Golden-Thompson inequality

- Lie Product Formula: $e^{A+B} = \lim\limits_{N\to\infty} \left(e^{A/N}\cdot e^{B/N}\right)^N$
- $\mathrm{tr}(XY) = \mathrm{tr}(YX)$
- $W, Z \in \mathbb{C}^{n\times n}$:

$$|\mathrm{tr}(WZ)| = \left|\sum_{j=1}^{n} (WZ)_{jj}\right| = \left|\sum_{j=1}^{n}\sum_{k=1}^{n} W_{jk}\cdot Z_{kj}\right|$$

$$\leq \left(\sum_{j,k=1}^{n} |W_{jk}|^2\right)^{1/2}\cdot\left(\sum_{j,k=1}^{n} |Z_{kj}|^2\right)^{1/2} \qquad = \|W\|_F\cdot\|Z\|_F$$

$$= \left(\sum_{j,k=1}^{n} W_{jk}\cdot\overline{W_{jk}}\right)^{1/2}\cdot\left(\sum_{j,k=1}^{n} Z_{kj}\cdot\overline{Z_{kj}}\right)^{1/2}$$

$$= \left(\sum_{j,k=1}^{n} W_{jk}\,(W^*)_{kj}\right)^{1/2}\cdot\left(\sum_{k,j=1}^{n} Z_{kj}\cdot(Z^*)_{jk}\right)^{1/2}$$

$$= \left(\sum_{j=1}^{n}\underbrace{\sum_{k=1}^{n} W_{jk}\cdot(W^*)_{kj}}_{(WW^*)_{jj}}\right)^{1/2}\cdot\left(\sum_{k=1}^{n}\underbrace{\sum_{j=1}^{n} Z_{kj}\cdot(Z^*)_{jk}}_{(ZZ^*)_{kk}}\right)^{1/2}$$

$$= \sqrt{\mathrm{tr}(WW^*)\cdot\mathrm{tr}(ZZ^*)}.$$

$$\Rightarrow |\mathrm{tr}(WZ)| \leq \sqrt{\mathrm{tr}(WW^*)\,\mathrm{tr}(ZZ^*)}$$

Theorem: Let $A, B \in \mathbb{C}^{n \times n}$. Then

$$|\operatorname{tr}(e^{A+B})| \leq \operatorname{tr}\underbrace{\left(e^{(A+A^*)/2} \cdot e^{(B+B^*)/2}\right)}_{>0} \qquad (*)$$

[If $A = A^*, B = B^*$: $\underbrace{\operatorname{tr}(e^{A+B})}_{>0} \leq \operatorname{tr}(e^A \cdot e^B)$]

Proof: Fix A, B, put $X = \exp(A/2^N)$, $Y = \exp(B/2^N)$, $N \in \mathbb{N}$.

$$(**): \quad |\operatorname{tr}[(XY)^{2^N}]| \leq \operatorname{tr}\left[(XX^*)^{2^{N-1}} (YY^*)^{2^{N-1}}\right]$$

[Step 1: $(**) \Rightarrow (*)$:

$$\left|\operatorname{tr}\left[\left(e^{A/2^N} \cdot e^{B/2^N}\right)^{2^N}\right]\right| \leq \operatorname{tr}\left[\left(\overbrace{e^{A/2^N}}^{X} \cdot \overbrace{e^{A^*/2^N}}^{X^*}\right)^{2^{N-1}} \cdot \left(\overbrace{e^{B/2^N}}^{Y} \cdot \overbrace{e^{B^*/2^N}}^{Y^*}\right)^{2^{N-1}}\right]$$

$$\left[z: [\exp(z)]^* = \left(\sum_{j=0}^{\infty} \frac{z^j}{j!}\right)^* = \sum_{j=0}^{\infty} \frac{(z^j)^*}{j!} = \sum_{j=0}^{\infty} \frac{(z^*)^j}{j!} = e^{(z^*)}\right.$$

$$\Big|\operatorname{tr}\Big[\underbrace{\left(e^{A/2^N} \cdot e^{B/2^N}\right)^{2^N}}_{\substack{N \to \infty \\ |\operatorname{tr}(e^{A+B})|}}\Big]\Big| \leq \operatorname{tr}\Big[\underbrace{\left(e^{(A/2)/2^{N-1}} \cdot e^{(A^*/2)/2^{N-1}}\right)^{2^{N-1}}}_{e^{A/2 + A^*/2}} \cdot \underbrace{\left(e^{(B/2)/2^{N-1}} \cdot e^{(B^*/2)/2^{N-1}}\right)^{2^{N-1}}}_{e^{B/2+B^*/2}}\Big]$$

$$\Rightarrow |\operatorname{tr}(e^{A+B})| \leq \operatorname{tr}\left(e^{(A+A^*)/2} \cdot e^{(B+B^*)/2}\right) \quad \ldots\ldots (*)$$
]

It remains to prove (**): $\left|\mathrm{tr}\left((XY)^{2^N}\right)\right| \le \mathrm{tr}\left\{(X^*X)^{2^{N-1}}\cdot(YY^*)^{2^{N-1}}\right\}$

We prove (***): $\left|\mathrm{tr}(A_1\cdot A_2\cdots A_{2^N})\right| \le \prod_{j=1}^{2^N}\left(\mathrm{tr}\left[(A_jA_j^*)^{2^{N-1}}\right]\right)^{1/2^N}$

First (***) $\Rightarrow$ (**)

- $A_1=A_2=\dots=A_{2^N}=Z$:

(***) $\Rightarrow$ $\left|\mathrm{tr}(Z^{2^N})\right| \le \prod_{j=1}^{2^N}\left(\mathrm{tr}\left[(ZZ^*)^{2^{N-1}}\right]\right)^{1/2^N} = \mathrm{tr}\left\{(ZZ^*)^{2^{N-1}}\right\}$

.... $|\mathrm{tr}(Z^2)| \le \mathrm{tr}(ZZ^*)$ ... $N=1$

- $Z=XY$

$$\left|\mathrm{tr}\left\{(XY)^{2^N}\right\}\right| \le \mathrm{tr}\left[(XY)(XY)^*\right]^{2^{N-1}} = \mathrm{tr}\left\{(XYY^*X^*)^{2^{N-1}}\right\}$$

trace is cyclic

$$= \mathrm{tr}\left\{(XYY^*X^*)(XYY^*X^*)\cdots(XYY^*X^*)\right\}$$

$$= \mathrm{tr}\left((X^*XYY^*)\cdot(X^*XYY^*)\cdots\right) = \mathrm{tr}\left((\underbrace{X^*XYY^*}_{Z})^{2^{N-1}}\right)$$

$$\le \mathrm{tr}\left[\left((X^*XYY^*)(X^*XYY^*)^*\right)^{2^{N-2}}\right]$$

$$= \mathrm{tr}\left\{(X^*XYY^*\cdot YY^*X^*X)^{2^{N-2}}\right\}$$

$(X^*X)^* = X^*X$

$$= \mathrm{tr}\left\{\left(\underbrace{(X^*X)^2\cdot(YY^*)^2}_{Z}\right)^{2^{N-2}}\right\} \le \mathrm{tr}\left\{(ZZ^*)^{2^{N-3}}\right\}$$

$$\le \mathrm{tr}\left\{\left[\overbrace{(X^*X)^2\cdot(YY^*)^2}^{Z}\cdot\overbrace{\left[(X^*X)^2\cdot(YY^*)^2\right]^*}^{Z^*}\right]^{2^{N-3}}\right\}$$

$$= \mathrm{tr}\left\{\left[(X^*X)^4\cdot(YY^*)^4\right]^{2^{N-3}}\right\} \;\dots\; \le \mathrm{tr}\left\{(X^*X)^{2^{N-1}}\cdot(YY^*)^{2^{N-1}}\right\} \;\text{-- (**)}$$

$$(***):\ \left|tr(A_1 \dots A_{2^m})\right| \leq \prod_{j=1}^{2^m} \left\{ tr\left[(A_j A_j^*)^{2^{m-1}}\right] \right\}^{1/2^m}$$

By induction:

$m=1$: $\left|tr(A_1 \cdot A_2)\right| \leq \sqrt{tr(A_1 \cdot A_1^*) \cdot tr(A_2 \cdot A_2^*)}$ ✓

Ind. step: $m \Rightarrow m+1$

$$\left|tr(A_1 A_2 \dots A_{2^{m+1}})\right| = \left|tr\ \underbrace{(A_1 A_2)(A_3 A_4) \dots (A_{2^{m+1}-1} \cdot A_{2^{m+1}})}_{2^m \text{ matrices}}\right|$$

(***) for m:

$$\leq \left(tr\left[(A_1 A_2)(A_1 A_2)^*\right]^{2^{m-1}}\right)^{1/2^m} \cdot \left\{tr\left[(A_3 A_4)(A_3 A_4)^*\right]^{2^{m-1}}\right\}^{1/2^m} \dots$$

$$= \left\{tr(A_1 A_2 A_2^* A_1^*)^{2^{m-1}}\right\}^{1/2^m} \dots$$

$$= \left\{tr(A_1^* A_1 A_2 A_2^*)^{2^{m-1}}\right\}^{1/2^m} \dots$$

$$= tr\left\{\underbrace{(A_1^* A_1)(A_2 A_2^*)(A_1^* A_1)(A_2 A_2^*) \dots}_{2^m \text{ matrices}}\right\}^{1/2^m} \dots$$

(***) for m

$$\leq \left\{tr\left[(A_1^* A_1)(A_1^* A_1)\right]^{2^{m-1}}\right\}^{2^{m-1}/2^m} \cdot \left\{tr\left[(A_2 A_2^*)(A_2 A_2^*)\right]^{2^{m-1}}\right\}^{2^{m-1}/2^m} \dots \quad \uparrow^{1/2^m}$$

$$= \left\{tr(A_1^* A_1)^{2^m}\right\}^{1/2} \cdot \left\{tr(A_2 A_2^*)^{2^m}\right\}^{1/2} \dots$$

$$\Rightarrow \left|tr(A_1 \dots A_{2^{m+1}})\right| \leq \left\{\left(tr(A_1 A_1^*)^{2^m}\right)^{1/2} \cdot \left(tr(A_2 A_2^*)^{2^m}\right)^{1/2} \dots\right\}^{\frac{1}{2^m}}$$

$$= \prod_{j=1}^{2^{m+1}} \left\{tr\left[(A_j A_j^*)^{2^m}\right]\right\}^{\frac{1}{2^{m+1}}}$$

... the end of (***)

$\Rightarrow$ the end of G.-T. ∎