• Protože $P(s)$ a $P(t)$ komutují pro všechna $s,t>0$

$$P(s)P(t) = P(s+t) = P(t)P(s),$$

komutují i G a $P(t)$

$$P(t)G\pi = P(t)\left[\lim_{h\to 0^+} \frac{P(h)-I}{h}\pi\right] = \lim_{h\to 0^+}\left(\frac{P(h)-I}{h}\right)\left(P(t)\pi\right)$$

$$= G P(t)\pi \ \ldots \ \text{tedy } P_{t_0}' = G P_{t_0} = P_{t_0} G.$$

Příklad: Yuleův proces

• Z každého jedince může v intervalu $(t, t+h]$ vzniknout nový jedinec s pravděpodobností $\lambda h + o(h)$, nezávisle na ostatních

• Jedinci nehynou; X_t je počet jedinců v čase $t\geq 0$; $X_0 = 1$ s.j.

Celkem je tedy $P_{j,j+1}(h) = j\cdot(\lambda h + o(h))(1-\lambda h + o(h))^{j-1}$

$$= j\lambda h + o(h)$$

$$P_{j,j+k}(h) = o(h),\ k\geq 2$$

$$P_{j,j}(h) = (1-\lambda h + o(h))^j = 1 - j\lambda h + o(h)$$

$$\hookrightarrow g_{jj} = \lim_{h\to 0^+} \frac{P_{jj}(h)-1}{h} = \lim_{h\to 0^+}\frac{1-j\lambda h - 1 + o(h)}{h} = -j\lambda$$

$$g_{j,j+1} = \lim_{h\to 0^+}\frac{P_{j,j+1}(h)-0}{h} = j\lambda\ ;\ g_{j,j+k} = 0,\ k\geq 2$$

$$G = \begin{pmatrix} -\lambda, & \lambda, & 0, & \ldots & \\ 0, & -2\lambda, & 2\lambda, & 0, & \ldots \\ 0, & 0, & -3\lambda, & 3\lambda, & 0\ldots \\ & & \ldots & & \end{pmatrix}$$

Rovnice $P' = GP = PG$ lze řešit dvěma způsoby

- algebraicky $P(t) = e^{tG}$, $P(0) = I$
- metodou "vytvořující funkce":

$$p'(t)^T = p(0)^T \cdot P'(t) = p(0)^T P(t) G = p(t)^T G$$

vede na $(p_1'(t), p_2'(t), p_3'(t), \_\_) = (p_1(t), p_2(t), p_3(t), \_\_) \begin{pmatrix} -\lambda, & \lambda, & 0, & 0, & \_ \\ 0, & -2\lambda, & 2\lambda, & 0, & \_ \\ 0, & 0, & -3\lambda, & 3\lambda, & \_ \\ & & \ddots & 0 & \end{pmatrix}$

$$\hookrightarrow p_1'(t) = -\lambda p_1(t) \,;\ p_2'(t) = \lambda p_1(t) - 2\lambda p_2(t), \ldots$$

$$p_j'(t) = \lambda (j-1) p_{j-1}(t) - \lambda j p_j(t), \ j > 1$$

j-tou rovnici vynásobíme s^j a sečteme

$$\sum_{j=1}^{\infty} p_j'(t) s^j = -\lambda \sum_{j=1}^{\infty} j p_j(t) s^j + \lambda \sum_{j=2}^{\infty} (j-1) p_{j-1}(t) s^j$$

$$= -\lambda s \sum_{j=1}^{\infty} j p_j(t) s^{j-1} + \lambda s^2 \sum_{j=1}^{\infty} j p_j(t) s^{j-1}$$

Pokud $\phi(s,t) = \sum_{j=1}^{\infty} p_j(t) s^j$, pak

$$\frac{\partial \phi(s,t)}{\partial t} = -\lambda s \frac{\partial \phi(s,t)}{\partial s} + \lambda s^2 \frac{\partial \phi(s,t)}{\partial s} = \lambda s \frac{\partial \phi(s,t)}{\partial s} \cdot (s-1)$$

$$\phi(s,0) = s$$

Hledáme řešení ve tvaru $\phi(s,t) = F(\varphi(s)\psi(t))$

$$F'(\varphi(s)\psi(t)) \cdot \varphi(s)\psi'(t) = \lambda s(s-1) F'(\varphi(s)\psi(t)) \cdot \varphi'(s)\psi(t)$$

$$\frac{\psi'(t)}{\psi(t)} = \lambda s(s-1) \frac{\varphi'(s)}{\varphi(s)} = \alpha$$

$$\psi(t) = K_1 e^{\alpha t}; \quad \ln(\varphi(s)) + c = \int \frac{\alpha/\lambda}{s(s-1)} ds = \frac{\alpha}{\lambda} \int \left\{ \frac{1}{s-1} - \frac{1}{s} \right\} ds = \frac{\alpha}{\lambda} \ln\left(\frac{s-1}{s}\right)$$

$$\varphi(s) = \left(\frac{s-1}{s}\right)^{\alpha/\lambda} \cdot K_2 \quad \ldots \ \phi(s,t) = F\left(\left(\frac{s-1}{s}\right)^{\alpha/\lambda} e^{\alpha t}\right) \ \ldots \ K_1 \cdot K_2 \text{ do } F$$

$$\Phi(s,t) = F\left(\left(\frac{s-1}{s}\right)^{\alpha/\lambda}\right) = s \;;$$

$$x = \left(\frac{s-1}{s}\right)^{\alpha/\lambda} \ldots \; x^{\lambda/\alpha} = \frac{s-1}{s} = 1-\frac{1}{s} \; \ldots \; \frac{1}{s} = 1 - x^{\lambda/\alpha}$$

$$s = \frac{1}{1-x^{\lambda/\alpha}}$$

$$F(x) = \frac{1}{1-x^{\lambda/\alpha}}$$

$$\phi(s,t) = \frac{1}{1-\left[\left(\frac{s-1}{s}\right)^{\alpha/\lambda} e^{\alpha t}\right]^{\lambda/\alpha}} = \frac{1}{1-\frac{s-1}{s}\cdot e^{\lambda t}} = \frac{s}{s-(s-1)e^{\lambda t}}$$

$$= \frac{s e^{-\lambda t}}{1-s+s e^{-\lambda t}} = s e^{-\lambda t}\sum_{k=0}^{\infty}\left[s(1-e^{-\lambda t})\right]^k = \sum_{k=1}^{\infty} s^k e^{-\lambda t}(1-e^{-\lambda t})^{k-1}$$

$$\Rightarrow P(X_t = k) = p_k(t) = e^{-\lambda t}(1-e^{-\lambda t})^{k-1}.$$

Z $\sum_{k=1}^{\infty} k\alpha^{k-1} = \frac{1}{(1-\alpha)^2}$ płynie $\mathbb{E}X_t = \sum_{k=1}^{\infty} k\,\mathbb{P}(X_t=k)$

$$= \sum_{k=1}^{\infty} k e^{-\lambda t}(1-e^{-\lambda t})^{k-1} = \frac{e^{-\lambda t}}{e^{-2t\lambda}} = e^{t\lambda}.$$