Arithmetic Complexity of Sturmian Words

Tom Hejda, tohe@centrum.cz
based on work of J. Cassaigne and A. Frid

Doppler Institute & Department of Mathematics,
FNSPE, Czech Technical University in Prague

Combinatorics on Words, Hojsova Straz 2010
Notation

- **alphabet** \( \{0, 1\} \)
- **(right) infinite word** \( s = s_0s_1s_2 \cdots \)
- **finite word** \( w = w_0w_1 \cdots w_{n-1}w_n \), length \( n + 1 \)
- **fractional part** of \( x \in \mathbb{R} \) is \( \{x\} = x - \lfloor x \rfloor \).
Complexity Functions

- **factor complexity** $C_u(n + 1) = \# \text{ of “subword” factors}

  \[ \mathcal{L}_u(n + 1) = \{ u_k u_{k+1} u_{k+2} \cdots u_{k+n} | k \geq 0 \} \]

- **Abelian complexity** $C_u^{ab}(n + 1) = \# \text{ of Parikh vectors}

  \[ \{(|w|_0, |w|_1) | w \in \mathcal{L}_u(n + 1)\} \]

- **arithmetic complexity** $C_u^{ar}(n + 1) = \# \text{ of arithmetic factors}

  \[ A_u(n + 1) = \{ u_k u_{k+d} u_{k+2d} \cdots u_{k+nd} | k \geq 0, d \geq 1 \} \]

Example: factors and arit. factors of $u = (01)^\omega$
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- **Abelian complexity** \( C^\text{ab}_u(n + 1) = \# \text{ of Parikh vectors} \)
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  \left\{(|w|_0, |w|_1) | w \in \mathcal{L}_u(n + 1)\right\}
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- **arithmetic complexity** \( C^\text{ar}_u(n + 1) = \# \text{ of arithmetic factors} \)
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  \mathcal{A}_u(n + 1) = \{u_k u_{k+d} u_{k+2d} \cdots u_{k+nd} | k \geq 0, d \geq 1\}
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- **Abelian complexity** \( C_u^{ab}(n + 1) = \# \) of Parikh vectors
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Complexity Functions

- **factor complexity** $C_u(n + 1) = \# \text{ of "subword" factors}$
  
  $$L_u(n + 1) = \{u_k u_{k+1} u_{k+2} \cdots u_{k+n} | k \geq 0\}$$

- **Abelian complexity** $C_{ab}^u(n + 1) = \# \text{ of Parikh vectors}$
  
  $$\{(|w|_0, |w|_1) | w \in L_u(n + 1)\}$$

- **arithmetic complexity** $C_{ar}^u(n + 1) = \# \text{ of arithmetic factors}$
  
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Example: factors and arit. factors of $u = (01)^\omega$
Sturmian Words

Many ways how to define Sturmian words — infinite $u$ is Sturmian, iff,

1. factor complexity satisfies $C_u(n) = n + 1 \quad \forall n \geq 0$
2. $u$ is aperiodic and balanced
3. $u$ is aperiodic and Abelian complexity satisfies $C_{u}^{ab}(n) = 2 \quad \forall n \geq 1$
4. $u$ is a rotation word with irrational slope $\alpha$
   - lower rotation word $s_{\alpha}(\rho) = s_0 s_1 \cdots$; $\alpha, \rho \in [0, 1)$
     $$s_k = \begin{cases} 
1 & \text{if } \{ (k+1)\alpha + \rho \} < \alpha, \\
0 & \text{otherwise},
\end{cases}$$
   - upper rotation word $s'_{\alpha}(\rho) \rightarrow \leq$ instead of $<$
5. $u$ codes irrational 2iet
6. $u$ is mechanical with irrational slope
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Theorem

Let $u$ be Sturmian word. Then its arithmetic complexity satisfies for all $n \geq 1$

$$\frac{n^3}{4\pi^2} + O(n^2) \leq C^a_u(n) \leq \left(\frac{1}{6} + \frac{1}{\pi^2}\right)n^3 + O(n^2).$$

We prove only upper bound (lower bound as well by Frid).
Theorem

Let \( u \) be Sturmian word. Then its arithmetic complexity satisfies for all \( n \geq 1 \)

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We prove only upper bound (lower bound as well by Frid).
Sturmian Words as Mechanical Words

- lower rotation word $s_\alpha(\rho) = s_0s_1 \cdots$, $\alpha, \rho \in [0, 1)$

$$s_k = \begin{cases} 1 & \text{if } \{k\alpha + \rho\} < \alpha, \\ 0 & \text{otherwise}, \end{cases}$$

- from now on, we fix $\alpha \in [0, 1) \setminus \mathbb{Q}$
- we define $w_\alpha(\beta, \gamma, n) = w_0 \cdots w_n$, $\beta, \gamma \in [0, 1)$, length $n + 1$, $0 \leq k \leq n$

$$w_i = \begin{cases} 1 & \text{if } \{i\beta + \gamma\} < \alpha, \\ 0 & \text{otherwise}, \end{cases}$$

Lemma

1. $L_{s_\alpha(\rho)}(n)$ and $A_{s_\alpha(\rho)}(n)$ depends only on $\alpha$ (denote $L_\alpha(n)$, $A_\alpha(n)$)
2. $w_\alpha(\beta, \gamma, n) \in A_\alpha \iff \exists k, d : \beta = \{d\alpha\}, \gamma = \{k\alpha + \rho\}$
3. $A_\alpha = \bigcup_{\beta, \gamma \in [0, 1)} w_\alpha(\beta, \gamma, n)$
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3. $\mathcal{A}_\alpha = \bigcup_{\beta, \gamma \in [0,1)} w_\alpha(\beta, \gamma, n)$
Planar Representation

- $w_\alpha(\beta, \gamma, n) = w_0 \cdots w_n$,

\[ w_i = \begin{cases} 1 & \text{if } \{i \beta + \gamma\} < \alpha, \\ 0 & \text{otherwise}, \end{cases} \quad 0 \leq k \leq n \]

- planar representation
  - line $y = \beta x + \gamma$
  - closest points below the line
  - sequence of $\bullet$ defines $w_\alpha(\beta, \gamma, n)$

- Question (not open):
  Can different sequences of $\bullet$ that came from some lines define the same word?
Planar Representation

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\begin{align*}
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Geometric Dual Method

Dual transformation:
- line \( l \equiv y = \beta x + \gamma \) maps to point \( l^* = (\beta, -\gamma) \)
- point \( p = (\beta, \gamma) \) maps to line \( p^* \equiv y = \beta x - \gamma \)

Lemma
1. \( l^{**} = l \) and \( p^{**} = p \).
2. Point \( p = (a, b) \) is below/above/on line \( l \equiv y = cx + d \) \( \iff \) point \( l^* = (c, -d) \) is below/above/on line \( p^* \equiv y = ax - b \).
3. Lines \( l_1, \ldots, l_k \) intersect in one point \( p \) \( \iff \) points \( l_1^*, \ldots, l_k^* \) lies on the same line \( p^* \).
Dual transformation:

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3. Lines $l_1, \ldots, l_k$ intersect in one point $p \iff$ points $l_1^*, \ldots, l_k^*$ lies on the same line $p^*$. 
• face of arrangement $\mathcal{D}_\alpha(n)$ defines arithmetic factor in $A_\alpha(n + 1)$
• it follows: $C^\text{ar}_\alpha(n + 1) \leq \# \text{ faces of } \mathcal{D}_\alpha(n)$
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- it follows: $C_{\alpha}^{ar}(n + 1) \leq \# \text{ faces of } \mathcal{D}_\alpha(n)$
• face of arrangement $D_\alpha(n)$ defines arithmetic factor in $A_\alpha(n+1)$
• it follows: $C^\text{ar}_\alpha(n+1) \leq \# \text{ faces of } D_\alpha(n)$
Theorem (Euler’s Formula)

Let \((V, E)\) be a planar graph with faces \(F\). Then

\[ \#F = \#E - \#V + 1. \]

4 types of vertices:

1. “boundary” vertices
2. crossings of lines of “0”-type
3. crossings of lines of “1”-type
4. crossings between both types
Counting Faces of $D_\alpha(n)$

Theorem (Euler’s Formula)

Let $(V, E)$ be a planar graph with faces $F$. Then

$$\#F = \#E - \#V + 1.$$
Counting Faces of $D_\alpha(n)$

1. “boundary” vertices
   \[ \# \text{new edges} - \# \text{“boundary” vertices} = O(\# \text{ lines}) = O(n^2) \]

2. crossings of lines of “0”-type
   Bestel, Pocchiola: \( \# \text{ new edges} - \# \text{ crossings} = \frac{1}{\pi^2} n^3 + O(n^2) \)

3. crossings of lines of “1”-type
   Bestel, Pocchiola: \( \# \text{ new edges} - \# \text{ crossings} = \frac{1}{\pi^2} n^3 + O(n^2) \)

4. crossings between both types
   - lines of both types: \( y = \{ ix \} - 1, \quad y = \{ jx - \alpha \} - 1, \quad i, j = 0, \ldots, n \)
   - equation \( \{ ix \} - 1 = \{ jx - \alpha \} - 1 \) has \( |j - i| \) solutions in \([0, 1)\)
   - \( \sum_{i,j=0}^{n} |j - i| = \frac{1}{3} n(n + 1)(n + 2) = \# \ \text{crossings} \)
   - a crossing generates 2 new edges (crossing points are unique)
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5. together

   \[ \#F = \#E - \#V + 1 = \left( \frac{1}{3} + \frac{2}{\pi^2} \right) n^3 + O(n^2) \]
Counting Faces of $D_\alpha(n)$

1. "boundary" vertices
   \[ \# \text{new edges} - \# \text{"boundary" vertices} = O(\# \text{ lines}) = O(n^2) \]

2. crossings of lines of "0"-type
   Bestel, Pocchiola: \[ \# \text{new edges} - \# \text{crossings} = \frac{1}{\pi^2} n^3 + O(n^2) \]

3. crossings of lines of "1"-type
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4. crossings between both types
   - lines of both types: \[ y = \{ix\} - 1, \quad y = \{jx - \alpha\} - 1, \]
     \[ i, j = 0, \ldots, n \]
   - equation \[ \{ix\} - 1 = \{jx - \alpha\} - 1 \] has \( |j - i| \) solutions in \([0, 1)\)
   - \[ \sum_{i,j=0}^{n} |j - i| = \frac{1}{3} n(n+1)(n+2) = \# \text{ crossings} \]
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Symmetry of Faces

what we got vs. what Theorem says

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Question (not open): Can different sequences of \( \bullet \) that came from some lines define the same word?

Answer: Yes, they can.
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Remarks

1. The upper bound is independent of $\alpha$.
2. Both lower and upper bound is $\sim n^3$ (upper is 10.58 larger).
3. The upper bound is satisfied for larger set of words than Sturmian:

$$s_\alpha(\beta, \rho), \quad \beta \notin \mathbb{Q}, \quad s_k = \begin{cases} 1 & \text{if } \{(k + 1)\beta + \rho\} < \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

4. Can be generalized to 3iet with permutation $0 \rightarrow 1$, $1 \rightarrow 2$, $2 \rightarrow 0$. 
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