Theorem (Lie Product Formula):

For every $A, B \in \mathbb{C}^{m\times m}$ it holds:

$$\underbrace{e^{A+B}}_{X_N^N} = \lim_{N\to\infty} \underbrace{(e^{A/N}\cdot e^{B/N})^N}_{Y_N^N}.$$

Proof: Fix $A, B \in \mathbb{C}^{m\times m}$. Let:

$$X_N = e^{(A+B)/N}; \quad Y_N = e^{A/N}\cdot e^{B/N}$$

in general: $X_N \neq Y_N$

Taylor's expansion: $\bullet\ X_N = I + \frac{A+B}{N} + \underbrace{\frac{(\frac{A+B}{N})^2}{2!} + \dots}_{O(N^{-2})}$ $\quad N \to \infty$

$\bullet\ Y_N = \left[I + \frac{A}{N} + \underbrace{\frac{(A/N)^2}{2} + \dots}_{O(N^{-2})}\right]\cdot\left[I + \frac{B}{N} + O(N^{-2})\right]$

$= I + \frac{A+B}{N} + O(N^{-2}) \qquad \Rightarrow X_N - Y_N = O(N^{-2})$

$\bullet\ e^{A+B} - (e^{A/N}\cdot e^{B/N})^N = X_N^N - Y_N^N$

$= (X_N^N - X_N^{N-1}Y_N) + (X_N^{N-1}Y_N - X_N^{N-2}Y_N^2) + \dots + (X_N Y_N^{N-1} - Y_N^N)$

$= X_N^{N-1}(X_N - Y_N) + X_N^{N-2}(X_N - Y_N)\cdot Y_N + \dots + (X_N - Y_N)Y_N^{N-1}$

$\bullet\ \|X_N^N - Y_N^N\| \le \|X_N - Y_N\|\left(\|X_N\|^{N-1} + \|X_N\|^{N-2}\cdot\|Y_N\| + \dots\right.$ $\quad \|CD\| \le \|C\|\cdot\|D\|$

$\left.\dots + \|Y_N\|^{N-1}\right)$

$\le N\cdot\|X_N - Y_N\|\cdot\max(\|X_N\|, \|Y_N\|)^{N-1}$

$\bullet$ By Taylor's expansion: $\bullet\ \|X_N\| = \|e^{(A+B)/N}\| \le e^{\|(A+B)/N\|} \le e^{\|A/N\|}\cdot e^{\|B/N\|}$

$\boxed{Z: \|e^Z\| = \|\sum_{m=0}^{\infty} \frac{Z^m}{m!}\| \le \sum_{m=0}^{\infty} \frac{\|Z^m\|}{m!} \le \sum_{m=0}^{\infty} \frac{(\|Z\|)^m}{m!} = e^{\|Z\|}.}$

$\bullet\ \|Y_N\| = \|e^{A/N}\cdot e^{B/N}\| \le \|e^{A/N}\|\cdot\|e^{B/N}\| \le e^{\|A/N\|}\cdot e^{\|B/N\|}.$

$\Rightarrow \|X_N^N - Y_N^N\| \le N\cdot O(N^{-2})\cdot\left[e^{\|A\|/N}\cdot e^{\|B\|/N}\right]^{N-1}$

$= \underbrace{N\cdot O(N^{-2})}_{\to 0}\cdot \underbrace{e^{(\|A\|+\|B\|)\cdot\frac{N-1}{N}}}_{\to e^{\|A\|+\|B\|}} \longrightarrow 0$ if $N \to +\infty$ ∎