) = \rho(H_{11}(X, \mathcal{O}), \ldots, H_{NN}(X, \mathcal{O}))$

Lze několika způsoby:

1, Lemma: Pro $A \in \mathbb{R}^{m \times N}$, $(\varphi_j)_{j=1}^N$, $(\psi_j)_{j=1}^N$ ort. báze $\mathbb{R}^N$

$$\text{je} \quad \sum_{j=1}^N \|A\varphi_j\|_2^2 = \sum_{j=1}^N \|A\psi_j\|_2^2 = \|A\|_F^2 = \sum_{i,j} A_{ij}^2$$

Důkaz: $\psi_j = \sum_{k=1}^N \langle \psi_j, \varphi_k \rangle \varphi_k$; $A\psi_j = \sum_{k=1}^N \langle \psi_j, \varphi_k \rangle A\varphi_k$

$$\text{a tedy} \quad \sum_{j=1}^N \|A\psi_j\|_2^2 = \sum_{j=1}^N \left\langle \sum_{k=1}^N \langle \psi_j, \varphi_k \rangle A\varphi_k, \sum_{l=1}^N \langle \psi_j, \varphi_l \rangle A\varphi_l \right\rangle$$

$$= \sum_{k=1}^N \sum_{l=1}^N \langle A\varphi_k, A\varphi_l \rangle \underbrace{\sum_{j=1}^N \langle \psi_j, \varphi_k \rangle \langle \psi_j, \varphi_l \rangle}_{= \langle \varphi_k, \varphi_l \rangle = \delta_{k,l}}$$

$$= \sum_{k=1}^N \langle A\varphi_k, A\varphi_k \rangle = \sum_{k=1}^N \|A\varphi_k\|_2^2 \quad \ldots \quad \varphi_k = e_k \text{ kan. báze}$$

$$\Rightarrow \quad = \|A\|_F^2$$

Důsledek: $A \in \mathbb{R}^{N \times N}$ symetrická $\Rightarrow \|A\|_F^2 = \sum_{j=1}^N \lambda_j(A)^2$

$\ldots (\varphi_k)_{k=1}^N$ je báze z vl. vektorů $\ldots A\varphi_k = \lambda_k \varphi_k$

$$\Rightarrow \|A\|_F^2 = \sum_{k=1}^N \|A\varphi_k\|_2^2 = \sum_{k=1}^N \lambda_k(A)^2.$$

Celkem je tedy $\rho(H) \cong \prod_{i,j=1}^{N} e^{-H_{ij}^2/2} = e^{-\|H\|_F^2/2} = e^{-\|x\|_2^2/2}$.

2, Pozn. Stejně se dokáže: $A \in \mathbb{R}^{N \times N}$, $(\varphi_j)_{j=1}^N$, $(\psi_j)_{j=1}^N$ onb. $\mathbb{R}^N$, pak

$$\sum_{j=1}^{N} \langle A\psi_j, \psi_j \rangle = \sum_{k=1}^{N} \langle A\varphi_k, \varphi_k \rangle = \mathrm{tr}(A)$$

Dále platí: $A, B \in \mathbb{R}^{m \times N}$: $\mathrm{tr}(A^T B) = \sum_{k=1}^{N} (A^T B)_{kk} =$

$$= \sum_{k=1}^{N} \sum_{j=1}^{m} (A^T)_{kj} B_{jk} = \sum_{j=1}^{m} \sum_{k=1}^{N} A_{jk} B_{jk} = \langle A, B \rangle_F$$

$= \langle B, A \rangle_F = \mathrm{tr}(B^T A)$ ... stopa je cyklická

- $\mathrm{tr}(H^2) = \sum_{k=1}^{N} (H^2)_{kk} = \sum_{k=1}^{N} \sum_{l=1}^{N} H_{kl} \cdot H_{lk} \overset{H=H^T}{=} \sum_{k,l=1}^{N} H_{kl}^2 = \|H\|_F^2$

- Celkem je tedy $\rho(H) \cong \exp(-\|H\|_F^2/2) = \exp(-\mathrm{tr}(H^2)/2)$

$$= \exp\left[-\mathrm{tr}((OXO^T)(OXO^T))/2\right] =$$

$$= \exp\left[-\mathrm{tr}(OX^2O^T)/2\right] = \exp(-\mathrm{tr}(O^T O X^2)/2)$$

$\mathrm{tr}(ABC) = \mathrm{tr}(CAB) = \mathrm{tr}(BCA)$ jsou-li součiny definovány

$$= \exp(-\mathrm{tr}(X^2)/2) = \exp(-\|x\|_2^2/2).$$

3, Poslední způsob - hrubou silou

$$H = OXO^T, \quad H_{ij} = (OXO^T)_{ij} = \sum_{l=1}^{N} O_{il} (XO^T)_{lj}$$

$$= \sum_{l=1}^{N} O_{il} \sum_{k=1}^{N} X_{lk} (O^T)_{kj} = \sum_{l,k=1}^{N} X_{lk} O_{il} O_{jk} \overset{l=k}{=} \sum_{k=1}^{N} x_k O_{ik} O_{jk}$$

a $$\sum_{i,j=1}^{N} H_{ij}^2 = \sum_{i,j=1}^{N} \left(\sum_{k=1}^{N} x_k O_{ik} O_{jk} \right)^2 =$$

$$= \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} x_k O_{ik} O_{jk} \, x_l O_{il} O_{jl} = \sum_{k,l=1}^{N} x_k x_l \sum_{i,j=1}^{N} O_{ik} O_{jk} O_{il} O_{jl}$$

$$= \sum_{k,l=1}^{N} x_k x_l \left(\sum_{i=1}^{N} O_{ik} O_{il} \right) \left(\sum_{j=1}^{N} O_{jk} O_{jl} \right) = \sum_{k=1}^{N} x_k^2 .$$

- Budeme muset vypočítat $J(H, (x, O))$ ... vyjde

$$|J(H, (x, O))| = \left| \prod_{j<k} (x_j - x_k) \right| = \prod_{j<k} |x_j - x_k|$$

... nezávisí na O!

V analogii s polárními souřadnicemi:

- Nechť B je „rotačně invariantní" podmnožina

$$\mathcal{M} = \{ M \in \mathbb{R}^{N \times N} : M = M^T \}$$

tedy $\forall M \in B \ \forall O : O M O^T \in B$

Pak

$$\int_B \rho(H)\, dH = \int_B \rho(H_{11}, \ldots, H_{NN})\, dH_{11} \ldots dH_{NN}$$

$$= \int_{\{(x,O) : \Psi(x,O) = O X O^T \in B\}} \rho(H_{11}(x,O), \ldots, H_{NN}(x,O)) \cdot |J(H, \{x, O\})| \cdot dx\, dO$$

$$= \int_{\Psi^{-1}(B)} e^{-\|x\|_2^2/2} \cdot \prod_{j<k} |x_j - x_k| \cdot dx\, dO \cdot (2\pi)^{-N/2} \cdot \pi^{-N(N-1)/4}$$

$\ldots \ \Psi^{-1}(B) = C \times \Omega_N$

$\Omega_N = \{ O : O O^T = O^T O = I \}$

$$= \int_C \int_{\Omega_N} e^{-\|x\|_2^2/2} \cdot \prod_{j<k} |x_j - x_k|\, dO\, dx \cdot c_N$$

$$= \int_C e^{-\|x\|_2^2/2} \prod_{j<k} |x_k - x_j|\, dx \cdot \underbrace{\int_{\Omega_N} 1\, dO}_{vol(\Omega_N)} \cdot c_N$$

$$\tau(x_1, \ldots, x_N) = e^{-\|x\|_2^2/2} \cdot \prod_{j<k} |x_j - x_k| \cdot \frac{c_N \cdot vol(\Omega_N)}{2^N \cdot N!}$$

- Faktor $2^N \cdot N!$ odpovídá počtu (x, O), které se zobrazují na stejné H ... permutace $x_1, \dots, x_N$, volba znaménka N vlastních vektorů.

- Pokud bychom v pol. souřadnic měli $\varphi \in [0, 4\pi]$, museli bychom hustotu vydělit dvěma ...

- Vzápětí ukážeme, že $\mathrm{vol}(\Omega_N) = \prod_{j=1}^{N} \frac{2\pi^{j/2}}{\Gamma(j/2)}$ ... Pak bude

$$\frac{1}{Z_N} = (2\pi)^{-N/2} \pi^{-N(N-1)/4} \cdot \left(\prod_{j=1}^{N} \frac{\cancel{2}\pi^{j/2}}{\Gamma(j/2)} \right) \cdot \frac{1}{\cancel{2^N} \cdot N!} =$$

$$(2\pi)^{-N/2} \underbrace{\pi^{-N(N-1)/4} \cdot \pi^{N(N+1)/4}}_{\pi^{N/2}} \cdot \frac{1}{\Gamma(1/2) \cdot 1/2 \cdots \Gamma(N/2) \cdot N/2 \cdot 2^N}$$

$$= (2\pi)^{-N/2} \cdot \frac{(\sqrt{\pi}/2)^N}{\Gamma(1/2+1) \cdots \Gamma(N/2+1)} \quad \dots \quad \sqrt{\pi}/2 = \Gamma(1 + \tfrac{1}{2})$$

$$\boxed{\begin{array}{ll} \text{Zbývá} & \cdot\ \mathrm{vol}(\Omega_N) \\ & \cdot\ |J(H, (x, O))| \end{array}}$$

Výpočet $vol(\Omega_N)$

- 1, jednotková koule a sféra v $\mathbb{R}^n$

$$B_n = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}; \quad S^{n-1} = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$$

$$\int_{\mathbb{R}^n} e^{-\|x\|_2^2/2} dx = \int_{\mathbb{R}^n} e^{-x_1^2/2} \dots e^{-x_n^2/2} dx_1 \dots dx_n = \left(\int_{\mathbb{R}} e^{-t^2/2} dt\right)^n = (2\pi)^{n/2}$$

$$\shortparallel$$

$$\int_0^\infty \int_{rS^{n-1}} e^{-r^2/2} dA\, dr = \int_0^\infty e^{-r^2/2} r^{n-1} A_{n-1} dr = A_{n-1} \int_0^\infty e^{-t} (2t)^{\frac{n-1}{2} - \frac{1}{2}} dt$$

$$t = r^2/2$$
$$dt = r\,dr$$

$$= A_{n-1} \cdot 2^{\frac{n}{2}-1} \Gamma(n/2)$$

$$\Rightarrow A_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}; \quad V_n = \int_0^1 \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1} dr = \frac{2\pi^{n/2}}{n\Gamma(n/2)} = \frac{\pi^{n/2}}{\Gamma(n/2+1)}.$$

Pozn.: $\bullet\ vol(R\cdot B) = \int_{R\cdot B} 1\, dx \underset{x=Ry}{=} \int_B R^n dy = R^n \cdot vol(B).$

$$\bullet\ A_{n-1} = \lim_{\varepsilon \to 0} \frac{vol[(1+\varepsilon)B] - vol[(1-\varepsilon)B]}{2\varepsilon} =$$

$$= \lim_{\varepsilon \to 0} \frac{(1+\varepsilon)^n - (1-\varepsilon)^n}{2\varepsilon} vol(B) = \lim_{\varepsilon \to 0} \frac{2n\varepsilon}{2\varepsilon} vol(B)$$

$$= n \cdot vol(B) = nV_n$$

$$\bullet \text{obecně } area(rS^{n-1}) = \frac{d}{dr} vol(rB_n)$$

• Formule $\int_{\mathbb{R}^n} f(x)\,dx = \int_0^\infty \int_{r\mathbb{S}^{n-1}} f(x)\,dA\,dr$

$f = \chi_{\{a \le \|x\| \le b\}}$: $vol(b\cdot B) - vol(a\cdot B) = [b^n - a^n]\cdot vol(B)$

& $\int_a^b area(r\mathbb{S}^{n-1})\,dr = area(\mathbb{S}^{n-1}) \int_a^b r^{n-1}\,dr = \frac{b^n - a^n}{n} area(\mathbb{S}^{n-1})$

platí pro jednoduché rad. symetrické funkce,

tedy i pro lin. kombinace ... tedy pro $f \in L_1(\mathbb{R}^n)$ rad. sym.

2) $\Omega_N = \{ O \in \mathbb{R}^{N\times N} : OO^T = O^T O = I \}$

počet stupňů volnosti ... dimenze Ω_N?

první řádek : $N-1$ ... N & $\|x\|_2^2 = 1$

druhý řádek : $N-2$ ... N & $\|a_2\|_2^2 = 1$ & $\langle a_1, a_2 \rangle = 0$

⋮

N-tý řádek : $(N-1)-(N-1) = 0$ ... jen 2 vektory možné

$D_N = dim(\Omega_N) = \frac{N(N-1)}{2}$

první řádek: $A_{N-1} = area(\mathbb{S}^{N-1}) = \frac{2\pi^{N/2}}{\Gamma(N/2)}$

druhý řádek : $A_{N-2} = \frac{2\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})}$

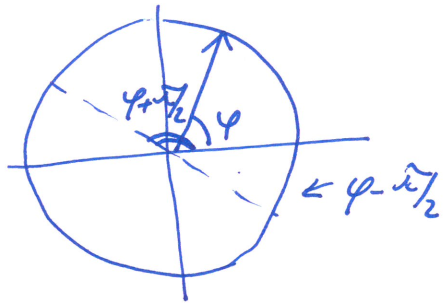
⋮

N-tý řádek : $2 = \frac{2\pi^{1/2}}{\Gamma(1/2)}$

$vol(\Omega_N) = \prod_{j=1}^{N} \frac{2\pi^{j/2}}{\Gamma(j/2)}$

Pro $N=2$ .... $\mathrm{vol}(\Omega_N) \overset{?}{=} \dfrac{2\pi^{1/2}}{\Gamma(1/2)} \cdot \dfrac{2\pi^{2/2}}{\Gamma(2/2)} = 2 \cdot 2\pi = 4\pi$

$$O = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix} : \quad \begin{aligned} O_{11}^2 + O_{12}^2 &= 1 \\ O_{21}^2 + O_{22}^2 &= 1 \\ O_{11}O_{21} + O_{12}O_{22} &= 0 \end{aligned}$$



$$\begin{pmatrix} \cos\varphi, & \sin\varphi \\ \sin\varphi, & -\cos\varphi \end{pmatrix}, \text{ nebo } \begin{pmatrix} \cos\varphi, & \sin\varphi \\ -\sin\varphi, & \cos\varphi \end{pmatrix} \ \ldots\ 2\pi \cdot 2 = 4\pi$$

$\varphi \in [0, 2\pi]$ $\qquad\qquad$ $\varphi \in [0, 2\pi]$

$$\mathrm{vol}(\Omega_2) = \int_{\mathbb{R}^4} \prod_{i,j=1}^{2} dO_{ij}\ \delta\left(\sqrt{O_{11}^2 + O_{12}^2} - 1\right) \delta\left(\sqrt{O_{21}^2 + O_{22}^2} - 1\right) \delta(O_{11}O_{21} + O_{12}O_{22})$$

$O_{11} = r\cos\varphi \qquad O_{21} = R\cos\psi$

$O_{12} = r\sin\varphi \qquad O_{22} = R\sin\psi$

$$\Rightarrow \int_0^{2\pi} d\varphi \int_0^{2\pi} d\psi \int_0^{\infty} dr \int_0^{\infty} dR \ \ r \cdot R \cdot \delta(r-1)\,\delta(R-1)\, \underbrace{\delta(rR(\cos\varphi\cos\psi + \sin\varphi\sin\psi))}_{=\delta(\cos(\varphi-\psi))}$$

$$= \int_0^{2\pi} \int_0^{2\pi} \delta(\cos(\varphi-\psi))\, d\varphi\, d\psi = \int_0^{2\pi} 2\, d\psi = 4\pi.$$

$\forall \psi \ \exists 2\varphi$

Ještě potřebujeme vypočítat Jakobián $J(H,\{x,O\})$

$H = O X O^T$

- Nejprve pro $N=2$

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix} = \begin{pmatrix} \cos\varphi, & \sin\varphi \\ -\sin\varphi, & \cos\varphi \end{pmatrix} \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} \cos\varphi; & -\sin\varphi \\ \sin\varphi; & \cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi, & \sin\varphi \\ -\sin\varphi, & \cos\varphi \end{pmatrix} \begin{pmatrix} x_1\cos\varphi, & -x_1\sin\varphi \\ x_2\sin\varphi, & x_2\cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} x_1\cos^2\varphi + x_2\sin^2\varphi, & (-x_1+x_2)\sin\varphi\cos\varphi \\ (x_2-x_1)\sin\varphi\cos\varphi, & x_1\sin^2\varphi + x_2\cos^2\varphi \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial H_{11}}{\partial x_1}, & \frac{\partial H_{11}}{\partial x_2}, & \frac{\partial H_{11}}{\partial\varphi} \\ \frac{\partial H_{12}}{\partial x_1}, & \frac{\partial H_{12}}{\partial x_2}, & \frac{\partial H_{12}}{\partial\varphi} \\ \frac{\partial H_{22}}{\partial x_1}, & \frac{\partial H_{22}}{\partial x_2}, & \frac{\partial H_{22}}{\partial\varphi} \end{pmatrix} = + \begin{pmatrix} \cos^2\varphi; & \sin^2\varphi; & +2(x_2-x_1)\cos\varphi\sin\varphi \\ -\sin\varphi\cos\varphi; & \sin\varphi\cos\varphi; & (x_2-x_1)(\cos^2\varphi-\sin^2\varphi) \\ \sin^2\varphi; & \cos^2\varphi; & (x_1-x_2)(2\sin\varphi\cos\varphi) \end{pmatrix}$$

$$\det \overset{I+III\to I}{=} (x_2-x_1)\cdot\det\begin{pmatrix} 1; & 1; & 0 \\ -\sin\varphi\cos\varphi, & \sin\varphi\cos\varphi, & \cos^2\varphi-\sin^2\varphi \\ \sin^2\varphi, & \cos^2\varphi, & -2\sin\varphi\cos\varphi \end{pmatrix}$$

$$\overset{II-I\to II}{=} (x_2-x_1)\det\begin{pmatrix} 1; & 0; & 0 \\ -\sin\varphi\cos\varphi; & 2\sin\varphi\cos\varphi, & \cos^2\varphi-\sin^2\varphi \\ \sin^2\varphi, & \cos^2\varphi-\sin^2\varphi, & -2\sin\varphi\cos\varphi \end{pmatrix}$$

$$= (x_2-x_1)\left\{-4\sin^2\varphi\cos^2\varphi - (\cos^2\varphi-\sin^2\varphi)\right\} = (x_2-x_1)\left\{-\cos^4\varphi-\sin^4\varphi-2\sin^2\varphi\cos^2\varphi\right\} = -(x_2-x_1)$$

Obecné $N \geq 2$: neschůdné

odvodíme v řeči diferenciálů

$$f(x+h) = f(x) + \alpha h + o(h) \ldots \qquad \Big| \quad df(x,y) = \alpha dh_1 + \beta dh_2$$

$$df = \alpha dh \quad \ldots \quad \alpha = \frac{\partial f}{\partial x} \qquad \Big| \quad \frac{\partial f}{\partial x} = \alpha, \ \frac{\partial f}{\partial y} = \beta$$

- $f(x+dx) - f(x) = \alpha dx$

$H = \sigma X \sigma^T$

- σ podléhá vazbě $\sigma\sigma^T = \sigma^T\sigma = I$

Tedy $(\sigma + \partial\sigma)(\sigma+\partial\sigma)^T = (\sigma+\partial\sigma)^T(\sigma+\partial\sigma) = I$

$(\sigma+\partial\sigma)(\sigma^T + (\partial\sigma)^T) = (\sigma^T + (\partial\sigma)^T)(\sigma+\partial\sigma) = I \qquad \cdot (\partial\sigma)^T = \partial(\sigma^T)$

$\partial\sigma \cdot \sigma^T + \sigma \cdot (\partial\sigma)^T = \partial\sigma^T \cdot \sigma + \sigma^T \partial\sigma = 0$

$\partial\sigma^T = -\sigma^T(\partial\sigma)\sigma^T$

- $H + \partial H = (\sigma + \partial\sigma)(X + \partial X)(\sigma + \partial\sigma)^T$

$= H + \partial\sigma \cdot X \cdot \sigma^T + \sigma \partial X (\sigma^T) + \sigma X (\partial\sigma)^T$

$\Rightarrow \partial H = \partial\sigma \cdot X \cdot \sigma^T + \sigma \cdot \partial X \cdot \sigma^T + \underbrace{\sigma X (\partial\sigma)^T}_{-\sigma X \sigma^T(\partial\sigma)\sigma^T}$

$= \sigma[\sigma^T \cdot \partial\sigma \cdot X + \partial X - X\sigma^T(\partial\sigma)]\sigma^T$

$\partial\Omega = \sigma^T\partial\sigma$: $\qquad \partial H = \sigma\underbrace{[\partial\Omega \cdot X + \partial X - X\partial\Omega]}_{\partial\hat{H}}\sigma^T$

- ∂H a $\partial\hat{H}$ se liší jen o ortonormální transformaci
- $\partial\Omega$ a $\partial\sigma$ se – " –

Místo ∂H podle $\partial X, \partial O$ budeme uvažovat pouze

$\partial \hat{H}$ podle $\partial X, \partial \Omega$!

$$\frac{d\hat{H}_{ij}}{dx_k} = \begin{cases} 1 & \text{pro } i=j=k \\ 0 & \dots \text{ jinak} \end{cases} \quad ; \quad \frac{d\hat{H}_{ij}}{d\Omega_{kl}} = \begin{cases} 0 & \text{pro } i \neq k, \text{ nebo } j \neq l \\ x_j - x_i & \text{pro } i=k,\ j=l \end{cases}$$

$$\begin{array}{c|ccc|ccc}
 & /dx_1 & \dots & /dx_N & /d\Omega_{12} & \dots & /d\Omega_{N-1,N} \\
\hline
d\hat{H}_{11} & 1 & & 0 & & & \\
\vdots & & \ddots & & & 0 & \\
d\hat{H}_{NN} & 0 & & 1 & & & \\
\hline
d\hat{H}_{12} & & & & x_2 - x_1 & & \\
\vdots & & 0 & & & \ddots & \\
d\hat{H}_{N-1,N} & & & & & & x_N - x_{N-1}
\end{array}$$

$$\Rightarrow \det = \prod_{i<j} x_j - x_i$$

- Pozn. $O^T \partial O$ je antisymetrická:

$$0 = O^T \cdot \partial O + \partial O^T \cdot O = O^T \cdot \partial O + (O^T \partial O)^T.$$

Obecný postup pro $N=2$

$$H \longleftrightarrow (x_1, x_2, \varphi)$$

$$X = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}; \quad O = \begin{pmatrix} \cos\varphi, & +\sin\varphi \\ -\sin\varphi, & \cos\varphi \end{pmatrix}$$

$$\partial X = \begin{pmatrix} dx_1 & 0 \\ 0 & dx_2 \end{pmatrix}; \quad \partial O = \begin{pmatrix} -\sin\varphi, & \cos\varphi \\ -\cos\varphi, & -\sin\varphi \end{pmatrix} d\varphi$$

$$\partial\Omega = O^T \cdot \partial O = \begin{pmatrix} \cos\varphi; & -\sin\varphi \\ \sin\varphi; & \cos\varphi \end{pmatrix} \cdot \begin{pmatrix} -\sin\varphi; & \cos\varphi \\ -\cos\varphi; & -\sin\varphi \end{pmatrix} d\varphi$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} d\varphi$$

$$\partial\hat{H} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} d\varphi + \begin{pmatrix} dx_1 & 0 \\ 0 & dx_2 \end{pmatrix} - \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} d\varphi$$

$$= \begin{pmatrix} dx_1 & (x_2 - x_1)\,d\varphi \\ (x_2 - x_1)\,d\varphi & dx_2 \end{pmatrix} \qquad \begin{pmatrix} \frac{\partial\hat{H}}{\partial x_1} & & \\ & \ddots & \\ & & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & x_2 - x_1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\det = x_1 - x_2$$