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Propriétés algébriques et combinatoires des numérations non-standard

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. . . numbers have neither substance, nor meaning, nor qualities. They are nothing but marks, and all that is in them we have put into them by the simple rule of straight succession.

Hermann Weyl
Mathematics and the Laws of Nature, 1959

Contents

Prologue

This thesis was written within the scope of *"Cotutelle agreement"* between two universities: Czech Technical University and Université Paris VII. To accomplish one of the articles of this agreement the author has to present a summary of the construed work in both universities' languages. Accordingly, this prologue gives requested summaries, which are essentially translations of the appropriate parts of the English introduction. A reader who does not speak any of these beautiful languages should therefore — without worry — move on directly to Introduction starting on page 1.

České shrnutí

Tato práce se věnuje numeračním systémům, jejich algebraickým a kombinatorickým vlastnostem a aritmetice s nimi spojené.

Nejznámějšími numeračními systémy jsou tak zvané poziční systémy, což jsou systémy dané celočíselnou bází (základem) $b > 1$ a abecedou cifer (koeficientů) A_b. Celé číslo x se v takové soustavě vyjadřuje jako

$$x = x_k b^k + \cdots + x_2 b^2 + x_1 b + x_0\,,$$

kde $x_i \in A_b$. Námi běžně používaná desítková soustava je příkladem pozičního systému s bází $b = 10$ a $A_{10} = \{0, \ldots, 9\}$.

V této práci studujeme obecnější numerační systémy, známé jako systémy *beta-numerační*. Klíčovým pojmem v této teorii je tzv. *beta-rozvoj*, který zavedl Rényi [94]. Beta-numerační systémy se od běžných pozičních systémů liší především tím, že připouštějí, aby bází bylo libovolné reálné číslo $\beta > 1$.

Pokud zvolíme β celé, dostaneme samozřejmě standardní poziční systém tak jak je popsán výše. Bude-li však β necelé, narazíme při zkoumání vlastností takovýchto soustav na několik neobvyklých a nepravidelných (tj. závisejících na základu β) jevů.

V první řadě zjistíme, že reálná čísla mají v těchto systémech více přípustných reprezentací. Abychom se vyhnuli případným problémům s nejednoznačností, musíme určit jednu reprezentaci, která bude „primus inter pares". Tuto roli hraje největší z reprezentací daného čísla při lexikografickém uspořádání; nazýváme ji *β-rozvojem* a získáme ji pomocí hladového algoritmu.

Analogicky běžným systémům definujeme dvě podmnožiny reálných čísel: množinu $\mathbb{Z}_\beta$ čísel, jejichž rozvoj má prázdnou zlomkovou část a množinu $\mathrm{Fin}(\beta)$ čísel, jejichž rozvoj má konečnou zlomkovou část, tedy od jistého indexu dál jsou všechny koeficienty v rozvoji rovny nule. Množinu $\mathbb{Z}_\beta$ můžeme chápat jako zobecnění množiny celých čísel, její prvky nazýváme *β-celá čísla*.

Tyto dvě množiny nemusí být při obecně zvoleném β uzavřené vůči aritmetickým operacím. Přesněji řečeno množina $\mathbb{Z}_\beta$ není uzavřená vůči aritmetickým operacím pro žádný necelý základ β; množina $\mathrm{Fin}(\beta)$ uzavřená vůči aritmetickým operacím být může a nemusí. V případě, že $\mathrm{Fin}(\beta)$ uzavřená je, řekneme, že β má tak zvanou „Finiteness property" (vlastnost konečnosti). V současné době není známa žádná algebraická charakteristika čísel β, která tuto vlastnost mají.

Poznamenejme, že naprosto zásadní roli v celé teorii β-numerace hrají Pisot čísla. Pisot číslo $\beta > 1$ je algebraické celé číslo, pro které jsou všechny ostatní kořeny jeho minimálního polynomu v absolutní hodnotě ostře menší než jedna. Nejznámějším Pisot číslem je zlatý řez.

Tato práce je rozdělena do pěti kapitol. První kapitola shrnuje potřebné matematické znalosti. Připomínáme pojmy z teorie algebraických čísel a všechny potřebné definice a známé výsledky týkající se kombinatoriky na slovech a konečných automatů.

V kapitole 2 začínáme vlastní studium numeračních systémů založených na β-rozvojích. Napřed zavedeme všechny potřebné pojmy a poté shrneme známé výsledky týkající se oblastí, kterými se budeme zabývat v následujících kapitolách. Jedná se o výsledky týkající se Finiteness property, odhadů maximálního počtu nenulových zlomkových koeficientů, které vznikají při aritmetice β-celých čísel, a složitosti nekonečných slov přidružených k těmto soustavám.

Kapitola 3 se zabývá aritmetickými vlastnostmi β-numeračních soustav. Definujeme zde *minimální zakázané řetězce* a s jejich pomocí odvodíme jednu nutnou podmínku (Tvrzení 3.1.2) a dvě podmínky postačující (Věty 3.1.3 a 3.1.5) pro to, aby β mělo Finteness property.

Poté se zaměříme na výše zmíněný problém zlomkových částí. Jak již víme, množina $\mathbb{Z}_\beta$ není uzavřená vůči aritmetickým operacím. Naším cílem je nalézt hodnoty $L_\oplus(\beta)$ a $L_\otimes(\beta)$, které udávají maximální možnou délku zlomkové části β-rozvoje součtu a součinu dvou β-celých čísel pro dané β.

Nejprve několika různými způsoby použijeme dříve známou metodu (Guimond et al. [69]) a získáme tak horní odhady na $L_\oplus(\beta)$ a $L_\otimes(\beta)$ pro zobecněné Tribonacci číslo, tj. Pisot číslo β s minimálním polynomem $x^3 - mz^2 - x - 1$, kde $m \geq 2$, a pro třídu kubických Pisot čísel β s minimálním polynomem $x^3 - az^2 - bx + 1$, kde $a \geq 2$ a $1 \leq b \leq a - 1$.

Poté diskutujeme nevýhody této metody a uvádíme metodu jinou (Věta 3.3.1), která je schopná v některých problematických případech pomoci. Pokaždé když odvodíme horní odhad na počet zlomkových koeficientů, uvádíme i odhad dolní, získaný pomocí počítačového programu `pisotarith` provádějícího aritmetické operace v Pisot soustavách.

V kapitole 4 studujeme jiný způsob reprezentace čísel, který ačkoli je mírně odlišný od β-rozvojů, je s nimi silně svázaný. Nazýváme tento způsob α-adická reprezentace a, v podstatě řečeno, je to reprezentace pomocí pozičního systému se základem α, kde α

je číslo algebraicky sdružené s nějakým Pisot číslem β.

Na základě známé věty o periodických β-rozvojích dokážeme, že číslo patří do tělesa $\mathbb{Q}(\alpha)$ tehdy a jen tehdy, když má posléze periodický α-adický rozvoj (Věta 4.2.3). Poté zkoumáme rozvoje čísel z okruhu $\mathbb{Z}[\alpha^{-1}]$ v případech, kdy β má Finiteness property. Ve speciálních případech, kdy je β kvadratická Pisot jednotka, navíc diskutujeme jednoznačnost/násobnost α-adických rozvojů prvků $\mathbb{Z}[\alpha^{-1}]$. Odvodíme také algoritmus, pomocí kterého je možné generovat α-adické rozvoje racionálních čísel.

Na konci kapitoly zkoumáme algoritmy pro aritmetické operace v jedné dané soustavě, a to v soustavě jejíž základ je číslo algebraicky sdružené se zlatým řezem τ.

Kapitola 5 se věnuje palindromické složitosti nekonečných aperiodických slov u_β. Tato slova jsou definována jako pevné body substituce přidružené k jednoduchým Parry číslům. Dokážeme nutnou podmínku pro to, aby slovo u_β obsahovalo nekonečně mnoho palindromů (Lemma 5.1.1). Pro systémy, které tuto podmínku splňují odvodníme vztah mezi klasickou a palindromickou složitostí (Věta 5.2.6) a poté kompletně popíšeme množinu palindromů v u_β, její strukturu a vlastnosti.

Résumé français

Ce travail est consacré aux systèmes de numération, aux propriétés algébriques et combinatoires de ces systèmes et à leur arithmétique.

Les systèmes de numération les plus connus sont les systèmes de position, c'est-à-dire les systèmes définis par une base b et un alphabet de chiffres A_b où $b > 1$ est un entier. On représente un entier x dans tel système de numération sous la forme

$$x = x_k b^k + \cdots + x_2 b^2 + x_1 b + x_0 \,,$$

où $x_i \in A_b$. Notre système décimal est un bon exemple du système de position avec $b = 10$ et $A_{10} = \{0, \dots, 9\}$.

L'objet de ce travail est une étude de systèmes de numération plus généraux réunis sous le nom de *beta-numération*, qui se base sur les beta-développements introduits par Rényi [94]. La beta-numération se distingue de la numération en base entière en ce qu'elle admet n'importe quel nombre réel $\beta > 1$ comme base.

Bien sûr si on prend β entier on obtient un système de position classique. Par contre si β n'est pas entier on rencontre plusieurs phénomènes extraordinaires et irréguliers dans ces systèmes.

En premier lieu, un nombre réel peut avoir plusieurs représentations en base β. On associe à tout nombre réel x une représentation canonique, appelé le β-développement de x. Le β-développement d'un nombre x est lexicographiquement plus grand que toute β-représentation de x et il peut être obtenu par l'algorithme glouton.

Tout comme dans les systèmes de numération en base entière on définit deux sous-ensembles de nombres réels : l'ensemble $\mathbb{Z}_\beta$ des nombres tels que leur β-développement a une partie fractionnaire vide (appelé nombres β-entiers) et l'ensemble $\mathrm{Fin}(\beta)$ des nombres tel que leur β-développement a une partie fractionnaire finie.

En général ces deux ensembles ne sont pas stables pour l'addition et la multiplication. Notamment, $\mathbb{Z}_\beta$ n'est stable pour ces opérations que si β est un entier. La stabilité de l'ensemble $\mathrm{Fin}(\beta)$ dépend de β. Dans le cas où $\mathrm{Fin}(\beta)$ est stable on dit que β a la propriété de finitude. Rappelons qu'il n'y a pas de caractéristique algébrique connue des nombres β ayant cette propriété. Notons que les nombres de Pisot (dont l'exemple le plus connu est le nombre d'or) jouent un rôle fondamental dans cette théorie.

Ce travail est organisé en cinq chapitres. Dans le premier, nous rappelons les connaissances mathématiques nécessaires pour cette thèse. On y trouve les notations et les définitions de la théorie algébrique des nombres, de la combinatoire des mots et aussi de la théorie des automates finis.

Dans le deuxième chapitre, nous commençons l'étude de la β-numération. Nous introduisons les notations nécessaires et nous récapitulons des résultats déjà connus pour les domaines qui seront étudiés dans les chapitres suivants. Nous rappelons les résultats concernant la propriété de finitude, les valeurs déjà connues de la fonction de complexité des mots associés à ces systèmes, et les estimations du nombre maximal des chiffres fractionnaires qui apparaissent lorsque on effectue la somme ou le produit de deux nombres β-entiers.

Le chapitre 3 s'occupe des propriétés arithmétiques de la β-numération. Nous dérivons une nouvelle condition nécessaire (Proposition 3.1.2) et deux conditions suffisantes (Théorème 3.1.3 et 3.1.5) pour la propriété de finitude. Ensuite nous nous concentrons sur le problème sus-mentionné des longueurs maximales des parties fractionnaires. Nous nous intéressons aux valeurs $L_\oplus(\beta)$ et $L_\otimes(\beta)$ qui indiquent la longueur maximale de la partie fractionnaire de la somme et du produit de deux β-entiers.

Nous appliquons une méthode connue (Guimond et al. [69]) pour calculer les estimations supérieures de $L_\oplus(\beta)$ et $L_\otimes(\beta)$ dans le cas de nombres de Tribonacci généralisés, c.-à-d. les nombres de Pisot dont le polynôme minimal est de la forme $x^3 - mx^2 - x - 1$, $m \geq 2$, et aussi dans une autre classe de nombres de Pisot avec polynôme minimal de la forme $x^3 - ax^2 - bx + 1$, $a \geq 2$, $1 \leq b \leq a - 1$.

Ensuite nous discutons les désavantages de cette méthode et nous donnons une autre méthode (Théorème 3.3.1) qui peut aider dans certains cas particuliers. Dans tous les cas quand nous trouvons une borne supérieure nous donnons aussi une borne inférieure obtenue à l'aide du programme `pisotarith` (voir Appendice B).

Dans le chapitre 4 nous étudions une autre façon de représenter les nombres. Elle est un peu différente de la β-numération mais y est relié en même temps. Il s'agit de la représentation α-adique, c'est-à-dire du système de position dont la base est un nombre α, qui est conjugué algébrique d'un nombre de Pisot β. Sur la base du résultat connu sur les β-développements périodiques nous montrons qu'un nombre x est un élément du corps $\mathbb{Q}(\alpha)$ si et seulement si son développement α-adique est ultimement périodique à gauche (Théorème 4.2.3).

Ensuite, nous nous intéressons au développement des éléments de l'anneau $\mathbb{Z}[\alpha^{-1}]$, de plus nous examinons soit leur univocité soit leur multiplicité dans le cas où β est un nombre de Pisot quadratique unitaire. On construit aussi un algorithme qui fait des calculs de développements α-adiques de nombres rationnels. En dernier lieu, nous étudions des algorithmes qui réalisent les opérations arithmétiques dans un système particulier dont la base est le conjugué du nombre d'or.

Le chapitre 5 est consacré à la complexité palindromique des mots infinis apériodiques u_β qui sont point fixe d'une substitution associée à un nombre de Parry simple β. Nous montrons une condition nécessaire pour que le mot u_β contiennent un nombre infini de palindromes (Lemme 5.1.1).

Pour les systèmes satisfaisant cette condition, nous donnons une relation entre la complexité en facteurs et la complexité palindromique (Théorème 5.2.6). Nous donnons aussi une description complète de l'ensemble des palindromes, de sa structure et de ses propriétés.

Introduction

This work is devoted to the study of some non-standard numeration systems from an algebraic and combinatorial point of view. Particular emphasis will be put on the algorithms performing arithmetic operations.

All the systems we will be dealing with are cases of the so-called *positional numeration systems*, which term is used to express that a number in such a system is represented by an ordered set of characters where the value of the character depends on the position. The most common representatives of these systems are systems given by an integer base $b > 1$ and by an alphabet of digits A_b. An integer x is in such a system expressed as

$$x = x_k b^k + \cdots + x_2 b^2 + x_1 b + x_0 \, ,$$

where the coefficients x_i are elements of the alphabet A_b. Our conventional decimal system is a particular case with $b = 10$ and $A_{10} = \{0, 1, \ldots, 9\}$.

Even though the decimal system is nearly the only one system used in our everyday lives, the history shows that it has not always been the case [102]. Other variants and slight modifications of positional system given above were used in mankind's history, e.g. the sexagesimal system (base 60) used by Babylonians, see Barrow [24]:

> *"... a positional numeration system appeared for the first time in Babylon about 3000 B.C. It was created as an extension of the old additive system with the base 60 so that it incorporated a positional information... this advance was not without problems, Babylonian system was in fact a hybrid of a positional and an additive system..."*

or a system with base 20 used by Maya Indians, see Knuth [78]:

> *"Fixed-point positional notation was apparently developed first by Maya Indians in central America 2000 years ago; their base 20 system was highly developed. But Spanish conquerors destroyed nearly all of the Maya books on history and science, so we do not know how sophisticated they had become at arithmetic."*

Some remains of these systems can be observed even nowadays — think over the way we measure time or angles.

The third independent invention of a positional numeration system was the Hindu-Arabic notation system (decimal system), which took place in India about 600 A.D.

That system eventually became the international standard for numeration. The first actual written zero as we know it today also appeared in India.

Note that the "leading role" of the decimal system is not true in at least one area of human activities, namely the area of computers, which is, for electronic reasons, internally addicted to the binary system.

All the systems whose base is a positive integer are essentially the same, their algebraic and arithmetic properties are all alike, not going against the "common sense" and corresponding to our experience with the decimal system. However, even in this domain interesting facts were observed in the past. For instance, addition in the usual binary system (base $b = 2$, alphabet $A_2 = \{0, 1\}$), takes a time proportional to the size of the data. But if we permit signed coefficients, taken in the alphabet $A = \{-1, 0, 1\}$, addition becomes realizable in parallel in a time independent of the size of the data, see Avizienis [18]. The benefits of using signed coefficients have already been observed by Cauchy [49]:

> *"Let us assume that in the notation of a number we place a sign — above a digit corresponding to a certain order, to express that this digit is taken with the minus sign."*
>
> *"When numbers are expressed as we have described before, [...] arithmetical operations will become much more simpler."*

However, as we have pointed out before, since all the systems having for base a positive integer are essentially the same, new interesting questions had not arisen until one considered some generalizations. Let us recall some of them briefly.

Mixed radix representations are obtained by taking a sequence of integers $(a_n)_{n\geq 0}$ such that $a_0 = 1$ and $a_n > 1$ for all $n \geq 1$. We then derive from $(a_n)_{n\geq 0}$ another sequence $(u_n)_{n\geq 0}$ defined by partial products of $(a_n)_{n\geq 0}$ by setting $u_n = a_0 a_1 \cdots a_n$. A positive integer N is represented as

$$N = \sum_{i=0}^{k} d_i u_i \,.$$

If the digits d_i are taken such that $0 \leq d_i < a_{i+1}$, the representation of any positive integer N is unique. This system was for example used to give a constructive proof of the generalized Chinese Reminder Theorem by Fraenkel [58].

The *factorial numeration system*, used e.g. by Lehmer (see Chapter 1 of [28]) for ranking permutations, is essentially a special case of mixed radix representation, obtained by taking $a_n = n + 1$ and hence having $u_n = (n + 1)!$ (here-from comes the name of the system).

Finally, there is a family of systems whose basis is given as a recurrent sequence $(u_n)_{n\geq 0}$, a famous example of which is the numeration system given by the sequence of Fibonacci numbers $(F_n)_{n\geq 0}$, where $F_0 = 1$, $F_1 = 2$ and $F_{n+2} = F_{n+1} + F_n$ for all non-negative n. By a result of Zeckendorf [107][1] we know that every positive integer

[1] This result is dated 1939, but was not published by Zeckendorf until 1972.

N can be represented in this system with coefficients in the alphabet $\{0, 1\}$ and that the representation is unique if we refrain from using two consecutive Fibonacci numbers anywhere in a representation, and if there are no leading zeroes.

All these systems are basically special cases of a general numeration system given by a strictly increasing infinite sequence of positive integers $(u_n)_{n\geq 0}$ whose first element is equal to one, as studied by Fraenkel [59]. Concerning representations of positive integers in such system, we have the following result.

Theorem ([59]). *Let $1 = u_0 < u_1 < u_2 < \cdots$ be a sequence of integers. Then any non-negative integer N has precisely one representation in the system given by $(u_n)_{n\geq 0}$ of the form $N = \sum_{i=0}^{k} d_i u_i$, where the d_i are non-negative digits satisfying $u_0 d_0 + \cdots + u_n d_n < u_{n+1}$ for all $n \geq 0$, and $d_k \neq 0$.*

So far, all the systems we were describing were designated for the representation of integers. However, it is not a difficult task to modify them to represent real numbers.

In the case of a system with an integer base b this modification is quite straightforward — one only needs to use negative powers of b. A real number x is then represented as

$$x = x_k b^k + \cdots + x_1 b + x_0 + x_{-1} b^{-1} + x_{-2} b^{-2} + \cdots .$$

Another example of a system representing real numbers is the so-called *Cantor numeration system*, based on the following theorem due to Cantor.

Theorem. *Let $(q_k)_{k=1}^{\infty}$ be a sequence of integers greater than one. Then any real number x can be uniquely expressed in the form*

$$x = c_0 + \sum_{k=1}^{\infty} \frac{c_k}{q_1 q_2 \cdots q_k} ,$$

where $c_0 \in \mathbb{Z}$ and c_k are integers such that $0 \leq c_k < q_k$ for $k = 1, 2, \ldots$ and $c_k < q_k - 1$ holds for an infinite number of indices k.

One can see these Cantor expansions as a generalization either of the mixed radix representation or of the positional system with base b for real numbers. Indeed, if $q_k = b \geq 2$ for all $k = 1, 2, \ldots$ we obtain nothing but the previously described positional system with the base b.

Cantor expansion have been used for example by Claude et al. [45] to decide the randomness of sequences with letters taken from different alphabets (i.e. from alphabets of different cardinality).

Numeration systems studied in this thesis are yet another generalization of the positional system customized to the representation of real numbers. They are based on the so-called *β-expansions*, introduced by Rényi [94]. The base in this theory is, in general, a real number $\beta > 1$. Indeed, if β is an integer, we get the standard positional system with an integer base. On the other hand, when β is not an integer new phenomena arise.

There is an algorithm due to Rényi [94], the so-called greedy algorithm, which allows to give a representation in a base $\beta > 1$ (called the β-expansion) to any non-negative real number. The digits (or coefficients) are integers in the interval $[0, \beta)$.

The first important property of this system is that there might exist other representations of a number, on the same canonical alphabet of digits. The representation given by the greedy algorithm is the greatest one in the radix order of all the β-representations of the same number. The study of the set of β-expansions has been carried out in many contributions, see in particular [1, 37, 40, 61, 65, 74, 87, 94, 98].

In this work, we will be concerned with the set $\mathbb{Z}_\beta$ of real numbers x such that the β-expansion of $|x|$ uses only non-negative powers of β. These numbers are called *β-integers*, and have been extensively studied as they appear in several fields, such that for instance mathematical formalization of quasicrystals, i.e. non-crystallographic materials displaying long-range order, since they define one-dimensional Delaunay sets with finite local complexity.

The first quasicrystal was discovered in 1984 [100]: it is a solid structure presenting a local symmetry of order 5, i.e. a local invariance under rotation of $2\pi/5$, and it is linked to an irrational number — the golden ratio τ — and to the Fibonacci substitution. The Fibonacci substitution φ_τ, is the canonical substitution associated with τ-numeration system, given by

$$0 \mapsto 01, \ 1 \mapsto 0\,.$$

It defines a quasiperiodic self-similar tiling of the positive real line, and it is a historical model of a one-dimensional quasicrystal. The description and the properties of this tiling use a τ-numeration system.

A more general theory has been elaborated with Pisot numbers for base, see [23, 44]. In this formalization, the set $\mathbb{Z}_\beta$ of β-integers labels the nodes of the quasi-periodic self-similar tiling associated with β.

Note that so far, all the quasicrystals discovered by physicists present local symmetry of order 5 or 10, 8, and 12, and are modelled using quadratic Pisot units, namely the golden ratio for order 5 or 10, $1+\sqrt{2}$ for order 8, and $2+\sqrt{3}$ for order 12.

Another important set is the set $\mathrm{Fin}(\beta)$ of real numbers x such that x has a finite β-expansion.

In general, the sets $\mathbb{Z}_\beta$ and $\mathrm{Fin}(\beta)$ are not closed under arithmetical operations. More precisely, when β is not an integer, $\mathbb{Z}_\beta$ is not closed and the closeness of $\mathrm{Fin}(\beta)$ under these operations depends on the base β.

When $\mathrm{Fin}(\beta) = \mathbb{Z}[\beta^{-1}]$, the number β is said to satisfy the *Finiteness property (F)*. It has been showed [65] that Property (F) implies that β is a Pisot number. Property (F) plays a significant role in in several domains of β-numeration, and hence this result stresses the importance of Pisot numbers in this theory.

Let us recall at least two reasons why we are so interested in Property (F). First it is an interesting property from the arithmetic point of view, which amounts to the question of whether $\mathrm{Fin}(\beta)$ has a ring structure or not, and, therefore, it characterizes the arithmetic simplicity of a given β-numeration system.

Secondly, it is connected with the fractal tiling generated by Pisot numbers. The

β-expansions when β is a Pisot number have a close connection with a tiling of the Euclidean space. Rauzy [93] constructed a domain with a fractal boundary connected with the Pisot number β whose minimal polynomial is $x^3 - x^2 - x - 1$. Later, Thurston [104] gave a construction of such a tiling for a general Pisot number. These results were later extended by other authors [72, 77].

In this framework, Property (F) was shown to be a sufficient condition for an important property of the fractal tiling: say that β is a Pisot number of degree m satisfying Property (F). Then the set of inner points of $\mathcal{T}_\omega$ — the domain given by Rauzy's and Thurston's construction — is dense in $\mathcal{T}_\omega$ and its boundary is nowhere dense in $\mathbb{R}^{m-1}$, see [2, 6]. This means that $\mathcal{T}_\omega$ is a *tile*.

This work is organized into six chapters. The first one is a preliminary chapter summarizing all necessary mathematical background. We recall there notions from the theory of algebraic numbers, we present definitions and results from the field of combinatorics on words, which are necessary especially for the fifth chapter, and also from the domain of finite automata, which is mostly needed for discussions of arithmetics in given numeration systems.

In Chapter 2 we begin the study of numeration systems based on β-expansions. After introducing β-expansions and all related notions, we give a survey of known results on topics which we will be dealing with in subsequent chapters. This comprises in particular the arithmetical properties of β-numeration systems, namely the Finiteness property and the estimates on the number of fractional digits arising under addition and multiplication of β-integers. We also recall the combinatorial properties, that is the study of the subword complexity of infinite words associated with β-numeration systems.

In Chapter 3 we continue the study of the arithmetic issues of β-numeration systems. At first we address the Finiteness property. We define a notion of *minimal forbidden words* and we give one necessary (Proposition 3.1.2) and two sufficient conditions (Theorem 3.1.3 and 3.1.5) for the Finiteness property in terms of these minimal forbidden words.

Then we turn ourselves to the arithmetics of elements in $\mathbb{Z}_\beta$. As we have said before this set is not closed under arithmetic operations for a non-integer base β. However, the β-expansion of the sum (or the product) of two β-integers may have only a finite number of fractional digits. We are interested in finding the maximal length of such an arising fractional part for a given base β. These maxima are denoted by $L_\oplus(\beta)$ and $L_\otimes(\beta)$ respectively.

First, we address a method to estimate these maxima due to Guimond at al. [69]. We apply it to the generalized Tribonacci number, i.e. to the algebraic integer β with minimal polynomial $x^3 - mx^2 - x - 1$, $m \geq 2$. We obtain the following results

$$\begin{aligned} 5 \leq L_\oplus(\beta) \leq 6 \qquad & \text{for } m = 2\,, \\ 4 \leq L_\oplus(\beta) \leq 5 \qquad & \text{for } m \geq 3\,, \\ 4 \leq L_\otimes(\beta) \leq 6 \qquad & \text{for } m \geq 2\,. \end{aligned}$$

Then we apply it to a class of totally real cubic Pisot units β, whose minimal polynomial is of the form $x^3 - ax^2 - b + 1$, where $a \geq 2$ and $1 \leq b \leq a-1$ and we obtain the following upper estimates

$$L_\oplus(\beta) \leq 2 \qquad \text{for all } a \geq a_0 \text{ for some } a_0\,,$$
$$L_\otimes(\beta) \leq 3 \qquad \text{for } a \geq 2\,.$$

We also find exact values for the "boundary cases" of this class, that is, for $b = 1$ and for $b = a - 1$

$$\begin{array}{lll} \text{for } b = 1,\ a \geq 3 & L_\oplus(\beta) = 2 \quad \text{and} & L_\otimes(\beta) = 3\,, \\ \text{for } b = a-1,\ a \geq 3 & L_\oplus(\beta) = 1 \quad \text{and} & L_\otimes(\beta) = 2\,. \end{array}$$

Then we discuss weaknesses of this method and we develop another one (Theorem 3.3.1), partly solving these issues. As an example illustrating the second method we propose a cubic Pisot number β with minimal polynomial $x^3 - 25x^2 - 15x - 2$ and we find exact values

$$L_\oplus(\beta) = 5 \qquad \text{and} \qquad L_\otimes(\beta) = 7\,,$$

for this number β.

In all cases, when we estimate an upper bound of a number of fractional digits, we also provide a lower bound obtained by means of the computer program `pisotarith` — the program performing arithmetics in β-numeration systems [11]. Naturally, we try to make the gap between these two bounds as small as possible.

The last section of this chapter is devoted to the algorithm for the addition of eventually periodic β-expansions in a general Pisot base. We describe the algorithm used in the program `pisotarith` and even though we did not succeed in providing a proof of the correctness of the algorithm we at least discuss ideas and directions of a possible proof.

In Chapter 4 we study another way of representation of numbers, different from the representations based on β-expansions, but strongly connected with them. It is called the α-adic representation and it is a representation of complex (or real) numbers in the positional numeration system with the base α, where α is an algebraic conjugate of a Pisot number β.

Based on a result of Bertrand and Schmidt, we prove that a number belongs to $\mathbb{Q}(\alpha)$ if and only if it has an eventually periodic α-adic expansion (Theorem 4.2.3). Then we consider α-adic expansions of elements of the extension ring $\mathbb{Z}[\alpha^{-1}]$ when β satisfies the Finiteness property (F). In the particular case that β is a quadratic Pisot unit, we inspect the unicity and/or multiplicity of α-adic expansions of elements of $\mathbb{Z}[\alpha^{-1}]$. We also provide an algorithm to generate α-adic expansions of rational numbers. Finally, we propose methods to perform arithmetic operations in one particular α-adic system, namely in the system whose base is given by the conjugate of the golden mean τ.

Chapter 5 is devoted to the study of palindromic structure of infinite aperiodic words u_β, which are fixed points of substitutions associated to simple Parry numbers. Palindromic complexity of an infinite word is strongly related to its factor complexity and hence this chapter extends the paper of Frougny et al. [63] studying the factor complexity of u_β.

We first show a necessary condition for the word u_β to contain infinitely many palindromes. Numbers β satisfying this condition have been introduced and studied in [60] from the point of view of linear numeration systems. Confluent linear numeration systems are exactly those for which there is no propagation of the carry to the right in the process of normalization, which consists of transforming a non-admissible representation on the canonical alphabet of a number into the admissible β-expansion of that number. Such a number β is known to be a Pisot number, and will be called a *confluent* Pisot number.

Then we determine the palindromic complexity of u_β when β is a confluent Pisot number, that is $\mathcal{P}(n)$, the number of palindromes in u_β of length n. In the description of $\mathcal{P}(n)$ we use the notions introduced in [63] for the factor complexity. The connection of the factor and palindromic complexity is not surprising. For example, in [9] the authors give an upper estimate of the palindromic complexity $\mathcal{P}(n)$ in terms of the factor complexity $\mathcal{C}(n)$.

We show that if the length of palindromes is not bounded, which is equivalent to $\limsup_{n\to\infty} \mathcal{P}(n) > 0$, then

$$\mathcal{P}(n+1) + \mathcal{P}(n) = \mathcal{C}(n+1) - \mathcal{C}(n) + 2\,, \qquad \text{for} \quad n \in \mathbb{N}\,.$$

We then give a complete description of the set of palindromes, its structure and properties. The exact palindromic complexity of the word u_β is given in Theorem 5.5.1.

In the last part of the chapter we study the occurrence of palindromes of an arbitrary length in the prefixes of the word u_β, when β is a confluent Pisot number.

Chapter 1

Preliminaries

In this chapter we provide an introduction to several fields of mathematics necessary in the rest of the work. We also establish a unified notation for most of the classical notions appearing elsewhere in the thesis.

Since the thesis is devoted to the study of algebraical and combinatorial properties of non-standard numeration systems, namely systems where the base is an algebraic number, we start with notions from algebraic number theory (Section 1.1), then we present definitions and results from combinatorics on words (Section 1.2) and also from the theory of finite automata (Section 1.3), which will be needed during the discussions of arithmetics in the studied numeration systems.

As usual, we use the following notations: $\mathbb{R}$ denotes the set of *real numbers*, $\mathbb{Q}$ denotes the set of *rational numbers*, $\mathbb{Z}$ denotes the set of *integers*, $\mathbb{C}$ denotes the set of *complex numbers* and $\mathbb{N}$ denotes the set of *non-negative integers*. The cardinality of a finite set S is denoted by $\#S$.

Let R be a ring. A polynomial ring $R[X]$ over R is defined to be the set of all polynomials of the form $a_0 + a_1X + a_2X^2 + \cdots a_nX^n$, where $a_i \in R$.

1.1 Algebraic numbers

In this section we essentially follow a classical textbook by Stewart and Tall [103].

We say that a complex number $\alpha \in \mathbb{C}$ is an *algebraic number* if it is algebraic over $\mathbb{Q}$, that is, if it is a root of a non-zero monic polynomial with coefficients in $\mathbb{Q}$. Moreover, a complex number α is said to be an *algebraic integer* if there exists a monic polynomial with integer coefficients, say $g \in \mathbb{Z}[X]$, such that $g(\alpha) = 0$. In other words

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0\,,$$

where $a_i \in \mathbb{Z}$ for all i.

Let α be an algebraic number. Then there exists a monic polynomial $f \in \mathbb{Q}[X]$ of least degree such that $f(\alpha) = 0$. This polynomial is uniquely determined, it is called the *minimal polynomial* of α and its degree is said to be the *degree* of α. If the constant-term a_0 of the minimal polynomial of an algebraic integer α is equal to ± 1, α is said to be a *unit*.

Note that the set of all algebraic numbers forms a subfield of the field of complex numbers $\mathbb{C}$, usually denoted by $\mathbb{A}$.

Let $\alpha \in \mathbb{C}$ be an algebraic number of degree n. Let us denote by $\mathbb{Z}[\alpha]$ the minimal subring of $\mathbb{C}$ containing α and $\mathbb{Z}$ and by $\mathbb{Q}(\alpha)$ the minimal subfield of $\mathbb{C}$ containing α and $\mathbb{Q}$. The following proposition gives an explicit description of this field.

Proposition 1.1.1. *Let α be an algebraic integer of degree n. Then*

$$\mathbb{Q}(\alpha) = \{a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{n-1}\alpha^{n-1} \mid a_0, \ldots, a_{n-1} \in \mathbb{Q}\}.$$

Given $\alpha \in \mathbb{A}$ of degree n, its minimal polynomial $f \in \mathbb{Q}[X]$ factors over $\mathbb{C}$ as

$$f(X) = \prod_{j=1}^{n} (X - \alpha^{(j)}),$$

where we put $\alpha = \alpha^{(1)}$. The numbers $\alpha^{(2)}, \ldots, \alpha^{(n)}$ are the *algebraic conjugates* of α. They are all algebraic numbers with the same minimal polynomial f, and, moreover, they are all distinct.

Since $\alpha^{(2)}, \ldots, \alpha^{(n)}$ are also algebraic numbers, they naturally generate their own fields $\mathbb{Q}(\alpha^{(j)})$. These fields are very similar to $\mathbb{Q}(\alpha)$ (and so one to each other). In fact they are isomorphic under isomorphisms induced by assignments $\alpha \mapsto \alpha^{(j)}$. Formally, one may define an isomorphism $\varrho_j : \mathbb{Q}(\alpha) \to \mathbb{Q}(\alpha^{(j)})$ by setting $\varrho_j(g(\alpha)) = g(\alpha^{(j)})$, where $g \in \mathbb{Q}[X]$. In the case that we consider only one resp. two conjugates of α we use instead of $\alpha^{(2)}$, resp. $\alpha^{(3)}$ the notation α', resp. α'', and similarly we use g', resp. g'' for $g(\alpha^{(2)})$, resp. $g(\alpha^{(3)})$.

There are several classes of algebraic integers that play an important role in number theory and its applications. The classification of an algebraic integer into one of these classes is derived from the modulus of its algebraic conjugates. An algebraic integer $\alpha > 1$ is said to be

- a *Pisot* number if $\max_{1<j\leq n} |\alpha^{(j)}| < 1$,
- a *Salem* number if $\max_{1<j\leq n} |\alpha^{(j)}| = 1$,
- a *Perron* number if $\max_{1<j\leq n} |\alpha^{(j)}| < \alpha$.

We conclude this section with a short note on extension rings. Let us consider extension rings $\mathbb{Z}[\beta]$, $\mathbb{Z}[\beta^{-1}]$ and $\mathbb{Z}[\beta, \beta^{-1}]$. Obviously, $\mathbb{Z}[\beta] \subset \mathbb{Z}[\beta, \beta^{-1}]$ and $\mathbb{Z}[\beta^{-1}] \subset \mathbb{Z}[\beta, \beta^{-1}]$. Moreover,

- if β is an algebraic integer, then $\mathbb{Z}[\beta, \beta^{-1}] \subset \mathbb{Z}[\beta^{-1}]$,
- if β is an algebraic unit, then $\mathbb{Z}[\beta, \beta^{-1}] \subset \mathbb{Z}[\beta]$ and so $\mathbb{Z}[\beta] = \mathbb{Z}[\beta^{-1}]$.

1.2 Combinatorics on words

For a survey of various aspects of combinatorics on words the reader is referred to the series of Lothaire's books [80, 81], more details on substitutions can be found in Pytheas Fogg [90].

1.2.1 Finite and infinite words

Let A be an ordered finite set, called *alphabet*. Its elements, most of the time denoted either by digits $\{0, 1, \dots, d-1\}$ or by characters $\{a_0, a_1, \dots, a_{d-1}\}$, are called *letters*. A *word* $w = w_1 w_2 \cdots w_n$ is a finite string of letters in A, the length n of a word w is denoted by $|w|$. The empty word is denoted by ε. Ordinarily, we define a *concatenation* of two words $u = u_0 u_1 \cdots u_k$, and $v = v_0 v_1 \cdots v_l$ by $uv = u_0 u_1 \cdots u_k v_0 v_1 \cdots v_l$. The set of all (finite) words over an alphabet A is denoted by A^*. This set with the concatenation as a binary operation and with the empty word as an identity forms a monoid, called the *free monoid* generated by A.

An infinite sequence $u = (u_n)_{n\in\mathbb{N}} = u_0 u_1 u_2 \cdots$ of letters in A is called a *right infinite word*. The set of infinite words, is denoted by $A^{\mathbb{N}}$. A word $u \in A^{\mathbb{N}}$ is said to be *eventually periodic* if it is of the form $u = vz^{\omega}$, where $v, z \in A^*$, $z \neq \varepsilon$ and $z^{\omega} = zzz\cdots$.

A *factor* of a (finite or infinite) word w is a finite word u such that $w = w_1 u w_2$ for some words w_1, w_2. If $w_1 = \varepsilon$ then u is called a *prefix* of w, if moreover $w_2 \neq \varepsilon$ then u is called a *proper prefix* of w. Similarly if $w_2 = \varepsilon$ then u is a *suffix* of w, and it is called *proper suffix* if in addition $w_1 \neq \varepsilon$. For an infinite word u we denote by $\mathcal{L}_n(u)$ the set of its factors of length n. The *language* of an infinite word u is defined as

$$\mathcal{L}(u) := \bigcup_{n\in\mathbb{N}} \mathcal{L}_n(u)\,.$$

An infinite word $u = (u_n)_{n\geq 0}$ is said to be *recurrent* if each of its factors occurs infinitely often; formally, for any factor $w = w_0 \cdots w_n$ of u the set $\{i \in \mathbb{N} \mid u_i \cdots u_{i+n} = w\}$ of *occurrences* of w is infinite. Moreover, if the set of occurrences has bounded gaps for any factor w, then u is called *uniformly recurrent* (or *minimal*). Equivalently, we say that u is uniformly recurrent if for any n there exists $R(n) > 0$ such that for any $i \in \mathbb{N}$ the set of factors of length n in the finite word $u_i u_{i+1} \cdots u_{i+R(n)}$ coincides with $\mathcal{L}_n(u)$.

The *lexicographic order*, sometimes called *alphabetic order*, on words is defined as follows. For two words u, v, we say that u is less in lexicographic order than v, denoted by $u <_{\text{lex}} v$, if u is a proper prefix of v or if there exist factorizations $u = wau'$ and $v = wbv'$ with $a, b \in A$ and $a < b$.

In the sequel, we will be using the three following operations on words.

- The *shift operator* σ acts on $A^{\mathbb{N}}$ by $\sigma(u_0 u_1 \cdots) = u_1 u_2 \cdots$.
- For a word $w = w_1 \cdots w_k w_{k+1} \cdots$ with a prefix $v = w_1 \cdots w_k$ we define $v^{-1}w := w_{k+1} w_{k+2} \cdots$.

- On the set A^* one defines the operation $\sim$ which to a finite word $w = w_1 \cdots w_n$ associates $\widetilde{w} = w_n \cdots w_1$, called the *reversal* of w. A finite word $w \in A^*$ for which $w = \widetilde{w}$ is called a *palindrome*.

All notions so far defined for right infinite words can be analogously introduced for *left infinite* words $u \in A^{\mathbb{N}}$, written in the form $u = \cdots u_2 u_1 u_0$, and for *two-sided infinite* words $u = \cdots u_{-2} u_{-1} u_0 u_1 u_2 \cdots$, the set of two-sided infinite words is denoted by $A^{\mathbb{Z}}$. If for a two-sided infinite words the position of the letter indexed by 0 is important, we introduce *pointed* two-sided infinite words, $u = \cdots u_{-2} u_{-1} | u_0 u_1 u_2 \cdots$.

1.2.2 Factor complexity and palindromic complexity

The *factor complexity* function is a classical way to quantify the diversity of an infinite sequence. The complexity function $\mathcal{C}_u : \mathbb{N} \to \mathbb{N}$ of a sequence u associates to any positive integer n the cardinality $\#\mathcal{L}_n(u)$, that is

$$\mathcal{C}_u(n) := \#\{w \in \mathcal{L}(u) \mid |w| = n\}\,.$$

Obviously, the complexity is a non-decreasing function and we can bound its values as $\mathcal{C}_u(1) \leq \mathcal{C}_u(n) \leq (\#A)^n$ for any positive integer n. The existence of sequences admitting both extremal values of the complexity is demonstrated by the following examples.

Example. A periodic sequence over the binary alphabet $u = 010101 \cdots$ has complexity $\mathcal{C}_u(n) = 2$ for every n.

Example. The Champernowe sequence $v = 011011100101110 \cdots$, obtained by concatenating the binary expansions of integers $0, 1, 2, \ldots$, has complexity $\mathcal{C}_u(n) = 2^n$ for every n.

Concerning the ability of the complexity function to measure the disorder of a sequence we have the following result on the complexity of periodic sequences by Coven and Hedlund.

Proposition 1.2.1 ([51]). *Let $u \in A^{\mathbb{N}}$. Then the following conditions are equivalent*

(i) u is eventually periodic,

(ii) $\mathcal{C}_u(n)$ is a bounded function,

(iii) there exists an integer n such that $\mathcal{C}_u(n) \leq n$.

By the previous Proposition the lowest possible value of the complexity for a non (eventually) periodic sequence is $\mathcal{C}_u(n) = n + 1$ for all $n \geq 1$. This is probably one of the facts that gave rise to a quite extensive study of the family of sequences having this precise value of complexity, see for example Chapter 2 in [81]. These sequences are known by the name of *Sturmian sequences (words)*, given to them by Morse and

Hedlund [86]. Since for a Sturmian word $\mathcal{C}_u(1) = 2$, any Sturmian word is necessarily over a two letter alphabet.

There is a natural question, evoked by Proposition 1.2.1: what are the functions from $\mathbb{N}$ to $\mathbb{N}$ which may be the complexity function of a sequence u. No definite answer to this question is known, but a couple of necessary and sufficient conditions has been already found. Some sufficient conditions established by listing examples can be found in [57], the *necessary conditions* are

- $\mathcal{C}_u(n)$ is non-decreasing,
- $\mathcal{C}_u(n+m) \leq \mathcal{C}_u(n)\mathcal{C}_u(m)$ for any $n, m \in \mathbb{N}$,
- whenever $\mathcal{C}_u(n+1) = \mathcal{C}_u(n)$ for some n then $\mathcal{C}_u(n)$ is bounded,
- $\mathcal{C}_u(n) \leq (\#A)^n$, if $\mathcal{C}_u(n) < (\#A)^n$ for some n, then there exists a real number $\kappa < \#A$ such that $\mathcal{C}_u(n) \leq \kappa^n$,
- if there exists a such that $\mathcal{C}_u(n) < an$ for all n, then the first difference of complexity $\Delta\mathcal{C}_u(n) = \mathcal{C}_u(n+1) - \mathcal{C}_u(n) \leq Ka^3$ for all n, for a universal constant K [47, 48],
- if $\Delta\mathcal{C}_u(n) = \mathcal{C}_u(n+1) - \mathcal{C}_u(n)$ is bounded, the set of n such that $\Delta\mathcal{C}_u(n+1) > \Delta\mathcal{C}_u(n)$ has density zero [8].

Before introducing the palindromic complexity let us denote by $\mathcal{P}al(u)$ the set of all palindromes contained in a word u. The *palindromic complexity* of u, is the function $\mathcal{P}_u : \mathbb{N} \to \mathbb{N}$ that associates to a nonnegative integer n the number of palindromes of length n in $\mathcal{L}(u)$,

$$\mathcal{P}_u(n) := \#\{w \in \mathcal{P}al(u) \mid |w| = n\}\,.$$

Obviously, we have $\mathcal{P}_u(n) \leq \mathcal{C}_u(n)$ for all $n \in \mathbb{N}$. Close relation between factor and palindromic complexity was given by Allouche et al.

Proposition 1.2.2 ([9]). *Let $u = u_0u_1u_2\cdots$ be an infinite non-eventually periodic sequence on a finite alphabet. Then, for all $n \geq 1$, we have*

$$\mathcal{P}_u(n) \leq \frac{16}{n}\,\mathcal{C}_u\Big(n + \Big\lfloor\frac{n}{4}\Big\rfloor\Big)\,.$$

When there is no room for confusion we will usually omit the subscript u in both $\mathcal{C}_u(n)$ and $\mathcal{P}_u(n)$.

A powerful tool for the determination of the complexity are the so-called left or right special factors, introduced by Cassaigne [48]. Let w be a factor of a sequence $u \in A^{\mathbb{N}}$, $w \in \mathcal{L}(u)$. A *right extension* (respectively *left extension*) of the factor w is a word wa (respectively aw), where $a \in A$, such that wa (respectively aw) is also element of $\mathcal{L}(u)$. The number of right (respectively left) extensions of a factor w is called the *right* (respectively *left*) *degree* of w, denoted by $\deg_{\mathrm{R}}(w)$ (respectively $\deg_{\mathrm{L}}(w)$). We say that w is a *left special factor* of the infinite word u if its left degree is at least 2.

Similarly, if the right degree of a factor w is at least 2, then w is a *right special factor* of u.

All these just introduced notions can be used to characterize Sturmian words. Let u be a binary infinite word. The following are equivalent

(i) u is Sturmian,

(ii) $\mathcal{C}_u(n) = n + 1$ for all $n \in \mathbb{N}$,

(iii) for any $n \in \mathbb{N}$ there exists exactly one right and one left special factor of length n,

(iv) there is exactly one palindrome of length n for any n even, and there are exactly two palindromes of length n for any n odd.

The characterization using palindromes is due to Droubay and Pirillo [54]. Other equivalent combinatorial definitions of a Sturmian word by means of balance, mechanical words or return words can be found for example in survey papers by Berstel [33, 81].

Infinite words which have for each $n \in \mathbb{N}$ at most one left special and at most one right special factor are called *episturmian* words, this class was introduced as a generalization of Sturmian words to an arbitrary alphabet by Droubay et al. [53].

Arnoux-Rauzy words (*AR words*) [17] of order d are special cases of episturmian words; they are defined as aperiodic words over a d-letter alphabet such that for each n there exist exactly one left special factor w_1 and exactly one right special w_2 factor of length n, and, moreover, these special factors satisfy $\deg_{\mathrm{L}}(w_1) = \deg_{\mathrm{R}}(w_2) = d$. This definition implies that the complexity of Arnoux-Rauzy word is equal to $(d-1)n + 1$.

The palindromic complexity of Arnoux-Rauzy words is also known [75, 52]: there is exactly one palindrome of length n if n is even and exactly d palindromes of length n if n is odd.

Another generalization of Sturmian words to words over a d-letter alphabet is the infinite words coding the *d-interval exchange transformation* [76, 92]. In a generic case the factor complexity is the same as for AR words [76], namely $\mathcal{C}(n) = (d-1)n + 1$ for $n \geq 1$. On the other hand the palindromic structure of these words is more complicated. The existence of palindromes of arbitrary length depends on the permutation which exchanges the intervals. For $d = 3$ the result is given in [52], for general d in [21].

Analogically to the case of factor complexity, for the study of the palindromic complexity it is important to define the palindromic extension. If for a palindrome $p \in \mathcal{P}al(u)$ there exists a letter $a \in A$ such that $apa \in \mathcal{P}al(u)$, then we call the word apa a *palindromic extension* of p.

1.2.3 Substitutions

Let us recall that *morphism* on a free monoid A^* is a mapping φ that fulfills $\varphi(uv) = \varphi(u)\varphi(v)$ for all $u, v \in A^*$. Obviously, for determining the morphism it suffices to define $\varphi(a)$ for all $a \in A$. The action of a morphism can be naturally extended on right infinite words by the prescription

$$\varphi(u_0u_1u_2\cdots) := \varphi(u_0)\varphi(u_1)\varphi(u_2)\cdots .$$

An infinite sequence $u \in A^{\mathbb{N}}$ is called a *fixed point* of φ if $\varphi(u) = u$. A morphism for which $\varphi(a) \neq \varepsilon$ for all $a \in A$ is called a *non-erasing morphism*. If, moreover, there exists a letter $a \in A$ such that $\varphi(a) = aw$ for some non-empty word $w \in A^*$, the non-erasing morphism φ is called a *substitution*. Any substitution has at least one fixed point, namely $\lim_{n\to\infty} \varphi^n(a)$.

For a substitution φ over a d-letter alphabet $A = \{a_0, \ldots, a_{d-1}\}$ we define its *incidence matrix* as a $d \times d$ integer matrix $\boldsymbol{M}_\varphi$ given by $(\boldsymbol{M}_\varphi)_{ij} := |\varphi(a_j)|_{a_i}$, i.e. the element (i, j) of the incidence matrix is equal to the number of occurrences of the letter a_i in $\varphi(a_j)$.

Note that for any morphism φ the mapping φ^n is also a morphism and for its incidence matrix we have $\boldsymbol{M}_{\varphi^n} = (\boldsymbol{M}_\varphi)^n$.

A substitution φ over an alphabet A is called *primitive* if there exists a positive integer k such that for every pair of letters $(a_i, a_j) \in A^2$ the letter a_i occurs in $\varphi^k(a_j)$. Equally, we can say that a substitution φ is primitive if there exists a positive integer k such that $\boldsymbol{M}_\varphi^k$ is a positive matrix.

Similarly, one can extend the action of a morphism to left infinite words. For a pointed two-sided infinite word $u = \cdots u_{-2}u_{-1}|u_0u_1u_2\cdots$ we define the action of a morphism φ by $\varphi(u) = \cdots\varphi(u_{-2})\varphi(u_{-1})|\varphi(u_0)\varphi(u_1)\varphi(u_2)\cdots$. One can also define analogically the notion of a fixed point.

1.3 Automata

The classical textbook on the theory of automata is the work of Eilenberg [55], the most recent summary book on finite automata was written by Sakarovitch [96], automata over infinite words are covered by Perrin and Pin [89, Chap. 1].

An *automaton* over an alphabet A, denoted $\mathcal{A} = \langle A, Q, E, I, F\rangle$, is a directed graph with labels in an alphabet A. The set Q is set of its vertices, called *states*, $I \subset Q$ is set of its *initial states*, $F \subset Q$ is set of *final states* and $E \subset Q \times A \times Q$ is the set of labeled edges, called *transitions*. If $(p, a, q) \in E$ one usually writes $p \xrightarrow{a} q$. The automaton is said to be *finite* if its set of states is finite.

A *computation* (or *path*) c in $\mathcal{A}$ is a finite sequence of transitions such that

$$c = q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \xrightarrow{a_3} \cdots \xrightarrow{a_n} q_n\,.$$

Such a path is usually denoted by $c = q_0 \xrightarrow{a} q_n$, where the finite word $a := a_1a_2\cdots a_n$ is the *label of the computation* c. The computation is *successful* if it starts in an initial state and ends in a final state. The *behavior* of $\mathcal{A}$, denoted by $|\mathcal{A}|$, is a subset of A^* of labels of successful computations of $\mathcal{A}$. Sometimes it is also said that an automaton $\mathcal{A}$ recognizes a language $\mathcal{L}(\mathcal{A}) \subset A^*$, where $\mathcal{L}(\mathcal{A}) = |\mathcal{A}|$.

An automaton $\mathcal{A}$ is called *unambiguous* if for all $p, q \in Q$ and for all words $w \in A^*$ there exist at most one path in $\mathcal{A}$ going from p to q labeled by w. An automaton $\mathcal{A}$ is called *deterministic* if for any pair $(p, a) \in Q \times A$ there exists at most one state $q \in Q$ such that $p \xrightarrow{a} q$ is a transition of $\mathcal{A}$.

Example. Let us illustrate the previous definitions on the automaton $\mathcal{A}_{\text{fib}}$ in Figure 1.1. $\mathcal{A}_{\text{fib}}$ has 2 states, denoted 0 and 1, both of them are final, the state 0 is also initial. It is a deterministic automaton and its behavior $|\mathcal{A}_{\text{fib}}| = 0^* \cup 0^*1(00^*1)^*0^*$ is the set of finite words over $\{0, 1\}$ not containing 11 as a factor.

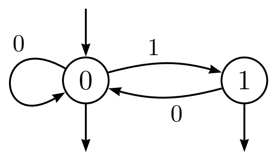


Figure 1.1: The Fibonacci automaton $\mathcal{A}_{\text{fib}}$.

An automaton $\mathcal{T} = \langle A^* \times B^*, Q, E, I, F\rangle$ over a monoid $A^* \times B^*$ is called a *transducer* from A^* to B^*. Its transitions are labeled by pairs of words $(u, v) \in A^* \times B^*$, u is called *input word* and v is called *output word.* If $(p, (u, v), q) \in E$ one usually writes $p \xrightarrow{u|v} q$. A transducer is said to be finite if the of its edges E is finite.

A computation c in $\mathcal{T}$ is a finite sequence

$$c = q_0 \xrightarrow{u_1|v_1} q_1 \xrightarrow{u_2|v_2} q_2 \xrightarrow{u_3|v_3} \cdots \xrightarrow{u_n|v_n} q_n\,.$$

The label of computation c is $(u, v) := (u_1u_2\cdots u_n, v_1v_2\cdots v_n)$. The behavior of a transducer $\mathcal{T}$ is defined as above, that is, as the set of successful computations of $\mathcal{T}$. In this case it is a relation $R \subset A^* \times B^*$ and $\mathcal{T}$ is said to *compute (realize)* R. If for any word $u \in A^*$ there exists at most one word $v \in B^*$ such that $(u, v) \in R$ then R is a function.

A transducer is called *real-time* if input words of all its transitions are letters in A (i.e. the transitions are labeled in $A \times B^*$). The *underlying input* (respectively *output*) *automaton* of a transducer $\mathcal{T}$ is obtained by omitting the output (respectively input) labels of each transition of $\mathcal{T}$.

In the usual setting a transducer $\mathcal{T}$ is assumed to operate "as we do", that is, it is reading its input (and writing its output) from left to right. Sometimes it may be more suitable to reverse the direction of the computation, that is, to have a machine reading/writing from right to left. In the latter case we stress this fact by saying that $\mathcal{T}$ is a *right transducer.*

A transducer is said to be *sequential* if it is real-time, it has a unique initial state and its underlying input automaton is deterministic. A function is called *sequential* if it can be realized by a sequential transducer. A *subsequential* transducer $(\mathcal{T}, \rho)$ is a pair of a sequential transducer $\mathcal{T}$ over $A^* \times B^*$ and of a function $\rho : F \to B^*$, called *final function*, where F is the set of final states of $\mathcal{T}$. A pair $(u, v) \in A^* \times B^*$ is in the behavior of a subsequential transducer $(\mathcal{T}, \rho)$ if there exists a successful computation $p \xrightarrow{u|v_1} q$ in $\mathcal{T}$ and a word $v_2 \in B^*$, $v_2 = \rho(q)$ such that $v = v_1v_2$. A function is called *subsequential* if it can be realized by a subsequential transducer.

Example. Transducer $\mathcal{T}_{\text{bin}}$ in Figure 1.2 is a right subsequential transducer with the final function ρ given by $\rho(0) = \varepsilon$ and $\rho(1) = 1$. It realizes addition in the binary numeration system, that is for an input word u over the alphabet $A = \{0, 1, 2\}$ the corresponding output word is a word v over the alphabet $B = \{0, 1\}$ having the same numerical value in the positional system with base 2.

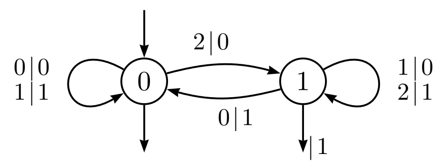


Figure 1.2: Transducer $\mathcal{T}_{\text{bin}}$ realizing addition in the binary numeration system.

The notion of an automaton (and of a transducer) can be generalized to the case of infinite words. A *Büchi automaton* $\mathcal{A} = \langle A, Q, E, I, F\rangle$ is an automaton such that a computation c in $\mathcal{A}$ is an infinite sequence of consecutive transitions

$$c = q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \xrightarrow{a_3} \cdots .$$

A computation is said to be successful if it starts in a initial state, $q_0 \in I$, and if it visits infinitely often the set of final states.

A transducer over infinite words is defined in a similar way, as a Büchi automaton over a monoid $A^{\mathbb{N}} \times B^{\mathbb{N}}$.

For automata and transducers, the set of states Q can be also infinite countable, see [89].

Chapter 2

Beta-numeration

In this chapter we give an introductory overview of the positional numeration systems based on the so-called β-expansions, first inspected by Rényi [94]. In the first part we present basic notions connected with the representation of real numbers in these systems, then we discuss problems connected with the arithmetic, that is, the question of whether the set of finite representations in a given system has a ring structure or not and also the problem of the number of fractional digits arising under the arithmetic of integers. The last section gives a survey of combinatorial properties of these systems — we recall results on the structure of the set of integers, the definition of the canonical substitution associated with these systems and the known values of subword complexity of infinite words generated by this substitution.

2.1 Beta-expansions

2.1.1 Definitions

Let $\beta > 1$ be a real number. A *representation in base β* (or simply a *β-representation*) of a real number $x \geq 0$ is an infinite sequence $(x_i)_{i \leq k}$, such that $x_i \in \mathbb{N}$ and

$$x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots + x_1\beta + x_0 + x_{-1}\beta^{-1} + x_{-2}\beta^{-2} + \cdots$$

for a certain $k \in \mathbb{Z}$. We denote a β-representation of a number x by $(x)_\beta$ and we use usual radix scale for it

$$(x)_\beta = x_k x_{k-1} \cdots x_1 x_0 \bullet x_{-1} x_{-2} \cdots \qquad \text{if } k \geq 0,$$

$$(x)_\beta = 0 \bullet \underbrace{00\cdots00}_{-k-1 \text{ digits}} x_k x_{k-1} \cdots \qquad \text{if } k < 0,$$

the symbol $\bullet$ is called the *fractional point.* If a β-representation of x ends in infinitely many zeros, it is said to be *finite* and the ending zeros are omitted.

The *β-value* is a function π_β defined on the set of β-representations $(x_i)_{i \leq k}$ by the prescription

$$\pi_\beta(x_k x_{k-1} \cdots) := \sum_{k \leq i} x_i \beta^i \,.$$

A particular β-representation — called *β-expansion* [94] — is computed by the so-called greedy algorithm.

Algorithm 2.1.1. *Let $x \in \mathbb{R}$ be a real number, denote by $\lfloor x \rfloor$, respectively by $\{x\}$, the integer part, respectively the fractional part, of the number x.*

1. *Find $k \in \mathbb{Z}$ such that $\beta^k \leq x < \beta^{k+1}$.*
2. *Set $x_k := \lfloor x/\beta^k \rfloor$ and $r_k := \{x/\beta^k\}$.*
3. *For $i < k$, let $x_i = \lfloor \beta r_{i+1} \rfloor$ and $r_i = \{\beta r_{i+1}\}$.*

Then $(x_i)_{i \leq k}$ is the β-expansion of x, denoted by $\langle x \rangle_\beta$.

The β-expansion of a number x is the greatest one among its β-representations in the "slightly modified" lexicographical order, usually called *radix order*: Let $(y_i)_{i \leq l}$ be a β-representation of a positive number x with $y_l \neq 0$ and $(x_i)_{i \leq k}$ be the β-expansion of x. Then either $l < k$ or $l = k$ and $y_k y_{k-1} \cdots$ is lexicographically smaller than or equal to $x_k x_{k-1} \cdots$.

The digits x_i obtained by the greedy algorithm are elements of the alphabet $A_\beta = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$, called the *canonical alphabet.*

Note that unlike the positional systems whose base is a positive integer, for general $\beta > 1$ a β-representation of x is not unique even on A_β.

For $x \in [0, 1)$ the β-expansion $\langle x \rangle_\beta$ can be generated using the *β-transformation* of the unit interval, which is a piecewise linear map $T_\beta : [0, 1] \to [0, 1)$ given by

$$T_\beta(x) := \{\beta x\} .$$

The sequence $\mathrm{d}_\beta(x) = x_1 x_2 \cdots$, where coefficients x_i are obtained by iterating the β-transformation as

$$x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor ,$$

coincide for each real $x \in [0, 1)$ with the β-expansion of x generated by the greedy algorithm.

The difference between these two methods appears in the case of the number $x = 1$. The natural expansion of 1 in an arbitrary positional system, and also the expansion generated by the greedy algorithm, is $\langle 1 \rangle_\beta = 1{\bullet}0^\omega$. On the other hand, the β-transformation generates

$$\mathrm{d}_\beta(1) = t_1 t_2 \cdots , \quad \text{where} \quad t_i = \lfloor \beta T_\beta^{i-1}(1) \rfloor .$$

$\mathrm{d}_\beta(1)$ is called the *Rényi expansion of unity.* Obviously, the numbers t_i are non-negative integers smaller than β, $t_1 = \lfloor \beta \rfloor$ and

$$1 = \sum_{i=1}^{\infty} t_i \beta^{-i} . \tag{2.1}$$

The notion of Rényi expansion of unity is very important for it plays a central role in the theory of β-expansions. Based on $\mathrm{d}_\beta(1)$ we define $\mathrm{d}^*_\beta(1)$ by putting

$$\mathrm{d}^*_\beta(1) = \begin{cases} \mathrm{d}_\beta(1) & \text{if } \mathrm{d}_\beta(1) \text{ is infinite,} \\ (t_1 \cdots t_{m-1}(t_m - 1))^\omega & \text{if } \mathrm{d}_\beta(1) = t_1 \cdots t_m \text{ is finite.} \end{cases}$$

Example. Let τ be the golden mean, that is, $\tau = \frac{1+\sqrt{5}}{2}$. Iterating the β-transformation and by the previous definition one has

$$\mathrm{d}_\tau(1) = 11 \qquad \text{and} \qquad \mathrm{d}^*_\tau(1) = (10)^\omega .$$

On the other hand, for example in the case of $\beta = \frac{3+\sqrt{5}}{2}$ both sequences coincide

$$\mathrm{d}_\beta(1) = \mathrm{d}^*_\beta(1) = 21^\omega .$$

Let $\mathrm{d}_\beta([0,1))$ be the set defined as follows

$$\mathrm{d}_\beta([0,1)) := \{\mathrm{d}_\beta(x) \mid x \in [0,1)\} .$$

The sequence $\mathrm{d}^*_\beta(1)$ is the supremum of this set. The set $\mathrm{d}_\beta([0,1))$ is shift invariant, that is, it is invariant with respect to the shift operator σ and we have the following commutative diagram

$$\begin{tikzcd} {[0,1)} \arrow[r, "T_\beta"] \arrow[d, "\mathrm{d}_\beta(1)"'] & {[0,1)} \arrow[d, "\mathrm{d}_\beta(1)"] \\ \mathrm{d}_\beta([0,1)) \arrow[r, "\sigma"'] & \mathrm{d}_\beta([0,1)) . \end{tikzcd}$$

By definition we have $\pi_\beta \circ \mathrm{d}_\beta(x) = x$, and hence $[0,1) \subset \pi_\beta(A_\beta^{\mathbb{N}})$. However, $A_\beta^{\mathbb{N}} \not\subset \mathrm{d}_\beta([0,1))$. If a word $a_1 a_2 \cdots \in A_\beta^{\mathbb{N}}$ is contained in $\mathrm{d}_\beta([0,1))$, it is said to be *admissible*.

The above mentioned role of $\mathrm{d}_\beta(1)$ (or better said of $\mathrm{d}^*_\beta(1)$) is that it permits us to distinguish β-expansions from non-admissible β-representations by characterizing those words, which are admissible. This is done by the so-called *Parry condition.*

Theorem 2.1.2 ([87]). *A β-representation $(x_i)_{i\le k}$ of a real number $x \ge 0$ is its β-expansion if and only if for all $j \le k$ the sequence $x_j x_{j-1} x_{j-2} \cdots$ is strictly lexicographically smaller than the sequence $\mathrm{d}^*_\beta(1)$.*

The previously alluded important role of Pisot numbers in the field of β-numeration (and in some sense also the simplicity of Pisot numbers) in this theory is given by the following Theorem due to Bertrand.

Theorem 2.1.3 ([37, 38]). *Let β be a Pisot number. Then $\mathrm{d}_\beta(1)$ is eventually periodic and the set $\mathrm{d}_\beta([0,1))$ is recognized by a finite automaton.*

Let C be a finite alphabet of digits. The *normalization* on C is the function ν_C which maps a β-representation $(w_i)_{i\le k}$ of a real number x with digits $w_i \in C$, to the β-expansion of x. By result of Frougny, it is known that the normalization is computable by a finite state automaton if β is a Pisot number.

Theorem 2.1.4 ([61]). *If β is a Pisot number, then the normalization function ν_C is computable by a finite letter-to-letter transducer on any finite alphabet C of digits.*

The reciprocal has been proved by Berend and Frougny.

Theorem 2.1.5 ([29]). *The normalization ν_C in base β is computable by a finite transducer on any finite alphabet of digits C if and only if β is a Pisot number.*

The set of all real numbers x for which the β-expansion $(x_i)_{k\geq i}$ of $|x|$ is finite is denoted by $\mathrm{Fin}(\beta)$. For $x \geq 0$ the number $\sum_{i=0}^{k} x_i\beta^i$ is called the *β-integer part* of x and $\sum_{i\leq -1} x_i\beta^i$ is called the *β-fractional part* of x.

We define also the so-called *β-integers* as the real numbers whose β-expansion has no fractional part, the set of β-integers is denoted by $\mathbb{Z}_\beta$,

$$\mathbb{Z}_\beta := \left\{x \in \mathbb{R} \mid \langle |x| \rangle_\beta = x_k \cdots x_1 x_0 \bullet \right\}.$$

Example. Let τ be the golden mean. Then the set of τ-integers is

$$\mathbb{Z}_\tau = \{\ldots, -\tau^3, -\tau^2-1, -\tau^2, -\tau, -1, 0, 1, \tau, \tau^2, \tau^2+1, \tau^3, \ldots\},$$

as drawn in Figure 2.1.

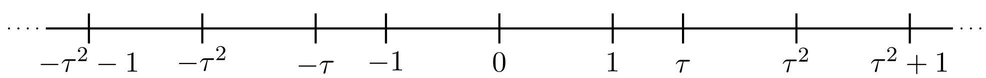


Figure 2.1: Some elements of the set $\mathbb{Z}_\tau$.

For $x \in \mathrm{Fin}(\beta)$ we define the length of the fractional part as the minimal $l \in \mathbb{N}$ such that $\beta^l x \in \mathbb{Z}_\beta$. This l is denoted by $\mathrm{fp}(x)$. From this definition we have for $x \in \mathrm{Fin}(\beta)$

(i) $\mathrm{fp}(x) = 0$ if and only if $x \in \mathbb{Z}_\beta$,

(ii) $\mathrm{fp}(x) = l \geq 1$ if and only if $\langle |x| \rangle_\beta = x_k \cdots x_0 \bullet x_{-1} \cdots x_{-l}$, with $x_{-l} \neq 0$.

2.1.2 Classification of Parry numbers

According to the form of Rényi expansion of unity we define two subclasses of real numbers, as introduced by Parry [87]. A real $\beta > 1$ is called a *Parry number* (or *β-number*) if $\mathrm{d}_\beta(1)$ is finite or eventually periodic and it is called a *simple Parry number* (or *simple β-number*) if $\mathrm{d}_\beta(1)$ is finite. Hence, by Theorem 2.1.3, a Pisot number is a Parry number.

Clearly a Parry number is an algebraic integer. On the other hand, it is a difficult task to characterize Parry numbers among the algebraic integers, however some partial results have been already found. We give in this section a summary on this topic.

The most general result on the question of the algebraic nature of Parry numbers is given by following theorem (proved e.g. in [81]).

Theorem 2.1.6. *Let $\beta > 1$ be a Parry number. Then β is a Perron number.*

The following result on Pisot numbers has been proved independently by Bertrand-Mathis and by Schmidt.

Theorem 2.1.7 ([37],[98]). *If β is a Pisot number then each element of $\mathbb{Q}(\beta) \cap [0, 1)$ has an eventually periodic β-expansion.*

Schmidt further gives a partial converse by proving the following theorem.

Theorem 2.1.8 ([98]). *Let $\beta > 1$ be a real number, and assume that each element of $\mathbb{Q} \cap [0, 1)$ has an eventually periodic β-expansion. Then β is either a Pisot or a Salem number.*

Finally, Schmidt also conjectures that Theorem 2.1.7 is true also for Salem numbers, that is, if β is a Salem number then each element of $\mathbb{Q}(\beta) \cap [0, 1)$ has an eventually periodic β-expansion.

Example. It is worth pointing out that there are Parry numbers that are neither Pisot nor Salem. For instance for the root $\beta \sim 3.616$ of the equation $x^4 - 3x^3 - 2x^2 - 3$ the Rényi expansion of unity is $\mathrm{d}_\beta(1) = 3203$, and β has a conjugate $\alpha \sim -1.096$.

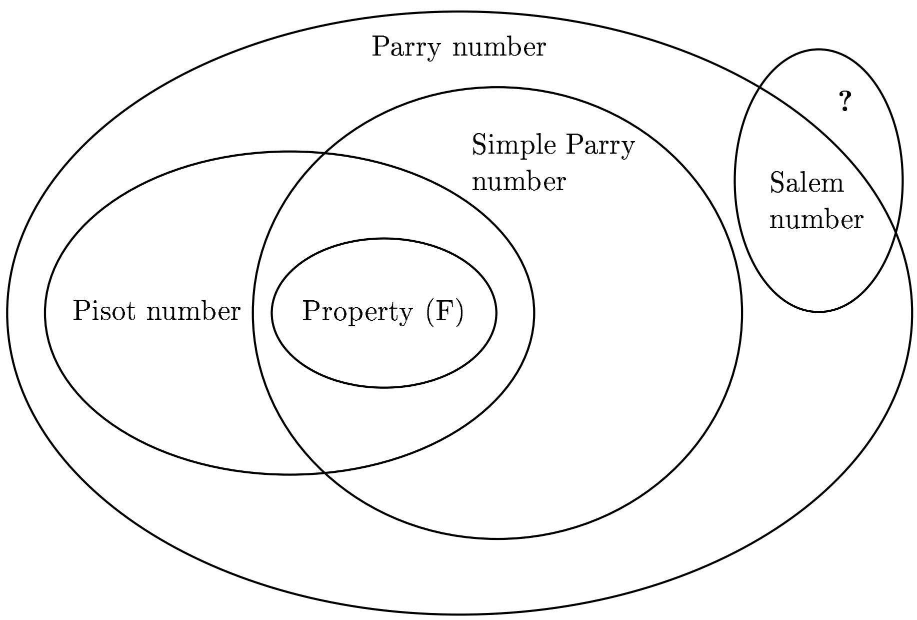


Figure 2.2: The classification of Parry numbers

The study of β-expansions in the case of Salem numbers — towards the conjecture of Schmidt — was pursued by Boyd [41, 42]. He gave an affirmative answer to a particular version of this conjecture in the case of Salem numbers of degree 4.

Theorem 2.1.9 ([41]). *Each Salem number of degree* 4 *is a Parry number.*

Further, he gave [42] a heuristic probabilistic argument that predicts that almost all Salem numbers of degree 6 are Parry numbers. Moreover, it predicts that for each fixed even degree $d \geq 8$ there should be a positive proportion of Salem numbers of degree d which are Parry numbers, as well as a positive proportion that are not Parry numbers. The last prediction given seems to impugn a legitimacy of Schmidt's conjecture.

***Remark* 2.1.10.** Boyd's consideration of only algebraic numbers of even degree is due to the result of Salem [97] stating that the degree d of any Salem number is even and $d \geq 4$.

A nice figure by Akiyama [5] summarizing the classification of Parry numbers given in this section is in Figure 2.2 (The Finiteness property (F) is discussed in Section 2.2.1).

2.2 Arithmetical properties

2.2.1 Finiteness property (F)

The so-called Finiteness property (F) of numeration systems is closely related not only to arithmetics on these systems, but for example also to the dual Pisot tiling generated by these systems, see papers by Akiyama [2, 4], Akiyama and Sadahiro [6] and Thurston [104].

Property (F) was introduced by Frougny and Solomyak [65], when they asked if

$$\mathrm{Fin}(\beta) = \mathbb{Z}[\beta^{-1}] \tag{2.2}$$

holds for a given β. We say that a number β satisfying (2.2) has Property (F). Other way to interpret Property (F) is to ask whether the set $\mathrm{Fin}(\beta)$ has a structure of a ring.

Clearly, one of the inclusions, namely $\mathrm{Fin}(\beta) \subset \mathbb{Z}[\beta^{-1}]$, holds by the definition of the set $\mathrm{Fin}(\beta)$ for any algebraic integer β. Concerning the other direction, authors showed in [65] the necessary condition.

Theorem 2.2.1 ([65]). *Let* $\mathbb{Z}[\beta^{-1}] \subset \mathrm{Fin}(\beta)$. *Then* β *is a Pisot number and* $\mathrm{d}_\beta(1)$ *is finite.*

However, the converse is not true; examples can be found by means of the following Proposition.

Proposition 2.2.2 ([1]). *Let* $\beta > 1$ *be a real algebraic integer with a positive real conjugate. Then* β *does not have Property (F).*

To find a simple algebraic characterization of Pisot numbers satisfying (F) is an open problem up to now, even though there is an algorithm by Akiyama to determine whether Property (F) holds for a given Pisot number β or not. Its essence is given by another result of [1].

Theorem 2.2.3 ([1]). *Let β be a Pisot number of degree m. Then β has Property (F) if and only if every element of*

$$C(\beta) := \left\{ x \in \mathbb{Z}[\beta] \;\middle|\; 0 < x = x^{(1)} < 1, |x^{(j)}| \leq \frac{\lfloor \beta \rfloor}{1 - |\beta^{(j)}|} \quad j = 2, 3, \ldots, m \right\}$$

has a finite β-expansion. Here $x^{(i)}$ with $i = 2, \ldots, m$ are the conjugates of $x \in \mathbb{Q}(\beta)$.

Indeed, the set $C(\beta)$ is finite. Therefore to check Property (F) it is enough to inspect the β-expansions of finitely many elements of $\mathbb{Z}[\beta]$.

Let β be a Pisot number with minimal polynomial

$$\mathrm{M}_\beta(x) = x^d - a_{d-1}x^{d-1} - \cdots a_1 x - a_0 \,. \tag{2.3}$$

Several authors have found some sufficient conditions in terms of coefficients in $\mathrm{M}_\beta(x)$ for β to have Property (F). We list them below.

Theorem 2.2.4 ([65]). *If the coefficients in* (2.3) *fulfill $a_{d-1} \geq a_{d-2} \geq \cdots \geq a_0 > 0$, then β has Property (F).*

Theorem 2.2.5 ([71]). *If the coefficients in* (2.3) *fulfill $a_{d-1} > a_{d-2} + \cdots + a_1 + a_0 > 0$ with $a_i \geq 0$, then β has Property (F).*

Theorem 2.2.6 ([3]). *Let β be a cubic Pisot unit with minimal polynomial $x^3 - a_2x^2 - a_1x - a_0$. Then the following statements are equivalent*

(i) β *has Property (F),*

(ii) $a_0 = 1$, $a_2 \geq 0$ *and* $-1 \leq a_1 \leq a_2 + 1$,

(iii) $\mathrm{d}_\beta(1)$ *is finite.*

2.2.2 Number of fractional digits

The second question arising when we consider the arithmetics on β-expansions is connected to the fact that if β is not an integer the set $\mathbb{Z}_\beta$ is not closed under arithmetic operations. In particular, $\lfloor \beta \rfloor \in \mathbb{Z}_\beta$, but $\lfloor \beta \rfloor + 1 \notin \mathbb{Z}_\beta$.

Example. Let τ be the golden mean. We have $\lfloor \tau \rfloor = 1$ and the number 2 (seen as the addition $1+1$) can be expanded as

$$2 = 1\bullet + 1\bullet = 1\bullet + 0\bullet 11 = 1\bullet 11 = 10\bullet 01 \,,$$

where the equality $1\bullet 0 = 0\bullet 11$ comes from the minimal polynomial $\tau^2 = \tau + 1$.

Hence, besides the question whether the result of an arithmetic operation on $\mathrm{Fin}(\beta)$ has a finite β-expansion, studied in previous section, we are also interested in describing the maximal length of the resulting fractional part. Since it is possible to convert $x, y \in \mathrm{Fin}(\beta)$ by multiplication by a common suitable factor β^k into elements of $\mathbb{Z}_\beta$, for the description of fractional parts of results of addition and multiplication it is enough to study the following quantities.

$$\begin{aligned}
L_\oplus = L_\oplus(\beta) &:= \max\left\{ \operatorname{fp}(x+y) \mid x,y \in \mathbb{Z}_\beta, x+y \in \operatorname{Fin}(\beta)\right\}, \\
L_\otimes = L_\otimes(\beta) &:= \max\left\{ \operatorname{fp}(x\cdot y) \mid x,y \in \mathbb{Z}_\beta, x\cdot y \in \operatorname{Fin}(\beta)\right\}.
\end{aligned}$$

The maximum of an unbounded set is defined to be $+\infty$. In [2] it is shown that for a Pisot number β, one has $L_\oplus(\beta) < +\infty$. In [69] the same result is given for $L_\otimes(\beta)$. The most recent result on finiteness of these bounds was given by Bernat.

Theorem 2.2.7 ([30]). *The quantities $L_\oplus(\beta)$ and $L_\otimes(\beta)$ are finite if β is a Perron number.*

There exists a method for determining upper estimates on the constants $L_\oplus(\beta)$ and $L_\otimes(\beta)$. It stems from a theorem given by Guimond et al. [69]. The idea of the theorem is quite simple and in some way it has already been used by Messaoudi [83], and Gazeau and Verger-Gaugry [66].

Theorem 2.2.8 ([69]). *Let β be an algebraic number, $\beta > 1$, with at least one conjugate β' satisfying*

$$\begin{aligned}
H &:= \sup\{|z'| \mid z \in \mathbb{Z}_\beta\} < +\infty, \\
K &:= \inf\{|z'| \mid z \in \mathbb{Z}_\beta \setminus \beta\mathbb{Z}_\beta\} > 0,
\end{aligned}$$

where z' denotes the image of $z \in \mathbb{Q}(\beta)$ under the field isomorphism $' : \mathbb{Q}(\beta) \to \mathbb{Q}(\beta')$. Then

$$\left(\frac{1}{|\beta'|}\right)^{L_\oplus(\beta)} < \frac{2H}{K} \qquad \textit{and} \qquad \left(\frac{1}{|\beta'|}\right)^{L_\otimes(\beta)} < \frac{H^2}{K}.$$

In the Theorem above we require the existence of at least one conjugate of β such that the constant H is finite and K is positive. To decide whether $K > 0$ or $K = 0$ is quite complicated. A sufficient condition for a number β so that $K = 0$ will be discussed later, cf. section 3.2.4.

On the other hand, a sufficient condition for a certain class of β so that $K > 0$ was given by Akiyama [2]. He proved that for β a Pisot unit satisfying Property (F), the origin is an inner point of the central tile in the conjugated plane, i.e. of the closure of the set $\{z' \mid z \in \mathbb{Z}_\beta\}$. This implies that K is positive for all conjugates of such a number β.

Let us summarize some known results on the values of $L_\oplus(\beta)$ and $L_\otimes(\beta)$.

- Knuth [79]: $L_\otimes(\beta) = 2$ for $\beta > 1$ root of $x^2 = x + 1$.
- Burdík et al. [44]

$$\begin{aligned}
L_\oplus(\beta) = L_\otimes(\beta) &= 1 \qquad \text{for } \beta \text{ root of equation } x^2 = mx - 1,\ m \ge 3, \\
L_\oplus(\beta) = L_\otimes(\beta) &= 2 \qquad \text{for } \beta \text{ root of equation } x^2 = mx + 1,\ m \ge 1.
\end{aligned}$$

- Guimond et al. [69]

  (i) β root of $x^2 = xm - n$, where $m, n \in \mathbb{N}$ and $m \geq n + 2$.

$$L_\oplus(\beta) \leq 3m \ln m\,,$$
$$L_\oplus(\beta) \leq 4m \ln m\,.$$

  (ii) β root of $x^2 = xm + n$, where $m, n \in \mathbb{N}$ and $m \geq n$.

$$\begin{aligned} L_\oplus(\beta) &= 2m && \text{for } m = n\,, \\ 2\left\lfloor \tfrac{m+1}{m-n+1} \right\rfloor \leq L_\oplus(\beta) &\leq 2\left\lceil \tfrac{m}{m-n+1} \right\rceil && \text{for } m > n\,, \end{aligned}$$

  and

$$L_\otimes(\beta) \leq 4 L_\oplus(\beta) \log_2(m+2)\,.$$

- Messaoudi [84]: $L_\otimes(\beta) \leq 5$ for β real root of the equation $x^3 = x^2 + x + 1$.

2.3 Combinatorial properties

2.3.1 Properties of $\mathbb{Z}_\beta$

Besides the fact that the set $\mathbb{Z}_\beta$ is not closed under arithmetic operations, there is another difference from the case where β is an integer. There is more than one possible gap (in terms of the length) between neighbors in $\mathbb{Z}_\beta$ (see Figure 2.1). The following theorem due to Thurston provides a complete characterization of these gaps.

Theorem 2.3.1 ([104]). *Let $\beta > 1$ be a Parry number. Then the lengths of gaps between neighbors in $\mathbb{Z}_\beta$ take values in the set $\{\Delta_0, \Delta_1, \ldots\}$ where*

$$\Delta_i = \sum_{k=1}^{\infty} \frac{t_{k+i}}{\beta^k}\,, \qquad \textit{for } i \in \mathbb{N}\,.$$

From the definition of $\mathrm{d}_\beta(1)$ it is obvious that the largest distance between neighboring β-integers is

$$\Delta_0 = \sum_{k=1}^{\infty} \frac{t_k}{\beta^k} = 1\,.$$

Obviously, by Theorem 2.3.1, there is only a finite number of different gaps between neighbors in $\mathbb{Z}_\beta$ for β being a Parry number, simple or not.

Let us denote the lengths of gaps by letters $a_i := \Delta_i$. Then the sequence of gaps between the neighbors in set $\mathbb{Z}_\beta^+$ uniquely defines an *infinite word associated with β*, denoted by u_β, over the alphabet $\{a_0, a_1, \ldots\}$.

2.3.2 Associated substitution

With every Parry number one can associate a canonical substitution $\varphi_\beta = \varphi$, such that the above defined word u_β is a unique fixed point of φ. Substitution φ is defined in the following way [56].

Simple case. Let β be a simple Parry number and let $\mathrm{d}_\beta(1) = t_1 \cdots t_m$. The alphabet of the substitution is $A_\varphi = \{a_0, \dots, a_{m-1}\}$ and we define

$$\varphi(a_i) = a_0^{t_{i+1}} a_{i+1} \quad \text{for all } 0 \le i < m-1\,, \tag{2.4a}$$

$$\varphi(a_{m-1}) = a_0^{t_m}\,. \tag{2.4b}$$

Non-simple case. Let β be a non-simple Parry number and let m, p be minimal such that $\mathrm{d}_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+p})^\omega$. The alphabet A_φ is $A_\varphi = \{a_0, \dots, a_{m+p-1}\}$ and we define

$$\varphi(a_i) = a_0^{t_{i+1}} a_{i+1} \quad \text{for all } 0 \le i < m+p-1\,, \tag{2.5a}$$

$$\varphi(a_{m+p-1}) = a_0^{t_{m+p}} a_m\,. \tag{2.5b}$$

As a consequence of this characterization we have the fact that the incidence matrix $\boldsymbol{M}_\varphi$ of the canonical substitution of any Parry number has one of the following forms

$$\boldsymbol{M}_\varphi = \begin{pmatrix} t_1 & t_2 & \cdots & \cdots & \cdots & t_m \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix} \qquad \boldsymbol{M}_\varphi = \begin{pmatrix} t_1 & t_2 & \cdots & \cdots & \cdots & t_{m+p} \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & 1 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}$$

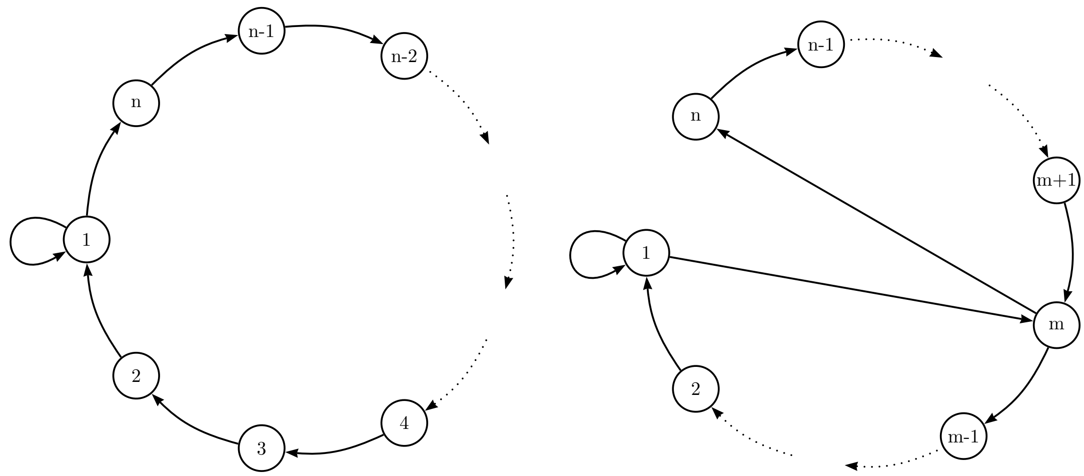
It has been proved by Canterini and Siegel [46] that any substitution of Pisot type (that is a substitution for which the characteristic polynomial of its incidence matrix is the minimal polynomial of a Pisot number) is primitive. We give below a proof that the same is true for all Parry numbers.

Proposition 2.3.2. *Let φ be a canonical substitution associated to a Parry number β. Then φ is primitive.*

Proof. Let $G = (V, E)$ be a graph with n vertices. Its adjacency matrix is defined to be a $n \times n$ matrix $\boldsymbol{B} = (b_{ij})_{n \times n}$ with rows and columns labeled by graph vertices. The element $b_{ij} = 1$ if there is an edge going from vertex i to vertex j and $b_{ij} = 0$ otherwise.

There is a result from graph theory saying that if $\boldsymbol{B} = (b_{ij})_{n\times n}$ is the adjacency matrix of a graph G, then $\boldsymbol{B}^k = (b'_{ij})_{n\times n}$ displays for all $i, j \le n$ the number b'_{ij} of walks of length k from the vertex i to the vertex j. Recall that a walk of length k in a graph $G = (V, E)$ is a sequence of vertices and edges $v_1 e_1 v_2 e_2 \cdots e_k v_{k+1}$ such that $v_i \in V$, $e_i \in E$ and $v_i \xrightarrow{e_i} v_{i+1}$ is an edge in G for all $i = 1, \dots, k$.

Consider graphs in Figure 2.3 and their adjacency matrices $\boldsymbol{B_1}$ and $\boldsymbol{B_2}$. In the first case there is a walk of length at most $n-1$ from vertex 1 to any other vertex and also a walk of length at most $n-1$ from any vertex to vertex 1. By virtue of the loop in vertex 1 (that is to say by using this loop repeatedly as many times as needed) we can



2.3.1: A graph with matrix $\boldsymbol{B_1}$. 2.3.2: A graph with matrix $\boldsymbol{B_2}$.

Figure 2.3: Graphs having adjacency matrices $\boldsymbol{B_1}$ and $\boldsymbol{B_2}$

find a walk of length exactly $2n-2$ between any pair of vertices. This implies that for all $i, j \le n$ the entry b'_{ij} of $\boldsymbol{B_1}^{2n-2}$ is $b'_{ij} \ge 1$. Hence

$$\boldsymbol{B_1}^{2n-2} > 0\,. \tag{2.6}$$

In the second case there is a walk of length at most $n-1$ from vertex 1 to any other vertex and also a walk of length at most $k := \max\{n-m+1, m+1\}$ from any vertex to vertex 1. Using the same argument as in the previous case, by virtue of the loop in vertex 1 we can find a walk of length exactly $n+k-1$ between any pair of vertices. This implies that for all $i, j \le n$ the entry b'_{ij} of $\boldsymbol{B_2}^{n+k-1}$ is $b'_{ij} \ge 1$. Hence

$$\boldsymbol{B_2}^{n+k-1} > 0\,. \tag{2.7}$$

Let β be a simple Parry number, φ its canonical substitution. Let us denote the dimension of $\boldsymbol{M}_\varphi$ by $n \times n$ (i.e. $n = m$). Then

$$\boldsymbol{M}_\varphi = \begin{pmatrix} t_1 & t_2 & \cdots & \cdots & \cdots & t_m \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix} \ge \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix} = \boldsymbol{B_1}$$

and hence $\boldsymbol{M}_\varphi^{2n-2} > 0$ by (2.6).

Let β be a non-simple Parry number with $\mathrm{d}_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+p})$, and let φ be its canonical substitution. Since m, p are minimal possible then either $t_{m+p} > 0$ or $t_{m+p} = 0$ and $t_m \geq 1$. In the first case we have $\boldsymbol{M}_\varphi \geq \boldsymbol{B_1}$ of corresponding dimensions, i.e. $n = m + p$. The primitivity is obtained by $\boldsymbol{M}_\varphi^{2n-2} > 0$ as in the simple case.

In the latter case (i.e. $t_{m+p} = 0$ and $t_m \geq 1$) we have

$$\boldsymbol{M}_\varphi = \begin{pmatrix} t_1 & t_2 & \cdots & t_m & \cdots & t_{m+p} \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & 1 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix} \geq \begin{pmatrix} 1 & 0 & \cdots & 1 & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & 1 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix} = \boldsymbol{B_2}$$

where $n = m + p$. Therefore $\boldsymbol{M}_\varphi^{n+k-1} > 0$ by (2.7), which completes the proof. $\square$

2.3.3 Subword complexity

The problem of the subword complexity of infinite words u_β associated with Parry numbers is so far not completely solved, however, at least an asymptotic bound is known, since all these words are fixed points of primitive substitutions, their factor complexity is at most linear [91]. Concerning the exact values of $\mathcal{C}_{u_\beta}$, obviously the simplest task is to determine its values for words u_β which are elements of some known class of infinite words.

Among the words u_β over a binary alphabet, i.e. in the cases $\mathrm{d}_\beta(1) = t_1 t_2$ and $\mathrm{d}_\beta(1) = t_1 t_2^\omega$, it is simple to determine, which u_β correspond to Sturmian words and hence have complexity $\mathcal{C}_{u_\beta}(n) = n + 1$. The matrices of canonical substitutions in these two cases are

$$\boldsymbol{M}_\varphi = \begin{pmatrix} t_1 & t_2 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{M}_\varphi = \begin{pmatrix} t_1 & t_2 \\ 1 & 1 \end{pmatrix}$$

It is follows from [85] that if a Sturmian word is a fixed point of a substitution φ with a matrix $\boldsymbol{M}_\varphi$ then

$$\boldsymbol{M}_\varphi \in \left\{ \boldsymbol{A} \in \mathbb{N}^{2\times 2} \mid \det \boldsymbol{A} = \pm 1 \right\}.$$

Therefore u_β being a Sturmian word implies that $\mathrm{d}_\beta(1) = t_1 1$ or $\mathrm{d}_\beta(1) = t_1(t_1 - 1)^\omega$. The converse is obvious from the form of corresponding canonical substitutions φ.

The second class of infinite words with known values of complexity is the one of Arnoux-Rauzy words (AR words). Let us recall that an AR word of order m has complexity $\mathcal{C}_{u_\beta}(n) = (m-1)n + 1$. It has been shown [63] that for a simple Parry number β, the word u_β is AR if and only if $\mathrm{d}_\beta(1) = s^{m-1}1$; a similar result concerning non-simple Parry numbers is as follows. Recall that the language of an Arnoux-Rauzy word is closed under reversal. By a result of Bernat [32] we have that a word u_β associated with a non-simple Parry number β is closed under reversal if and only if $\mathrm{d}_\beta(1) = st^\omega$.

Other results on the complexity of the sequences u_β for β being a Parry number were obtained by Frougny et al. [63]. Their results, covering a class of simple Parry numbers, are as follows.

Theorem 2.3.3 ([63]). *Let $\beta > 1$ such that $\mathrm{d}_\beta(1) = t_1 t_2 \cdots t_m$, and either $t_1 > \max\{t_2, \ldots, t_{m-1}\}$ or $t_1 = t_2 = \cdots = t_{m-1}$.*

1. *Suppose that $t_m = 1$. Then we have*

$$\mathcal{C}(n) = (m-1)n + 1\,.$$

2. *Suppose that $t_m > 1$. Then the complexity of the infinite word u_β satisfies*

$$(m-1)n + 1 \le \mathcal{C}(n) \le mn\,.$$

In the later case the authors gave the precise value in terms of the coefficients of the linear recurrent system associated with β, see Bertrand [39].

Concerning the subword complexity of u_β associated with non-simple Parry numbers, much less is known. Exact values of complexity are known only for some special cases, cf. [63, 22, 64].

Chapter 3

Arithmetics of β-expansions

This chapter pursues further the arithmetic issues of the β-numeration systems. At first we inspect the Finiteness property (F). We obtain new necessary and sufficient conditions so that β satisfies it; these conditions are expressed in terms of the so-called minimal forbidden words.

Then we turn ourselves to the previously discussed problem of the maximal number of fractional digits arising under arithmetic operations of β-integers. At first, we use a method due to Guimond et al. (Theorem 2.2.8). We apply it to obtain upper estimates in the case where β is a generalized Tribonacci number, that is, the algebraic integer with minimal polynomial of the form $x^3 - mx^2 - x - 1$, $m \geq 2$. In the Tribonacci case ($m = 1$), we obtain that $5 \leq L_\oplus(\beta) \leq 6$ and $4 \leq L_\otimes(\beta) \leq 5$. The exact bound $L_\oplus(\beta) = 5$ has been obtained by Bernat [31]. For $m \geq 2$ we get

$$\begin{aligned} 5 \leq L_\oplus(\beta) &\leq 6 \qquad \text{for } m = 2\,, \\ 4 \leq L_\oplus(\beta) &\leq 5 \qquad \text{for } m \geq 3\,, \\ 4 \leq L_\otimes(\beta) &\leq 6 \qquad \text{for } m \geq 2\,. \end{aligned}$$

We then consider the case of totally real cubic Pisot units with minimal polynomial of the form $x^3 - ax^2 - bx + 1$, $a \geq 2$ and $1 \leq b \leq a - 1$. We show that

$$\begin{aligned} L_\oplus(\beta) &\leq 2 \qquad \text{for all } a \geq a_0 \text{ for some } a_0\,, \\ L_\otimes(\beta) &\leq 3 \qquad \text{for } a \geq 2\,. \end{aligned}$$

We also find exact values for the "boundary cases" of this class, that is, for $b = 1$ and for $b = a - 1$

$$\begin{aligned} &\text{for } b = 1,\ a \geq 3 && L_\oplus(\beta) = 2 \quad \text{and} \quad L_\otimes(\beta) = 3\,, \\ &\text{for } b = a-1,\ a \geq 3 && L_\oplus(\beta) = 1 \quad \text{and} \quad L_\otimes(\beta) = 2\,. \end{aligned}$$

Even though the assumptions of Theorem 2.2.8 are not very strict, it is easy to see that this method is not able to provide estimates in all the cases. We discuss these problematic cases and then we provide another method, partly solving these issues (on the other hand this second method is limited to Pisot numbers only).

At the end of the chapter we study the algorithm performing addition in an arbitrary Pisot numeration system. Note that this algorithm has been implemented in the form of the program `pisotarith`, see Appendix B.

The first section (On the Finiteness property (F)) and the third section (Bounds on $L_\oplus(\beta)$, $L_\otimes(\beta)$ for Pisot numbers (Cut-and-project method)) of this chapter have been published in the Bulletin of the Belgian Mathematical Society [14]. A part of the second section, namely the application of the first method for finding the upper bounds on $L_\oplus(\beta)$ and $L_\otimes(\beta)$ for the generalized Tribonacci base, is the subject of a paper accepted for publication in Journal of Automata, Languages and Combinatorics [16].

3.1 On the Finiteness property (F)

In this section we shall investigate some necessary and some sufficient conditions for β, in order that $\mathrm{Fin}(\beta)$ is a ring. According to the definition, $\mathrm{Fin}(\beta)$ contains both positive and negative numbers. Therefore we first justify why, in order to decide about $\mathrm{Fin}(\beta)$ being a ring, it is enough to study only the question of addition of positive numbers.

Proposition 3.1.1. *Let $\beta > 1$.*

(i) If $\mathrm{d}_\beta(1)$ is infinite, then $\mathrm{Fin}(\beta)$ is not a ring.

(ii) If $\mathrm{d}_\beta(1)$ is finite, then $\mathrm{Fin}(\beta)$ is a ring if and only if $\mathrm{Fin}(\beta)$ is closed under addition of positive elements.

Proof. (i) Let $\mathrm{d}_\beta(1) = t_1t_2t_3\ldots$ be infinite. Then (2.1) implies

$$1 - \frac{1}{\beta} = \frac{t_1 - 1}{\beta} + \frac{t_2}{\beta^2} + \frac{t_3}{\beta^3} + \cdots \tag{3.1}$$

Since $(t_1 - 1)t_2t_3\ldots <_{\mathrm{lex}} \mathrm{d}_\beta(1)$, the expression on the right hand side of (3.1) is the β-expansion of $1 - \beta^{-1}$ which therefore does not belong to $\mathrm{Fin}(\beta)$.

(ii) Let

$$1 = \frac{t_1}{\beta} + \frac{t_2}{\beta^2} + \cdots + \frac{t_m}{\beta^m}\,, \tag{3.2}$$

and let $\mathrm{Fin}(\beta)$ be closed under addition of positive numbers. Consider arbitrary $x \in \mathrm{Fin}(\beta)$ and arbitrary $\ell \in \mathbb{Z}$ such that $x > \beta^\ell$. Then the β-expansion of x has the form $x = \sum_{i=-N}^{n} x_i\beta^i$, where $n \geq \ell$. Repeated application of (3.2) allows us to create a representation of x, say $x = \sum_{i=-M}^{\ell} \tilde{x}_i\beta^i$ such that $\tilde{x}_\ell \geq 1$. Then

$$(\tilde{x}_\ell - 1)\beta^\ell + \sum_{i=-M}^{\ell-1} \tilde{x}_i\beta^i$$

is a finite β-representation of $x - \beta^\ell$. Such a representation can be interpreted as a sum of a finite number of positive elements of $\mathrm{Fin}(\beta)$, which is, according to the assumption, again in $\mathrm{Fin}(\beta)$.

It suffices to realize that subtraction $x - y$ of arbitrary $x, y \in \mathrm{Fin}(\beta)$, $x > y > 0$ is a finite number of subtractions of some powers of β. Therefore $\mathrm{Fin}(\beta)$ being closed under addition of positive elements implies being closed under addition of arbitrary $x, y \in \mathrm{Fin}(\beta)$.

Since multiplication of numbers $x, y \in \mathrm{Fin}(\beta)$ is by the distributive law addition of a finite number of summands from $\mathrm{Fin}(\beta)$, the proposition is proved. $\square$

Let us mention that $\mathrm{d}_\beta(1)$ infinite does not exclude $\mathrm{Fin}(\beta)$ closed under addition of positive elements, see Remark 3.1.6.

From now on, we focus on addition $x + y$ for $x, y \in \mathrm{Fin}(\beta)$, $x, y \geq 0$. As we have already explained before, since the sum and the product of two numbers from $\mathrm{Fin}(\beta)$ can be converted by multiplication by a suitable factor β^k into the sum or the product of two β-integers, it suffices to consider $x, y \in \mathbb{Z}_\beta$.

Let $x, y \in \mathbb{Z}_\beta$, $x, y \geq 0$ with β-expansions $x = \sum_{k=0}^{n} x_k \beta^k$, $y = \sum_{k=0}^{n} y_k \beta^k$. Then $\sum_{k=0}^{n} (x_k + y_k)\beta^k$ is a β-representation of the sum $x + y$. If the sequence of coefficients $(x_n + y_n)(x_{n-1} + y_{n-1}) \cdots (x_0 + y_0)$ verifies the Parry condition (Theorem 2.1.2), we have directly the β-expansion of $x + y$. In the opposite case, the sequence must contain a non-admissible (also called *forbidden*) word.

Special role in our consideration play the so-called minimal forbidden words.

Definition. Let $\beta > 1$. A forbidden word $u_k u_{k-1} \cdots u_0$ of non-negative integers is called *minimal*, if

(i) $u_{k-1} \cdots u_0$ and $u_k \cdots u_1$ are admissible, and

(ii) $u_i \geq 1$ implies $u_k \cdots u_{i+1}(u_i - 1)u_{i-1} \cdots u_0$ is admissible, for all $i = 0, 1, \ldots, k$.

Obviously, a minimal forbidden word $u_k u_{k-1} \cdots u_0$ contains at least one non-zero digit, say $u_i \geq 1$. The word is a β-representation of the addition of two β-integers $z = u_k \beta^k + \cdots + u_{i+1}\beta^{i+1} + (u_i - 1)\beta^i + u_{i-1}\beta^{i-1} + \cdots + u_0$ and $w = \beta^i$. The β-expansion of a number is lexicographically the greatest among all its β-representations, and thus if the sum $z + w$ belongs to $\mathrm{Fin}(\beta)$, then there exists a finite β-representation of $z + w$ lexicographically strictly greater than $u_k u_{k-1} \cdots u_0$, (the β-expansion of $z + w$).

We have thus shown the following necessary condition.

Proposition 3.1.2 (Property T). *If $\mathrm{Fin}(\beta)$ is closed under addition of two positive numbers, then β must satisfy the following property:*

For every minimal forbidden word $u_k u_{k-1} \cdots u_0$ there exists a finite sequence $v_n \cdots v_\ell$ of non-negative integers, such that

1. $k, \ell \leq n$,
2. $v_n \beta^n + \cdots + v_\ell \beta^\ell = u_k \beta^k + \cdots + u_1 \beta + u_0$,
3. $v_n v_{n-1} \cdots v_\ell >_{\mathrm{lex}} \underbrace{00 \cdots 0}_{(n-k) times} u_k \cdots u_0$.

The rewriting of the β-representation $z = u_k\beta^k + \cdots + u_1\beta + u_0$ on a lexicographically strictly greater β-representation $z = v_n\beta^n + \cdots + v_\ell\beta^\ell$ will be called a *transcription*.

In general we shall apply the transcription on a β-representation of a number z in the following way. Every β-representation of z which contains a forbidden word can be written as a sum of a minimal forbidden word $\beta^j(u_k\beta^k + \cdots + u_1\beta + u_0)$ and of a β-representation of some number $\tilde{z}$. The new transcribed β-representation of z is obtained by digit-wise addition of the transcription $\beta^j(v_n\beta^n + \cdots + v_\ell\beta^\ell)$ of the minimal forbidden word and the β-representation of $\tilde{z}$.

Obviously, the transcribed β-representation of z is lexicographically strictly greater than the original one. This transcription may be repeated until the β-representation does not contain any forbidden word. In general, it can happen that the procedure may be repeated infinitely many times. Following two theorems provide sufficient conditions, in order that this situation is avoided.

Theorem 3.1.3. *Let $\beta > 1$ satisfy Property T, and suppose that for every minimal forbidden word $u_k u_{k-1} \cdots u_0$ we have the following condition:*

If $v_n v_{n-1} \cdots v_\ell$ is the lexicographically greater word corresponding to $u_k u_{k-1} \cdots u_0$ in the sense of Property T, then

$$v_n + v_{n-1} + \cdots + v_l \leq u_k + u_{k-1} + \cdots + u_0 \,.$$

Then $\operatorname{Fin}(\beta)$ *is closed under addition of positive elements. Moreover, for every positive $x, y \in \operatorname{Fin}(\beta)$, the β-expansion of $x+y$ can be obtained from any β-representation of $x + y$ using finitely many transcriptions.*

Proof. Without loss of generality, it suffices to decide about finiteness of the sum $x+y$, where $x, y \in \mathbb{Z}_\beta$, $\langle x\rangle_\beta = x_n \cdots x_1 x_0 \bullet$ and $\langle y\rangle_\beta = y_n \cdots y_1 y_0 \bullet$.

We prove the theorem by contradiction, i.e. suppose that we can apply a transcription to the β-representation $(x + y)_\beta$ infinitely many times.

We find $M \in \mathbb{N}$ such that $x+y < \beta^{M+1}$. Then the β-representation of $x+y$ obtained after the k-th transcription is of the form

$$x + y = \sum_{i=\ell_k}^{M} c_i^{(k)} \beta^i \,,$$

where ℓ_k is the smallest index of non-zero coefficient in the β-representation after the k-th step.

Since for every exponent $i \in \mathbb{Z}$ there exists a non-negative integer f_i such that $x + y \leq f_i\beta^i$, we have that $c_i^{(k)} \leq f_i$ for every step k.

Realize that for every index $p \in \mathbb{Z}$, $p \leq M$, there are only finitely many sequences $c_M c_{M-1} \cdots c_p$ satisfying $0 \leq c_p \leq f_i$ for all $i = M, M-1, \ldots, p$. Since in every step k the sequence $c_M^{(k)} c_{M-1}^{(k)} \cdots$ lexicographically increases, we can find for every index p the step κ, so that the digits $c_M^{(k)}$, $c_{M-1}^{(k)}$, $\ldots$, $c_p^{(k)}$ are constant for $k \geq \kappa$. Formally, we have

$$(\forall p \in \mathbb{Z},\, p \leq M)(\exists \kappa \in \mathbb{N})(\forall k \in \mathbb{N},\, k \geq \kappa)(\forall i \in \mathbb{Z},\, M \geq i \geq p)(c_i^{(k)} = c_i^{(\kappa)}) \qquad (3.3)$$

Since by assumption of the proof, the transcription can be performed infinitely many times, it is not possible that the digits $c_i^{(\kappa)}$ for $i < p$ are all equal to 0. Let us denote by r the maximal index $r < p$ with non-zero digit, i.e. $c_r^{\kappa} \geq 1$.

In order to obtain the contradiction, we use the above idea (3.3) repeatedly. For $p = 0$ we find $\kappa =: \kappa_1$ and $r =: r_1$ satisfying

$$x + y = \sum_{i=0}^{M} c_i^{(\kappa_1)} \beta^i + c_{r_1}^{(\kappa_1)} \beta^{r_1} + \sum_{i=\ell_{\kappa_1}}^{r_1 - 1} c_i^{(\kappa_1)} \beta^i .$$

In further steps $k \geq \kappa_1$ the digit sum $\sum_{i=0}^{M} c_i^{(k)}$ remains constant, since the digits $c_i^{(k)}$ remain constant. The digit sum $\sum_{i=r_1}^{(-1)} c_i^{(k)} \geq 1$, because the sequence of digits lexicographically increases. For every $k \geq \kappa_1$ we therefore have

$$\sum_{i=r_1}^{M} c_i^{(k)} \quad \geq \quad 1 + \sum_{i=0}^{M} c_i^{(k)} .$$

We repeat the same considerations for $p = r_1$. Again, we find the step $\kappa =: \kappa_2 > \kappa_1$ and the position $r =: r_2 < r_1$, so that for every $k \geq \kappa_2$

$$\sum_{i=r_2}^{M} c_i^{(k)} \quad \geq \quad 1 + \sum_{i=r_1}^{M} c_i^{(k)} .$$

In the same way we apply (3.3) and find steps $\kappa_3 < \kappa_4 < \kappa_5 < \cdots$ and positions $r_3 > r_4 > r_5 > \cdots$ such that the digit sum $\sum_{i=r_s}^{M} c_i^{(\kappa_s)}$ increases with s at least by 1. Since there are infinitely many steps, the digit sum increases with s to infinity, which contradicts the fact that we started with the finite digit sum $\sum_{k=0}^{n}(x_k + y_k)$ and the transcription we use do not increase the digit sum. $\square$

Let us comment on the consequences of the proof for β satisfying the assumptions of the theorem. The β-expansion of the sum of two β-integers can be obtained by finitely many transcriptions where the order in which we transcribe the forbidden words in the β-representation of $x + y$ is not important. However, the proof does not provide an estimate on the number of steps needed. Recall that for rational integers the number of steps depends only on the number of digits of the summed numbers. It is an interesting open problem to determine the complexity of the summation algorithm for β-integers.

In order to check whether β satisfies Property T we have to know all the minimal forbidden words. When $\mathrm{d}_\beta(1) = t_1 t_2 \cdots t_m$, a minimal forbidden word has one of the forms

$$(t_1 + 1), \quad t_1(t_2 + 1), \quad t_1 t_2 (t_3 + 1), \quad \ldots, \quad t_1 t_2 \ldots t_{m-2}(t_{m-1} + 1), \quad t_1 \ldots t_{m-1} t_m .$$

Note that not all the above forbidden words must be minimal. For example if β has the Rényi expansion of unit being $\mathrm{d}_\beta(1) = 111$, the above list of words is equal to 2, 12, 111. However, 12 is not minimal.

Theorem 2.2.4 on closeness of $\mathrm{Fin}(\beta)$ under addition in the case where $d_\beta(1)$ is finite with decreasing digits, is a consequence of our Theorem 3.1.3.

Corollary 3.1.4 ([65]). *Let* $d_\beta(1) = t_1 \cdots t_m$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 1$. *Then* $\mathrm{Fin}(\beta)$ *is closed under addition of positive elements.*

Proof. We shall verify the assumptions of Theorem 3.1.3. Consider the forbidden word $t_1 t_2 \cdots t_{i-1}(t_i + 1)$, for $1 \leq i \leq m-1$. Clearly, the following equality is verified

$$\begin{aligned} t_1\beta^{i-1} + \cdots + (t_{i-2})\beta^2 + t_{i-1}\beta + (t_i+1) = & \\ = \beta^i + (t_1 - t_{i+1})\beta^{-1} + (t_2 - t_{i+2})\beta^{-2} + \cdots + (t_{m-i} - t_m)\beta^{-m+i} + & \\ + t_{m-i+1}\beta^{-m+i+1} + \cdots + t_m\beta^{-m}\,. & \end{aligned}$$

The assumption of the corollary assures that the coefficients on the right hand side are non-negative. The digit sum on the left and on the right is the same. Thus

$$1\underbrace{0\,0\cdots 0}_{i \text{ times}}(t_1 - t_{i+1})(t_2 - t_{i+2})\cdots(t_{m-i} - t_m)$$

is the desired finite word lexicographically strictly greater than $0\,t_1 t_2 \cdots t_{i-1}(t_i + 1)$.

It remains to transcribe the word $t_1 t_2 \ldots t_{m-1} t_m$ into the lexicographically greater word $1\underbrace{0\,0 \ldots 0}_{m \text{ times}}$. ☐

The conditions of Theorem 3.1.3 are however satisfied also for other irrationals that do not fulfil assumptions of 3.1.4. As an example we may consider the minimal Pisot number. It is known that the smallest among all Pisot numbers is β solution of the equation $x^3 = x + 1$, and $d_\beta(1) = 10001$. The number β thus satisfies the relations

$$\beta^3 = \beta + 1 \qquad \text{and} \qquad \beta^5 = \beta^4 + 1\,.$$

The minimal forbidden words are 2, 11, 101, 1001, and 10001. Their transcription according to Property T is the following:

$$\begin{aligned} 2 &= \beta^2 + \beta^{-5} \\ \beta + 1 &= \beta^3 \\ \beta^2 + 1 &= \beta^3 + \beta^{-3} \\ \beta^3 + 1 &= \beta^4 + \beta^{-5} \\ \beta^4 + 1 &= \beta^5 \end{aligned}$$

The digit sum in every transcription is smaller than or equal to the digit sum of the corresponding minimal forbidden word. Therefore $\mathrm{Fin}(\beta)$ is according to Theorem 3.1.3 closed under addition of positive numbers. Since $d_\beta(1)$ is finite, by Proposition 3.1.1 $\mathrm{Fin}(\beta)$ is a ring. This was shown already in [3].

In the assumptions of Theorem 3.1.3 the condition of non-increasing digit sum can be replaced by another requirement.We state it in the following theorem. Its proof uses the idea and notation of the proof of Theorem 3.1.3.

Theorem 3.1.5. *Let $\beta > 1$ be an algebraic integer satisfying Property T, and suppose that at least one of its conjugates, say β', belongs to $(0,1)$. Then $\mathrm{Fin}(\beta)$ is closed under addition of positive elements. Moreover, for every positive $x, y \in \mathrm{Fin}(\beta)$, the β-expansion of $x+y$ can be obtained from any β-representation of $x+y$ using finitely many transcriptions.*

Proof. If it was possible to apply a transcription on the β-representation of $x+y$ infinitely many times, then we obtain the sequence of β-representations

$$x + y = \sum_{i=\ell_k}^{M} c_i^{(k)} \beta^i ,$$

where the smallest indices of the non-zero digits ℓ_k satisfy $\lim_{k\to\infty} \ell_k = -\infty$. Here we have used the notation of the proof of Theorem 3.1.3. Now we use the isomorphism between algebraic fields $\mathbb{Q}(\beta)$ and $\mathbb{Q}(\beta')$ to obtain

$$(x+y)' = x' + y' = \sum_{i=\ell_k}^{M} c_i^{(k)} (\beta')^i \geq (\beta')^{\ell_k} .$$

The last inequality follows from the fact that $\beta' > 0$ and $c_i^{(k)} \geq 0$ for all k and i. Since $\beta' < 1$ we have $\lim_{k\to\infty} (\beta')^{\ell_k} = +\infty$, which is a contradiction. ☐

***Remark* 3.1.6.** Let us point out that an algebraic integer β with at least one conjugate in the interval $(0,1)$ must have an infinite Rényi expansion of unit. Such β has necessarily infinitely many minimal forbidden words. The only examples known up to now of β satisfying Property T and having a conjugate $\beta' \in (0,1)$ have been treated in [65], namely those which have eventually periodic $\mathrm{d}_\beta(1)$ with period of length 1,

$$\mathrm{d}_\beta(1) = t_1 t_2 \cdots t_{m-1} (t_m)^\omega , \qquad \text{with } t_1 \geq t_2 \geq \cdots \geq t_{m-1} > t_m \geq 1 . \tag{3.4}$$

In such a case every minimal forbidden word has a transcription with digit sum strictly smaller than its own digit sum. Thus closure of $\mathrm{Fin}(\beta)$ under addition of positive elements follows already by Theorem 3.1.3. This means that we don't know any β for which Theorem 3.1.5 would be necessary.

From the above remark one could expect that the closure of $\mathrm{Fin}(\beta)$ under addition forces that the digit sum of the transcriptions of minimal forbidden words is smaller than or equal to the digit sum of the corresponding forbidden word. It is not so. For example let β be the solution of $x^3 = 2x^2 + 1$. Then $\mathrm{d}_\beta(1) = 201$ and the minimal forbidden word 3 has the β-expansion

$$3 = \beta + \frac{1}{\beta^2} + \frac{1}{\beta^3} + \frac{1}{\beta^4} ,$$

The digit sum of this transcription of 3 is equal to 4. If there exists another transcription of 3 with digit sum ≤ 3, it must be lexicographically strictly larger than 3•

and strictly smaller than $10\bullet0111$, because the β-expansion is lexicographically greatest among all representations of a number. It can be shown easily that a word with the above properties does not exist. In the same time $\mathrm{Fin}(\beta)$ is closed under addition by Theorem 2.2.5.

On the other hand, Property T is not sufficient for $\mathrm{Fin}(\beta)$ to be closed under addition of positive elements. As an example we can mention a β with $\mathrm{d}_\beta(1) = 100001$. Such β satisfies $\beta^6 = \beta^5 + 1$. Among the conjugates of β there is a pair of complex conjugates, say β', $\beta'' = \overline{\beta'}$, with absolute value $|\beta'| = |\beta''| \doteq 1.0328$. Thus β is not a Pisot number and according to Theorem 2.2.1, $\mathrm{Fin}(\beta)$ cannot be closed under addition of positive elements.

However, Property T is satisfied for β. All minimal forbidden words can be transcribed as follows:

$$\begin{aligned}
2 &= \beta + \beta^{-6} + \beta^{-7} + \beta^{-8} + \beta^{-9} + \beta^{-10} \\
\beta + 1 &= \beta^2 + \beta^{-6} + \beta^{-7} + \beta^{-8} + \beta^{-9} \\
\beta^2 + 1 &= \beta^3 + \beta^{-6} + \beta^{-7} + \beta^{-8} \\
\beta^3 + 1 &= \beta^4 + \beta^{-6} + \beta^{-7} \\
\beta^4 + 1 &= \beta^5 + \beta^{-6} \\
\beta^5 + 1 &= \beta^6
\end{aligned}$$

The expressions on the right hand side are desired transcriptions, since they are finite and lexicographically strictly greater than the corresponding minimal forbidden words.

3.2 Bounds on $L_\oplus(\beta)$ and $L_\otimes(\beta)$ (One conjugate method)

As we have mentioned earlier, $\mathrm{Fin}(\beta)$ can be a ring only for β a Pisot number. However, it is meaningful to study upper bounds on the number of fractional digits that appear as a result of addition and multiplication of β-integers also in the case that $\mathrm{Fin}(\beta)$ is not a ring.

In this section, we address the method for determining upper estimates on $L_\oplus(\beta)$ and $L_\otimes(\beta)$ due to Guimond et al. [69], which was stated in the previous chapter.

We use it to obtain upper estimates on $L_\oplus(\beta)$ and $L_\otimes(\beta)$ for the Tribonacci number, for generalized Tribonacci numbers and for a class of totally real cubic Pisot units. Note that this method is applicable even in the case that β is not a Pisot number.

For a convenience, let us recall the theorem providing the method for determining the upper estimates.

Theorem 3.2.1 ([69]). *Let β be an algebraic number, $\beta > 1$, with at least one conjugate β' satisfying*

$$\begin{aligned}
H &:= \sup\{|z'| \mid z \in \mathbb{Z}_\beta\} < +\infty\,, \\
K &:= \inf\{|z'| \mid z \in \mathbb{Z}_\beta \setminus \beta\mathbb{Z}_\beta\} > 0\,,
\end{aligned}$$

where z' denotes the image of $z \in \mathbb{Q}(\beta)$ under the field isomorphism $' : \mathbb{Q}(\beta) \to \mathbb{Q}(\beta')$. Then

$$\left(\frac{1}{|\beta'|}\right)^{L_\oplus} < \frac{2H}{K} \qquad \text{and} \qquad \left(\frac{1}{|\beta'|}\right)^{L_\otimes} < \frac{H^2}{K}\,.$$

***Remark* 3.2.2.** One can do two very simple observations on H and K.

- Since $H \geq \sup\{|\beta'|^k \mid k \in \mathbb{N}\}$, the condition $H < +\infty$ implies that $|\beta'| < 1$. Moreover, in this case we have
$$H \leq \sum_{i=0}^{\infty} \lfloor\beta\rfloor |\beta'|^i = \frac{\lfloor\beta\rfloor}{1-|\beta'|}\,.$$
- If $\beta' \in (0,1)$, we have for $z \in \mathbb{Z}_\beta \setminus \beta\mathbb{Z}_\beta$ that $|z'| = \sum_{i=0}^{n} z_i(\beta')^i \geq z_0 \geq 1$. The value 1 is achieved for $z = 1$. Therefore $K = 1$.

If the considered algebraic conjugate β' of β is negative or complex, it is complicated to determine the value of K and H. However, for obtaining bounds on $L_\oplus(\beta)$, $L_\otimes(\beta)$ it suffices to have a "reasonable" estimates on K and H. In order to determine a good approximation of K and H we introduce some notation. For $n \in \mathbb{N}$ we shall consider the set

$$E_n := \{z \in \mathbb{Z}_\beta \mid 0 \leq z < \beta^n\}\,.$$

In fact this is the set of all $a_0 + a_1\beta + \cdots + a_{n-1}\beta^{n-1}$ where $a_{n-1}\cdots a_1 a_0$ is an admissible β-expansion. We denote

$$\mathrm{Min}_n := \min\{|z'| \mid z \in E_n, z \notin \beta\mathbb{Z}_\beta\}\,, \tag{3.5a}$$
$$\mathrm{Max}_n := \max\{|z'| \mid z \in E_n\}\,. \tag{3.5b}$$

Lemma 3.2.3. *Let $\beta > 1$ be an algebraic number with at least one conjugate $|\beta'| < 1$. Then*

(i) For all $n \in \mathbb{N}$ we have $H \leq H_n := \frac{\mathrm{Max}_n}{1-|\beta'|^n}$.

(ii) For all $n, m \in \mathbb{N}$ we have $K \geq \widehat{K}_n := \mathrm{Min}_n - |\beta'|^n H$ and $K \geq K_{n,m} := \mathrm{Min}_n - |\beta'|^n H_m$.

(iii) $K > 0$ if and only if there exists $n, m \in \mathbb{N}$ such that $K_{n,m} > 0$.

Proof. (i) Let $z \in \mathbb{Z}_\beta$. Then $z = \sum_{i=0}^{N} b_i\beta^i$ and we can write

$$z \leq (b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1}) + \beta^n(a_n + \cdots + a_{2n-1}\beta^{n-1}) + \cdots\,.$$

Using triangle inequality we have

$$|z'| \leq (1 + |\beta'|^n + |\beta'|^{2n} + \cdots)\mathrm{Max}_n = \frac{\mathrm{Max}_n}{1-|\beta'|^n} =: H_n\,.$$

(ii) Let $z \in \mathbb{Z}_\beta \setminus \beta\mathbb{Z}_\beta$. Then $z = \sum_{i=0}^{N} b_i\beta^i$, $b_0 \neq 0$. Again, by the triangle inequality

$$|z'| \geq \left|\sum_{i=0}^{n-1} b_i\beta'^i\right| - \left|\sum_{i=n}^{N} b_i\beta'^i\right| \geq \mathrm{Min}_n - |\beta'|^n \left|\sum_{i=n}^{N} b_i\beta'^{i-n}\right| > \mathrm{Min}_n - |\beta'|^n H = \widehat{K}_n \,.$$

Hence taking the infimum on both sides we obtain $K \geq \widehat{K}_n$. Using the same inequality and the fact that

$$\mathrm{Min}_n - |\beta'|^n H \geq \mathrm{Min}_n - |\beta'|^n H_m$$

for all $m \in \mathbb{N}$ we have $K \geq K_{n,m}$.

(iii) From the definition of Min_n it follows that Min_n is a decreasing sequence with $\lim_{n\to\infty} \mathrm{Min}_n = K$. If there exists $n, m \in \mathbb{N}$ such that $\mathrm{Min}_n - |\beta'|^n H_m > 0$ we have $K > 0$ from (ii). The opposite implication follows easily from the fact that $\lim_{n\to\infty} K_{n,m} = K$ for any m. □

For a fixed β, the determination of Max_n resp. of Min_n for small n is relatively easy. It suffices to find the maximum resp. minimum of a finite set with small number of elements. If for such n we have $\widehat{K}_n = \mathrm{Min}_n - |\beta'|^n H > 0$ (or $K_{n,m} = \mathrm{Min}_n - |\beta'|^n H_m > 0$ for some m), we obtain the bounds on $L_\oplus(\beta)$ and $L_\otimes(\beta)$ using Theorem 2.2.8. We illustrate this procedure on β solution of $x^3 = x^2 + x + 1$, the so-called Tribonacci number in the next section.

3.2.1 $L_\oplus(\beta)$, $L_\otimes(\beta)$ for the Tribonacci number

Let β be the real root of $x^3 = x^2 + x + 1$, that is, the so-called Tribonacci number. The arithmetics on β-expansions was already studied in [84, 83]. Messaoudi finds the upper bound on the number of β-fractional digits for the Tribonacci multiplication as 9 and later improves the estimation to 5. Arnoux, see [83], conjectures that $L_\otimes(\beta) = 3$. We refute the conjecture of Arnoux and we find the bound for $L_\oplus(\beta)$, as well.

It turns out that the best estimates on $L_\oplus(\beta)$, $L_\otimes(\beta)$ are obtained by Theorem 2.2.8 with approximation of K by $\widehat{K}_n$ for $n = 9$. By inspection of the set E_9 we obtain

$$\begin{aligned} \mathrm{Min}_9 &= |1 + \beta'^2 + \beta'^4 + \beta'^7| \doteq 0.5465\,, \\ \mathrm{Max}_9 &= |1 + \beta'^3 + \beta'^6| \doteq 1.5444\,. \end{aligned}$$

Consider $y \in \mathbb{Z}_\beta$, $y = \sum_{k=0}^{N} a_k\beta^k$. Then from the triangle inequality

$$\begin{aligned} |y'| &\leq \left|\sum_{k=0}^{8} a_k\beta'^k\right| + |\beta'|^9 \left|\sum_{k=9}^{17} a_k\beta'^{k-9}\right| + |\beta'|^{18} \left|\sum_{k=18}^{26} a_k\beta'^{k-18}\right| + \cdots \\ &< \mathrm{Max}_9 \left(1 + |\beta'|^9 + |\beta'|^{18} + \cdots\right) = \frac{\mathrm{Max}_9}{1 - |\beta'|^9}\,. \end{aligned}$$

In this way we have obtained an upper estimate on H, i.e. $H \leq \frac{\mathrm{Max}_9}{1-|\beta'|^9}$. This implies

$$\widehat{K}_9 = \mathrm{Min}_9 - |\beta'|^9 H \geq \mathrm{Min}_9 - |\beta'|^9 \frac{\mathrm{Max}_9}{1 - |\beta'|^9}\,.$$

Hence

$$\left(\frac{1}{|\beta'|}\right)^{L_\oplus} < \frac{2H}{K} \le 2\frac{\mathrm{Max}_9}{1-|\beta'|^9}\left(\mathrm{Min}_9 - |\beta'|^9\frac{\mathrm{Max}_9}{1-|\beta'|^9}\right)^{-1} \doteq 7.5003\,,$$
$$\left(\frac{1}{|\beta'|}\right)^{L_\otimes(\beta)} < \frac{H^2}{K} \le \left(\frac{\mathrm{Max}_9}{1-|\beta'|^9}\right)^2\left(\mathrm{Min}_9 - |\beta'|^9\frac{\mathrm{Max}_9}{1-|\beta'|^9}\right)^{-1} \doteq 6.1908\,.$$

Since

$$\left(\frac{1}{|\beta'|}\right)^5 \doteq 4.5880\,,\qquad \left(\frac{1}{|\beta'|}\right)^6 \doteq 6.2222\,,\qquad \left(\frac{1}{|\beta'|}\right)^7 \doteq 8.4386\,,$$

we conclude that $L_\oplus(\beta) \le 6$, $L_\otimes(\beta) \le 5$.

In order to determine the lower bounds on $L_\oplus(\beta)$, $L_\otimes(\beta)$ we have used a computer program [11] to perform additions and multiplications on a large set of β-expansions. As a result we have obtained examples of a sum with 5 and product with 4 fractional digits, namely:

$$1001011010 + 1001011011 = 10100100100\bullet 10101$$
$$110100100101101 \times 110100100101101 = 11001000100010000100100101011011\bullet 0011$$

We can thus sum up our results as

$$5 \le L_\oplus(\beta) \le 6 \qquad \text{and} \qquad 4 \le L_\otimes(\beta) \le 5\,.$$

Let us note that recently the exact bound for addition, $L_\oplus(\beta) = 5$, was obtained by Bernat [31].

3.2.2 $L_\oplus(\beta)$, $L_\otimes(\beta)$ in generalized Tribonacci base

In this section we will give result on $L_\oplus(\beta)$ and $L_\otimes(\beta)$ in the case of cubic units $\beta > 1$, roots of

$$x^3 = mx^2 + x + 1\,, \qquad m \ge 1\,.$$

We call such base β the generalized Tribonacci number, since the Tribonacci case is obtained for $m = 1$. The equation $x^3 = mx^2 + x + 1$ has a unique real solution β. It satisfies $m < \beta < m + 1$. The other two roots β' and β'' are mutually complex conjugates $|\beta'| = |\beta''| < 1$. Obviously, we have

$$\beta\beta'\beta'' = 1\,, \qquad \beta + \beta' + \beta'' = m\,.$$

We have $\mathrm{d}_\beta(1) = m11$. Therefore the digits in a β-expansion take values in $\{0, 1, \dots, m\}$ and the Parry condition implies for $a_0, a_1, \dots, a_k \in \mathbb{N}$ that

$$\sum_{i=0}^{k} a_i\beta^i \text{ is a } \beta\text{-expansion} \iff a_ia_{i-1}a_{i-2} <_{\mathrm{lex}} m11 \quad \text{for all } i = 2, 3, \dots, k\,.$$

Such numbers β have Property (F) by Theorem 2.2.4 or 2.2.6.

Our aim is to provide estimates on the quantities $L_\oplus(\beta)$, $L_\otimes(\beta)$ for these generalized Tribonacci numbers β. Since the Tribonacci case $m = 1$ has already been solved, we consider $m \geq 2$.

Lower bounds on $L_\oplus(\beta)$, $L_\otimes(\beta)$ can be clearly obtained by determining the length of the fractional part of the sum, respectively product, of some suitably chosen elements $x, y \in \mathbb{Z}_\beta$.

Proposition 3.2.4. *Let β be the real root of $x^3 = mx^2 + x + 1$. Then*

$$L_\oplus(\beta) \geq \begin{cases} 5 & \text{for } m = 2\,, \\ 4 & \text{for } m \geq 3\,, \end{cases} \qquad L_\otimes(\beta) \geq 4\,, \quad \text{for } m \geq 2\,.$$

Proof. We have for $m = 2$,

$$(\beta^4 + \beta^2 + 2\beta) - (\beta^3 + 2) = 2\beta^3 + 2 + \beta^{-3} + \beta^{-4} + \beta^{-5}\,,$$

$$2 \times (\beta^6 + 2\beta^5 + 2\beta^3 + \beta^2 + 2) = \beta^7 + \beta^6 + 2\beta^4 + 2\beta + \beta^{-2} + 2\beta^{-3} + \beta^{-4}\,,$$

and for $m \geq 3$,

$$\begin{aligned} (m\beta^3 + m) + (m\beta^3 + m) \;&=\; 2 \times (m\beta^3 + m) = \\ &=\; \beta^4 + (m-1)\beta^3 + (m-1)\beta^2 + 2\beta \,+ \\ &\qquad + (m-3)\beta^{-1} + (m-1)\beta^{-2} + 2\beta^{-3} + \beta^{-4}\,. \end{aligned}$$

□

Below, we are going to derive the estimates on the upper bounds of $L_\oplus(\beta)$ and $L_\otimes(\beta)$. The method used to find these estimates is the same one as in the previous section, using Max_p and Min_q, with the exception that here we compute values of Max_p and Min_q in a more general way — for a class of numbers and not for a fixed number β.

Recall from the definition of Max_p and Min_q (equations (3.5b) and (3.5a)) that both of them are calculated using the modulus $|z'|$ of the image of β-integers $|z| < \beta^p$ under the field morphism. It turns out to be more convenient to study the square of this quantity.

Let us consider a β-integer z, such that $|z| < \beta^p$, i.e. z is of the form $z = a_0 + a_1\beta + a_2\beta^2 + \cdots + a_{p-1}\beta^{p-1}$, where the integer coefficients a_i satisfy the admissibility condition $a_i a_{i-1} a_{i-2} <_{\text{lex}} m11$. Since the conjugates of β are mutually complex conjugates we have

$$|z'|^2 = z'z'' = \sum_{i=0}^{p-1} a_i \beta'^i \sum_{j=0}^{p-1} a_j \beta''^j = \sum_{i=0}^{p-1}\sum_{j=0}^{p-1} a_i a_j \beta'^i \beta''^j\,.$$

We use the relation $\beta\beta'\beta'' = 1$ to derive for $j > i$,

$$a_i a_j \beta'^i \beta''^j + a_j a_i \beta'^j \beta''^i = a_i a_j \beta'^{j-i} \frac{1}{\beta^i} + a_i a_j \beta''^{j-i} \frac{1}{\beta^i} = \frac{a_i a_j}{\beta^i} 2\Re(\beta'^{j-i})\,.$$

Substituting this to the expression for $|z'|^2$, we obtain

$$|z'|^2 = \sum_{k=0}^{p-1} \frac{a_k^2}{\beta^k} + \sum_{0\le i<j\le p-1} \frac{a_i a_j}{\beta^i}\, 2\Re(\beta'^{j-i})\,. \tag{3.6}$$

Therefore, $|z'|^2$ is a real quadratic form of integer variables $a_0, a_1, \ldots, a_{p-1}$. In order to express it in a simpler form, we need to determine the coefficients $2\Re(\beta'^{j-i})$. Let us denote $c_k := 2\Re(\beta'^k)$. For calculation of c_k we find a recurrent formula,

$$\begin{aligned} c_k c_1 = (\beta'^k + \beta''^k)(\beta' + \beta'') &= \beta'^{k+1} + \beta'^k\beta'' + \beta''^k\beta' + \beta''^{k+1} \\ &= c_{k+1} + \beta'\beta''(\beta'^{k-1} + \beta''^{k-1}) \\ &= c_{k+1} + \frac{1}{\beta} c_{k-1}\,. \end{aligned}$$

Let us enumerate several initial coefficients,

$$c_0 = 2\,, \tag{3.7a}$$

$$c_1 = -\left(\frac{1}{\beta} + \frac{1}{\beta^2}\right)\,, \tag{3.7b}$$

$$c_2 = -\left(\frac{2}{\beta} - \frac{1}{\beta^2} - \frac{2}{\beta^3} - \frac{1}{\beta^4}\right)\,, \tag{3.7c}$$

$$c_3 = \frac{3}{\beta^2} + \frac{2}{\beta^3} - \frac{3}{\beta^4} - \frac{3}{\beta^5} - \frac{1}{\beta^6}\,, \tag{3.7d}$$

$$c_4 = \frac{2}{\beta^2} - \frac{4}{\beta^3} - \frac{7}{\beta^4} + \frac{6}{\beta^6} + \frac{4}{\beta^7} + \frac{1}{\beta^8}\,. \tag{3.7e}$$

Note that for $m \ge 3$ the coefficients satisfy $c_1 < 0$, $c_2 < 0$, $c_3 > 0$ and $c_4 > 0$.

Our aim is to find Max_p and Min_q for suitable p, q so that the bounds on $L_\oplus(\beta)$, $L_\otimes(\beta)$ obtained by Theorem 2.2.8 are the best possible. Computer experiments show that optimal constants are H_3, $K_{5,3}$, that is to say the values of H_n and $K_{m,n}$ change too little with increasing n, m and do not provide better estimates on the values of $L_\oplus(\beta)$ and $L_\otimes(\beta)$. Thus we need to calculate Max_3 and Min_5.

Since we are interested in Max_3, the real quadratic form $|z'|^2$ to be examined simplifies to

$$Q_1(a_2, a_1, a_0) := |z'|^2 = a_0^2 + \frac{a_1^2}{\beta} + \frac{a_2^2}{\beta^2} + a_0 a_1 c_1 + a_0 a_2 c_2 + \frac{a_1 a_2}{\beta} c_1\,. \tag{3.8}$$

In order to calculate the value of Max_3 we have to find the maximum of the quadratic form $Q_1(a_2, a_1, a_0)$ on the set

$$\mathcal{S}_1 := \{(a_2, a_1, a_0) \in \mathbb{Z}^3 \mid 0 \le a_i \le m,\ a_2 a_1 a_0 <_{\mathrm{lex}} m11\}\,. \tag{3.9}$$

Similarly, for calculation of Min_5, we use the quadratic form

$$\begin{aligned} Q_2(a_4, a_3, a_2, a_1, a_0) := |z'|^2 = a_0^2 + \frac{a_1^2}{\beta} + \frac{a_2^2}{\beta^2} + \frac{a_3^2}{\beta^3} + \frac{a_4^2}{\beta^4} + a_0 a_1 c_1 + a_0 a_2 c_2 + \\ + a_0 a_3 c_3 + a_0 a_4 c_4 + \frac{a_1 a_2}{\beta} c_1 + \frac{a_1 a_3}{\beta} c_2 + \frac{a_1 a_4}{\beta} c_3 + \frac{a_2 a_3}{\beta^2} c_1 + \frac{a_2 a_4}{\beta^2} c_2 + \frac{a_3 a_4}{\beta^3} c_1\,. \end{aligned} \tag{3.10}$$

We have to find a minimum of this quadratic form on the set

$$\mathcal{S} := \{(a_4, a_3, a_2, a_1, a_0) \in \mathbb{Z}^5 \mid m \geq a_i \geq 0, a_0 > 0 \text{ and } a_{k+2}a_{k+1}a_k \prec m11\}\,. \tag{3.11}$$

The condition $a_0 \neq 0$ in the definition of set $\mathcal{S}$ corresponds to the condition $z \in \mathbb{Z}_\beta \backslash \beta\mathbb{Z}_\beta$ in the definition of Min_p.

For finding the extremal values of the quadratic forms, we inspect the first differences of the quadratic form for each variable a_i, each time fixing all yet appointed values of the variables. The difference in a variable a_i will be denoted

$$\Delta_{a_i} := Q(\dots, a_i + 1, \dots) - Q(\dots, a_i, \dots)\,,$$

where $a_i \leq m - 1$ and the variables a_j, $j \neq i$ take values at most m. Using this method we prove the following result.

Proposition 3.2.5. *Let $\beta > 1$ be the real root of the equation $x^3 = mx^2 + x + 1$ for $m \in \mathbb{N}$, $m \geq 2$. Then*

$$\begin{aligned}
\mathrm{Max}_3 &= \max\{\, |z'| \;\big|\; z \in \mathbb{Z}_\beta,\ |z| < \beta^3 \} = m\,,\\
\mathrm{Min}_5 &= \min\{\, |z'| \;\big|\; z \in \mathbb{Z}_\beta \setminus \beta\mathbb{Z}_\beta,\ |z| < \beta^5 \} = \begin{cases} \beta^{-2}(\beta+1) & \text{for } m \geq 3 \\ |1 + 2\beta'^2 + 2\beta'^4| & \text{for } m = 2\,. \end{cases}
\end{aligned}$$

Proof. **Determining $\mathbf{Min_5}$.** In the case $m = 2$ the set $\mathcal{S}_2$ defined in (3.11) has only 79 elements. It is easy to enumerate them one by one and to find the value of Min_5. Therefore, from now on we will consider $m \geq 3$.

1) Difference Δ_{a_0}. The inspected first difference of the quadratic form Q_2 is

$$\Delta_{a_0} = 2a_0 + 1 + c_1a_1 + c_2a_2 + c_3a_3 + c_4a_4\,.$$

Since $a_0 \geq 1$, $c_1 < 0$, $c_2 < 0$, $c_3 > 0$, $c_4 > 0$ and $a_1, \dots, a_4 \in \{0, \dots, m\}$ we have

$$\begin{aligned}
\Delta_{a_0} &\geq 3 + mc_1 + mc_2 = 3 - m\left(\tfrac{1}{\beta} + \tfrac{1}{\beta^2}\right) - m\left(\tfrac{2}{\beta} - \tfrac{1}{\beta^2} - \tfrac{2}{\beta^3} - \tfrac{1}{\beta^4}\right)\\
&\geq 3 - \tfrac{3m}{\beta} > 0\,,
\end{aligned}$$

where the last inequality follows from the fact that $m = \lfloor \beta \rfloor$.

Therefore, the form is increasing in the variable a_0 and so the minimum is reached at the smallest possible value of a_0, i.e. at $a_0 = 1$. From now on we will consider $a_0 = 1$.

2) Difference Δ_{a_1}. The inspected difference is

$$\Delta_{a_1} = \frac{2a_1 + 1}{\beta} + c_1 + c_1\frac{a_2}{\beta} + c_2\frac{a_3}{\beta} + c_3\frac{a_4}{\beta}\,.$$

Since $c_1 < 0$, $c_2 < 0$, $c_3 > 0$ we have

$$\Delta_{a_1} \geq \frac{2a_1}{\beta} - \frac{1}{\beta^2} - \frac{m}{\beta}\left(\frac{1}{\beta} + \frac{1}{\beta^2}\right) - \frac{m}{\beta}\left(\frac{2}{\beta} - \frac{1}{\beta^2} - \frac{2}{\beta^3} - \frac{1}{\beta^4}\right)\frac{2a_1}{\beta} - \frac{3m+1}{\beta^2}\,.$$

The right side of the inequality is strictly greater than zero for $a_1 \geq 2$. The minimum is therefore reached for either $a_1 = 0$, $a_1 = 1$ or $a_1 = 2$.

3) Difference Δ_{a_2}. The inspected difference is

$$\Delta_{a_2} = \frac{2a_2+1}{\beta^2} + c_2 + c_1\frac{a_1}{\beta} + c_1\frac{a_3}{\beta^2} + c_2\frac{a_4}{\beta^2}\,.$$

For the difference $\Delta_{a_2} := Q_2(a_4, a_3, a_2+1, a_1, a_0) - Q_2(a_4, a_3, a_2, a_1, a_0)$ one needs to consider only $a_2 \leq m-1$. Since $c_1, c_2 < 0$, we obtain

$$\Delta_{a_2} \leq \frac{2m-1}{\beta^2} + c_2 = \frac{2m-1}{\beta^2} - \frac{2m-1}{\beta^2} - \frac{1}{\beta^4} = -\frac{1}{\beta^4} < 0.$$

The quadratic form is decreasing in the variable a_2 and the minimum is reached for the highest possible (w.r.t. Parry's condition) value of a_2.

4) Difference Δ_{a_3}. The inspected difference is

$$\Delta_{a_3} = \frac{2a_3+1}{\beta^3} + c_3 + c_2\frac{a_1}{\beta} + c_1\frac{a_2}{\beta^2} + c_1\frac{a_4}{\beta^3}\,.$$

We have shown that the minimum of the quadratic form is reached for $a_1 \in \{0,1,2\}$. It is suitable to discuss the cases $a_1 \in \{0,1\}$ and $a_1 = 2$ separately.

Let us assume $a_1 \in \{0,1\}$. Since $a_2, a_4 \leq m < \beta$ and $c_1, c_2 < 0$ we have

$$\Delta_{a_3} \geq \frac{1}{\beta^3} + c_3 + \frac{c_2}{\beta} + \frac{c_1}{\beta} + \frac{c_1}{\beta^2} = \frac{2(m-1)}{\beta^4} + \frac{1}{\beta^6} > 0\,.$$

This means that for $a_1 \in \{0,1\}$ the minimum of the form is reached for $a_3 = 0$. Moreover, we already know that the minimum is reached for the highest possible value of a_2 (w.r.t. Parry's condition), therefore, we will inspect following candidates for the minimum $(a_4, 0, m-1, 1, 1)$ and $(a_4, 0, m, 0, 1)$.

The discussion for $a_1 = 2$ will be postponed for this moment.

5) Difference Δ_{a_4}. The inspected difference is

$$\Delta_{a_4} = \frac{2a_4+1}{\beta^4} + c_4 + c_3\frac{a_1}{\beta} + c_2\frac{a_2}{\beta^2} + c_1\frac{a_3}{\beta^3}\,.$$

For the quintuple $(a_4, 0, m-1, 1, 1)$ we obtain

$$\Delta_{a_4} \geq \frac{1}{\beta^4} + c_4 + c_3\frac{1}{\beta} + c_2\frac{m-1}{\beta^2} > 0\,.$$

Hence we have the first candidate for the point where the form $Q_2(a_4, a_3, a_2, a_1, a_0)$ reaches its minimum

$$q_1 := (0, 0, m-1, 1, 1)\,.$$

For the quintuple $(a_4, 0, m, 0, 1)$ we obtain

$$\Delta_{a_4} \le \frac{2m+1}{\beta^4} + c_4 + c_2\frac{m}{\beta^2} < 0\,.$$

Second candidate for the point where the form $Q_2(a_4, a_3, a_2, a_1, a_0)$ reaches its minimum is

$$q_2 := (m, 0, m, 0, 1)\,.$$

Here, we get back to the case $a_1 = 2$. We inspect the difference Δ_{a_4} for the quintuple $(a_4, a_3, a_2, 2, 1)$. Note that due to the Parry's condition $a_2 \le m-1$. We can estimate

$$\Delta_{a_4} \ge \frac{1}{\beta^4} + c_4 + c_3\frac{2}{\beta} + c_2\frac{m-1}{\beta^2} + c_1\frac{m}{\beta^3} = \frac{2}{\beta^3} + \frac{2m-1}{\beta^4} - \frac{3}{\beta^5} > 0\,.$$

Hence we have to inspect the candidate $(0, a_3, a_2, 2, 1)$.

We treat separately the the case $a_3 = m$, which implies $a_2 = 0$. So we have the third candidate

$$q_3 := (0, m, 0, 2, 1)$$

and the cases $a_3 \le m-1$, where according to the negativity of Δ_{a_2} the value of corresponding variable is $a_2 = m-1$. The difference Δ_{a_3} for the quintuple $(0, a_3, m-1, 2, 1)$ is

$$\Delta_{a_3} = \frac{2a_3+1}{\beta^3} + c_3 + c_2\frac{2}{\beta} + c_1\frac{m-1}{\beta^2} = \frac{2a_3-2m+6}{\beta^3} + \frac{1-m}{\beta^4} - \frac{2}{\beta^5} - \frac{1}{\beta^6}\,.$$

It is easy to see that for $a_3 \le m-3$ the $\Delta_{a_3} < 0$ and that for $a_3 \ge m-2$ the $\Delta_{a_3} > 0$. Therefore the fourth candidate for the minimum is

$$q_4 := (0, m-2, m-1, 2, 1)\,.$$

By computing the values of the quadratic form Q_2 in the points q_1, q_2, q_3 and q_4 we obtain

$$\frac{(\beta+1)^2}{\beta 4} = Q_2(q_1) < \min\{Q_2(q_2), Q_2(q_3), Q_2(q_4)\}\,.$$

Therefore, $\mathrm{Min}_5^2 = \beta^{-4}(\beta+1)^2$ for $m \ge 3$.

Determining $\mathbf{Max_3}$. Similarly as in the previous part of the proof we will now inspect the first differences of the quadratic form Q_1 in the individual variables.

From the definition of Max_3 we need to determine the maximal value of Q_1 over the set $\mathcal{S}_1$, which allows the coefficients a_0, a_1, a_2 take any values in $\{0, 1, \dots, m\}$ with the admissibility condition. However, it is sufficient to consider $a_0 > 0$, since Q_1 reaches its maximum on such a point. Otherwise, we would have

$$\mathrm{Max}_3 = |0 + a_1\beta' + a_2(\beta')^2| = |\beta'||a_1 + a_2\beta'| \le |\beta'|\mathrm{Max}_2 < \mathrm{Max}_2\,,$$

which is in contradiction with $\mathrm{Max}_2 \le \mathrm{Max}_3$.

The inspection of the difference Δ_{a_0} of the quadratic form Q_1 is very similar to the one of Q_2 in the first part of the proof. We have

$$\Delta_{a_0} = 2a_0 + 1 + a_1 c_1 + a_2 c_2 \,,$$

$a_0 \geq 1$, $c_1 < 0$ and $c_2 < 0$. Hence

$$\Delta_{a_0} \geq 3 + mc_1 + mc_2 \geq 3 - \frac{3m}{\beta} > 0 \,,$$

the form is increasing in the variable a_0 and so the maximum is reached for the highest possible value of a_0 (w.r.t. Parry's condition).

Since

$$\Delta_{a_1} = \frac{2a_1 + 1}{\beta} + c_1 a_0 + c_1 \frac{a_2}{\beta}$$

is a linear function in the variable a_1 and the coefficient $\frac{2}{\beta}$ at a_1 is positive, the difference Δ_{a_1} is either positive for all values of a_1 or negative for all values of a_1 or negative for some initial values of a_1 and then positive, regardless of the values of a_0 and a_2.

Anyway, the maximum in the variable a_1 is reached for some extremal value of a_1. The same reasoning works also for the maximum in the variable a_2.

One can easily see that there are only six candidates fulfilling preceding conditions on the values of the variables a_2, a_1 and a_0:

	a_2	a_1	a_0
q_1	0	0	m
q_2	0	$m-1$	m
q_3	0	m	1
q_4	$m-1$	$m-1$	m
q_5	$m-1$	m	1
q_6	m	0	m

Finally, by computing the values of the quadratic form $Q_1(a_2, a_1, a_0)$ in the points $q_1, \ldots, q_6$ we have

1. $Q_1(0,0,m) = m^2$.
2. $Q_1(0, m-1, m) = m^2 + \frac{(m-1)^2}{\beta} - m(m-1)\left(\frac{1}{\beta} + \frac{1}{\beta 2}\right) < m^2$.
3. $Q_1(0, m, 1) = 1 + \frac{m^2}{\beta} - m\left(\frac{1}{\beta} + \frac{1}{\beta^2}\right) < m^2$.
4. $Q_1(m-1, m-1, m) = m^2 + \frac{(m-1)^2}{\beta} + \frac{(m-1)^2}{\beta^2} - m(m-1)\left(\frac{1}{\beta} + \frac{1}{\beta^2}\right) -$
$- (m-1)^2\left(\frac{1}{\beta^2} + \frac{1}{\beta^3}\right) - m(m-1)\left(\frac{2}{\beta} - \frac{1}{\beta^2} - \frac{2}{\beta^3} - \frac{1}{\beta^4}\right) < m^2$.
5. $Q_1(m-1, m, 1) = 1 + \frac{m^2}{\beta} + \frac{(m-1)^2}{\beta^2} - m\left(\frac{1}{\beta} + \frac{1}{\beta^2}\right) - m(m-1)\left(\frac{1}{\beta^2} + \frac{1}{\beta^3}\right) -$
$- (m-1)\left(\frac{2}{\beta} - \frac{1}{\beta^2} - \frac{2}{\beta^3} - \frac{1}{\beta^4}\right) < m^2$.

6. $Q_1(m,0,m) = m^2 + \frac{m^2}{\beta^2} - m^2\left(\frac{2}{\beta} - \frac{1}{\beta^2} - \frac{2}{\beta^3} - \frac{1}{\beta^4}\right) < m^2$.

Therefore, $\mathrm{Max}_3^2 = m^2$ and the proposition is proved.

□

***Remark* 3.2.6.** It is noteworthy that a similar computation as in the proof of Proposition 3.2.5 using real quadratic forms and their differences was the bottleneck of [6] and [67] in studying topology of Thurston-Rauzy fractals associated with β-expansions.

Preceding Proposition gives us the values of Max_3 and Min_5. According to the definitions of H_n and $K_{p,q}$ we have for $m \geq 3$,

$$H_3 = \frac{\mathrm{Max}_3}{1-|\beta'|^3} = \frac{m\beta^{3/2}}{\beta^{3/2}-1}$$

and

$$K_{5,3} = \mathrm{Min}_5 - |\beta'|^5 H_3 = \frac{\beta+1}{\beta^2} - \frac{m}{\beta(\beta^{3/2}-1)}\,.$$

Now, we will use these values to obtain upper estimates on the value of $L_\oplus(\beta)$.

Theorem 3.2.7. *Let $\beta > 1$ be the real root of the equation $x^3 = mx^2 + x + 1$ for $m \in \mathbb{N}$. Then*

$$\begin{aligned} 4 \leq L_\oplus(\beta) \leq 5 \qquad &\textit{for } m \geq 3\,,\\ 5 \leq L_\oplus(\beta) \leq 6 \qquad &\textit{for } m = 2\,. \end{aligned}$$

Proof. The upper bound $L_\oplus(\beta) \leq 6$ for $m = 2$ can be easily checked numerically. Since

$$\frac{2H}{K} < \frac{2H_3}{K_{5,3}} \simeq 18.4596 < \left(\frac{1}{|\beta'|}\right)^7 \simeq 26.3628\,,$$

according to the Theorem 2.2.8 we have $L_\oplus(\beta) \leq 6$.

Let us assume $m \geq 3$. Again according to the Theorem 2.2.8 we have

$$\left(\frac{1}{|\beta'|}\right)^{L_\oplus} < \frac{2H}{K} \leq \frac{2H_3}{K_{5,3}}\,.$$

We will prove the inequality

$$\frac{2H_3}{K_{5,3}} = \frac{2m\beta^{3/2}}{\beta^{3/2}-1}\frac{1}{\frac{\beta+1}{\beta^2} - \frac{m}{\beta(\beta^{3/2}-1)}} < \beta^3 = \frac{1}{|\beta'|^6}\,, \tag{3.12}$$

which implies $L_\oplus(\beta) \leq 5$.

After a few simple operations the inequality (3.12) is transformed into

$$2m\beta^{1/2} + m\beta < (\beta+1)(\beta^{3/2}-1)\,. \tag{3.13}$$

For $m \geq 4$ the proof of the inequality (3.13) will be based on the fact that $\beta > \beta^{1/2} + 2$, which holds since

$$\beta > \beta^{1/2} + 2 \ \Leftrightarrow\ (\beta-2)^2 > \beta \ \Leftrightarrow\ m\beta + 1 + \frac{1}{\beta} - 4\beta + 4 > \beta\,.$$

Now we estimate the left hand side of (3.13)

$$m\beta^{1/2}(2+\beta^{1/2}) < \beta^{1/2} m\beta < \beta^{1/2}\left(m\beta + \frac{1}{\beta}\right) = \beta^{1/2}(\beta^2-1) =$$
$$= \beta^{1/2}(\beta-1)(\beta+1) = (\beta+1)(\beta^{3/2}-\beta^{1/2}) < (\beta+1)(\beta^{3/2}-1)\,,$$

which proves (3.13).

In the omitted case $m = 3$, the inequality (3.13) can be proved directly. □

The following theorem yields the main result for multiplication — the estimates of the value of $L_\otimes(\beta)$.

Theorem 3.2.8. *Let $\beta > 1$ be the real root of the equation $x^3 = mx^2 + x + 1$ for $m \in \mathbb{N}$, $m \geq 2$. Then*

$$4 \leq L_\otimes(\beta) \leq 6\,.$$

Proof. Let us assume $m \geq 3$. According to the Theorem 2.2.8 it suffices to show

$$\frac{H^2}{K} \leq \frac{H_3^2}{K_{5,3}} = \frac{m2}{(1-\beta^{-3/2})^2}\ \frac{1}{\frac{\beta+1}{\beta^2} - \beta^{-5/2}\frac{m}{1-\beta^{-3/2}}} < \frac{1}{|\beta'|^7}\,,$$

which can be transformed into

$$(1+m)\beta^3 + \beta^2 + (m+2)\beta + 1 > \left((m+2)\beta + m^2 + 2\right)\beta^{3/2}\,. \tag{3.14}$$

By $m \leq \beta < m+1$ we can estimate the left hand side of (3.14) as

$$(1+m)\beta^3 + \beta^2 + (m+2)\beta + 1 \geq$$
$$(1+m)m^3 + m^2 + (m+2)m + 1 \geq m^3(1+m) + m^2\,,$$

whereas the right hand side as

$$\left((m+2)\beta + m^2 + 2\right)\beta^{3/2} <$$
$$((m+2)(m+1) + m^2 + 2)(m+1)^{3/2} \leq 3m^2(m+1)^{3/2}\,,$$

where the last inequality holds for $m \geq 4$. The omitted cases will be treated separately at the end of the proof.

Using last two estimates we have

$$3m^2(m+1)^{3/2} \leq m^2 + m^3(1+m)\,.$$

It is easy to check that this inequality holds for $m \geq 10$.

Since our estimates were too rough to check the validity of (3.14) for $m \leq 9$, we verified the proposition numerically for these cases as well as for the case $m = 2$ which was omitted at the beginning of the proof, see Appendix A. □

Let us mention that computer experiments support the hypothesis that $L_\otimes(\beta) = 4$ for $m \geq 2$.

3.2.3 $L_\oplus(\beta)$, $L_\otimes(\beta)$ for a class of totally real cubic Pisot units

We conclude the study of $L_\oplus(\beta)$, $L_\otimes(\beta)$ by means of the method 2.2.8 by providing results for several other cubic Pisot units.

Let us recall results by Akiyama and Bassino giving complete characterization of minimal polynomials and Rényi expansions of unity for cubic Pisot units.

Lemma 3.2.9 (Akiyama [3]). *Let $\beta > 1$ be a cubic number with minimal polynomial*

$$M(\beta) = x^3 - ax^2 - bx - c. \tag{3.15}$$

Then β is a Pisot number if and only if both inequalities

$$|b-1| < a + c \qquad \textit{and} \qquad (c^2 - b) < \operatorname{sgn}(c)(1 + ac)$$

hold.

Proposition 3.2.10 (Bassino [25]). *Let $\beta > 1$ be a cubic Pisot unit. Then the Rényi expansion of unit is given by the following table.*

	coefficients in $M(\beta)$		$\mathrm{d}_\beta(1)$
$c = 1$	$0 \leq a$	$b = a+1$	$(a+1)(0)(0)(a)(1)$
	$1 \leq a$	$0 \leq b \leq a$	$(a)(b)(1)$
	$2 \leq a$	$b = -1$	$(a-1)(a-1)(0)(1)$
	$3 \leq a$	$-a+1 \leq b \leq -2$	$(a-1)(a+b-1)(a+b)^\omega$
$c = -1$	$2 \leq a$	$1 \leq b \leq a-1$	$(a)((b-1)(a-1))^\omega$
	$3 \leq a$	$-a+3 \leq b \leq 0$	$(a-1)(a+b-1)(a+b-2)^\omega$

The results obtained in the previous two sections refer to the second line of the table from Proposition 3.2.10. Below we derive the estimates on the values of $L_\oplus(\beta)$ and $L_\otimes(\beta)$ for cubic Pisot numbers β whose minimal polynomial is of the form

$$x^3 - ax^2 - bx + 1 \qquad \text{where } 2 \leq a \text{ and } 1 \leq b \leq -2.$$

Note that since the product of two complex conjugated numbers is a non-negative number, the fact that $c = -1$ implies that the algebraic conjugates β', β'' of β are real numbers having different sign, i.e. $\beta' \in (0,1)$, $\beta'' \in (-1,0)$. Hence, by Remark 3.2.2, K $= 1$ for β'.

Proposition 3.2.10 gives us $\mathrm{d}_\beta(1) = a((b-1)(a-1))^\omega$. For $z \in \mathbb{Z}_\beta$, $z = \sum_{i=0}^n z_i \beta^i$ we have

$$\begin{aligned} z' = \sum_{i=0}^{n} z_i(\beta')^i \leq a + (a-1)\beta' + \cdots + (a-1)(\beta')^n < \\ < a + (a-1)\beta' + (a-1)(\beta')^2 + \cdots = 1 + (a-1)\frac{1}{1-\beta'} = H\,. \end{aligned} \tag{3.16}$$

From the minimal polynomial (3.15) one can derive in the usual way the following relations between roots and coefficients, namely

$$\beta + \beta' + \beta'' = a\,, \tag{3.17a}$$
$$\beta\beta' + \beta\beta'' + \beta'\beta'' = -b\,, \tag{3.17b}$$
$$\beta\beta'\beta'' = -1\,. \tag{3.17c}$$

Let us recall that $a+1 > \beta > a$, $\beta' \in (0,1)$ and $\beta'' \in (-1,0)$. From (3.17a) we have

$$\beta' + \beta'' = a - \beta < 0 \quad \Rightarrow \quad |\beta'| < |\beta''| \quad \Rightarrow \quad |\beta'|^2 < |\beta'\beta''| = \frac{1}{\beta} < \frac{1}{a}\,.$$

Hence

$$\lim_{a\to\infty} \beta'(a,b) = 0\,, \tag{3.18}$$

independently on b.

Case $b = 1$

We will at first treat the case $b = 1$ separately; the minimal polynomial of β is of the form

$$x^3 - ax^2 - x + 1\,,$$

and therefore from (3.16) we obtain

$$H = 1 + (a-1)\frac{1}{1-\beta'} = \frac{1}{(\beta')^2}\,.$$

For $a \geq 3$ the positive root β' lies in $(0, \frac{1}{2})$. Therefore using the Theorem 2.2.8 we can write

$$2\left(\frac{1}{\beta'}\right)^{L_\oplus(\beta)-1} < \left(\frac{1}{\beta'}\right)\left(\frac{1}{\beta'}\right)^{L_\oplus(\beta)-1} = \left(\frac{1}{\beta'}\right)^{L_\oplus(\beta)} < 2H = 2\frac{1}{(\beta')^2}$$

hence

$$L_\oplus(\beta) \leq 2.$$

For $a = 2$ we have $L_\oplus(\beta) \leq 3$, directly from Theorem 2.2.8.

The constant $L_\otimes(\beta)$ can be estimated independently on a, since

$$\left(\frac{1}{\beta'}\right)^{L_\otimes(\beta)} < H^2 = \left(\frac{1}{\beta'}\right)^4\,,$$

hence

$$L_\otimes(\beta) \leq 3.$$

The lower bounds are obtained due to

for $a \geq 2$ $\qquad (a-1)\ 1 + a = 1\ 0\ 0 \bullet 0\ 1$

and

$$\begin{array}{ll} \text{for } a=2 & 2\times 1\ 2 = 1\ 1\ 1 \bullet 0\ 1 \\ \text{for } a\geq 3 & (a-1)\ a \times 1\ 0\ (a-1)\ 1\ (a-1)\ 0\ (a-1)\ (a-1) = \\ & \qquad = 1\ 0\ (a-2)\ 1\ 0\ a\ 0\ (a-2)\ 0\ 2 \bullet 0\ (a-3)\ 1 \end{array}$$

The results are summarized in the following table

$\mathrm{d}_\beta(1) = a(0(a-1))^\omega$		
	$L_\oplus(\beta)$	$L_\otimes(\beta)$
$a=2$	$2 \leq L_\oplus \leq 3$	$2 \leq L_\otimes \leq 3$
$a \geq 3$	$L_\oplus = 2$	$L_\otimes = 3$

Case $2 \leq b \leq a-1$

The inequalities from Theorem 2.2.8, using (3.16) for H,

$$\left(\frac{1}{\beta'}\right)^{L_\oplus(\beta)} < 2\frac{a-1}{1-\beta'} + 2 \quad \text{and} \quad \left(\frac{1}{\beta'}\right)^{L_\otimes(\beta)} < \left(\frac{a-1}{1-\beta'} + 1\right)^2$$

can be transformed as follows

$$L_\oplus(\beta) < \mathrm{bnd}_\oplus(\beta') := \frac{\ln\left(2\frac{a-\beta'}{1-\beta'}\right)}{\ln\left(\frac{1}{\beta'}\right)} \quad \text{and} \quad L_\otimes(\beta)) < \mathrm{bnd}_\otimes(\beta') := \frac{\ln\left(\frac{(a-\beta')^2}{(1-\beta')^2}\right)}{\ln\left(\frac{1}{\beta'}\right)}.$$

Concerning values of $\mathrm{bnd}_\oplus(\beta')$ and $\mathrm{bnd}_\otimes(\beta')$ we have following results.

Proposition 3.2.11. *Let $\mathrm{d}_\beta(1) = a((b-1)(a-1))^\omega$ be the Rényi expansion of unit with $a \geq 2$ and $2 \leq b \leq a-1$. Then there exists a_0 such that for all $a \geq a_0$ we have $L_\oplus(\beta) \leq 2$.*

Proof. Using $\frac{1}{\beta'^2} > a$ we can estimate

$$\mathrm{bnd}_\oplus(\beta') = \frac{\ln\left(2\frac{a-\beta'}{1-\beta'}\right)}{\ln\left(\frac{1}{\beta'}\right)} < \frac{\ln\left(2\frac{a-\beta'}{1-\beta'}\right)}{\frac{1}{2}\ln(a)} = \frac{\ln 2 + \ln a + \ln\left(\frac{1-\beta'/a}{1-\beta'}\right)}{\frac{1}{2}\ln(a)} =$$

$$= \frac{\frac{\ln 2}{\ln a} + 1 + \frac{\ln\left(\frac{1-\beta'/a}{1-\beta'}\right)}{\ln(a)}}{1/2} \xrightarrow{a\to\infty} 2\,.$$

$\square$

Proposition 3.2.12. *Let $\mathrm{d}_\beta(1) = a((a-2)(a-1))^\omega$ be the Rényi expansion of unit with $a \geq 2$ and $2 \leq b \leq a-1$. Then $L_\otimes(\beta) \leq 3$ for all $a \geq 2$.*

Proof. To prove the proposition it is enough to show that $\mathrm{bnd}_\oplus \le 4$, i.e.

$$\ln\left(\frac{(a-\beta')^2}{(1-\beta')^2}\right) \le 4\ln\left(\frac{1}{\beta'}\right) \qquad \Longrightarrow \qquad \frac{a-\beta'}{1-\beta'} \le \frac{1}{(\beta')^2}\,. \tag{3.19}$$

Since $\beta' \in (0,1)$ we have

$$\frac{a-\beta'}{1-\beta'} < \frac{a}{1-\beta'}$$

and if we show that

$$\frac{a}{1-\beta'} < \frac{1}{(\beta')^2}\,, \tag{3.20}$$

the inequality (3.19) will be proved.

The inequality (3.20), or equivalently $a(\beta')^2 \le 1-\beta'$, can be shown by using the minimal polynomial of β', i.e.

$$a(\beta')^2 + \beta' - 1 = (\beta')^3 - (b-1)\beta' = \beta'((\beta')^2 - (b-1)) \le \beta'((\beta')^2 - 1) < 0\,.$$

□

The bounds obtained in Propositions 3.2.11 and 3.2.12 above are not always the best ones. We have seen already that for $b=1$ in almost all cases values of $L_\oplus(\beta)$ and $L_\otimes(\beta)$ actually reach the estimates. On the other hand in the second extremal case, i.e. $b = a-1$, the opposite is true.

Case $b = a - 1$

We will show $L_\oplus(\beta) \le 1$, that is, $\mathrm{bnd}_\oplus < 2$ or

$$2\frac{a-\beta'}{1-\beta'} < \frac{1}{(\beta')^2}\,. \tag{3.21}$$

From (3.17b) we can estimate

$$\begin{aligned}
\beta(-\beta'-\beta'') + \frac{1}{\beta} &= a-1 \\
-\beta'-\beta'' &= \frac{1}{\beta}\left(a-1-\frac{1}{\beta}\right) > \frac{1}{a+1}\left(a-1-\frac{1}{a}\right) \\
1 > -\beta'' &> \frac{1}{a+1}\left(a-1-\frac{1}{a}\right) + \beta'
\end{aligned}$$

hence

$$\beta' < 1 - \frac{1}{a+1}\left(a-1-\frac{1}{a}\right) = \frac{2+\frac{1}{a}}{a+1}\,. \tag{3.22}$$

The terms on both sides of (3.21) can be estimated as

$$2\frac{a-\beta'}{1-\beta'} < \frac{2a(a+1)}{\left(a-1-\frac{1}{a}\right)} \qquad \text{and} \qquad \left(\frac{a+1}{2+\frac{1}{a}}\right)^2 < \frac{1}{(\beta')^2}.$$

and since the inequality

$$\begin{aligned}
\frac{2a(a+1)}{\left(a-1-\frac{1}{a}\right)} &< \left(\frac{a+1}{2+\frac{1}{a}}\right)^2 \\
2a(a+1)\left(2+\frac{1}{a}\right)^2 &< (a+1)^2\left(a-1-\frac{1}{a}\right) \\
8a+8+\frac{2}{a} &< a^2-2-\frac{1}{a} \\
0 &< a^2-8a-10-\frac{3}{a}
\end{aligned}$$

holds on the interval $[10,\infty)$, the estimate $L_\oplus(\beta) \leq 1$ is valid for all $a \geq 10$.

To solve the remaining cases we numerically checked the inequality (obtained from Theorem 2.2.8)

$$\left(\frac{1}{\beta'}\right)^{L_\oplus} \leq \left(\frac{1}{\beta'}\right) < \frac{2H}{K},$$

for $a = 3,\ldots,9$, see Appendix A.

The estimate on $L_\otimes(\beta) \leq 2$, i.e. $\mathrm{bnd}_\otimes < 3$ is obtained in a similar way. The inequality to be proved is

$$\left(\frac{a-\beta'}{1-\beta'}\right)^2 < \frac{1}{(\beta')^3}.$$

Again, using (3.22), the term on the left, respectively on the right side of the inequality can be estimated by

$$\left(\frac{a-\beta'}{1-\beta'}\right)^2 < \frac{a^2(a+1)^2}{\left(a-1-\frac{1}{a}\right)^2} \qquad \text{and} \qquad \left(\frac{a+1}{2+\frac{1}{a}}\right)^3 < \frac{1}{(\beta')^3}.$$

The inequality

$$\begin{aligned}
\frac{a^2(a+1)^2}{\left(a-1-\frac{1}{a}\right)^2} &< \left(\frac{a+1}{2+\frac{1}{a}}\right)^3 \\
a^2\left(2+\frac{1}{a}\right)^3 &< (a+1)\left(a-1-\frac{1}{a}\right)^2 \\
8a^2+12a+6+\frac{1}{a} &< a^3-a^2-a+3+\frac{3}{a}+\frac{1}{a^2} \\
0 &< a^3-9a^2-13a-3+\frac{2}{a}+\frac{1}{a^2}
\end{aligned}$$

holds on the interval $[11,\infty)$ and therefore the estimate $L_\otimes(\beta) \le 2$ holds for all $a \ge 11$.

As before, to solve the remaining cases the inequality (from Theorem 2.2.8)

$$\left(\frac{1}{\beta'}\right)^{L_\otimes} \le \left(\frac{1}{\beta'}\right)^2 < \frac{H^2}{K},$$

was checked numerically for $a = 3, \dots, 10$, see Appendix A.

The following examples of addition

$$\begin{array}{ll} \text{for } a = 2 & 1 + 1\ 2 = 1\ 0\ 0 \bullet 0\ 1 \\ \text{for } a \ge 3 & a\ (a-2) + 1 = 1\ 0\ 0 \bullet 1 \end{array}$$

and multiplication

$$\text{for } a \ge 2 \qquad a \times (a-1)\ a = (a-1)\ 1\ (a-1) \bullet 0\ 1$$

give us lower bounds on values of $L_\oplus(\beta)$ and $L_\otimes(\beta)$.

The results are summarized in the following table

$\mathrm{d}_\beta(1) = a((a-2)(a-1))^\omega$		
	$L_\oplus(\beta)$	$L_\otimes(\beta)$
$a = 2$	$2 \le L_\oplus \le 3$	$2 \le L_\otimes \le 3$
$a \ge 3$	$L_\oplus = 1$	$L_\otimes = 2$

3.2.4 Case $K = 0$

The method of estimation of $L_\oplus(\beta)$ and $L_\otimes(\beta)$ based on Theorem 2.2.8 cannot be used in case that $K = 0$. It is however difficult to prove $K = 0$ for a given algebraic β and its conjugate β'. Particular situation is solved by the following proposition.

Proposition 3.2.13. *Let $\beta > 1$ be an algebraic number and $\beta' \in (-1, 0)$ its conjugate such that $\frac{1}{\beta'^2} < \lfloor\beta\rfloor$. Then $K = 0$.*

Proof. Set $\gamma := \beta'^{-2}$. Digits in the γ-expansion take values in the set $\{0, 1, \dots, \lfloor\gamma\rfloor\}$. Since $\lfloor\gamma\rfloor \le \lfloor\beta\rfloor - 1$ and the Rényi expansion of unit $\mathrm{d}_\beta(1)$ is of the form $\mathrm{d}_\beta(1) = \lfloor\beta\rfloor t_2 t_3 \cdots$, every sequence of digits in $\{0, 1, \dots, \lfloor\gamma\rfloor\}$ is lexicographically smaller than $\mathrm{d}_\beta(1)$ and thus is an admissible β-expansion.

Since $1 < -\beta'^{-1} < \gamma$, the γ-expansion of $-\beta'^{-1}$ has the form

$$-\beta'^{-1} = c_0 + c_1\gamma^{-1} + c_2\gamma^{-2} + c_3\gamma^{-3} + \cdots \tag{3.23}$$

where all coefficients $c_i \le \lfloor\beta\rfloor - 1$.

Let us define the sequence $z_n := 1 + c_0\beta + c_1\beta^3 + c_2\beta^5 + \cdots + c_n\beta^{2n+1}$. Clearly, $z_n \in \mathbb{Z}_\beta \setminus \beta\mathbb{Z}_\beta$ and $z'_n := 1 + \beta'(c_0 + c_1\beta'^2 + c_2\beta'^4 + \cdots + c_n\beta'^{2n})$. According to (3.23) we have $\lim_{n\to\infty} z'_n = 0 = \lim_{n\to\infty} |z'_n|$. Finally, this implies $K = 0$. $\square$

As an example of an algebraic number satisfying assumptions of Propositions 3.2.13 is $\beta > 1$ solution of the equation $x^3 = 25x^2 + 15x + 2$. The algebraic conjugates of $\beta \doteq 25.5892$ are $\beta' \doteq -0.38758$ and $\beta'' \doteq -0.20165$, and so $K = 0$ for both of them. Hence Theorem 2.2.8 cannot be used for determining the bounds on $L_\oplus(\beta)$, $L_\otimes(\beta)$. We thus present another method for finding these bounds and illustrate it further on the above mentioned example.

Note that similar situation happens infinitely many times, for example for a class of totally real cubic numbers, solutions to $x^3 = p^6x^2 + p^4x + p$, for $p \geq 3$. Theorem 2.2.8 cannot be applied to any of them which justifies utility of a new method.

3.3 Bounds on $L_\oplus(\beta)$, $L_\otimes(\beta)$ for Pisot numbers (Cut-and-project method)

The second method for determining upper bounds on $L_\oplus(\beta)$, $L_\otimes(\beta)$ studied here is applicable to β being a Pisot number. This method is based on the so-called cut-and-project scheme.

Let $\beta > 1$ be an algebraic integer of degree d, let $\beta^{(2)}, \ldots, \beta^{(s)}$ be its real conjugates and let $\beta^{(s+1)}$, $\beta^{(s+2)} = \overline{\beta^{(s+1)}}$, $\ldots$, $\beta^{(d-1)}$, $\beta^{(d)} = \overline{\beta^{(d-1)}}$ be its non-real conjugates. Then there exists a basis $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_d$ of the space $\mathbb{R}^d$ such that every $\vec{x} = (a_0, a_1, \ldots, a_{d-1}) \in \mathbb{Z}^d$ has in this basis the form

$$\vec{x} = \alpha_1\vec{x}_1 + \alpha_2\vec{x}_2 + \cdots + \alpha_d\vec{x}_d\,,$$

where

$$\alpha_1 = a_0 + a_1\beta + a_2\beta^2 + \cdots + a_{d-1}\beta^{d-1} =: z \in \mathbb{Q}(\beta)$$

and

$$\begin{aligned} \alpha_i &= z^{(i)} && \text{for } i = 2, 3, \ldots, s,\\ \alpha_j &= \Re(z^{(j)}) && \text{for } s < j \leq d,\ j \ \text{ odd},\\ \alpha_j &= \Im(z^{(j)}) && \text{for } s < j \leq d,\ j \ \text{ even}. \end{aligned}$$

Technical details of the construction of the basis $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_d$ can be found in [2, 69, 66].

Note that the ring $\mathbb{Z}[\beta]$ can be geometrically interpreted as a projection of the lattice $\mathbb{Z}^d$ on a suitable chosen straight line in $\mathbb{R}^d$. The correspondence $(a_0, a_1, \ldots, a_{d-1}) \mapsto a_0 + a_1\beta + a_2\beta^2 + \cdots + a_{d-1}\beta^{d-1}$ is a bijection of the lattice $\mathbb{Z}^d$ on the ring $\mathbb{Z}[\beta]$.

In the following, we shall consider β an irrational Pisot number. Important property that will be used is the inclusion

$$\mathbb{Z}_\beta \subset \mathbb{Z}[\beta]\,. \tag{3.24}$$

Let us recall that $\mathbb{Z}_\beta$ is a proper subset of $\mathbb{Z}[\beta]$, since $\mathbb{Z}[\beta]$ is dense in $\mathbb{R}$ as a projection of the lattice $\mathbb{Z}^d$, whereas $\mathbb{Z}_\beta$ has no accumulation points. Since $\mathbb{Z}[\beta]$ is a ring,

$$\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}[\beta] \qquad \text{and} \qquad \mathbb{Z}_\beta \times \mathbb{Z}_\beta \subset \mathbb{Z}[\beta]\,.$$

Consider an $x \in \mathbb{Z}_\beta$ with the β-expansion $x = \sum_{k=0}^{n} a_k\beta^k$. Then

$$|x^{(i)}| = \left|\sum_{k=0}^{n} a_k(\beta^{(i)})^k\right| < \sum_{k=0}^{\infty}[\beta]\left|\beta^{(i)}\right|^k = \frac{[\beta]}{1 - |\beta^{(i)}|}\,,$$

for every $i = 2, 3, \dots, d$. Therefore we can define

$$H_i := \sup\{|x^{(i)}| \mid x \in \mathbb{Z}_\beta\} \tag{3.25}$$

The inclusion (3.24) thus can be precised to

$$\mathbb{Z}_\beta \subset \{x \in \mathbb{Z}[\beta] \mid |x^{(i)}| < H_i,\ i = 2, 3, \dots, d\}\,.$$

Another important property needed for determination of bounds on $L_\oplus(\beta)$, $L_\otimes(\beta)$ is the finiteness of the set

$$C(l_1, l_2, \dots, l_d) := \{x \in \mathbb{Z}[\beta] \mid |x| < l_1,\ |x^{(i)}| < l_i,\ i = 2, 3, \dots, d\}\,,$$

for every choice of positive $l_1, l_2, \dots, l_d$. A point $a_0 + a_1\beta + \cdots + a_{d-1}\beta^{d-1}$ belongs to $C(l_1, l_2, \dots, l_d)$ only if the point $(a_0, a_1, \dots, a_{d-1})$ of the lattice $\mathbb{Z}^d$ has all coordinates in the basis $\vec{x}_1, \dots, \vec{x}_d$ in a bounded interval $(-l_i, l_i)$, i.e. $(a_0, a_1, \dots, a_{d-1})$ belongs to a centrally symmetric parallelepiped. Every parallelepiped contains only finitely many lattice points.

Let us mention that notation $C(l_1, l_2, \dots, l_d)$ is kept in accordance with [2], where Akiyama finds some conditions for $\mathrm{Fin}(\beta)$ to be a ring according to the properties of $C(l_1, l_2, \dots, l_d)$. Our aim here is to use this set for determining the bounds on the length of the β-fractional part of the results of additions and multiplications in $\mathbb{Z}_\beta$.

Theorem 3.3.1. *Let β be a Pisot number of degree d, and let $H_2, H_3, \dots, H_d$ be defined by* (3.25)*. Then*

$$L_\oplus(\beta) \le \max\{\mathrm{fp}(r) \mid r \in \mathrm{Fin}(\beta) \cap C(1, 3H_2, 3H_3, \dots, 3H_d)\}\,,$$

$$L_\otimes(\beta) \le \max\{\mathrm{fp}(r) \mid r \in \mathrm{Fin}(\beta) \cap C(1, H_2^2 + H_2, \dots, H_d^2 + H_d)\}\,.$$

Proof. Consider $x, y \in \mathbb{Z}_\beta$ such that $x + y > 0$, $x + y \in \mathrm{Fin}(\beta)$. Set $z := \max\{w \in \mathbb{Z}_\beta \mid w \le x + y\}$. Then $r := x + y - z$ is the β-fractional part of $x + y$ and thus $r \in \mathrm{Fin}(\beta)$ and $\mathrm{fp}(r) = \mathrm{fp}(x+y)$, and $0 \le r < 1$. Numbers x, y, z belong to the ring $\mathbb{Z}[\beta]$ and hence also $r \in \mathbb{Z}[\beta]$. From the triangle inequality

$$|r^{(i)}| = |x^{(i)} + y^{(i)} - z^{(i)}| \le 3H_i$$

for all $i = 2, 3, \dots, d$. Therefore r belongs to the finite set $C(1, 3H_2, 3H_3, \dots, 3H_d)$, which together with the definition of $L_\oplus(\beta)$ gives the statement of the theorem for addition. The upper bound on $L_\otimes(\beta)$ is obtained analogically. $\square$

3.3.1 Application to β solution of $x^3 = 25x^2 + 15x + 2$

We apply Theorem 3.3.1 on $\beta > 1$ solution of the equation $x^3 = 25x^2 + 15x + 2$. Recall that such β satisfies the conditions of Proposition 3.2.13 for both conjugates β', β'' and thus Theorem 2.2.8 cannot be used for determining the bounds on $L_\oplus(\beta)$, $L_\otimes(\beta)$.

The Rényi expansion of unit is $\mathrm{d}_\beta(1) = (25)(15)(2)$. Since the minimal polynomial of β satisfies the assumptions of Theorem 3.1.3, the set $\mathrm{Fin}(\beta)$ is a ring.

In case that some of the algebraic conjugates of β is a real number, the bounds from Theorem 3.3.1 can be refined. In our case β is totally real. Let $x \in \mathbb{Z}_\beta$, $x = \sum_{i=0}^{n} a_i\beta^i$. Since $\beta' < 0$, we have

$$x' = \sum_{i=0}^{n} a_i(\beta')^i \leq \sum_{\substack{i=0,\\ i \text{ even}}}^{n} a_i(\beta')^i < \sum_{i=0}^{\infty}(25)(\beta')^{2i} = \frac{25}{1-\beta'^2} =: H_1\,.$$

The lower bound on x' is

$$x' = \sum_{i=0}^{n} a_i(\beta')^i \geq \sum_{\substack{i=0,\\ i \text{ odd}}}^{n} a_i(\beta')^i > \beta' H_1\,.$$

Similarly for x'' we obtain

$$\beta'' H_2 < x'' < \frac{25}{1-\beta''^2} =: H_2\,.$$

Consider $x, y \in \mathbb{Z}_\beta$ such that $x + y > 0$. Again, the β-fractional part of $x + y$ has the form $r = x + y - z$ for some $z \in \mathbb{Z}_\beta$. Thus

$$\begin{gathered}(2\beta'-1)H_1 = \beta' H_1 + \beta' H_1 - H_1 < \ r' = x' + y' - z' \ < H_1 + H_1 - \beta' H_1 = (2-\beta')H_1\\ (2\beta''-1)H_2 < r'' = x'' + y'' - z'' < (2-\beta'')H_2\end{gathered}$$

We have used a computer to calculate explicitly the set of remainders $r = A + B\beta + C\beta^2$, $A, B, C \in \mathbb{Z}$, satisfying

$$\begin{aligned} 0 <& \ A + B\beta + C\beta^2 \ < 1\\ (2\beta'-1)H_1 <& \ A + B\beta' + C\beta'^2 \ < (2-\beta')H_1\\ (2\beta''-1)H_2 <& \ A + B\beta'' + C\beta''^2 < (2-\beta'')H_2 \end{aligned}$$

where for β', β'' we use numerical values, (see Section 3.2.4). The set has 93 elements, which we shall not list here. For every element of the set we have found the corresponding β-expansion. The maximal length of the β-fractional part is 5. Thus $L_\oplus(\beta) \leq 5$.

On the other hand, for $\langle x\rangle_\beta = \langle y\rangle_\beta = (25)(0)(25)\bullet$ one have

$$\langle x + y\rangle_\beta = (1)(24)(12)(11)\bullet(23)(0)(14)(13)(2)\,.$$

Thus we have found the exact value

$$L_\oplus(\beta) = 5\,.$$

In order to obtain bounds on $L_\otimes(\beta)$, we have computed the list of all $r = A + B\beta + C\beta^2$, $A, B, C \in \mathbb{Z}$, satisfying the inequalities

$$\begin{aligned} 0 <& \ A + B\beta + C\beta^2 \ < 1\\ \beta' H_1^2 - H_1 <& \ A + B\beta' + C\beta'^2 \ < H_1^2 - \beta' H_1\\ \beta'' H_2^2 - H_2 <& \ A + B\beta'' + C\beta''^2 < H_2^2 - \beta'' H_2 \end{aligned}$$

In this case we have obtained 8451 candidates on the β-fractional part of multiplication. The longest of them has 7 digits. The lower bound is obtained via multiplying $\langle x\rangle_\beta = \langle y\rangle_\beta = (25)_\bullet$ because

$$\langle x \times y\rangle_\beta = (24)(10)_\bullet(21)(24)(16)(7)(16)(13)(2)\,.$$

Therefore

$$L_\otimes(\beta) = 7\,.$$

Let us mention that the above method can be applied also for example to the case of Tribonacci number, but the bounds obtained in this way are not better than those from Theorem 2.2.8. We get $L_\oplus(\beta) \le 6$, $L_\otimes(\beta) \le 6$.

3.4 Addition in Pisot numeration systems

As we have seen earlier, suppose that β satisfies the assumptions of Theorem 3.1.3. As a consequence, the sum of two β-integers can be obtained by a finite number of transcriptions, as described in the proof. On the other hand, we know (Theorem 2.1.7) that every $x \in \mathbb{Q}(\beta)$ has finite or eventually periodic β-expansion for any Pisot number β. Therefore it was quite reasonable to try to find an algorithm performing addition, which would comply with eventually periodic expansions, and, moreover, which would work in any Pisot numeration system.

In this section we present such an algorithm — or strictly speaking — we present a general normalization algorithm supplemented by a procedure telling how to compute a β-representation of the result beforehand.

3.4.1 Computing a β-representation of the sum

As we have mentioned above the algorithm is composed of two parts: the first one consist in computing a β-representation of the result (usually non-admissible and not over the canonical alphabet), the second one consists in normalization of such a representation. The heart of the process used during the first step lies in the following two facts.

Fact 3.4.1. *Let $x_k \cdots x_0{\bullet}x_{-1}\cdots x_{-m}(x_{-m-1}x_{-m-2}\cdots x_{-m-p})^\omega$ be a β-representation of a real number x. Then $x_k\cdots x_0{\bullet}x_{-1}\cdots x_{-m}x_{-m-1}(x_{-m-2}\cdots x_{-m-p}x_{-m-1})^\omega$ is also a β-representation of x.*

Fact 3.4.2. *Let $x_k \cdots x_0{\bullet}x_{-1}\cdots x_{-m}(x_{-m-1}x_{-m-2}\cdots x_{-m-p})^\omega$ be a β-representation of a real number x. Then $x_k\cdots x_0{\bullet}x_{-1}\cdots x_{-m}((x_{-m-1}x_{-m-2}\cdots x_{-m-p})^l)^\omega$ is also a β-representation of the number u, for any positive integer $l \in \mathbb{N}$.*

Using period transformations described by both previous facts we are able to obtain the desired representation of $x+y$ for any x, y with eventually periodic β-expansions. First we shift the period of x or y so that they start with a coefficient belonging to the same power of β. Then we stretch the periods to the length equal to the least common multiple of their original lengths. Finally, the result is obtained by a simple digit-wise addition.

Example. Let x and y have following representations

$$\langle x\rangle_\beta = 1\bullet 13(312)^\omega\,,$$
$$\langle y\rangle_\beta = 2\bullet 0212(10)^\omega\,.$$

According to the instructions, we the shift period of $\langle x\rangle_\beta$ 2 positions to the left

$$\langle x\rangle_\beta = 1\bullet 1331(231)^\omega\,,$$
$$\langle y\rangle_\beta = 2\bullet 0212(10)^\omega\,,$$

and stretch both periods to the length 6

$$\langle x\rangle_\beta = 1\bullet 1331(231231)^\omega\,,$$
$$\langle y\rangle_\beta = 2\bullet 0212(101010)^\omega\,.$$

Representation of $x+y$ is then obtained by digit-wise addition

$$(x+y)_\beta = 3\bullet 1543(332241)^\omega\,.$$

3.4.2 Normalization algorithm

For the simplification of the following description of the algorithm, we additionally request two rather technical conditions on the representation, namely, we want the input representation as well as the output expansion to have empty integer part, which can be simply accomplished by multiplication by a suitable power of β.

Algorithm 3.4.3. *Input: a β-representation, that is, a sequence $s = 0\bullet s_1 s_2 s_3 \cdots$ over a finite alphabet of digits.*
Output: A sequence $\hat{s} = 0\bullet \widehat{s_1}\widehat{s_2}\widehat{s_3}\cdots$ over the canonical alphabet A_β such that $\pi_\beta(s) = \pi_\beta(\hat{s})$.
Note that for clarity reason we omit $0\bullet$ in both s and $\hat{s}$.

begin
    while not *admissible*(s) **do**
        $s :=$ *step_A*(s);
        test_reoccurrence(s);
        while $\exists\, i : s_i < 0$ **do**
            $s :=$ *step_B*(s);
            test_reoccurrence(s);
    $\hat{s} := s$;
end

Now we will separately describe the functions appearing in the algorithm. Let $s^{(i)}$ denote the sequence s in its state before the i-th transcription (transcriptions are done by functions *step_A* and *step_B*). During the description of the functions we will use the following notation: if $w = w_1 w_2 \cdots$ is a word then $s \oplus_k w$ denotes the word obtained by digit wise addition of w to s starting from the coefficient s_k, that is

$$s \oplus_k w := s_1 s_2 \cdots s_{k-1}(s_k + w_1)(s_{k+1} + w_2) \cdots .$$

Function *step_A*(s)**.** Let us note that due to the condition on the line 5 of the algorithm, the sequence $s^{(i)}$ has only non-negative coefficients every time it is used as a parameter of the function *step_A*. This implies that the non-admissibility of $s^{(i)}$ has to be caused by a factor that is lexicographically greater than $\mathrm{d}^*_\beta(1)$.

Let A_i be the smallest integer such that

$$\sigma^{A_i}(s^{(i)}) \geq_{\text{lex}} \mathrm{d}^*_\beta(1) , \tag{3.26}$$

then the transcription used to banish non-admissibility (3.26) is

$$s^{(i+1)} := s^{(i)} \oplus_{A_i} 1\, \overline{t_1}\, \overline{t_2} \cdots . \tag{3.27}$$

Obviously, the transcription (3.27) may create some negative coefficients in s. The algorithm deals with them by using another transcription, given by the function *step_B*. One can see (line 5 of the algorithm) that this function is used (after each transcription (3.27)) repeatedly as long as there are any negative coefficients.

Function *step_B*(s)**.** Let us inspect the first step of type B, directly following a step of type A. Let B_{i+1} be the smallest integer such that $s^{(i)}_{B_{i+1}+1} < 0$. Then the transcription used is

$$s^{(i+2)} := s^{(i+1)} \oplus_{B_{i+1}} \overline{1}\, t_1\, t_2 \cdots . \tag{3.28}$$

Of course, one single use of this transcription does not have to banish all the negative coefficients, it can even create one new (namely the coefficient at $\beta^{-B_{i+1}}$). Nevertheless, it follows from (3.26) that there exists $r \geq A_i + 1$ such that

$$s^{(i)}_{A_i+1} \cdots s^{(i)}_{r-1} \equiv t_1 t_2 \cdots t_{r-A_i-1} \qquad \text{and} \qquad s^{(i)}_r > t_{r-A_i} ,$$

hence, after applying (3.27) to $s^{(i)}$

$$s^{(i+1)}_{A_i+1} = 0, \ \ldots \ , s^{(i+1)}_{r-1} = 0 \qquad \text{and} \qquad s^{(i+1)}_r > 0 , \tag{3.29}$$

where necessarily $B_{i+1} \geq r$. If there remains a negative coefficient in $s^{(i+2)}$ then it is either one of those created in *step_A* (or strictly speaking it is a negative coefficient created by (3.27), possibly increased afterwards by some t_p by (3.28)), or it is a coefficient -1 at $\beta^{-B_{i+1}}$ (this is possible in the case when we had $s^{(i+1)}_{B_{i+1}} = 0$). In the later case several numbers of successive applications of (3.28) will banish the mentioned coefficient -1. Since the transcription (3.28) will every time shift this -1 one position to the left as long as there is a zero in front of it, the exact number of these steps is given by the number of zeros between $s^{(i+1)}_{B_{i+1}+1}$ and its first non-zero neighbor to the left. The existence of such a non-zero coefficient is ensured by (3.29).

Finally, since we created only a finite number of negative coefficients in *step_A*, after a finite number of employment of *step_B* we will obtain a representation, say $s^{(j)}$, formed only by non-negative coefficients. Consequently, $s^{(j)}$ is either admissible and the algorithm stops (by the condition on line 2), or another step of type A follows.

We give below an example illustrating the so far described functions of the algorithm. The function *test_reoccurrence*, which is used to stop the running of the algorithm in more complicated cases, is not necessary here, so we postpone its description for the moment.

Example. Let $\mathrm{d}^*_\beta(1) = 3(12)^\omega$. We will add the numbers $\langle x\rangle_\beta = 1$ and $\langle y\rangle_\beta = 312$. The β-representation of the result is $(x+y)_\beta = s^{(1)} = 0313$. The process of transcriptions follows

1. In the first step $A_1 = 2$, and we obtain $s^{(2)} = 1001(\overline{1}\,\overline{2})^\omega$.
2. There is a negative coefficient in the representation, hence the condition on line 5 is passed and the next step is of the type B. We have $B_2 = 4$ and $s^{(3)} = 10002(\overline{1}1)^\omega$.
3. Since there is still a negative coefficient ($s^{(3)}_6 = -1$), the condition on line 5 is passed again. This time $B_3 = 5$ and hence $s^{(4)} = 100012(21)^\omega$.
4. Representation $s^{(4)}$ computed in the previous step does not contain any negative coefficient, and, moreover, it does not pass the non-admissibility condition on the line 2. Therefore it is the wanted β-expansion of $x+y$.

Function *test_reoccurrence*(s)**.** The algorithm may stop in two possible ways. Either by fulfilment of the admissibility condition on the line 2, or by finding a "re-occurrence" on line 4 or 7. The idea of the re-occurrence is based upon the following fact.

Fact 3.4.4. *Let $s^{(i)}$ and $s^{(j)}$ be two β-representations obtained in course of the normalization, such that*

$$\begin{aligned} s^{(i)} &= w\ u\ v^\omega\,, \\ s^{(j)} &= w\ z\ u\ v^\omega\,, \end{aligned}$$

where $w \in \mathcal{A}^*_\beta$ is the longest common prefix of $s^{(i)}$ and $s^{(j)}$, $u, v \in \mathbb{Z}^*$, $z \in \mathcal{A}^*_\beta$ and $A_j > |wz|$ or $B_j > |wz|$ (depending on the type of the j-th step). Then we will subsequently obtain

$$\begin{aligned} s^{(i+2(j-i))} &= w\ z\ z\ u\ v^\omega\,, \\ s^{(i+3(j-i))} &= w\ z\ z\ z\ u\ v^\omega\,, \\ &\vdots \end{aligned}$$

Hence the re-occurrence during the normalization will come into being if for two steps, say i and j with $i < j$, following is fulfilled

- both steps are of the same type, either of the type A with $A_i < A_j$ (and we denote $H_i = A_i$ and $H_j = A_j$), or of the type B with $B_i < B_j$ (and we denote $H_i = B_i$ and $H_j = B_j$).
- if w is the longest common prefix of $s^{(i)}$ and $s^{(j)}$ then
$$s^{(i)}_{|w|+1} s^{(i)}_{|w|+2} \cdots = s^{(j)}_{H_j-(H_i-|w|)+1} s^{(j)}_{H_j-(H_i-|w|)+2} \cdots$$
- the word
$$\hat{s} := s^{(j)}_1 s^{(j)}_2 \cdots s^{(j)}_{|w|} \big(s^{(j)}_{|w|+1} \cdots s^{(j)}_{H_j-(H_i-|w|)}\big)^{\omega}$$
is admissible in the β-numeration system.

If all three conditions are satisfied we can stop the algorithm and return $\hat{s}$ as the result, that is, as the β-expansion of $\pi_\beta(s^{(1)})$.

We provide here another example, this time more complicated and requiring the test of re-occurrence for the algorithm to be stopped.

Example. Let $\mathrm{d}_\beta(1) = 2011$, i.e. $\mathrm{d}^*_\beta(1) = (2010)^\omega$. We want to find the sum of $\langle 1\rangle_\beta = 1$ and $\langle y\rangle_\beta = 2$. The starting representation is $(x+y)_\beta = s^{(1)} = 03$. The running of the algorithm is the following one

$$\begin{array}{ll}
 & s^{(1)} = 03 \\
A_1 = 1 & s^{(2)} = 11(0\,\overline{1}\,0\,\overline{2}\,)^\omega \\
B_2 = 3 & s^{(3)} = 11\overline{1}(10\overline{1}0)^\omega \\
B_3 = 2 & s^{(4)} = 101(11\overline{1}2)^\omega \\
B_4 = 5 & s^{(5)} = 10110(1221)^\omega \\
A_5 = 6 & s^{(6)} = 101102(0201)^\omega \\
A_6 = 5 & s^{(7)} = 101110(010\overline{1})^\omega \\
B_7 = 9 & s^{(8)} = 10111001\overline{1}(1020)^\omega \\
B_8 = 8 & s^{(9)} = 101110001(1122)^\omega \\
A_9 = 11 & s^{(10)} = 10111000112(0201)^\omega \\
A_{10} = 10 & s^{(11)} = 10111000120(010\overline{1})^\omega \\
B_{11} = 14 & \quad\cdots
\end{array}$$

Indeed, $s^{(11)}$ is not an admissible β-expansion, but if we look at the 7. step and the 11. step we see that

- both are of the same type, with $9 = B_7 < B_{11} = 14$,

- the longest common prefix of $s^{(7)}$ and $s^{(11)}$ is $w = 1011100$, i.e. $|w| = 7$,
- relevant suffixes $s_8^{(7)} s_9^{(7)} \cdots = (10\overline{1}0)^\omega$ and $s_{13}^{(11)} s_{14}^{(11)} \cdots = (10\overline{1}0)^\omega$ are identical,
- the word $\hat{s} = 1011100(01200)^\omega$ is admissible.

Since all the conditions of re-occurrence are fulfilled, $\hat{s}$ is the final result.

Finiteness of the normalization algorithm. Let $s^{(i)}$ and $\hat{s}$ be as in the description of Algorithm 3.4.3. Then we denote by $R(i)$ the longest common prefix of $s^{(i)}$ and $\hat{s}$, that is, "the already correct part of the representation before the k-th step."

We can do following observations on the length of $R(i)$.

- At the beginning of each step of type A, we have $A_i \geq |R(i)|$. Otherwise, $s^{(i)}$ would be of the form drawn in Figure 3.4.2.

$$\underbrace{s_1^{(i)} s_2^{(i)} \cdots \overset{\text{non-admissible factor}}{\overrightarrow{s_{A_i+1}^{(i)} \cdots s_{|R(i)|}^{(i)}}}}_{\text{prefix common with } \hat{s}} \overrightarrow{s_{|R(i)|+1}^{(i)}\, s_{|R(i)|+2}^{(i)}} \cdots$$

Figure 3.1: The situation where $A_i < |R(i)|$

However, under the assumption that the coefficients in $s^{(i)}$ are non-negative before each application of the transcription of the type A, a representation of the form drawn in Figure 3.4.2 is lexicographically greater than the β-expansion $\hat{s}$. This is in contradiction with the definition of β-expansions.

- In addition to previous fact we can exclude the possibility $A_i = |R(i)|$ at the beginning of each step of the type A. Let us suppose the converse, i.e. we have steps I and J, such that $I < J$, both are of the type A, all the step between them are of the type B, and $A_I = |R(I)|$.

  It follows from the description of the algorithm that no step of type B between I and J can affect coefficients with indices less than of equal to A_I. Hence we have $s^{(I)}$ and $s^{(J)}$ of the form drawn in Figure 3.4.2.

  Again, due to the assumption of the non-negativity of all the coefficients in $s^{(J)}$, this representation is lexicographically greater than β-expansion $\hat{s}$, which is the contradiction.

These two observations allow us to formulate following lemma.

Lemma 3.4.5. *Let $i = 1, 2, \ldots$ be the indices of the steps in the normalization algorithm. Then the length of $R(i)$ is a non-decreasing function of i.*

However, to prove that the algorithm can be stopped after a finite number of steps, we would need to prove a stronger statement.

$$I: \quad \underbrace{s_1^{(I)} \cdots s_{|R(I)|-1}^{(I)}\, s_{|R(I)|}^{(I)}}_{\text{prefix common with } \hat{s}} \Big| \overset{\text{non-admissible factor}}{\overrightarrow{s_{|R(I)|+1}^{(I)}\, s_{|R(I)|+2}^{(I)} \cdots}}$$

$$J: \quad s_1^{(J)} \cdots s_{|R(I)|-1}^{(J)}\, (s_{|R(I)|}^{(J)} + 1)\, s_{|R(I)|+1}^{(J)}\, s_{|R(I)|+2}^{(J)} \cdots$$

Figure 3.2: The situation where $A_i = |R(i)|$

Conjecture 3.4.6. *Let Let $i = 1, 2, \ldots$ be the indices of the steps in the normalization algorithm. Then the length of $R(i)$ is not eventually stationary, that is, there does not exists an index I such that $|R(I)| = |R(i)|$ for all $i \geq I$.*

Concerning a possible proof of Conjecture 3.4.6, note that each step of Type A lexicographically increases the transcribed β-representation. In the case of finite representations this fact is clearly enough to prove that the algorithm would eventually stop. However, we are working with representation which are, on principle, right-infinite and one can imagine a situation in which the algorithm would perform transcription every time more far to the right without returning back "to the end of $R(i)$" for some i.

On the other hand, considerable amount of computations was done using this algorithm (cf. Appendix B) and no case which would not eventually stop was observed.

Now let us assume that we were able to show the validity of Conjecture 3.4.6. This would mean that after a finite number of steps one would have $|R(i)| > m + p$, where m and p are lengths of the pre-period, respectively of the period of $\langle x + y \rangle_\beta$. Therefore the algorithm would thereafter produce a periodic output and so it would be stopped by finding the re-occurence.

This implies that the algorithm would work also for β not being a Pisot number provided that $\langle x + y \rangle_\beta$ is finite or eventually periodic.

Chapter 4

Alpha-adic expansions

In this chapter we study another mode of representation of numbers, different from the representations based on β-expansions, but strongly connected with them. It is called the α-adic representation and, roughly speaking, is a representation of a complex (or real) number in a form of (possibly) left infinite power series in α, where α is a complex (or real) number of modulus less than 1.

We have two sources of inspiration — the β-numeration systems on one hand and the p-adic numbers (representations of numbers in the form of left infinite power series in a prime p) on the other hand. However, contrary to the usual p-adic numbers the base of the α-adic system is in modulus smaller than one. This fact implies an important advantage over the usual p-adic expansions, since we do not have to introduce any special valuation for the series to converge.

As seen before, in the β-numeration numbers are right infinite power series. The deployment of left infinite power series has been used by several authors for different purposes. Vershik [105] (probably the first use of the term fibadic expansion) and Sidorov and Vershik [101] use two-sided expansions to show a connection between symbolic dynamics of toral automorphisms and arithmetic expansions associated with their eigenvalues and for study of the Erdös measure (more precisely two-sided generalization of Erdös measure). Two-sided beta-shifts have been studied in full generality by Schmidt [99]. Ito and Rao [73], and Berthé and Siegel [36] use representations of two-sided β-shift in their study of purely periodic expansions with Pisot unit and non-unit base. The realization by a finite automaton of the odometer on the two-sided β-shift has been studied by Frougny [62].

Left-sided extensions of numeration systems defined by a sequence of integers, like the Fibonacci numeration system, have been introduced by Grabner, Liardet and Tichy [68], and studied from the point of view of the odometer function. The use (at least implicit) of representations infinite to the left is contained in every study of the Rauzy fractal [93], especially in a study of its border, see e.g. Akiyama [2], Akiyama and Sadahiro [6] or Messaoudi [82].

Finally, there is a recent paper by Sadahiro [95] on multiply covered points in the conjugated plane in the case of cubic Pisot units having complex conjugates. Sadahiro's approach to the left infinite expansions is among all mentioned works the closest one to our own.

This chapter is organized as follows. First we define α-adic expansions in the case where α is an algebraic conjugate of a Pisot number β. Recall that, by the results of Bertrand and Schmidt a positive real number belongs to the extension field $\mathbb{Q}(\beta)$ if and only if its β-expansion (which is right infinite) is eventually periodic (Theorem 2.1.7). Thus it is natural to try to get a similar result for the α-adic expansions. We prove that a number belongs to the field $\mathbb{Q}(\alpha)$ if and only if its α-adic expansion is eventually periodic to the left with a finite fractional part. Note that the fields $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ are identical, but our result includes also negative numbers, that means one can represent by α-adic expansions with positive digits also negative numbers without utilization of the sign.

Further on, we consider α-adic expansions of elements of the ring $\mathbb{Z}[\alpha^{-1}]$ in the case when β satisfies the Finiteness property (F). We give two algorithms for computing these expansions — one for positive and one for negative numbers. Finally, in the case of quadratic Pisot units, we study the unicity of the expansions of elements of the ring $\mathbb{Z}[\alpha^{-1}]$. We give an algorithm for computing an α-adic representation of a rational number and we discuss normalization of such representation by means of a finite transducer.

In the last section we discuss the arithmetics of eventually periodic α-adic expansions from a more practical point of view. We limit ourselves to one particular irrationality — the golden mean τ — and we provide arithmetic algorithms on the ring of left infinite eventually periodic expansions. Similarly to the addition in a general Pisot base, an algorithm of an arithmetic operation is composed of two parts: at first we compute an α-adic representation of the result, which is normalized afterwards.

The normalization is carried out by a non-sequential transducer, which is constructed for this purpose. To this transducer one then applies the determinization algorithm (the usual subset construction), however, the resulting machine is an infinite transducer with a countable number of states. We show that with a suitable preprocessing of input words only a finite portion of the transducer is used. This and the fact that the output is an eventually periodic word permits us to stop the computation after a finite number of steps.

The first part (the four sections concerning the algebraic properties of α-adic expansions) have been submitted for publication in Theoretical Computer Science [13], while a preliminary version of the same work has been published in the proceedings of the conference Words'05 [12]. The last section (the arithmetics of τ-adic expansions) has been published in the proceedings of the conference CANT 2006 [10].

4.1 Alpha-adic expansions

Arrangement. Throughout the whole chapter β will stand for a Pisot number such that $\mathrm{d}_\beta(1) = t_1 \cdots t_\ell$ and α will be one of its algebraic conjugates.

Definition. An *α-adic representation* of a number $x \in \mathbb{C}$ is a left infinite sequence $(x_i)_{i \geq -k}$ such that $x_i \in \mathbb{Z}$ and

$$x = \cdots + x_2\alpha^2 + x_1\alpha + x_0 + x_{-1}\alpha^{-1} + \cdots + x_{-k}\alpha^{-k}$$

for a certain $k \in \mathbb{Z}$. It is denoted ${}_\alpha(x) = \cdots x_1 x_0 {\bullet} x_{-1} \cdots x_{-k}$.

Definition. A (finite, right infinite or left infinite) sequence is said to be *weakly admissible* if all its finite factors are lexicographically less than or equal to $\mathrm{d}^*_\beta(1)$, which is equivalent to the fact that each factor of length ℓ is less than $t_1 \cdots t_\ell$ in the lexicographic order.
If an α-adic representation $(x_i)_{i \geq -k}$ of a number x is weakly admissible it is said to be an *α-adic expansion* of x, denoted

$${}_\alpha\langle x\rangle = \cdots x_1 x_0 {\bullet} x_{-1} \cdots x_{-k}\,.$$

Example. Let β be the golden mean, that is the Pisot number with minimal polynomial $x^2 - x - 1$. We have $\mathrm{d}_\beta(1) = 11$ and $\mathrm{d}^*_\beta(1) = (10)^\omega$. Hence the sequence $(10)^\omega$ is a forbidden factor of any β-expansion. On the other hand, ${}^\omega(10)010{\bullet}1$ is an α-adic expansion of -2.

***Remark* 4.1.1.** Although the β-expansion of a number is unique, an α-adic expansion is not. For instance in the α-adic system associated with the golden mean, the number -1 has two α-adic expansions

$$\begin{aligned} {}_\alpha\langle -1\rangle &= {}^\omega(10){\bullet} \\ {}_\alpha\langle -1\rangle &= {}^\omega(10)0{\bullet}1 \end{aligned}$$

Analogous to the case of β-representations we define for α-adic representations the *α-value* function π_α and the normalization function ν_C on a digit alphabet C.

4.2 Eventually periodic α-adic expansions

In order to prove the main theorem about eventually periodic expansions, we need two technical lemmas.

Lemma 4.2.1. *Let $y \in (0,1)$ be a real number with the purely periodic β-expansion $\langle y\rangle_\beta = 0{\bullet}(y_{-1}\cdots y_{-p})^\omega$. Then ${}_\alpha\langle -y'\rangle = {}^\omega(y_{-1}\cdots y_{-p}){\bullet}0$.*

Proof. Suppose $y = \frac{y_{-1}}{\beta} + \cdots + \frac{y_{-p}}{\beta^p} + \frac{y_{-1}}{\beta^{p+1}} + \cdots$, which can be also written $y = \frac{y_{-1}}{\beta} + \cdots + \frac{y_{-p}}{\beta^p} + \frac{y}{\beta^p}$. Conjugating the equation we obtain $y' = \frac{y_{-1}}{\alpha} + \cdots + \frac{y_{-p}}{\alpha^p} + \frac{y'}{\alpha^p}$. Hence $-y' = y_{-1}\alpha^{p-1} + \cdots + y_{-p} - y'\alpha^p$ that is ${}_\alpha\langle -y'\rangle = {}^\omega(y_{-1}\cdots y_{-p}){\bullet}0$, which completes the proof. □

Lemma 4.2.2. *Let $x \in (0,1)$ be a number with finite β-expansion $\langle x\rangle_\beta = 0{\bullet}x_{-1}\cdots x_{-p}$, then ${}_\alpha\langle x'\rangle$ is of the form ${}^\omega(t_1\cdots t_{\ell-1}(t_\ell - 1))u_n\cdots u_0{\bullet}u_{-1}\cdots u_{-m}$.*

Proof. Let $\langle x\rangle_\beta = 0\bullet x_{-1}\cdots x_{-p}$ with $x_{-p}\neq 0$. By conjugating it and by changing the sign of its coefficients we obtain an α-adic representation of $-x'$, ${}_\alpha(-x') = 0\bullet\overline{x_{-1}}\cdots\overline{x_{-p}}$, where $\bar{d}$ denotes the signed digit $-d$. If we subtract -1 from the last non-zero coefficient $\overline{x_{-p}}$ and replace it by an α-adic expansion of -1, ${}_\alpha\langle -1\rangle = {}^\omega(t_1\cdots t_{\ell-1}(t_\ell-1))\bullet$, we obtain another representation, which is eventually periodic with a pre-period of the form of a finite word over the alphabet $\{-\lfloor\beta\rfloor,\ldots,\lfloor\beta\rfloor\}$. Finally, an α-adic expansion of $-x'$ is simply obtained by the normalization of the pre-period (cf. Algorithm 4.3.2 and Example 4.3). ☐

Lemma 4.2.1 and 4.2.2 allow us to derive from Theorem 2.1.7 a characterization of numbers with eventually periodic α-adic expansions. The main difference with Theorem 2.1.7 is that the version for α-adic expansions includes also negative numbers, that is one can represent by α-adic expansions with positive digits also negative numbers without the necessity of utilization of the sign.

Theorem 4.2.3. *Let α be a conjugate of a Pisot number β. A number x' has an eventually periodic α-adic expansion if and only if $x'\in\mathbb{Q}(\alpha)$.*

Proof. **$\Leftarrow$:** Let x' have an eventually periodic α-adic expansion, say

$$ {}_\alpha\langle x'\rangle = {}^\omega(x_{k+p}\cdots x_{k+1})x_k\cdots x_0\bullet x_{-1}\cdots x_{-j}\,. $$

Let $u' := \sum_{i=-j}^{k} x_i\alpha^i$ and $v' := \sum_{i=k+1}^{k+p} x_i\alpha^i$. Then $u',v'\in\mathbb{Z}[\alpha^{-1}]$ and

$$ x' = u' + \frac{v'}{1-\alpha^p}\,, $$

which proves the implication.

$\Rightarrow$: Let $x\in\mathbb{Q}(\beta)\cap[0,1)$. According to Theorem 2.1.7 the β-expansion of x is eventually periodic, say $\langle x\rangle_\beta = 0\bullet x_{-1}\cdots x_{-n}(x_{-n-1}\cdots x_{-n-p})^\omega$. In the case where the period of $\langle x\rangle_\beta$ is empty, an eventually periodic α-adic expansion of $-x'$ is obtained by Lemma 4.2.2.
Let us assume that the period of $\langle x\rangle_\beta$ is non-empty and let us denote its value by

$$ y := \pi_\beta(0\bullet(x_{-(n+1)}\cdots x_{-(n+p)})^\omega)\,, $$

therefore

$$ x = \frac{x_{-1}}{\beta} + \cdots\frac{x_{-n}}{\beta^n} + \frac{y}{\beta^n}\,. $$

Conjugating the equation we obtain

$$ \begin{aligned} x' &= \frac{x_{-1}}{\alpha} + \cdots + \frac{x_{-n}}{\alpha^n} + \frac{y'}{\alpha^n}\,,\\ -x' &= -\frac{y'}{\alpha^n} - \frac{x_{-1}}{\alpha} - \cdots - \frac{x_{-n}}{\alpha^n}\,. \end{aligned} $$

According to Lemma 4.2.1 we know how to obtain an α-adic expansion of $-y'$, hence an α-adic representation of $-x'$ can be obtained by digit wise addition

$$\begin{array}{lcccc}
{}^{\omega}(x_{-(n+1)}\cdots x_{-(n+p)})x_{-(n+1)}\cdots x_{-p} \bullet & x_{-(p+1)} & \cdots & x_{-(n+p)} \\
\bullet & (-x_{-1}) & \cdots & (-x_{-n}) \\
\hline
{}^{\omega}(x_{-(n+1)}\cdots x_{-(n+p)})x_{-(n+1)}\cdots x_{-p} \bullet & (x_{-(p+1)}-x_{-1}) & \cdots & (x_{-(n+p)}-x_{-n})
\end{array}$$

Therefore we have ${}_{\alpha}\langle -x' \rangle$ of the form ${}^{\omega}(c_1 \cdots c_p)u$, where u is a finite word, obtained by the normalization of the pre-period $x_{-(n+1)} \cdots x_{-p} \bullet (x_{-(p+1)} - x_{-1}) \cdots (x_{-(n+p)} - x_{-n})$. Note that this pre-period can be seen as a difference between two finite expansions and so the normalization will not interfere with the period.

Now let $x \geq 1$, $x \in \mathbb{Q}(\beta)$. Indeed, there exists a positive integer N such that $x < \beta^N$. Hence $t := 1 - \frac{x}{\beta^N} \in \mathbb{Q}(\beta) \cap [0, 1)$. As we have proved before the number $-t = \frac{x'}{\alpha^N} - 1$ has an eventually periodic α-adic expansion. Therefore an eventually periodic α-adic expansion ${}_{\alpha}\langle x' \rangle$ is simply obtained by adding 1 to ${}_{\alpha}\langle \frac{x'}{\alpha^N} - 1 \rangle$, followed by shifting the fractional point N positions to the left. ☐

4.3 Expansions in bases with Finiteness property (F)

In the previous section we proved a general theorem characterizing α-adic expansions of elements of the extension field $\mathbb{Q}(\alpha)$. If we add one additional condition on β, namely that it fulfills Property (F), we are able to characterize the expansions of elements of the ring $\mathbb{Z}[\alpha^{-1}]$ more precisely.

Proposition 4.3.1. *Let α be a conjugate of a Pisot number β satisfying Property (F). For any $x \in \mathbb{Z}[\beta^{-1}]_+$ its conjugate x' has at least one α-adic expansion. This expansion is finite and ${}_{\alpha}\langle x' \rangle = \langle x \rangle_{\beta}$.*

Proof. Since β has Property (F), $\mathrm{Fin}(\beta) = \mathbb{Z}[\beta^{-1}]$ and any $x \in \mathbb{Z}[\beta^{-1}]_+$ has a finite β-expansion, say $x = \sum_{i=-j}^{k} x_i \beta^i$. By conjugation we have $x' = \sum_{i=-j}^{k} x_i \alpha^i$. ☐

The proof of Proposition 4.3.1 shows us a way how to compute an α-adic expansion of a number x' which is a conjugate of $x \in \mathbb{Z}[\beta^{-1}]_+$. The same task is a little bit more complicated in the case where x' is a conjugate of an $x \in \mathbb{Z}[\beta^{-1}]_-$. An α-adic expansion of such a negative number x' is computed by Algorithm 4.3.2 below.

Algorithm 4.3.2. *Let $x \in \mathbb{Z}[\beta^{-1}]_-$. An α-adic expansion of x' is obtained as follows.*

1. *Use the greedy algorithm to find the β-expansion of $-x$, say*

   $$\langle -x \rangle_{\beta} = {}_{\alpha}\langle -x' \rangle = x_k \cdots x_0 \bullet x_{-1} \cdots x_{-j}\,,$$

   with $x_{-j} \neq 0$. This expansion is finite since β satisfies Property (F).

2. *By changing the sign of coefficients in the expansion, $x_i \mapsto -x_i$, we obtain an α-adic representation of x' in the form of a finite word over the alphabet $\{-\lfloor \beta \rfloor, \ldots, -1, 0\}$.*

3. *Subtract -1 from the rightmost non-zero coefficient x_{-j} and replace it by an α-adic expansion of -1, ${}_\alpha\langle -1\rangle = {}^\omega(t_1\cdots t_{\ell-1}(t_\ell-1))$. The representation of x' has now a periodic part ${}^\omega(t_1\cdots t_{\ell-1}(t_\ell-1))$ and a pre-period, which is a finite word over the alphabet $\{-\lfloor\beta\rfloor,\dots,\lfloor\beta\rfloor\}$.*

4. *Finally, the α-adic expansion of x' is simply obtained by the normalization of the pre-period. Note that the pre-period can be seen as a difference between two finite expansions and so the normalization will not interfere with the period.*

Example. Let β be the golden mean, α its conjugate. Recall that for example ${}_\alpha\langle -1\rangle = {}^\omega(10)\bullet$. We compute an α-adic expansion of the number -4. The β-expansion of 4 is $101\bullet 01$, so $\bar{1}0\bar{1}\bullet 0\bar{1}$ is an α-adic representation of the number -4. Now we subtract -1 from the rightmost non-zero coefficient and replace it by ${}_\alpha\langle -1\rangle$ as follows

$$\begin{array}{r} \bar{1}\,0\,\bar{1}\bullet 0\,\bar{1} \\ \bullet\ \ 1 \\ {}^\omega(1\,0)\,1\,0\,1\,0\bullet 1\,0 \\ \hline {}^\omega(1\,0)\,1\,\bar{1}\,1\,\bar{1}\bullet 1\,0 \end{array}$$

Since the normalization of the pre-period $1\bar{1}1\bar{1}\bullet 10$ gives $0100\bullet 001$, the expansion is ${}_\alpha\langle -4\rangle = {}^\omega(10)0100\bullet 001$.

Proposition 4.3.3. *Let α be a conjugate of a Pisot number β satisfying Property (F). For any $x \in \mathbb{Z}[\beta^{-1}]_-$, its conjugate x' has at least ℓ different α-adic expansions, which are eventually periodic to the left with the period ${}^\omega(t_1\cdots t_{\ell-1}(t_\ell-1))$.*

Proof. First, we show that the number -1 has ℓ different α-adic expansions. Recall that $-1+\pi_\beta(\mathrm{d}_\beta(1)) = 0$, hence $-\alpha^\ell + \alpha^\ell\pi_\alpha(\mathrm{d}_\beta(1)) - 1 = -1$. Therefore we have the first expansion

$$ {}_\alpha\langle -1\rangle = {}^\omega(t_1\cdots t_{\ell-1}(t_\ell-1)) \,. \tag{4.1}$$

Now we successively use the equality $-\alpha^j + \alpha^j\pi_\alpha(\mathrm{d}_\beta(1)) - 1 = -1$ for $j = \ell-1,\dots,1$ to obtain the other $\ell-1$ representations. For given j this equation is

$$-\alpha^j + t_1\alpha^{j-1} + \cdots + t_{j+1}\alpha + (t_j-1) + t_{j-1}\alpha^{-1} + \cdots + t_\ell\alpha^{j-\ell} = -1\,.$$

If we replace the coefficient -1 at α^j by its expansion (4.1) we have

$$ {}_\alpha\langle -1\rangle = {}^\omega(t_1\cdots t_{\ell-1}(t_\ell-1))t_1\cdots t_{j+1}(t_j-1)\bullet t_{j-1}\cdots t_\ell\,. \tag{4.2}$$

Note that periods of expansions obtained in (4.2) are mutually shifted, they are situated on all possible ℓ positions. That is why all these expansions are essentially distinct.
The only difficulty would arise if $t_j = 0$ for some j and hence we would obtain the coefficient -1 at α^0 by equation (4.2). If this is the case we take the pre-period and normalize it

$$t_1\cdots t_{j+1}(t_j-1)\bullet t_{j-1}\cdots t_\ell \overset{\nu_C}{\mapsto} u_1\cdots u_j\bullet u_{j+1}\cdots u_i\,,$$

where $C = \{-1, 0, \dots, \lfloor\beta\rfloor\}$. An α-adic expansion of -1 then will be

$$_\alpha\langle -1\rangle = {}^\omega(t_1 \cdots t_{\ell-1}(t_\ell - 1))u_1 \cdots u_j\bullet u_{j+1} \cdots u_i\,.$$

Then we consider an $x \in \mathbb{Z}[\beta^{-1}]_-$. Using the ℓ different expansions of -1 in Algorithm 4.3.2 gives us ℓ different α-adic expansions of the number x'. ☐

Note that, conversely, if an expansion of a number z' is of the form ${}^\omega(t_1 \cdots t_{\ell-1}(t_\ell - 1))u\bullet v$, then z belongs to $\mathbb{Z}[\beta^{-1}]_-$.

Example. Let β of minimal polynomial $x^3 - x^2 - 1$; such a number is Pisot and satisfies Property (F) [3]. We have $\mathrm{d}_\beta(1) = 101$ and $\mathrm{d}^*_\beta(1) = (100)^\omega$. Let α be one of its (complex) conjugates. The number -1 has three different α-adic expansion

$$\begin{aligned} _\alpha\langle -1\rangle &= {}^\omega(100)\bullet \\ _\alpha\langle -1\rangle &= {}^\omega(100)0\bullet 01 \\ _\alpha\langle -1\rangle &= {}^\omega(100)01\bullet 00001 \end{aligned}$$

Since $t_2 = 0$ the normalization of the pre-period was necessary to obtain an admissible expansion for $j = 2$.

4.4 Quadratic Pisot units

This section is devoted to quadratic Pisot units, i.e. to the algebraic units β, with minimal polynomials of the form $x^2 - ax - 1$, $a \in \mathbb{Z}_+$. Then $\alpha = -\beta^{-1}$. The Rényi expansion of unity is $\mathrm{d}_\beta(1) = a1$ for such a number β, which satisfies Property (F), by Theorem 2.2.4. The canonical alphabet is $\mathcal{A} = \{0, \dots, a\}$.

4.4.1 Unicity of expansions in $\mathbb{Z}[\beta]$

We first establish a technical result.

Proposition 4.4.1. *Let α be the conjugate of a quadratic Pisot unit β. Let $_\alpha\#(x) : \mathbb{R} \to \mathbb{N}$ be the function counting the number of different α-adic expansions of a number x. Then $_\alpha\#(x) < +\infty$ for any $x \in \mathbb{R}$.*

Proof. Let $x \in \mathbb{R}$ and let $_\alpha\langle x'\rangle = u\bullet v$ be an α-adic expansion of x'. Then

$$\begin{aligned} &\pi_\beta(\bullet v) \in \mathbb{Z}[\beta] \cap [0,1)\,, \\ &|x' - \pi_\alpha(\bullet v)| = |\pi_\alpha(u\bullet)| < \frac{\lfloor\beta\rfloor}{1 - |\alpha|}\,. \end{aligned} \tag{4.3}$$

Let $D_x := \{(\pi_\beta(\bullet v), \pi_\alpha(\bullet v)) \mid {}_\alpha\langle x'\rangle = u\bullet v\}$. Clearly by (4.3), D_x is a subset of $[0,1) \times \mathbb{R}$ with uniformly bounded cardinality, that is to say there exists a constant B such that $\#D_x \le B$ for all $x \in \mathbb{R}$.

Now suppose that there is a number $y \in \mathbb{R}$ such that y' has an infinite number of α-adic expansions. Indeed, there exists a constant N such that $\alpha^{-N}y'$ has $B + 1$ different fractional parts. This is in contradiction with the above proved fact that the number of different fractional parts is uniformly bounded for $x \in \mathbb{R}$. ☐

Let us note that Proposition 4.4.1 is conjectured to be valid for all Pisot numbers with Property (F). In the case that β is a cubic Pisot unit with complex conjugates satisfying Property (F), Sadahiro [95] has proved that the above result holds true.

Proposition 4.4.2. *Let β be a quadratic Pisot unit. Let $x \in \mathbb{Z}[\beta]_+$. Then x' has a unique α-adic expansion, which is finite and such that ${}_\alpha\langle x'\rangle = \langle x\rangle_\beta$.*

Proof. By Proposition 4.3.1 any number $x' \in \mathbb{Z}[\beta]_+$ has a finite expansion ${}_\alpha\langle x'\rangle = x_k \cdots x_0 \bullet x_{-1} \cdots x_{-j}$. Let us suppose that x' has another α-adic expansion (necessarily infinite)

$$ {}_\alpha\langle x'\rangle = \cdots u_n \cdots u_0 \bullet u_{-1} \cdots u_{-m} \,. $$

Subtracting these two expansions of x' and normalizing the result we obtain an admissible expansion of zero of the form $\cdots u_{k+3}u_{k+2}v_{k+1} \cdots v_0 \bullet v_{-1} \cdots v_{-p}$, with $v_{-p} \neq 0$. By shifting and relabeling

$$ 0 = \sum_{i\geq 0} \alpha^i z_i \,, \tag{4.4} $$

where $(z_i)_{i\geq 0}$ is an admissible sequence with $z_0 \neq 0$. The admissibility condition implies $z_1 \in \{0, \ldots, a-1\}$. Since $\alpha = -\beta^{-1}$ one can rewrite (4.4) as

$$ \underbrace{z_0 + \frac{z_2}{\beta^2} + \frac{z_4}{\beta^4} + \cdots}_{LS:=} = \underbrace{\frac{z_1}{\beta} + \frac{z_3}{\beta^3} + \frac{z_5}{\beta^5} + \cdots}_{RS:=} \,. \tag{4.5} $$

The coefficients z_i for $i \geq 1$ belong to $\{0, \ldots, a\}$, hence by summing the geometric series on both sides of (4.5) we obtain $LS \in [1, a + \frac{1}{\beta}]$ and $RS \in [0, 1 - \frac{1}{\beta}]$ which is absurd. □

To prove an analogue of Proposition 4.4.2 stating the unicity of α-adic expansions for the elements of $\mathbb{Z}[\beta]_-$ we first need the following Lemma.

Lemma 4.4.3. *If a number z has an eventually periodic α-adic expansion then all its α-adic expansions are eventually periodic.*

Proof. We have already shown that if a number x has a finite α-adic expansion then this expansion is unique.

Let us consider a number x' with an eventually periodic expansion

$$ {}_\alpha\langle x'\rangle = {}^\omega(x_{k+p} \cdots x_{k+1}) x_k \cdots x_0 \bullet x_{-1} \cdots x_{-j} \,. \tag{4.6} $$

For the sake of contradiction let us assume that x' has another α-adic expansion, which is infinite and non-periodic

$$ {}_\alpha\langle x'\rangle = \cdots u_1 u_0 \bullet u_{-1} \cdots u_{-m} \,. \tag{4.7} $$

Put $y' := \alpha^{-(k+1)}x' - \pi_\alpha(0\bullet x_k \cdots x_0 x_{-1} \cdots x_{-j})$. Hence from (4.6) we have

$$ {}_\alpha\langle y'\rangle = {}^\omega(x_{k+p} \cdots x_{k+1})\bullet 0 \,. \tag{4.8} $$

From (4.7), defining $v_{k+1}\bullet v_k \cdots v_0 v_{-1} \cdots v_{-q}$ as the word obtained by normalization of the result of digit-wise subtraction $u_{k+1}\bullet u_k \cdots u_1 u_0 u_{-1} \cdots u_{-m} - 0\bullet x_k \cdots x_0 x_{-1} \cdots x_{-j}$, we have

$$_\alpha\langle y' \rangle = \cdots u_{k+3} u_{k+2} v_{k+1}\bullet v_k \cdots v_0 v_{-1} \cdots v_{-q}\,, \tag{4.9}$$

which is non-periodic.

Equation (4.8) gives us another formula for y', namely

$$y' = \alpha^{-p} y' - \pi_\alpha(0\bullet x_{k+p} \cdots x_{k+1})\,.$$

Iterating this formula on the non-periodic expansion (4.9) yields infinitely many different α-adic expansions of the number y'. This is in the contradiction with the statement of Lemma 4.4.1. ☐

Proposition 4.4.4. *Let β be a quadratic Pisot unit. Let $x \in \mathbb{Z}[\beta]_-$. Then x' has exactly two eventually periodic α-adic expansions with period ${}^\omega(a0)$.*

Proof. At first, we prove that the number -1 has no other α-adic expansions than those from Proposition 4.3.3. Since all α-adic expansions of -1 have to be eventually periodic, we will discuss only two cases: when the period is ${}^\omega(a0)$ and when it is different.

1. Consider an α-adic expansion of -1 with the period ${}^\omega(a0)$

$$\begin{aligned} _\alpha\langle -1 \rangle &= {}^\omega(a0) d_k \cdots d_0\bullet d_{-1} \cdots d_{-j}\,, \\ -1 &= -\alpha^{k+1} + \pi_\alpha(d_k \cdots d_0\bullet d_{-1} \cdots d_{-j})\,. \end{aligned}$$

   The number $-1+\alpha^{k+1}$ is the conjugate of $\beta^{k+1} - 1 \in \mathbb{Z}[\beta]_+$ and as such has a unique α-adic expansion. Therefore there cannot be two different pre-periods for a given position of the period.

2. Suppose that -1 has an α-adic expansion with a different period

$$_\alpha\langle -1 \rangle = {}^\omega(d_{k+p} \cdots d_{k+1}) d_k \cdots d_0\bullet d_{-1} \cdots d_{-j}.$$

   Let $P' := \pi_\alpha(d_{k+p} \cdots d_{k+1})$. Then

$$-1 = \alpha^{k+1} \frac{P'}{1-\alpha^p} + \pi_\alpha(d_k \cdots d_0\bullet d_{-1} \cdots d_{-j})\,,$$

   and by taking the conjugate we obtain

$$-1 = \beta^{k+1} \frac{P}{1-\beta^p} + \pi_\beta(d_k \cdots d_0\bullet d_{-1} \cdots d_{-j})\,.$$

   Therefore

$$\underbrace{\pi_\beta(d_k \cdots d_0\bullet d_{-1} \cdots d_{-j}) + 1}_{\in \mathbb{Z}[\beta]_+} = \underbrace{\beta^{k+1} \frac{P}{\beta^p - 1}}_{\notin \mathbb{Z}[\beta]_+}\,,$$

   which is a contradiction.

Validity of the statement for numbers $x' \in \mathbb{Z}[\alpha]_-$, $x' \neq -1$, is then a simple consequence of Algorithm 4.3.2. ☐

4.4.2 Representations of rational numbers

In this subsection we inspect α-adic expansions of rational numbers. We give below an algorithm for computing an α-adic representation of a rational number $q \in \mathbb{Q}$, $|q| < 1$. The algorithm for computing ${}_\alpha\langle q\rangle$ is a sort of a right to left normalization — it consists of successive transformations of a representation of q, and it gives as a result a left infinite sequence on the canonical alphabet $\mathcal{A}$.

Let x_1, x_2 and x_3 be rational digits, and define the following transformation

$$\psi : (x_3)(x_2)(x_1) \mapsto \ (x_3 - (\lceil x_1 \rceil - x_1))(x_2 + a(\lceil x_1 \rceil - x_1))(\lceil x_1 \rceil)\,. \tag{4.10}$$

Note that this transformation preserves the α-value.

Algorithm 4.4.5. *Input:* $q \in \mathbb{Q} \cap (-1,1)$.
Output: a sequence $s = (s_i)_{i\geq 0}$ *of* $\mathcal{A}^{\mathbb{N}}$ *such that* $\sum_{i\geq 0} s_i\alpha^i = q$.
begin
  $s_0 := q;$
  for $i \geq 1$ **do** $s_i := 0;$
  $i := 0;$
  repeat
    $s_{i+2}s_{i+1}s_i := \psi(s_{i+2}s_{i+1}s_i);$
    $i := i+1;$
end

Since the starting point of the whole process is a single rational number, after each step there will be at most two non-integer coefficients — rational numbers with the same denominator as q.

Denote $s^{(i+1)}$ the resulting sequence after step i; thus $s^{(0)} = {}^{\omega}0q$ and, for $i \geq 0$, $s^{(i+1)} = \cdots s^{(i+1)}_{i+4}s^{(i+1)}_{i+3}s^{(i+1)}_{i+2}s^{(i+1)}_{i+1}s^{(i+1)}_{i}\cdots s^{(i+1)}_{0}$ where the digits $s^{(i+1)}_0 = s_0, \ldots, s^{(i+1)}_i = s_i$ are integer digits of the output, and the factor $s^{(i+1)}_{i+3}s^{(i+1)}_{i+2}s^{(i+1)}_{i+1}$ is under consideration. Note that for $j \geq i+3$, the coefficients $s^{(i+1)}_j$ are all equal to 0. Thus the next iteration of the algorithm gives $\psi(s^{(i+1)}_{i+3}s^{(i+1)}_{i+2}s^{(i+1)}_{i+1}) = s^{(i+2)}_{i+3}s^{(i+2)}_{i+2}s^{(i+2)}_{i+1}$.

Lemma 4.4.6. *After every step i of the algorithm, the coefficients satisfy:*

- $s^{(i+1)}_0 = s_0, \ldots, s^{(i+1)}_i = s_i$ *belong to* $\mathcal{A}$
- $s^{(i+1)}_{i+1} \in (-1, a)$
- $s^{(i+1)}_{i+2} \in (-1, 0]$.

Proof. We will prove the statement by induction on the number of steps of the algorithm. The statement is valid for $i = 0$ due to the assumption $|q| < 1$.

By Transformation (4.10) we have $\psi(s_{i+3}^{(i+1)} s_{i+2}^{(i+1)} s_{i+1}^{(i+1)}) = s_{i+3}^{(i+2)} s_{i+2}^{(i+2)} s_{i+1}^{(i+2)}$, thus

$$\begin{aligned}
s_{i+1}^{(i+2)} &= \lceil s_{i+1}^{(i+1)} \rceil \in \mathbb{Z} \cap [0, a]\,, \\
s_{i+2}^{(i+2)} &= s_{i+2}^{(i+1)} + a(\lceil s_{i+1}^{(i+1)} \rceil - s_{i+1}^{(i+1)}) \in (-1, a)\,, \\
s_{i+3}^{(i+2)} &= -(\lceil s_{i+1}^{(i+1)} \rceil - s_{i+1}^{(i+1)}) \in (-1, 0]\,.
\end{aligned}$$

□

Since the factor $s_{i+3}^{(i+2)} s_{i+2}^{(i+2)} s_{i+1}^{(i+2)}$ after step $i+1$ is uniquely determined by the factor $s_{i+3}^{(i+1)} s_{i+2}^{(i+1)} s_{i+1}^{(i+1)}$, and the coefficients $s_{i+1}^{(i+1)}$ and $s_{i+2}^{(i+1)}$ are uniformly bounded, as a corollary we get the following result.

Proposition 4.4.7. *Algorithm 4.4.5 generates an α-adic representation of q which is eventually periodic.*

Example. We compute an α-adic representation of the number $\frac{1}{2}$ in the case $\mathrm{d}_\beta(1) = 31$.

$$\begin{array}{cccccc}
 & & & & & \frac{1}{2} \\
 & & & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\
\hline
 & & & -\frac{1}{2} & \frac{3}{2} & 1 \\
 & & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \\
\hline
 & & -\frac{1}{2} & 1 & 2 & 1 \\
-\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & & & \\
\hline
-\frac{1}{2} & \frac{3}{2} & 0 & 1 & 2 & 1
\end{array}$$

Because the prefix $(-\frac{1}{2})(\frac{3}{2})$ which arises after step 3 is the same as the one which arises after step 0, the same sequence of steps (with the same results) will follow from now on. Therefore the α-adic representation computed by the algorithm is ${}_\alpha\langle\frac{1}{2}\rangle = {}^\omega(012)1\bullet$. It happens that, in this particular case, this is an α-adic expansion of ${}_\alpha\langle\frac{1}{2}\rangle$.

4.4.3 Normalization

Unfortunately, Algorithm 4.4.5 does not give directly an admissible α-adic expansion in general. In this section we discuss the normalization of such a non-admissible output. Since the output word is a word on the canonical alphabet, its only possible non-admissible factors are either of the type $a^n b$ with $n \geq 1$, $b \neq a$ or of the type ${}^\omega a$. The following result shows that the latter case will not appear.

Proposition 4.4.8. *The number of consecutive letters a's in an output word of Algorithm 4.4.5 is bounded for all $q \in \mathbb{Q} \cap (-1, 1)$.*

Proof. We will prove the result by contradiction. Let us assume that from some step on, say from step i, the output of the algorithm is composed only of letters a's. This means that the output is of the form $\cdots \lceil V_4 \rceil \lceil V_3 \rceil \lceil V_2 \rceil \lceil V_1 \rceil v$, where v has length $i+1$, and for each $k \geq 1$, $\lceil V_k \rceil = a$. We have $V_1 = s_{i+1}^{(i+1)}$, and $V_2 = s_{i+2}^{(i+1)} + a(\lceil V_1 \rceil - V_1)$. Iterating twice the transformation ψ, we get

$$V_k = -(\lceil V_{k-2} \rceil - V_{k-2}) + a(\lceil V_{k-1} \rceil - V_{k-1}) \quad \text{for} \quad k \geq 3. \tag{4.11}$$

From Relation (4.11) and the fact that $V_k > a - 1$ one get

$$1 - \frac{1}{a} + \frac{1}{a}(\lceil V_{k-2} \rceil - V_{k-2}) < (\lceil V_{k-1} \rceil - V_{k-1})\,. \tag{4.12}$$

Then iterating (4.12) we obtain an explicit estimate for $(\lceil V_k \rceil - V_k)$

$$\begin{aligned}
(\lceil V_k \rceil - V_k) &> 1 - \frac{1}{a} + \frac{1}{a}(\lceil V_{k-1} \rceil - V_{k-1}) \\
&> 1 - \frac{1}{a} + \frac{1}{a}\Big(1 - \frac{1}{a} + \frac{1}{a}(\lceil V_{k-2} \rceil - V_{k-2})\Big) \\
&= 1 - \frac{1}{a^2} + \frac{1}{a^2}(\lceil V_{k-2} \rceil - V_{k-2}) \\
&> 1 - \frac{1}{a^3} + \frac{1}{a^3}(\lceil V_{k-3} \rceil - V_{k-3}) \\
&> \cdots \\
&> 1 - \frac{1}{a^{k-1}} + \frac{1}{a^{k-1}}(\lceil V_1 \rceil - V_1)
\end{aligned}$$

Since $s_{i+2}^{(i+1)} \in (-1, 0]$ we can estimate

$$a - 1 < V_2 = s_{i+2}^{(i+1)} + a(\lceil V_1 \rceil - V_1) \leq a(\lceil V_1 \rceil - V_1)\,,$$

which gives $1 - \frac{1}{a} < (\lceil V_1 \rceil - V_1)$. Therefore we have

$$-(\lceil V_k \rceil - V_k) < -1 + \frac{1}{a^{k-1}} - \frac{1}{a^{k-1}}\Big(1 - \frac{1}{a}\Big) = \frac{1}{a^k} - 1\,. \tag{4.13}$$

Finally, by inequality (4.13) and the fact that $a - 1 < V_k \leq a$, we obtain a bound on V_k

$$V_k = \underbrace{-(\lceil V_{k-2} \rceil - V_{k-2})}_{<\frac{1}{a^{k-2}}-1} \; \underbrace{+a\lceil V_{k-1} \rceil}_{=a^2} \; \underbrace{-aV_{k-1}}_{<a-a^2} < a - 1 + \frac{1}{a^{k-2}}\,. \tag{4.14}$$

Suppose that we are computing an α-adic expansion of a rational number q with denominator $p \in \mathbb{N}$. Find the smallest K such that $\frac{1}{p} > \frac{1}{a^{K-2}}$. Since any V_k is a fraction with denominator p, by (4.14) we have $V_K = \frac{t}{p} < a - 1 + \frac{1}{a^{K-2}}$, which implies $V_K < a - 1$. This is in contradiction with the assumption that $a - 1 < V_k$ for all $k \geq 1$. ◻

Proposition 4.4.9. *Let w be an output of Algorithm 4.4.5 for a number $q \in \mathbb{Q} \cap (-1, 1)$ and let $\widehat{w}$ be the image of w under the normalization function, $\nu_{\mathcal{A}}(w) = \widehat{w}$. Then $\widehat{w}$ is left eventually periodic with no fractional part.*

Proof. First of all, a number β such that $\mathrm{d}_\beta(1) = a1$ is a so-called *confluent Pisot number* (cf. [60]). For these numbers, it is known that the normalization on the canonical alphabet does not produce a carry to the right. This assures that $\widehat{w}$ will have no fractional part and that we can perform normalization starting from the fractional point and then just read and write from right to left.

We have shown earlier that for a given rational number q the number of consecutive letters a's in an output word w is bounded, moreover the proof of Proposition 4.4.8 gives us this upper bound. We give here a construction of a right sequential transducer $\mathcal{T}$ performing the normalization of such a word w.

Define $\mathcal{A}_\emptyset := \mathcal{A} \setminus \{0\}$, and let C be the bound on the number of consecutive letters a in a word w. Because the result of the normalization of non-admissible factors of w depends on the parity of the length of blocks of consecutive a's, the transducer $\mathcal{T}$ has to count this parity. This is done by memorizing the actually processed forbidden factors; the states of the transducer are labeled by these memorized factors.

Transducer $\mathcal{T}$ is constructed as follows

- The initial state is labeled by the empty word ε, and there is a loop $\varepsilon \xrightarrow{0|0} \varepsilon$.
- There are states labeled by a single letter $h \in \mathcal{A}_\emptyset$ connected with the initial state by edges $\varepsilon \xrightarrow{h|\varepsilon} h$ and $h \xrightarrow{0|0h} \varepsilon$. These states are also connected one with each other by edges $i \xrightarrow{j|i} j$ where $i, j \in \mathcal{A}_\emptyset$, $j \neq a$. Finally there is a loop $h \xrightarrow{h|h} h$ on each state $h \in \mathcal{A}_\emptyset$, $h \neq a$.
- For each $h \in \mathcal{A}_\emptyset$ there is a chain of consecutive states $a^k h$, where $k = 1, \ldots, C-1$, linked by edges $a^k h \xrightarrow{a|\varepsilon} a^{k+1}h$. Moreover, there are edges $a^k h \xrightarrow{i|u} i+1$ where $u = (0a)^m 0(h-1)$ for $k = 2m+1$ and $u = (0a)^m 0(a-1)h$ for $k = 2m+2$.

The edges $a^k h \xrightarrow{a|\varepsilon} a^{l+1}h$ are these which count the number of consecutive letters a in a forbidden factor, whereas the edges $a^k h \xrightarrow{i|u} i+1$ are these which, depending on the parity of the length k of a run a^k, replace a forbidden factor by its normalized equivalent.

One can easily check that the transducer is input deterministic, and thus right sequential. Clearly the output word is admissible. Since the image by a sequential function of an eventually periodic word is eventually periodic (cf. [55]), the image $\hat{w}$ is eventually periodic. $\square$

The following is just a rephrasing.

Theorem 4.4.10. *Let β be a quadratic Pisot unit. Any rational number $q \in \mathbb{Q} \cap (-1, 1)$ has an eventually periodic α-adic expansion with no fractional part.*

Remark that there exist rational numbers larger than 1 such that the α-adic expansion has no fractional part. We have shown in Example 4.4.2 that for $\mathrm{d}_\beta(1) = 31$, ${}_\alpha\langle \frac{1}{2} \rangle = {}^\omega(012)1\bullet$. Thus ${}_\alpha\langle \frac{3}{2} \rangle = {}^\omega(012)2\bullet$ has no fractional part.

***Remark* 4.4.11.** Let us stress out that the analogue of Propositions 4.4.2 and 4.4.4 has been proved by Sadahiro for the case that β is a cubic Pisot unit with complex conjugates satisfying Property (F). The extension of these results to other Pisot units satisfying Property (F) is an open problem.

4.5 Arithmetics of eventually periodic tau-adic expansions

In this last section we discuss more practically the arithmetics of eventually periodic α-adic expansions. We consider one particular irrationality, the golden mean τ. We denote by $\mathcal{F}_{\text{ep}}(\tau')$ the set of all real numbers which have their tau-adic expansions eventually periodic to the left. By Theorem 4.2.3, $\mathcal{F}_{\text{ep}}(\tau')$ is a ring. We give below algorithms to perform ring arithmetic operations in $\mathcal{F}_{\text{ep}}(\tau')$. More precisely, we construct a transducer with a countable number of states to perform addition, subtraction is reduced to two additions, and for multiplication, we give an algorithm which uses additions and subtractions.

4.5.1 Determinization of a transducer

In the construction which follows, we will need to determinize a transducer. Note that there is a known algorithm (cf. Choffrut [50], Béal and Carton [26]) to determinize a real-time transducer realizing a sequential function. We recall here the description of this algorithm due to Béal and Carton, slightly changed for the case of right subsequential transducer.

Let $\mathcal{T} = \langle A \times B^*, Q, E, I, F\rangle$ be a real-time transducer labeled in $A \times B^*$. Its deterministic equivalent — the subsequential transducer $(\mathcal{T}_{\text{det}}, \rho)$ — is defined as follows. A state P of $\mathcal{T}_{\text{det}}$ is a set $P = \{(q_i, w_i) \in Q \times B^* |\ i \in \mathbb{N}\}$, where q_i is a state of $\mathcal{T}$ and w_i is a word over B. The transition (P, a), where $a \in A$ is a letter from the input alphabet is determined by the set $R_a(P)$ defined by

$$R_a(P) := \{(q_i', uw_i) \mid \text{there exist } (q_i, w_i) \in P \text{ and } q_i \xrightarrow{a|u} q_i' \in E\}\,.$$

If $R_a(P)$ is empty, there is no transition from P input labeled by a. Otherwise, let v be the longest common suffix of words uw_i for all pairs $(q_i', uw_i) \in R_a(P)$ and

$$P' = \{(q_i', uw_iv^{-1}) \mid (q_i', uw_i) \in R_a(P)\}\,.$$

Then there is a transition $P \xrightarrow{a|v} P'$. The initial state of $\mathcal{T}_{\text{det}}$ is $J = \{(i, \varepsilon) \mid i \in I\}$. Finally, a state P is final if it contains at least one pair (q_i, w_i) where q_i is a final state of $\mathcal{T}$. The final function ρ then maps such final state P to the word w_i.

In the case of a sequential transducer over infinite words there exists also an algorithm to determinize such a transducer (cf. Béal and Carton [26], [27]). This algorithm (in the simplified case of a transducer having all its states final) differs from the previous one in two points. First, it specially deals with the so-called constant states (a constant state is a state $q \in Q$ such that all infinite paths starting in q have the same output label regardless their input labels); second, it uses a deterministic Büchi automaton recognizing the domain of the function realized by the transducer to insure that the output is infinite only when the input belongs to the domain of the function.

4.5.2 Addition

General principle. Similarly to the addition in a general Pisot base, all three arithmetic algorithms for the τ-adic system are composed of two parts. Again, the first one consists in computing a representation of the result and the second one consists in normalization of such a representation.

The computation of a representation is very similar to that one described in the section 3.4, with the exception that Facts 3.4.1 and 3.4.2 have to be changed in the sense of the "direction" of the period.

Fact 4.5.1. *Let* ${}^{\omega}(u_{m+p}\cdots u_{m+2}u_{m+1})u_m\cdots u_0{\bullet}u_{-1}\cdots u_{-k}$ *be a* τ*-adic representation of a real number* u. *Then* ${}^{\omega}(u_{m+1}u_{m+p}\cdots u_{m+2})u_{m+1}u_m\cdots u_0{\bullet}u_{-1}\cdots u_{-k}$ *is also a* τ*-adic representation of* u.

Fact 4.5.2. *Let* ${}^{\omega}(u_{m+p}\cdots u_{m+1})u_m\cdots u_0{\bullet}u_{-1}\cdots u_{-k}$ *be a* τ*-adic representation of a real number* u. *Then* ${}^{\omega}((u_{m+p}\cdots u_{m+1})^l)u_m\cdots u_0{\bullet}u_{-1}\ldots u_{-k}$ *is also a* τ*-adic representation of the number* u, *for any positive integer* $l\in\mathbb{N}$.

Let $x,y\in\mathcal{F}_{\mathrm{ep}}(\tau')$. We want to find a representation of $z=x+y$. First we shift period of x or of y so that they start with a coefficient belonging to the same power of τ'. Then we stretch the periods to the length equal to the least common multiple of their original lengths. Finally the result is obtained by a simple digit-wise addition.

Normalization. We describe a transducer $\mathcal{T}$ — a finite non-deterministic transducer performing right to left normalization on alphabet $\{0,1,2\}$ in the τ-numeration system (hence also in τ-adic numeration system), with the additional condition on the input word that every coefficient 2 is surrounded by at least one 0 from each side. This condition is equivalent to the input word being a digit wise sum of two expansions in the τ-numeration system.

It is known that the sum of two tau-integers in the τ-numeration system is a number with at most two fractional digits in its τ-expansion [44]. Hence if z_k is the rightmost non-zero coefficient in the τ-adic representation of z, obtained in the previous step, the process of normalization will affect at most coefficients with indices greater than or equal to $k-2$. To avoid technical difficulties we, without loss of generality, request that the input words for $\mathcal{T}$ begin with 00 (once more we recall that the transducer is working from right to left, hence the factor 00 is supposed to be on the right end of the representation of z). In what follows coefficients of the input word will be denoted $\ldots a_k a_{k-1}\ldots a_1 a_0$ (hence $a_0=a_1=0$) and coefficients of the output word will be $\ldots b_k b_{k-1}\ldots b_1 b_0$.

We denote the initial state of $\mathcal{T}$ by q_0. Then we have the starting part of the transducer

$$
\begin{array}{ccc}
 & q_0 \xrightarrow{00|\varepsilon} q_1 & \\
q_1 \xrightarrow{0|100} \bar{1}1 & q_1 \xrightarrow{0|000} 00_{(a)} & q_1 \xrightarrow{1|100} 00_{(a)} \\
q_1 \xrightarrow{0|000} 00_{(b)} & q_1 \xrightarrow{1|000} 1\bar{1}_{(a)} & q_1 \xrightarrow{1|001} \bar{1}2 \\
q_1 \xrightarrow{0|010} 1\bar{2} & q_1 \xrightarrow{2|001} 01_{(d)} & q_1 \xrightarrow{2|100} 1\bar{1}_{(c)} \,.
\end{array}
$$

We now describe the synchronous part of the transducer. Its states are denoted by words of length two, with d_1d_0 representing the polynomial $d_1\tau+d_0$ and the signed digit -1 being denoted by $\bar{1}$. Transitions are of the form $d_1d_0 \xrightarrow{a_k|b_k} d_1'd_0'$ if $a_k + (d_0 + d_1\tau) = b_k + (d_0'\tau + d_0'\tau^2)$.

Moreover, there are several states having the same numerical value, but different conditions on incoming edges. This is mainly due to the fact that there is a condition on the input word too. In those cases the states having same numerical value are distinguished by subscripts — the subscript (a) is used for the state whose incoming edges are input-labeled by zeros and ones, subscript (b) for the state with entirely zero input-labeled incoming edges, subscript (c) for a state with incoming edge labeled by $2|1$ and subscript (d) for a state with incoming edge labeled by $2|0$. Note that two edges belonging to the starting part of the transducer $\mathcal{T}$ are exceptions from these rules, since their output labels are of the length three — in this case we have to see a label $a_k|b_kb_{k-1}b_{k-2}$ as a label $a_k|b_k$.

All the states of the transducer are final. The synchronous part of the transducer $\mathcal{T}$ is drawn in Figure 4.1. Following proposition easily follows.

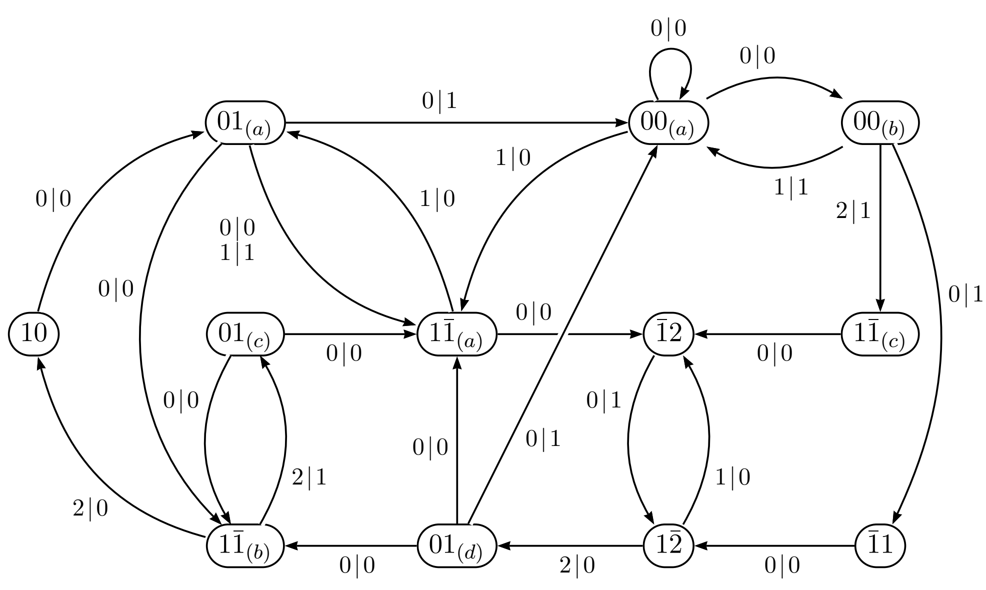


Figure 4.1: The synchronous part of the transducer $\mathcal{T}$.

Proposition 4.5.3. *Apart from the starting part the rest of $\mathcal{T}$ is a letter-to-letter transducer with zero delay (i.e. when the transducer reads the input letter a_k it writes the output letter b_k). Moreover, if $d_1'd_0'$ is the state reached after the k-th step (i.e. after writing the coefficient b_k of the output word) we have*

$$\sum_{i=0}^{k} a_i(\tau')^i = \left(d_1'(\tau')^2 + d_0'\tau'\right)(\tau')^k + \sum_{i=0}^{k} b_i(\tau')^i\,.$$

***Remark* 4.5.4.** Should the transducer $\mathcal{T}$ be used to normalize finite words, for simplicity we would suppose that the input and output word are of the same length, i.e. there is enough zeros in the front (recall that we are working from right to left) of each input word for the automaton arrive into the state $00_{(a)}$, this state would be the only one final state.

Now let us assume that we would like to really use the transducer $\mathcal{T}$ to perform the normalization (that is to say to implement the adding machine). A non-deterministic transducer over infinite words does not seem to be the best suitable machine for this task.

In Subsection 4.5.1, we recalled two algorithms to determinize a sequential transducer. It was suitable for determinization of a transducer over finite words (in the first case) and over infinite words (in the second case). As we have mentioned before, they differ mainly by two things — the second one specially deals with the so-called constant states and it also uses a deterministic Büchi automaton recognizing the domain of the function realized by the transducer to insure that the output is infinite only when the input belongs to the domain of the function. Nevertheless, it can be easily checked that our transducer $\mathcal{T}$ has no constant states and, moreover, we will employ the transducer only on words surely belonging to the domain of the normalization function (hence there is no need of the above mentioned deterministic Büchi automaton). Therefore we do not have to apply the second algorithm.

Let us for a while forget that our function is not sequential and apply the first determinization algorithm to transducer $\mathcal{T}$. We obtain a transducer, say $\mathcal{T}_{\mathrm{det}}$, with an infinite countable number of states. However, while performing the algorithm it is easy to see that the resulting transducer $\mathcal{T}_{\mathrm{det}}$ is virtually composed of two parts: the "non-repeating part" (counting 11 states) and the "repeating part" (the rest of the transducer).

The "non-repeating" part of the transducer $\mathcal{T}_{\mathrm{det}}$ is in Figure 4.2. The states $P_{10}^{(4)}$ to $P_{14}^{(4)}$ are the first states of the repeating part, the state J is the only one initial state and the double circled states P_4, P_7 and P_8 are the states where edges returning from the repeating part reenter the non-repeating part (see below).

The states of the repeating part of the transducer $\mathcal{T}_{\mathrm{det}}$ are denoted by symbols $P_j^{(i)}$. The subscript indicates the type of the state, i.e. the set of states of transducer $\mathcal{T}$ "contained" in state P_j, whereas the superscript indicates the length of memorized words, $i = |w_k|$ for all pairs (q_k, w_k) in $P_j^{(i)}$.

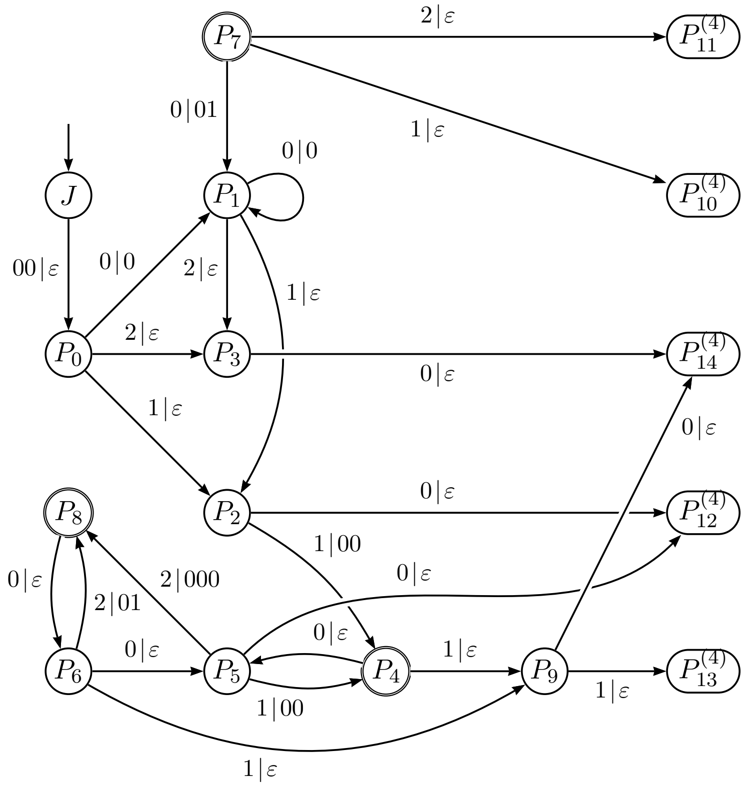


Figure 4.2: The synchronous part of the transducer $\mathcal{T}$.

There are five different types of states P_j in the repeating part[1]

$$\begin{aligned}
P_{10}^{(i)} &= \{(00_{(a)}, \star), (1\bar{1}_{(a)}, \star), (\bar{1}2, \star)\} \\
P_{11}^{(i)} &= \{(1\bar{1}_{(c)}, \star), (01_{(d)}, \star)\} \\
P_{12}^{(i)} &= \{(00_{(a)}, \star), (00_{(b)}, \star), (\bar{1}2, \star), (1\bar{2}, \star)\} \\
P_{13}^{(i)} &= \{(1\bar{1}_{(a)}, \star), (01_{(a)}, \star)\} \\
P_{14}^{(i)} &= \{(00_{(a)}, \star), (1\bar{1}_{(a)}, \star), (\bar{1}2, \star), (1\bar{1}_{(b)}, \star)\}\,.
\end{aligned}$$

We can organize the repeating part of $\mathcal{T}_{\text{det}}$ into the levels according to the superscripts of the states within.

All the transitions between the states inside the repeating part have empty output

[1] The symbol $\star$ stands for some nonempty word of length i admissible in the τ-numeration system

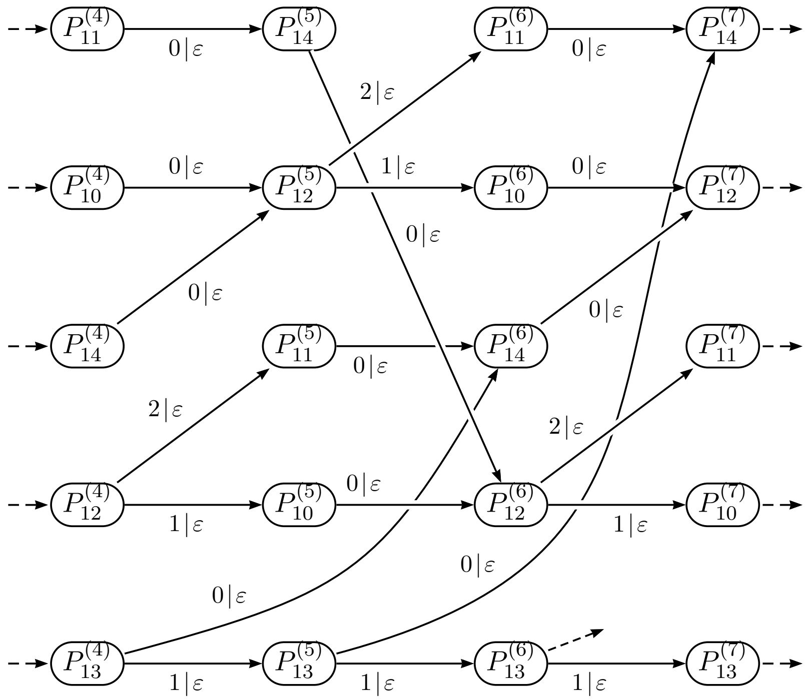


Figure 4.3: First four levels of the periodic part of the transducer $\mathcal{T}_{\text{det}}$.

word, transitions of the repeating part between those levels are then the following ones

$$\begin{array}{ll} P_{10}^{(i)} \xrightarrow{0|\varepsilon} P_{12}^{(i+1)} & P_{11}^{(i)} \xrightarrow{0|\varepsilon} P_{14}^{(i+1)} \\ P_{12}^{(i)} \xrightarrow{1|\varepsilon} P_{10}^{(i+1)} & P_{12}^{(i)} \xrightarrow{2|\varepsilon} P_{11}^{(i+1)} \\ P_{13}^{(i)} \xrightarrow{0|\varepsilon} P_{14}^{(i+2)} & P_{13}^{(i)} \xrightarrow{1|\varepsilon} P_{13}^{(i+1)} \end{array}$$
$$P_{14}^{(i)} \xrightarrow{0|\varepsilon} P_{12}^{(i+1)}\,.$$

The first four levels of the repeating part are drawn in Figure 4.3.

In addition to the edges inside the repeating part, there are also the so-called "return edges", i.e. the edges aiming back to the non-repeating part (note that for clarity reasons these are not drawn in Figure 4.3).

$$\begin{array}{ll} P_{10}^{(i)} \xrightarrow{1|w_2} P_4 & P_{12}^{(i)} \xrightarrow{0|w_3} P_7 \\ P_{14}^{(i)} \xrightarrow{1|w_2} P_4 & P_{14}^{(i)} \xrightarrow{2|w_4} P_8\,, \end{array}$$

where

$$w_2 = \begin{cases} {}^{(k-2)}(0\ 1)\ 0\ 0\ 1 & \text{for } i = 2k \\ {}^{(k-1)}(0\ 1)\ 0\ 0 & \text{for } i = 2k+1 \end{cases}$$

$$w_3 = \begin{cases} {}^{(k-2)}(0\ 1)\ 0\ 0 & \text{for } i = 2k \\ {}^{(k-2)}(0\ 1)\ 0\ 0\ 1 & \text{for } i = 2k+1 \end{cases}$$

$$w_4 = \begin{cases} 0\ {}^{(k-2)}(0\ 1)\ 0\ 0\ 1 & \text{for } i = 2k \\ 0\ 0\ {}^{(k-2)}(0\ 1)\ 1\ 0\ 0 & \text{for } i = 2k+1\,. \end{cases}$$

Example. We will normalize the representation ${}_{\tau'}(z) = {}^{\omega}(0111010)02000$ using the transducer $\mathcal{T}_{\text{det}}$. The path in $\mathcal{T}_{\text{det}}$ is the following one[2]

Used transition	Output
$J \xrightarrow{00\vert\varepsilon} P_0 = \{q_1, 00\}$	ε
$P_0 \xrightarrow{0\vert 0} P_1 = \{(00_{(a)}, 00), (00_{(b)}, 00), (\bar{1}1, 10), (1\bar{2}, 01)\}$	0
$P_1 \xrightarrow{2\vert\varepsilon} P_3 = \{(1\bar{1}_{(c)}, 100), (01_{(d)}, 001)\}$	0
$P_3 \xrightarrow{0\vert\varepsilon} P_{14}^{(4)} = \{(00_{(a)}, 1001), (1\bar{1}_{(a)}, 0001), (\bar{1}2, 0100), (1\bar{1}_{(b)}, 0001)\}$	0
$P_{14}^{(4)} \xrightarrow{0\vert\varepsilon} P_{12}^{(5)} = \{(00_{(a)}, 01001), (00_{(b)}, 01001), (\bar{1}2, 00001), (1\bar{2}, 10100)\}$	0
$P_{12}^{(5)} \xrightarrow{1\vert\varepsilon} P_{10}^{(6)} = \{(00_{(a)}, 101001), (1\bar{1}_{(a)}, 001001), (\bar{1}2, 010100)\}$	0
$P_{10}^{(6)} \xrightarrow{0\vert\varepsilon} P_{12}^{(7)} =$ $\{(00_{(a)}, 0101001), (00_{(b)}, 0101001), (\bar{1}2, 0001001), (1\bar{2}, 1010100)\}$	0
$P_{12}^{(7)} \xrightarrow{1\vert\varepsilon} P_{10}^{(8)} = \{(00_{(a)}, 10101001), (1\bar{1}_{(a)}, 00101001), (\bar{1}2, 01010100)\}$	0
$P_{10}^{(8)} \xrightarrow{1\vert 0101001} P_4 = \{(1\bar{1}_{(a)}, 01), (01_{(a)}, 00)\}$	01010010
$P_4 \xrightarrow{1\vert\varepsilon} P_9 = \{(1\bar{1}_{(a)}, 100), (01_{(a)}, 001)\}$	01010010
$P_9 \xrightarrow{1\vert\varepsilon} P_{14}^{(4)} \xrightarrow{0\vert\varepsilon} P_{12}^{(5)} \cdots$ from now on steps 5 to 11 are reoccurring	01010010

The τ-adic expansion of z is therefore ${}_{\tau'}\langle z\rangle = {}^{\omega}(0101001)0$.

The transducer $\mathcal{T}_{\text{det}}$ realizes the same function as the transducer $\mathcal{T}$. Unfortunately, since the normalization function ν_C is not sequential, it has an infinite number of states; we have to deal with this fact.

It turns out that if we get rid of a few particular cases, which can be treated separately, only a finite portion of the transducer $\mathcal{T}_{\text{det}}$ will be actually used during the normalization.

Obviously, the cases resulting in the use of the whole transducer $\mathcal{T}_{\text{det}}$ (or strictly speaking use of an infinite number of different states of $\mathcal{T}_{\text{det}}$) are those for which a prefix of the input word is an input label of some path in the repeating part of $\mathcal{T}_{\text{det}}$, which never uses any return edge (i.e. never returns back to the non-repeating part). We will treat these cases first.

[2] In the column Output is not written the output label of the actual transition, but the whole so far generated output word.

Since all the edges inside the repeating part of $\mathcal{T}_{\mathrm{det}}$ have empty output and so the input using infinite part of $\mathcal{T}_{\mathrm{det}}$ will cause the transducer only to read and write nothing from some coefficient on, we will call those cases as "infinite reading" cases. They are simply characterized by the following lemma.

Lemma 4.5.5. *The prefix of an eventually periodic input word triggering the infinite reading in $\mathcal{T}_{\mathrm{det}}$ is one of the following words ${}^{\omega}(1)$, ${}^{\omega}(01)$, ${}^{\omega}(002)$, ${}^{\omega}(\{01, 002\}^*)$.*

Proof. Let us denote the discussed prefix by w. We will successively inspect all possible infinite paths in the repeating part of $\mathcal{T}_{\mathrm{det}}$, according to the state in which they begin. Input labels of those paths will be prefixes w triggering the infinite reading.

1. The path starts in state $P_{10}^{(i)}$. There is only one transition leaving $P_{10}^{(i)}$ in the repeating part of transducer, namely $P_{10}^{(i)} \xrightarrow{0|\varepsilon} P_{12}^{(i+1)}$. Indeed the prefix $w = v0$ is a concatenation of the letter zero and a word v — some prefix triggering infinite reading, starting in the state of type P_{12}.

2. The path starts in state $P_{11}^{(i)}$. Just as in the case of state $P_{10}^{(i)}$ there is only one transition leaving $P_{11}^{(i)}$, namely $P_{11}^{(i)} \xrightarrow{0|\varepsilon} P_{14}^{(i+1)}$. Hence $w = v0$, where v is some prefix triggering infinite reading, starting in the state of type P_{14}.

3. The path starts in state $P_{12}^{(i)}$. There are two possibilities for the path leaving this state. It can either be $P_{12}^{(i)} \xrightarrow{1|\varepsilon} P_{10}^{(i+1)} \xrightarrow{0|\varepsilon} P_{12}^{(i+2)}$ which means there is a factor 01 in w (recall that the automaton is working from right to left) or $P_{12}^{(i)} \xrightarrow{2|\varepsilon} P_{11}^{(i+1)} \xrightarrow{0|\varepsilon} P_{14}^{(i+2)} \xrightarrow{0|\varepsilon} P_{12}^{(i+3)}$ implying the factor 002 in w. Therefore the path advancing from state of type P_{12} to another state of the same type using only the first possibility will have ${}^{\omega}(01)$ as its input label, while the path using only the second possibility will have ${}^{\omega}(002)$ as its input. Indeed there are also paths using both possibilities and hence having ${}^{\omega}(\{01, 002\}^*)$ as their input.

4. The path starts in $P_{13}^{(i)}$. Then it either proceeds with an infinite number of transitions $P_{13}^{(i)} \xrightarrow{1|\varepsilon} P_{13}^{(i+1)}$ and hence $w = {}^{\omega}(1)$ or the transition $P_{13}^{(j)} \xrightarrow{0|\varepsilon} P_{14}^{(j+2)}$ is used for some j. In the later case the prefix is $w = v(1)^*$, where v is a prefix triggering infinite reading, starting in the state of type P_{14}.

5. The path starts in state $P_{14}^{(i)}$. Also in this case there is only one transition $P_{14}^{(i)} \xrightarrow{0|\varepsilon} P_{12}^{(i+1)}$ and so $w = v0$, where v is a prefix triggering infinite reading, starting in the state of type P_{12}.

□

Now we will show how to pre-process the input words having one of the prefixes listed in Lemma 4.5.5.

1. The input word z (τ-adic representation of $x + y$) has prefix ${}^{\omega}(01)$, or one can say ${}^{\omega}(10)1$. We distinguish two sub-cases

a) There are less than two zeros in front of ${}^{\omega}(10)1$. Indeed, there are three possibilities, namely ${}^{\omega}(10)102u$, ${}^{\omega}(10)110u$, ${}^{\omega}(10)111u$. Since ${}^{\omega}(10)11$ is τ-adic expansion of zero, all those three possibilities lead to the normalization of a finite word, which is again a finite word (the coefficient 2 in the first case has to be substituted by $10{\bullet}01$ to create zero-valued prefix ${}^{\omega}(10)11$).

b) There are at least two zeros in front of the prefix ${}^{\omega}(10)1$ in the representation of z, i.e. z is of the form ${}^{\omega}(10)100u$. Normalizing the word $00u$ we obtain either $10v_1$ or a word lexicographically smaller or equal to $01v_0$ where v_1 is an admissible word possibly starting with 1, whereas v_0 is an admissible word starting with 0. In the first case by concatenating the prefix we have ${}^{\omega}(10)110v_1$ which has been already discussed in Item a), hence the τ-adic expansion of z is ${}_{\tau'}\langle z\rangle = v_1$, while in the second case the word ${}^{\omega}(10)10v_0$ is already admissible τ-adic expansion.

2. The input word z has prefix composed of the factors 01 and 002. The following operation shows how to shift a block 002 "one position" to the left

$$\begin{array}{ccccccc}
0 & 1 & \multicolumn{3}{|c|}{0\quad 0\quad 2} & 0 & 1 \\
 & \bar{1} & 1 & 1 & & & \\
 & & 1 & \bar{1} & \bar{1} & & \\
\hline
\multicolumn{3}{|c|}{0\quad 0\quad 2} & 0 & 1 & 0 & 1
\end{array}$$

while the following operation is used to modify two consecutive 002 blocks

$$\begin{array}{cccccccccc}
0 & 1 & \multicolumn{3}{|c|}{0\quad 0\quad 2} & \multicolumn{3}{|c|}{0\quad 0\quad 2} & 0 & 1 \\
 & & & 1 & \bar{1} & \bar{1} & & & & \\
 & & & & \bar{1} & 1 & 1 & & & \\
 & & & & & 1 & \bar{1} & \bar{1} & & \\
\hline
0 & 1 & \multicolumn{2}{|c|}{0\quad 1} & \multicolumn{2}{|c|}{0\quad 1} & \multicolumn{2}{|c|}{0\quad 1} & 0 & 1
\end{array}$$

Indeed, blocks 002 are treated by pairs — we shift the right block next to the other one and then turn them into sequence of blocks 01. If the number of blocks 002 is odd, the last one can be paired with its next occurrence (i.e. with the occurrence in the next period). By turning all the factors 002 into factors 01 we transform a representation of z into the form discussed in the Case 1.

3. The input word z has a prefix composed of factors 002. Obviously, using the above mentioned operation for turning two consecutive blocks 002 into three blocks 01 transforms the representation into the representation of the form discussed in the Case 1.

4. The input word z has prefix ${}^{\omega}(1)$. We use the following transformation

$$\begin{array}{c}
{}^{\omega}(\,1\;1\,)\;1\;1 \\
{}^{\omega}(\,\bar{1}\;0\,)\;\bar{1}\;\bar{1} \\
\hline
{}^{\omega}(\,1\;0\,)\;0\;0
\end{array}$$

which once again leads back to the Case 1.

There are four simple pre-processing transformations that turn any input word triggering infinite reading either directly into its τ-adic expansion or into a finite word, which can be simply normalized.

In all other cases, only a finite portion of the transducer $\mathcal{T}_{\text{det}}$ is used during the normalization. Obviously, the computation can enter the repeating part of the transducer, but since the prefix of the input word is not any of those from Lemma 4.5.5, every time the computation enters the repeating part it eventually uses some return edge to go back to the non-repeating part. Indeed, we need only a finite portion of $\mathcal{T}_{\text{det}}$ to normalize these representations.

Therefore, in fact, we have a deterministic transducer performing normalization. Since the result is eventually periodic, the computation done by the transducer can be stopped after a finite number of steps.

We can formulate the following Algorithm.

Algorithm 4.5.6. *Let $x, y \in \mathcal{F}_{\text{ep}}(\tau')$. The τ-adic expansion of sum $x + y$ is obtained as follows.*

1. *Compute a τ-adic representation of $z = x + y$, using Facts 4.5.1 and 4.5.2.*

2. *Check whether the prefix of the obtained representation is one of the prefixes triggering infinite reading (Lemma 4.5.5). If so, pre-process the prefix in the corresponding way.*

3. *Check whether the representation begins with* 00*, otherwise append zeros in front of it.*

4. *Normalize the representation, using deterministic transducer $\mathcal{T}_{\text{det}}$. Since the result will be eventually periodic, the computation done by the transducer can be stopped after a finite number of steps.*

4.5.3 Subtraction

Let $x, y \in \mathcal{F}_{\text{ep}}(\tau')$. We want to find the τ-adic expansion of $x - y$. The first step is the same as for addition. By simple digit wise subtraction (using Facts 4.5.1 and 4.5.2) we find a τ-adic representation of $x - y$, denoted by $z = {}_{\tau'}(x - y)$. Obviously, the coefficients of z are from the alphabet $\{-1, 0, 1\}$. Without loss of generality, we can suppose that z has no fractional part. Normalization is then done using the following algorithm.

Algorithm 4.5.7. *Let z be a τ-adic representation of $x - y$.*

1. *Define a partition of z into three other representations u, v_{odd} and v_{even}, such that this partition preserves the numerical value $\pi_{\tau'}(z) = \pi_{\tau'}(u) + \pi_{\tau'}(v_{\text{odd}}) + \pi_{\tau'}(v_{\text{even}})$ and*

    - *u is obtained from z by putting all the negative coefficients equal to zero*

- v_{odd} *is obtained from* z *by keeping only negative coefficients which belong to the odd powers of* τ'
- v_{even} *is obtained from* z *by keeping only negative coefficients which belong to the even powers of* τ'

2. *Modify* v_{odd} *and* v_{even} *by transformation*

$$\widehat{v}_{\text{odd}} := v_{\text{odd}} + {}^{\omega}(10)_{\bullet}11 \qquad \textit{and} \qquad \widehat{v}_{\text{even}} := v_{\text{even}} + {}^{\omega}(01)_{\bullet}1\,, \tag{4.15}$$

which does not change numerical values of the representations, since both added sequences are representations of zero. Hence we have $u, \widehat{v}_{\text{odd}}, \widehat{v}_{\text{even}} \in \{0,1\}^{\mathbb{N}}$ *and* $\pi_{\tau'}(u), \pi_{\tau'}(\widehat{v}_{\text{odd}}), \pi_{\tau'}(\widehat{v}_{\text{even}}) \in \mathcal{F}_{\text{ep}}(\tau')$.

3. *A* τ*-adic expansion of* $x - y$ *is obtained by performing two consecutive additions* $(u + \widehat{v}_{\text{odd}}) + \widehat{v}_{\text{even}}$.

Example. Let ${}_{\tau'}\langle x\rangle = {}^{\omega}(0100)10_{\bullet}0$ and ${}_{\tau'}\langle y\rangle = {}^{\omega}(0100)1_{\bullet}0$. Then by digit wise subtraction we obtain ${}_{\tau'}(x-y) = z = {}^{\omega}(01\bar{1}0)1\bar{1}_{\bullet}0$. The partition is following one.

$$\begin{aligned} u &= {}^{\omega}(0100)10_{\bullet}0 \\ v_{\text{odd}} &= {}^{\omega}(00\bar{1}0)00_{\bullet}0 \\ v'_{\text{odd}} &= {}^{\omega}(1000)10_{\bullet}11 \\ v_{\text{even}} &= \qquad\quad \bar{1}_{\bullet}0 \\ v'_{\text{even}} &= \quad {}^{\omega}(01)0_{\bullet}1\,. \end{aligned}$$

Corollary 4.5.8. *The partition construction in Algorithm 4.5.7 allows us to compute a* τ*-adic expansion from an eventually periodic* τ*-adic representation over any finite alphabet of digits:*

1. *Dispart the* τ*-adic representation to be normalized into* u*,* v_{odd} *and* v_{even}.
2. *Use transformation* (4.15) *on* v_{odd} *and* v_{even} *as many times as needed to get rid of all negative coefficients.*
3. *The result is obtained by adding* $(u + \widehat{v}_{odd}) + \widehat{v}_{even}$*, each of them seen as a sum of finite number of elements from the set* $\mathcal{F}_{\text{ep}}(\tau')$.

4.5.4 Multiplication

The operation of multiplication is different from addition and subtraction. Even though the usual naive way to perform the multiplication — a series of successive additions — seems to be an infinite process, in the case of eventually periodic expansions it can be used, but it needs more careful investigation.

We start with the simplest case of two purely periodic τ-adic expansions, say ${}_{\tau'}\langle x\rangle = {}^{\omega}(x_k \ldots x_0)_{\bullet}$ and ${}_{\tau'}\langle y\rangle = {}^{\omega}(y_l \ldots y_0)_{\bullet}$.

At first, let us assume that there is only one non-zero coefficient in ${}_{\tau'}\langle y\rangle$, say y_n, $0 \le n \le l$. In this case the multiplication will consist of successive summation of x multiplied by $(\tau')^{n+li}$ for $i \ge 0$ (i.e. summation of copies of ${}_{\tau'}\langle x\rangle$ every time shifted l positions to the left). This process produces a representation with a "re-occurring pattern": after summation of k shifted copies of ${}_{\tau'}\langle x\rangle$, the period of $(k+1)^{\text{st}}$ copy of ${}_{\tau'}\langle x\rangle$ is exactly aligned with the period of the 1^{st} copy, while in the copies in-between it appears in all other possible "shift positions"; the period of the $(k+2)^{\text{nd}}$ copy is aligned with the period of the 2^{nd} copy and so on. Therefore, the sum will be composed of blocks of the length m, where $m = \operatorname{lcm}(k, l)$, such that each coefficient in a block is by ς greater than the coefficient at the same position one block to the right, where $\varsigma = x_k + \cdots + x_0$ is the sum of the coefficients in the period of ${}_{\tau'}\langle x\rangle$

$$ {}_{\tau'}(xy) = \cdots \underbrace{(z_m + 2\varsigma)\cdots(z_1 + 2\varsigma)}_{3^{\text{rd}}\text{ block}} \underbrace{(z_m + \varsigma)\cdots(z_2 + \varsigma)(z_1 + \varsigma)}_{2^{\text{nd}}\text{ block}} \underbrace{z_m \cdots z_2 z_1}_{1^{\text{st}}\text{ block}} \bullet \tag{4.16} $$

Example. Let ${}_{\tau'}\langle x\rangle = {}^{\omega}(10010)$ and ${}_{\tau'}\langle y\rangle = {}^{\omega}(01)$. Then $\varsigma = 2$ and the process of computing of a block-representation (4.16) starts as follows.

...	1 0 0 1 0 1 0 0 1 0	1 0 0 1 0 $\boxed{1\ 0\ 0\ 1\ 0}$	1 0 0 1 0 1 0 0 1 0
...	0 1 0 1 0 0 1 0 1 0	0 1 0 1 0 0 1 0 1 0	0 1 0 1 0 0 1 0
...	0 1 0 0 1 0 1 0 0 1	0 1 0 0 1 0 1 0 0 1	0 1 0 0 1 0
...	0 0 1 0 1 0 0 1 0 1	0 0 1 0 1 0 0 1 0 1	0 0 1 0
...	1 0 1 0 0 1 0 1 0 0	1 0 1 0 0 1 0 1 0 0	1 0
...	1 0 0 1 0 1 0 0 1 0	1 0 0 1 0 $\boxed{1\ 0\ 0\ 1\ 0}$	
...	0 1 0 1 0 0 1 0 1 0	0 1 0 1 0 0 1 0	
...	0 1 0 0 1 0 1 0 0 1	0 1 0 0 1 0	
...	0 0 1 0 1 0 0 1 0 1	0 0 1 0	
...	1 0 1 0 0 1 0 1 0 0	1 0	
	⋰		
...	6 6 5 6 5 5 5 4 5 4	4 4 3 4 3 3 3 2 3 2	2 2 1 2 1 1 1 0 1 0

Boxes emphasize the alignment of periods of the 1^{st} and of the 6^{th} copy of the representation.

Recall that we have two non-admissible τ-adic representations of zero (obtained by adding 1 to representations of -1), namely ${}^{\omega}(10)\bullet 11$ and ${}^{\omega}(01)\bullet 011$. Hence their digit-wise sum — as well as any multiple of this sum by a constant integer — is also a τ-adic representation of zero. Among others ${}_{\tau'}(0) = {}^{\omega}(\varsigma\varsigma)\bullet\varsigma(2\varsigma)\varsigma$.

Let us take the block-shaped representation (4.16) and successively subtract from it shifted representations $\sigma^{mi}({}^{\omega}(\varsigma\varsigma)\bullet\varsigma(2\varsigma)\varsigma)$ for $i \ge 1$. One can easily see that the fractional point of the i-th subtracted representation is aligned with the barrier between blocks i and $i+1$, for all $i \ge 1$.

After first subtraction, the first block (the right most block) will be changed into $(z_m - \varsigma)(z_{m-1} - 2\varsigma)(z_{m-2} - \varsigma)(z_{m-3})\cdots(z_1)$, the second block into $z_m \cdots z_2 z_1$ (thus into the form of the first block prior to the subtraction), the third block into $(z_m + \varsigma) \cdots (z_2 + \varsigma)(z_1 + \varsigma)$ and so on.

The second subtraction will not affect the first block, the second block will be changed into $(z_m - \varsigma)(z_{m-1} - 2\varsigma)(z_{m-2} - \varsigma)(z_{m-3}) \cdots (z_1)$, the third block into $z_m \cdots z_2 z_1$ and so on. Indeed, after all the subtractions we will have an eventually periodic τ-adic representation

$$_{\tau'}(xy) = {}^{\omega}\big((z_m - \varsigma)(z_{m-1} - 2\varsigma)(z_{m-2} - \varsigma)(z_{m-3}) \cdots (z_2)(z_1)\big)_{\bullet},$$

which obviously still does not have to be an admissible τ-adic expansion. However, normalization of such a representation can be done as stated in Corollary 4.5.8.

***Example* (Continuation).** Let us take the representation obtained in the previous example and successively subtract $\sigma^{mi}({}^{\omega}(2)\bullet 2(4)2)$, $i \geq 1$ from it.

...	6 6 5 6 5 5 5 4 5 4	4 4 3 4 3 3 3 2 3 2	2 2 1 2 1 1 1 0 1 0
...	$\bar{2}\ \bar{2}\ \bar{2}\ \bar{2}\ \bar{2}\ \bar{2}\ \bar{2}\ \bar{2}\ \bar{2}\ \bar{2}$	$\bar{2}\ \bar{2}\ \bar{2}\ \bar{2}\ \bar{2}\ \bar{2}\ \bar{2}\ \bar{2}\ \bar{2}\ \bar{2}$	$\bar{2}\ \bar{4}\ \bar{2}$
...	4 4 3 4 3 3 3 2 3 2	2 2 1 2 1 1 1 0 1 0	$0\ \bar{2}\ \bar{1}\ 2\ 1\ 1\ 1\ 0\ 1\ 0$
...	$\bar{2}\ \bar{2}\ \bar{2}\ \bar{2}\ \bar{2}\ \bar{2}\ \bar{2}\ \bar{2}\ \bar{2}\ \bar{2}$	$\bar{2}\ \bar{4}\ \bar{2}$	
...	2 2 1 2 1 1 1 0 1 0	$0\ \bar{2}\ \bar{1}\ 2\ 1\ 1\ 1\ 0\ 1\ 0$	$0\ \bar{2}\ \bar{1}\ 2\ 1\ 1\ 1\ 0\ 1\ 0$
...	$\bar{2}\ \bar{4}\ \bar{2}$		
...	$0\ \bar{2}\ \bar{1}\ 2\ 1\ 1\ 1\ 0\ 1\ 0$	$0\ \bar{2}\ \bar{1}\ 2\ 1\ 1\ 1\ 0\ 1\ 0$	$0\ \bar{2}\ \bar{1}\ 2\ 1\ 1\ 1\ 0\ 1\ 0$
			${}^{\omega}(0\ \bar{2}\ \bar{1}\ 2\ 1\ 1\ 1\ 0\ 1\ 0)$

Now suppose that there are more than one non-zero coefficient in $_{\tau'}\langle y\rangle$. Indeed, we can treat them one by one each time pursuing the above described algorithm for the case where $_{\tau'}\langle y\rangle$ has only one non-zero coefficient. Doing this we transform a multiplication of two elements form $\mathcal{F}_{\text{ep}}$ into a sum of a finite number of elements from $\mathcal{F}_{\text{ep}}$.

Finally, let us suppose that x or y (or both) are not purely periodic. In the case where y is not purely periodic we just have to add a finite number of shifted copies of x (i.e. of elements of $\mathcal{F}_{\text{ep}}(\tau')$) to the result, whereas in the case where x is not periodic we can treat its pre-period and its period separately, that is, the period is treated as described above, whereas the pre-period is done by adding a finite number of shifted copies of y.

Chapter 5

Palindromic complexity of infinite words associated with simple Parry numbers

One of the reasons motivating the study of combinatorial properties of infinite words associated with β-numeration systems is the fact that, in general, infinite aperiodic words over a finite alphabet are suitable models for one-dimensional quasicrystals, i.e. non-crystallographic materials displaying long-range order, since they define one-dimensional Delaunay sets with finite local complexity.

For the description of physical properties of these materials it is important to know the combinatorial properties of the infinite aperiodic words, such as the factor complexity, which corresponds to the number of local configurations of atoms in the material, or the palindromic structure of the aperiodic words, describing local symmetry of the material. The palindromic structure of the infinite words has been proved important for the description of the spectra of Schrödinger operators with potentials adapted to aperiodic structures [70].

This chapter is devoted to the description of the palindromic structure of the infinite words u_β associated with a simple Parry number β. We first show a necessary condition on $\mathrm{d}_\beta(1)$ for the word u_β to contain infinitely many palindromes (Lemma 5.1.1). Numbers β satisfying this condition are the so-called confluent Pisot numbers [60]. Then we determine the palindromic complexity of u_β when β is a confluent Pisot number.

We show that if the length of palindromes is not bounded, which is equivalent to $\limsup_{n\to\infty} \mathcal{P}(n) > 0$, then

$$\mathcal{P}(n+1) + \mathcal{P}(n) = \mathcal{C}(n+1) - \mathcal{C}(n) + 2\,, \qquad \text{for} \quad n \in \mathbb{N}\,. \tag{5.1}$$

In general it is has been shown [20] that for a uniformly recurrent word u with $\limsup_{n\to\infty} \mathcal{P}(u) > 0$ the inequality

$$\mathcal{P}(n+1) + \mathcal{P}(n) \le \mathcal{C}(n+1) - \mathcal{C}(n) + 2$$

holds for all $n \in \mathbb{N}$. Moreover, the authors proved Formula (5.1) to be valid for infinite words coding the r-interval exchange. Finally, it is known that the Formula (5.1) holds also for Arnoux-Rauzy words [9] and for complementation-symmetric sequences [52].

We then give a complete description of the set of palindromes, its structure and properties. The exact palindromic complexity of the word u_β is given in Theorem 5.5.1.

In the last part of the chapter we study the occurrence of palindromes of an arbitrary length in the prefixes of the word u_β, when β is a confluent Pisot number. It is known [53] that every word w of length n contains at most $n+1$ different palindromes. The value by which the number of palindromes differs from $n+1$ is called the *defect* of the word w. Infinite words whose every prefix has defect 0 are called *full*. We show that whenever $\limsup_{n\to\infty} \mathcal{P}(n) > 0$, the infinite word u_β is full.

The content of this chapter is essentially the same as the content of the article accepted for publication in Annales de l'Institut Fourier [15].

5.1 Words u_β with a bounded number of palindromes

The infinite word u_β associated with a Parry number β is a fixed point of a primitive substitution. This implies that the word u_β is uniformly recurrent [90].

Lemma 5.1.1. *If the language $\mathcal{L}(u)$ of a uniformly recurrent word u contains infinitely many palindromes, then $\mathcal{L}(u)$ is closed under reversal.*

Proof. From the definition of a uniformly recurrent word u it follows that for every $n \in \mathbb{N}$ there exists an integer $R(n)$ such that every arbitrary factor of u of length $R(n)$ contains all factors of u of length n. Since we assume that $\mathcal{P}al(u)$ is an infinite set, it must contain a palindrome p of length $\geq R(n)$. Since p contains all factors of u of length n, and p is a palindrome, it contains with every w such that $|w| = n$ also its reversal $\widetilde{w}$. Thus $\widetilde{w} \in \mathcal{L}(u)$. This consideration if valid for all n and thus the statement of the lemma is proved. ☐

Note that this result was first stated, without proof, in [53]. The fact that the language is closed under reversal is thus a necessary condition so that a uniformly recurrent word has infinitely many palindromes. The converse is not true [34].

For infinite words u_β associated with simple Parry numbers β the invariance of $\mathcal{L}(u_\beta)$ under reversal was studied by Frougny et al.

Proposition 5.1.2 ([63]). *Let $\beta > 1$ be a simple Parry number such that* $\mathrm{d}_\beta(1) = t_1 t_2 \cdots t_m$.

(i) The language $\mathcal{L}(u_\beta)$ is closed under reversal, if and only if

$$\textit{Condition (C):} \qquad t_1 = t_2 = \cdots = t_{m-1}\,.$$

(ii) The infinite word u_β is an Arnoux-Rauzy word if and only if Condition (C) is satisfied and $t_m = 1$.

Corollary 5.1.3. *Let β be a simple Parry number which does not satisfy Condition (C). Then there exists $n_0 \in \mathbb{N}$ such that $\mathcal{P}(n) = 0$ for $n \geq n_0$.*

Numbers β satisfying Condition (C) have been introduced and studied in [60] from the point of view of linear numeration systems. Confluent linear numeration systems are exactly those for which the normalization on the canonical alphabet does not produce a carry to the right. A number β satisfying Condition (C) is known to be a Pisot number, and will be called a *confluent* Pisot number.

Let β be a confluent Pisot number with $\mathrm{d}_\beta(1) = t_1 t_2 \cdots t_m$. Set

$$t := t_1 = t_2 = \cdots = t_{m-1} \qquad \text{and} \qquad s := t_m\,.$$

From the Parry condition for the Rényi expansion of 1 it follows that $t \geq s \geq 1$. Then $A_\varphi = \{0, 1, \ldots, m-1\}$ and the canonical substitution φ is of the form

$$\begin{aligned} \varphi(i) &= 0^t(i+1) \qquad \text{for all } 0 \leq i < m-1\,, \\ \varphi(m-1) &= 0^s\,. \end{aligned} \tag{5.2}$$

Note that in the case $s = 1$, the number β is an algebraic unit, and the corresponding word u_β is an Arnoux-Rauzy word, for which the palindromic complexity is known. Therefore in this chapter we often treat separately the cases $s \geq 2$ and $s = 1$.

5.2 Palindromic extensions in u_β

Let us recall that for an Arnoux-Rauzy word u (and thus also for a Sturmian word) it has been shown that for every palindrome $p \in \mathcal{L}(u)$ there is exactly one letter a in the alphabet, such that $apa \in \mathcal{L}(u)$, i.e. any palindrome in an Arnoux-Rauzy word has exactly one palindromic extension [52]. Since the length of the palindromic extension apa of p is $|apa| = |p| + 2$, we have for Arnoux-Rauzy words $\mathcal{P}(n+2) = \mathcal{P}(n)$ and therefore

$$\mathcal{P}_{u_\beta}(2n) = \mathcal{P}_{u_\beta}(0) = 1 \qquad \text{and} \qquad \mathcal{P}_{u_\beta}(2n+1) = \mathcal{P}_{u_\beta}(1) = \#A_\varphi\,.$$

Determining the number of palindromic extensions for a given palindrome of u_β is essential also for our considerations here. However, let us first introduce the following notion.

Definition. We say that a palindrome p_1 is a *central factor* of a palindrome p_2 if there exists a finite word $w \in \mathcal{A}^*$ such that $p_2 = wp_1\widetilde{w}$.

For example, a palindrome is a central factor of its palindromic extensions.

The following simple result can be easily obtained from the form of the substitution (5.2), and is a special case of a result given in [63].

Lemma 5.2.1 ([63]). *All factors of u_β of the form $X0^nY$ for $X, Y \neq 0$ are the following*

$$X0^t1,\ 10^tX \text{ with } X \in \{1, 2, \ldots, m-1\}, \quad \text{and} \quad 10^{t+s}1\,. \tag{5.3}$$

***Remark* 5.2.2.** 1. Every pair of non-zero letters in u_β is separated by a block of at least t zeros. Therefore every palindrome $p \in \mathcal{L}(u_\beta)$ is a central factor of a palindrome with prefix and suffix 0^t.

2. Since $\varphi(A_\varphi)$ is a suffix code, the coding given by the substitution φ is uniquely decodable. In particular, if $w_1 \in \mathcal{L}(u_\beta)$ is a factor with the first and the last letter non-zero, then there exist a factor $w_2 \in \mathcal{L}(u_\beta)$ such that $0^t w_1 = \varphi(w_2)$.

Proposition 5.2.3.

(i) Let $p \in \mathcal{L}(u_\beta)$. Then $p \in \mathcal{P}al(u_\beta)$ if and only if $\varphi(p)0^t \in \mathcal{P}al(u_\beta)$.

(ii) Let $p \in \mathcal{P}al(u_\beta)$. The number of palindromic extensions of p and $\varphi(p)0^t$ is the same, i.e.

$$\#\{a \in A_\varphi \mid apa \in \mathcal{P}al(u_\beta)\} = \#\{a \in A_\varphi \mid a\varphi(p)0^t a \in \mathcal{P}al(u_\beta)\}\,.$$

Proof. (i) Let $p = w_0 w_1 \cdots w_{n-1} \in \mathcal{L}(u_\beta)$. Let us study under which conditions the word $\varphi(p)0^t$ is also a palindrome, i.e. when

$$\varphi(w_0)\varphi(w_1)\cdots\varphi(w_{n-1})0^t = 0^t\widetilde{\varphi(w_{n-1})}\cdots\widetilde{\varphi(w_1)}\widetilde{\varphi(w_0)}\,. \tag{5.4}$$

The substitution φ has the property that for each letter $a \in A_\varphi$ it satisfies $\widetilde{\varphi(a)} = 0^{-t}\varphi(a)0^t$. Using this property, the equality (5.4) can be equivalently written as

$$\varphi(p) = \varphi(w_0)\cdots\varphi(w_{n-1}) = \varphi(w_{n-1})\cdots\varphi(w_0) = \varphi(\widetilde{p})\,.$$

As a consequence of unique decodability of φ we obtain that (5.4) is valid if and only if $p = \widetilde{p}$.

(ii) We show that for a palindrome p it holds that

$$apa \in \mathcal{P}al(u_\beta) \qquad \Longleftrightarrow \qquad b\varphi(p)0^t b \in \mathcal{P}al(u_\beta)\,, \quad \text{where } b \equiv a+1 \pmod{m}\,,$$

which already implies the equality of the number of palindromic extensions of palindromes p and $\varphi(p)0^t$.

Let $apa \in \mathcal{P}al(u_\beta)$. Then

$$\varphi(a)\varphi(p)\varphi(a)0^t = \begin{cases} 0^t(a+1)\varphi(p)0^t(a+1)0^t\,, & \text{for } a \neq m-1\,,\\ 0^s\varphi(p)0^{t+s}\,, & \text{for } a = m-1\,, \end{cases}$$

is, according to (i) of this proposition, also a palindrome, which has a central factor $(a+1)\varphi(p)0^t(a+1)$ for $a \neq m-1$, and $0\varphi(p)0^t 0$ for $a = m-1$.

On the other hand, assume that $b\varphi(p)0^t b \in \mathcal{P}al(u_\beta)$. If $b \neq 0$, then using point 1. of Remark 5.2.2, we have $0^t b\varphi(p)0^t b 0^t = \varphi\big((b-1)p(b-1)\big)0^t \in \mathcal{P}al(u_\beta)$. Point (i) implies that $(b-1)p(b-1) \in \mathcal{P}al(u_\beta)$ and thus $(b-1)p(b-1)$ is a palindromic extension of p. If $b = 0$, then Lemma 5.2.1 implies that $10^s\varphi(p)0^t 0^s 1 \in \mathcal{L}(u_\beta)$ and so $1\varphi\big((m-1)p(m-1)0\big) \in \mathcal{L}(u_\beta)$, which means that $(m-1)p(m-1)$ is a palindromic extension of p. $\square$

Unlike Arnoux-Rauzy words, in the case of infinite words u_β with $\mathrm{d}_\beta(1) = tt\cdots ts$, $t \geq s \geq 2$, it is not difficult to see using Lemma 5.2.1 that there exist palindromes which do not have any palindromic extension. Such a palindrome is for example the word 0^{t+s-1}.

Definition. A palindrome $p \in \mathcal{P}al(u_\beta)$ which has no palindromic extension is called a *maximal* palindrome.

It is obvious that every palindrome is either a central factor of a maximal palindrome, or is a central factor of palindromes of arbitrary length.

Proposition 5.2.3 allows us to define a sequence of maximal palindromes starting from an initial maximal palindrome. Put

$$U^{(1)} := 0^{t+s-1}, \qquad U^{(n)} := \varphi(U^{(n-1)})0^t, \quad \text{for } n \geq 2\,. \tag{5.5}$$

Lemma 5.2.1 also implies that the palindrome 0^t has for $s \geq 2$ two palindromic extensions, namely 00^t0 and 10^t1. Using Proposition 5.2.3 we create a sequence of palindromes, all having two palindromic extensions. Put

$$V^{(1)} := 0^t, \qquad V^{(n)} := \varphi(V^{(n-1)})0^t, \quad \text{for } n \geq 2\,. \tag{5.6}$$

***Remark* 5.2.4.** It is necessary to mention that the factors $U^{(n)}$ and $V^{(n)}$ defined above play an important role in the description of factor complexity of the infinite word u_β. Let us cite several results for u_β invariant under the substitution (5.2) with $s \geq 2$, taken from [63], which will be used in the sequel.

(1) Any prefix w of u_β is a left special factor which can be extended to the left by any letter of the alphabet, that is, $aw \in \mathcal{L}(u_\beta)$ for all $a \in A_\varphi$, or equivalently $\deg_{\mathrm{L}}(w) = \#A_\varphi$.

(2) Any left special factor w which is not a prefix of u_β is a prefix of $U^{(n)}$ for some $n \geq 1$ and such w can be extended to the left by exactly two letters.

(3) The words $U^{(n)}$, $n \geq 1$ are maximal left special factors of u_β, i.e. $U^{(n)}a$ is not a left special factor for any $a \in A_\varphi$. The infinite word u_β has no other maximal left special factors.

(4) The word $V^{(n)}$ is the longest common prefix of u_β and $U^{(n)}$, moreover, for every $n \geq 1$ we have
$$|V^{(n)}| < |U^{(n)}| < |V^{(n+1)}| \tag{5.7}$$

(5) For the first difference of factor complexity we have
$$\Delta\mathcal{C}(n) = \begin{cases} m & \text{if } |V^{(k)}| < n \leq |U^{(k)}| \text{ for some } k \geq 1\,, \\ m-1 & \text{otherwise}\,. \end{cases}$$

Now we are in position to describe the palindromic extensions in u_β. The main result is the following one.

Proposition 5.2.5. *Let u_β be the fixed point of the substitution φ given by* (5.2) *with parameters $t \geq s \geq 2$, and let p be a palindrome in u_β. Then*

(i) p is a maximal palindrome if and only if $p = U^{(n)}$ for some $n \geq 1$;

(ii) p has two palindromic extensions in u_β if and only if $p = V^{(n)}$ for some $n \geq 1$;

(iii) p has a unique palindromic extension if and only if $p \neq U^{(n)}$, $p \neq V^{(n)}$ for all $n \geq 1$.

Proof. (i) Proposition 5.2.3, point (ii) and the construction of $U^{(n)}$ imply that $U^{(n)}$ is a maximal palindrome for every n. The proof that no other palindrome p is maximal will be done by induction on the length $|p|$ of the palindrome p.

Let p be a maximal palindrome. If p does not contain a non-zero letter, then using Lemma 5.2.1, obviously $p = U^{(1)}$. Assume therefore that p contains a non-zero letter. Point 1. of Remark 5.2.2 implies that $p = 0^t\hat{p}0^t$, where $\hat{p}$ is a palindrome. Since p is a maximal palindrome, $\hat{p}$ ends and starts in a non-zero letter. Otherwise, p would be extendible to a palindrome, which contradicts maximality. From point 2. of Remark 5.2.2 we obtain that $p = 0^t\hat{p}0^t = \varphi(w)0^t$ for some factor w. Proposition 5.2.3, point (i), implies that w is a palindrome. Point (ii) of the same proposition implies that w has no palindromic extension, i.e. w is a maximal palindrome, with clearly $|w| < |p|$. The induction hypothesis implies that $w = U^{(n)}$ for some $n \geq 1$ and $p = \varphi(U^{(n)})0^t = U^{(n+1)}$.

(ii) and (iii) From what we have just proved it follows that every palindrome $p \neq U^{(n)}$, $n \geq 1$, has at least one palindromic extension. Since we know that $V^{(n)}$ has exactly two palindromic extensions, for proving (ii) and (iii) it remains to show that if a palindrome p has more than one extension, then $p = V^{(n)}$, for some $n \geq 1$.

Assume that ipi and jpj are in $\mathcal{L}(u_\beta)$ for $i, j \in A_\varphi$, $i \neq j$. Obviously, p is a left special factor of u_β. We distinguish two cases, according to whether p is a prefix of u_β, or not.

- Let p be a prefix of u_β. Then there exists a letter $k \in A_\varphi$ such that pk is a prefix of u_β and using (1) of Remark 5.2.4, the word $apk \in \mathcal{L}(u_\beta)$ for every letter $a \in A_\varphi$, in particular ipk and jpk belong to $\mathcal{L}(u_\beta)$. We have either $k \neq i$, or $k \neq j$; without loss of generality assume that $k \neq i$. Since $\mathcal{L}(u_\beta)$ is closed under reversal, we must have $kpi \in \mathcal{L}(u_\beta)$. Since ipi and kpi are in $\mathcal{L}(u_\beta)$, we obtain that pi is also a left special factor of u_β, and pi is not a prefix of u_β. By (2) of Remark 5.2.4, p is the longest common prefix of u_β and some maximal left special factor $U^{(n)}$, therefore using (4) of Remark 5.2.4 we have $p = V^{(n)}$.

- If p is a left special factor of u_β, which is not a prefix of u_β, then by (2) of Remark 5.2.4, p is a prefix of some $U^{(n)}$ and the letters i, j are the only possible left extensions of p. Since $p \neq U^{(n)}$, there exists a unique letter k such that pk is a left special factor of u_β and pk is a prefix of $U^{(n)}$, i.e. the possible left extensions of pk are the letters i, j. Since by symmetry $kp \in \mathcal{L}(u_\beta)$, we have $k = i$ or $k = j$, say $k = i$. Since $jpk = jpi \in \mathcal{L}(u_\beta)$, we have also $ipj \in \mathcal{L}(u_\beta)$. Since by assumption

ipi and jpj are in $\mathcal{L}(u_\beta)$, both pi and pj are left special factors of u_β. Since p is not a prefix of u_β, neither pi nor pj are prefixes of u_β. This contradicts the fact that k is a unique letter such that pk is left special.

Thus we have shown that if a palindrome p has at least two palindromic extensions, then $p = V^{(n)}$. □

From the above result it follows that if $n \neq |V^{(k)}|$, $n \neq |U^{(k)}|$ for all $k \geq 1$, then every palindrome of length n has exactly one palindromic extension, and therefore $\mathcal{P}(n+2) = \mathcal{P}(n)$. Inequalities in (4) of Remark 5.2.4 further imply that $|V^{(i)}| \neq |U^{(k)}|$ for all $i, k \geq 1$. Therefore the statement of Proposition 5.2.5 can be reformulated in the following way

$$\mathcal{P}(n+2) - \mathcal{P}(n) = \begin{cases} 1 & \text{if } \ n = |V^{(k)}|\,, \\ -1 & \text{if } \ n = |U^{(k)}|\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Point (5) of Remark 5.2.4 can be used for deriving for the second difference of factor complexity

$$\Delta^2\mathcal{C}(n) = \Delta\mathcal{C}(n+1) - \Delta\mathcal{C}(n) = \begin{cases} 1 & \text{if } \ n = |V^{(k)}|\,, \\ -1 & \text{if } \ n = |U^{(k)}|\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Therefore we have for $s \geq 2$ that

$$\mathcal{P}(n+2) - \mathcal{P}(n) = \Delta\mathcal{C}(n+1) - \Delta\mathcal{C}(n)\,,$$

for all $n \in \mathbb{N}$. We thus can derive the following theorem.

Theorem 5.2.6. *Let u_β be the fixed point of the substitution* (5.2). *Then*

$$\mathcal{P}(n+1) + \mathcal{P}(n) = \Delta\mathcal{C}(n) + 2\,, \qquad \textit{for} \ \ n \in \mathbb{N}\,.$$

Proof. Let the parameter s in the substitution (5.2) be $s = 1$. Then u_β is an Arnoux-Rauzy word, for which

$$\mathcal{P}(n+2) - \mathcal{P}(n) = 0 = \Delta\mathcal{C}(n+1) - \Delta\mathcal{C}(n)\,.$$

For $s \geq 2$ we use $\mathcal{P}(n+2) - \mathcal{P}(n) = \Delta\mathcal{C}(n+1) - \Delta\mathcal{C}(n)$ derived above. We have

$$\begin{aligned} \mathcal{P}(n+1) + \mathcal{P}(n) = \mathcal{P}(0) + \mathcal{P}(1) + &\sum_{i=1}^{n}\big(\mathcal{P}(i+1) - \mathcal{P}(i-1)\big) \\ &= 1 + m + \sum_{i=1}^{n}\big(\Delta\mathcal{C}(i) - \Delta\mathcal{C}(i-1)\big) = 1 + m + \Delta\mathcal{C}(n) - \Delta\mathcal{C}(0) = \\ &= 1 + m + \Delta\mathcal{C}(n) - \mathcal{C}(1) + \mathcal{C}(0) = \Delta\mathcal{C}(n) + 2\,, \end{aligned}$$

where we have used $\mathcal{P}(0) = \mathcal{C}(0) = 1$ and $\mathcal{P}(1) = \mathcal{C}(1) = m = \#A_\varphi$. □

***Remark* 5.2.7.** According to (5) of Remark 5.2.4, we have $\Delta\mathcal{C}(n) \leq \#A_\varphi$. This implies $\mathcal{P}(n+1) + \mathcal{P}(n) \leq \#A_\varphi + 2$, and thus the palindromic complexity is bounded.

5.3 Centers of palindromes

We have seen that the set of palindromes of u_β is closed under the mapping $p \mapsto \varphi(p)0^t$. We study the action of this mapping on the centers of the palindromes. Let us mention that the results of this section are valid for β a confluent Pisot number with $t \geq s \geq 1$, i.e. also for the Arnoux-Rauzy case.

Definition. Let p be a palindrome of odd length. The *center* of p is a letter a such that $p = wa\widetilde{w}$ for some $w \in \mathcal{A}^*$. The center of a palindrome p of even length is the empty word.

If palindromes p_1, p_2 have the same center, then also palindromes $\varphi(p_1)0^t$, $\varphi(p_2)0^t$ have the same center. This is a consequence of the following lemma.

Lemma 5.3.1. *Let $p, q \in \mathcal{P}al(u_\beta)$ and let q be a central factor of p. Then $\varphi(q)0^t$ is a central factor of $\varphi(p)0^t$.*

Proof. Since $p = wq\widetilde{w}$ for some $w \in A_\varphi^*$, we have $\varphi(p)0^t = \varphi(w)\varphi(q)\varphi(\widetilde{w})0^t$, which is a palindrome by point (i) of Proposition 5.2.3. It suffices to realize that 0^t is a prefix of $\varphi(\widetilde{w})0^t$. Therefore we can write $\varphi(p)0^t = \varphi(w)\varphi(q)0^t0^{-t}\varphi(\widetilde{w})0^t$. Since $|\varphi(w)| = |0^{-t}\varphi(\widetilde{w})0^t|$, the word $\varphi(q)0^t$ is a central factor of $\varphi(p)0^t$. □

Note that the statement is valid also for q being the empty word.

The following lemma describes the dependence of the center of the palindrome $\varphi(p)0^t$ on the center of the palindrome p. Its proof is a simple application of properties of the substitution φ, we will omit it here.

Lemma 5.3.2. *Let $p_1 \in \mathcal{P}al(u_\beta)$ and let $p_2 = \varphi(p_1)0^t$.*

(i) If $p_1 = w_1a\widetilde{w}_1$, where $a \in A_\varphi$, $a \neq m-1$, then $p_2 = w_2(a+1)\widetilde{w}_2$, where $w_2 = \varphi(w_1)0^t$.

(ii) If $p_1 = w_1(m-1)\widetilde{w}_1$ and $s+t$ is odd, then $p_2 = w_2 0\widetilde{w}_2$, where $w_2 = \varphi(w_1)0^{\frac{s+t-1}{2}}$.

(iii) If $p_1 = w_1(m-1)\widetilde{w}_1$ and $s+t$ is even, then $p_2 = w_2\widetilde{w}_2$, where $w_2 = \varphi(w_1)0^{\frac{s+t}{2}}$.

(iv) If $p_1 = w_1\widetilde{w}_1$ and t is even, then $p_2 = w_2\widetilde{w}_2$, where $w_2 = \varphi(w_1)0^{\frac{t}{2}}$.

(v) If $p_1 = w_1\widetilde{w}_1$ and t is odd, then $p_2 = w_2 0\widetilde{w}_2$, where $w_2 = \varphi(w_1)0^{\frac{t-1}{2}}$.

Lemmas 5.3.1 and 5.3.2 allow us to describe the centers of palindromes $V^{(n)}$ which are in case $s \geq 2$ characterized by having two palindromic extensions.

Proposition 5.3.3. *Let $V^{(n)}$ be palindromes defined by equations* (5.6)*.*

(i) If t is even, then for every $n \geq 1$ the empty word ε is the center of $V^{(n)}$, and $V^{(n)}$ is a central factor of $V^{(n+1)}$.

(ii) If t is odd and s is even, then for every $n \geq 1$ the letter $i \equiv n-1 \pmod{m}$ is the center of $V^{(n)}$, and $V^{(n)}$ is a central factor of $V^{(n+m)}$.

(iii) If t is odd and s is odd, then for every $n \geq 1$, $V^{(n)}$ has the empty word ε for center if $n \equiv 0 \pmod{(m+1)}$, otherwise it has the letter $i \equiv n-1 \pmod{(m+1)}$ for the center. Moreover, $V^{(n)}$ is a central factor of $V^{(m+n+1)}$.

Proof. If t is even, then the empty word ε is the center of $V^{(1)} = 0^t$. Using Lemma 5.3.1 we have that $\varphi(\varepsilon)0^t = V^{(1)}$ is a central factor of $\varphi(V^{(1)})0^t = V^{(2)}$. Repeating the utilization of Lemma 5.3.1 we obtain that $V^{(n)}$ is a central factor of $V^{(n+1)}$. Since ε is the center of $V^{(1)}$, it is also the center of $V^{(n)}$ for all $n \geq 1$.

If t is odd, the palindrome $V^{(1)}$ has center 0 and using Lemma 5.3.2, $V^{(2)}$ has center 1, $V^{(3)}$ has center 2, …, $V^{(m)}$ has center $m-1$. If moreover s is even, then $V^{(m+1)}$ has again center 0. From (ii) of Lemma 5.3.2 we see that 0^{s+t} is a central factor of $V^{(m+1)}$, which implies that $V^{(1)} = 0^t$ is a central factor of $V^{(m+1)}$. In case that s is odd, then $V^{(m)}$ having center $m-1$ implies that $V^{(m+1)}$ has center ε and $V^{(m+2)}$ has center 0. Using (v) of Lemma 5.3.2 we see that $V^{(1)} = 0^t$ is a central factor of $V^{(m+2)}$. Repeated application of Lemma 5.3.1 implies the statement of the proposition. $\square$

As we have said, every palindrome p is either a central factor of a maximal palindrome $U^{(n)}$, for some $n \geq 1$, or p is a central factor of palindromes with increasing length. An example of such a palindrome is $V^{(n)}$, for $n \geq 1$, which is according to Proposition 5.3.3 the central factor of palindromes of arbitrary length. According to the notation introduced by Cassaigne in [48] for left and right special factors extendible to arbitrary length special factors, we introduce the notion of infinite palindromic branch. We will study infinite palindromic branches in the next section.

5.4 Infinite palindromic branches

Definition. Let $v = \cdots v_3 v_2 v_1$ be a left infinite word in the alphabet $\mathcal{A}_\varphi$. Denote by $\widetilde{v}$ the right infinite word $\widetilde{v} = v_1 v_2 v_3 \cdots$.

- Let $a \in \mathcal{A}_\varphi$. If for every index $n \geq 1$, the word $p = v_n v_{n-1} \cdots v_1 a v_1 v_2 \cdots v_n \in \mathcal{P}al(u_\beta)$, then the two-sided infinite word $va\widetilde{v}$ is called an *infinite palindromic branch* of u_β with *center* a, and the palindrome p is called a *central factor* of the infinite palindromic branch $va\widetilde{v}$.

- If for every index $n \geq 1$, the word $p = v_n v_{n-1} \cdots v_1 v_1 v_2 \cdots v_n \in \mathcal{P}al(u_\beta)$, then the two-sided infinite word $v\widetilde{v}$ is called an *infinite palindromic branch* of u_β with *center* ε, and the palindrome p is called a *central factor* of the infinite palindromic branch $v\widetilde{v}$.

Since for Arnoux-Rauzy words every palindrome has exactly one palindromic extension, we obtain for every letter $a \in A_\varphi$ exactly one infinite palindromic branch with center a; there is also one infinite palindromic branch with center ε.

Obviously, every infinite word with bounded palindromic complexity $\mathcal{P}(n)$ has only a finite number of infinite palindromic branches. This is therefore valid also for u_β.

Proposition 5.4.1. *The infinite word u_β invariant under the substitution* (5.2) *has for each center $c \in A_\varphi \cup \{\varepsilon\}$ at most one infinite palindromic branch with center c.*

Proof. Lemma 5.3.2 allows us to create from one infinite palindromic branch another infinite palindromic branch. For example, if $va\widetilde{v}$ is an infinite palindromic branch with center $a \neq m-1$, then using (i) of Lemma 5.3.2, the two-sided word $\varphi(v)0^t(a+1)0^t\widetilde{\varphi(v)}$ is an infinite palindromic branch with center $(a+1)$. Similarly for the center $m-1$ or ε. Obviously, this procedure creates from distinct palindromic branches with the same center $c \in A_\varphi \cup \{\varepsilon\}$ again distinct palindromic branches, for which the length of the maximal common central factor is longer than the length of the maximal common central factor of the original infinite palindromic branches. This would imply that u_β has infinitely many infinite palindromic branches, which is in contradiction with the boundedness of the palindromic complexity of u_β, see Remark 5.2.7. ☐

***Remark* 5.4.2.** Examples of infinite palindromic branches can be easily obtained from Proposition 5.3.3 as a centered limit of palindromes $V^{(k_n)}$ for a suitably chosen subsequence $(k_n)_{n\in\mathbb{N}}$ and n going to infinity, namely

- If t is even, then the centered limit of palindromes $V^{(n)}$ is an infinite palindromic branch with center ε.
- If t is odd and s even, then the centered limit of palindromes $V^{(k+mn)}$ for $k = 1, 2, \dots, m$ is an infinite palindromic branch with center $k-1$.
- If t is odd and s odd, then the centered limit of palindromes $V^{(k+(m+1)n)}$ for $k = 1, 2, \dots, m$ is an infinite palindromic branch with center $k-1$, and for $k = m+1$ it is an infinite palindromic branch with center ε.

Corollary 5.4.3.

(i) If s is odd, then u_β has exactly one infinite palindromic branch with center c for every $c \in A_\varphi \cup \{\varepsilon\}$.

(ii) If s is even and t is odd, then u_β has exactly one infinite palindromic branch with center c for every $c \in A_\varphi$, and u_β has no infinite palindromic branch with center ε.

(iii) If s is even and t is even, then u_β has exactly one infinite palindromic branch with center ε, and u_β has no infinite palindromic branch with center $a \in A_\varphi$.

Proof. According to Proposition 5.4.1, u_β may have at most one infinite palindromic branch for each center $c \in A_\varphi \cup \{\varepsilon\}$. Therefore it suffices to show existence/non-existence of such a palindromic branch. We distinguish four cases according to the parity of s and t.

(a) Let s be odd and t odd. Then an infinite palindromic branch with center c exists for every $c \in \mathcal{A} \cup \{\varepsilon\}$, by Remark 5.4.2.

(b) Let s be odd and t even. The existence of an infinite palindromic branch with center ε is ensured again by Remark 5.4.2. For determining the infinite palindromic branches with other centers, we define a sequence of words

$$W^{(1)} = 0\,, \qquad\qquad W^{(n+1)} = \varphi(W^{(n)})0^t\,, \quad n \in \mathbb{N}, n \geq 1\,.$$

Since $s + t$ is odd, using (i) and (ii) of Lemma 5.3.2, we know that $W^{(n)}$ is a palindrome with center $i \equiv n - 1 \pmod m$. In particular, we have that $0 = W^{(1)}$ is a central factor of $W^{(m+1)}$. Using Lemma 5.3.1, also $W^{(n)}$ is a central factor of $W^{(m+n)}$ for all $n \geq 1$. Therefore we can construct the centered limit of palindromes $W^{(k+mn)}$ for n going to infinity, to obtain an infinite palindromic branch with center $k - 1$ for all $k = 1, 2, \ldots, m$.

(c) Let s be even and t be odd. Then an infinite palindromic branch with center c exists for every $c \in \mathcal{A}$, by Remark 5.4.2. A palindromic branch with center ε does not exist, since using Lemma 5.2.1 two non-zero letters in the word u_β are separated by a block of 0's of odd length, which implies that palindromes of even length must be shorter than $t + s$.

(d) Let s and t be even. The existence of an infinite palindromic branch with center ε is ensured again by Remark 5.4.2. Infinite palindromic branches with other centers do not exist. The reason is that in this case the maximal palindrome $U^{(1)} = 0^{t+s-1}$ has center 0 and using Lemma 5.3.2 the palindromes $U^{(2)}$, $U^{(3)}$, $\ldots$, $U^{(m)}$ have centers $1, 2, \ldots, m-1$, respectively. For all $n > m$ the center of $U^{(n)}$ is the empty word ε. If there existed an infinite palindromic branch $va\widetilde{v}$, then the maximal common central factor p of $va\widetilde{v}$ and $U^{(a+1)}$ would be a palindrome with center a and with two palindromic extensions. Using Proposition 5.2.5, $p = V^{(k)}$ for some k. Proposition 5.3.3 however implies that for t even the center of $V^{(k)}$ is the empty word ε, which is a contradiction.

□

***Remark* 5.4.4.** The proof of the previous corollary implies the following facts.

1. In case t odd, s even, u_β has only finitely many palindromes of even length, all of them being central factors of $U^{(1)} = 0^{t+s-1}$.

2. In case t and s are even, u_β has only finitely many palindromes of odd length and all of them are central factors of one of the palindromes $U^{(1)}$, $U^{(2)}$, $\ldots$, $U^{(m)}$, with center $0, 1, \ldots, m-1$, respectively.

5.5 Palindromic complexity of u_β

The aim of this section is to give explicit values of the palindromic complexity of u_β. We shall derive them from Theorem 5.2.6, which expresses $\mathcal{P}(n) + \mathcal{P}(n+1)$ using the first difference of factor complexity; and from (5) of Remark 5.2.4, which recalls the results about $\mathcal{C}(n)$ of [63].

Theorem 5.5.1. *Let u_β be the fixed point of the substitution (5.2), with $t \geq s \geq 2$.*

(i) Let s be odd and let t be even. Then

$$\mathcal{P}(2n+1) = m$$

$$\mathcal{P}(2n) = \begin{cases} 2\,, & \textit{if } \ |V^{(k)}| < 2n \leq |U^{(k)}| \ \textit{ for some } k, \\ 1\,, & \textit{otherwise.} \end{cases}$$

(ii) Let s and t be odd. Then

$$\mathcal{P}(2n+1) = \begin{cases} m+1\,, & \textit{if } \ |V^{(k)}| < 2n+1 \leq |U^{(k)}| \ \textit{ for some } k \\ & \qquad \textit{with } k \not\equiv m \pmod{(m+1)}, \\ m\,, & \textit{otherwise.} \end{cases}$$

$$\mathcal{P}(2n) = \begin{cases} 2\,, & \textit{if } \ |V^{(k)}| < 2n \leq |U^{(k)}| \ \textit{ for some } k \\ & \qquad \textit{with } k \equiv m \pmod{(m+1)}, \\ 1\,, & \textit{otherwise.} \end{cases}$$

(iii) Let s be even and t be odd. Then

$$\mathcal{P}(2n+1) = \begin{cases} m+2\,, & \textit{if } \ |V^{(k)}| < 2n+1 \leq |U^{(k)}| \ \textit{ for some } k \geq 2\,, \\ m\,, & \textit{if } \ 2n+1 \leq |V^{(1)}|\,, \\ m+1\,, & \textit{otherwise.} \end{cases}$$

$$\mathcal{P}(2n) = \begin{cases} 1\,, & \textit{if } \ 2n \leq |U^{(1)}|\,, \\ 0\,, & \textit{otherwise.} \end{cases}$$

(iv) Let s and t be even. Then

$$\mathcal{P}(2n+1) = \begin{cases} \#\{k \leq m \mid 2n+1 \leq |U^{(k)}|\}\,, & \textit{if } \ 2n+1 \leq |U^{(m)}|\,, \\ 0\,, & \textit{otherwise.} \end{cases}$$

$$\mathcal{P}(2n) = \begin{cases} m+2\,, & \textit{if } \ |V^{(k)}| < 2n \leq |U^{(k)}| \\ & \qquad \textit{for some } k \geq m+1\,, \\ \#\{k \leq m \mid 2n \leq |V^{(k)}|\}\,, & \textit{if } \ 2n \leq |V^{(m+1)}|\,, \\ m+1\,, & \textit{otherwise.} \end{cases}$$

Proof. We prove the statement by cases.

(i) Let s be odd and t be even. It is enough to show that $\mathcal{P}(2n+1) = m$ for all $n \in \mathbb{N}$. The value of $\mathcal{P}(2n)$ can then be easily calculated from Theorem 5.2.6 and (5) of Remark 5.2.4.

From (i) of Corollary 5.4.3 we know that there exists an infinite palindromic branch with center c for all $c \in A_\varphi$. This implies that $\mathcal{P}(2n+1) \geq m$. In order to show the equality, it suffices to show that all maximal palindromes $U^{(k)}$ are of even length, or equivalently, have ε for center. Since both t and $t+s-1$ are even, $0^t = V^{(1)}$ is a central factor of $0^{t+s-1} = U^{(1)}$. Using Lemma 5.3.1, $V^{(k)}$ is a central factor of $U^{(k)}$ for all $k \geq 1$. According to (i) of Proposition 5.3.3, $V^{(k)}$ are palindromes of even length, and thus also the maximal palindromes $U^{(k)}$ are of even length. Therefore they do not contribute to $\mathcal{P}(2n+1)$.

(ii) Let s and t be odd. We shall determine $\mathcal{P}(2n)$ and the values of $\mathcal{P}(2n+1)$ can be deduced from Theorem 5.2.6 and (5) of Remark 5.2.4.

From (i) of Corollary 5.4.3 we know that there exists an infinite palindromic branch with center ε. Thus $\mathcal{P}(2n) \geq 1$ for all $n \in \mathbb{N}$. Again, $V^{(1)} = 0^t$ is a central factor of $U^{(1)} = 0^{t+s-1}$, and thus $V^{(k)}$ is a central factor of $U^{(k)}$ for all $k \geq 1$. A palindrome of even length, which is not a central factor of an infinite palindromic branch must be a central factor of $U^{(k)}$ for some k, and longer than $|V^{(k)}|$. Since $|U^{(k)}| < |V^{(k+1)}| < |U^{(k+1)}|$ (cf. (5) of Remark 5.2.4), at most one such palindrome exists for each length. We have $\mathcal{P}(2n) \leq 2$. It suffices to determine for which k, the maximal palindrome $U^{(k)}$ is of even length, which happens exactly when its central factor $V^{(k)}$ is of even length and that is, using (iii) of Proposition 5.3.3, for $k \equiv 0 \pmod{m+1}$.

(iii) Let s be even and t be odd. According to (1) of Remark 5.4.4, all palindromes of even length are central factors of $U^{(1)} = 0^{t+s-1}$. Therefore $\mathcal{P}(2n) = 1$ if $2n \leq |U^{(1)}|$ and 0 otherwise. The value of $\mathcal{P}(2n+1)$ can be calculated from Theorem 5.2.6 and (5) of Remark 5.2.4.

(iv) Let s and t be even. Using (2) of Remark 5.4.4, the only palindromes of odd length are central factors of $U^{(k)}$ for $k = 1, 2, \ldots, m$. Therefore $\mathcal{P}(2n+1) = 0$ for $2n+1 > |U^{(m)}|$. If $2n+1 \leq |U^{(m)}|$, the number of palindromes of odd length is equal to the number of maximal palindromes longer than $2n+1$. The value of $\mathcal{P}(2n)$ can be calculated from Theorem 5.2.6 and (5) of Remark 5.2.4.

□

For the determination of the value $\mathcal{P}(n)$ for a given n, we have to know lengths $|V^{(k)}|$ and $|U^{(k)}|$. In [63] it is shown that

$$|V^{(k)}| = t\sum_{i=0}^{k-1} G_i\,, \qquad \text{and} \qquad |U^{(k)}| = |V^{(k)}| + (s-1)G_{k-1}\,,$$

where G_n is a sequence of integers defined by the recurrence

$$\begin{aligned} G_0 &= 1\,, && \\ G_n &= t(G_{n-1}+\cdots+G_0)+1\,, && \text{for } 1\le n\le m-1\,,\\ G_n &= t(G_{n-1}+\cdots+G_{n-m+1})+sG_{n-m}\,, && \text{for } n\ge m\,. \end{aligned}$$

The sequence $(G_n)_{n\in\mathbb{N}}$ defines the canonical linear numeration system associated with the number β, see [39] for general results on these numeration systems. In this particular case, $(G_n)_{n\in\mathbb{N}}$ defines a confluent linear numeration system, see [60] for its properties.

5.6 Substitution invariance of palindromic branches

Infinite words u_β are invariant under the substitution (5.2). One can ask whether also their infinite palindromic branches are invariant under a substitution. In case that an infinite palindromic branch has as its center the empty word ε, we can use the notion of invariance under substitution as defined for pointed two-sided infinite words. We restrict our attention to infinite palindromic branches of such type.

Recall that an infinite palindromic branch of u_β with center ε exists, (according to Corollary 5.4.3), only if in the Rényi expansion of unit is $d_\beta(1)=tt\cdots ts$, t is even, or both t and s are odd. Therefore we shall study only such parameters.

Let us first study the most simple case, $d_\beta(1)=t1$ for $t\ge 1$. Here β is a quadratic unit, and the infinite word u_β is a Sturmian word, expressible in the form of the mechanical word $\mu_{\alpha,\varrho}$,

$$\mu_{\alpha,\varrho}(n)=\lfloor(n+1)\alpha+\varrho\rfloor-\lfloor n\alpha+\varrho\rfloor\,,\quad n\in\mathbb{N}\,,$$

where the irrational slope α and the intercept ρ satisfy $\alpha=\rho=\frac{\beta}{\beta+1}$. The infinite palindromic branch with center ε of the above word $u_\beta=\mu_{\alpha,\varrho}$ is a two-sided Sturmian word with the same slope $\alpha=\frac{\beta}{\beta+1}$, but intercept $\frac{1}{2}$. Indeed, two mechanical words with the same slope have the same set of factors independently on their intercepts, and moreover the Sturmian word $\mu_{\alpha,\frac{1}{2}}$ is an infinite palindromic branch of itself, since

$$\mu_{\alpha,\frac{1}{2}}(n)=\mu_{\alpha,\frac{1}{2}}(-n-1)\,,\quad \text{for all } n\in\mathbb{Z}\,.$$

Therefore if $v=\mu_{\alpha,\frac{1}{2}}(0)\mu_{\alpha,\frac{1}{2}}(1)\mu_{\alpha,\frac{1}{2}}(2)\cdots$, then $\widetilde{v}v$ is the infinite palindromic branch of u_β with the center ε.

Since the Sturmian word $\mu_{\alpha,\varrho}$ coincides with u_β, it is invariant under the substitution φ. As a consequence of [88], the slope α is a Sturm number, i.e. a quadratic number in $(0,1)$ such that its conjugate α' satisfies $\alpha'\notin(0,1)$, (using the equivalent definition of Sturm numbers given in [7]).

The question about the substitution invariance of the infinite palindromic branch $\widetilde{v}v$ is answered using the result of [19] (or also [35, 106]). It says that a Sturmian word whose slope is a Sturm number, and whose intercept is equal to $\frac{1}{2}$, is substitution invariant as a two-sided pointed word, i.e. there exists a substitution ψ such that $\widetilde{v}|v=\psi(\widetilde{v})|\psi(v)$.

Example. The Fibonacci word u_β for $d_\beta(1) = 11$ is a fixed point of the substitution

$$\varphi(0) = 01, \quad \varphi(1) = 0\,.$$

Its infinite palindromic branch with center ε is

$$\widetilde{v}v \qquad \text{for} \qquad v = 010100100101001001010\cdots$$

which is the fixed point $\lim_{n\to\infty} \psi^n(0)|\psi^n(0)$ of the substitution

$$\psi(0) = 01010, \quad \psi(1) = 010\,.$$

Let us now study the question whether infinite palindromic branches in u_β for general $d_\beta(1) = tt\cdots ts$ with t even, or t and s odd, are also substitution invariant. It turns out that the answer is positive. For construction of a substitution ψ under which a given palindromic branch is invariant, we need the following lemma.

Lemma 5.6.1. *Let $v\widetilde{v}$ be an infinite palindromic branch with center ε. Then the left infinite word $v = \cdots v_3v_2v_1$ satisfies*

$$\begin{aligned} v &= \varphi(v)0^{\frac{t}{2}} && \textit{for } t \textit{ even,} \\ v &= \varphi^{m+1}(v)\varphi^m(0^{\frac{t+1}{2}})0^{\frac{t-s}{2}} && \textit{for } t \textit{ and } s \textit{ odd.} \end{aligned}$$

Proof. Let t be even and let $v\widetilde{v}$ be the unique infinite palindromic branch with center ε. Recall that $v\widetilde{v}$ is a centered limit of $V^{(n)}$. Consider arbitrary suffix v_{suf} of v, i.e. $v_{\text{suf}}\widetilde{v}_{\text{suf}}$ is a palindrome of u_β with center ε. Denote $w := \varphi(v_{\text{suf}})0^{\frac{t}{2}}$. Using (iv) of Lemma 5.3.2 the word $p = w\widetilde{w}$ is a palindrome of u_β with center ε. We show by contradiction that w is a suffix of v.

Suppose that $p = w\widetilde{w}$ is not a central factor of $v\widetilde{v}$, then there exists a unique n such that p is a central factor of $U^{(n)}$. Then according to Proposition 5.2.5, p is uniquely extendible into a maximal palindrome. In that case we take a longer suffix v'_{suf} of v, so that the length of the palindrome $p' = w'\widetilde{w}'$, $w' := \varphi(v'_{\text{suf}})0^{\frac{t}{2}}$ satisfies $|p'| > |U^{(n)}|$. However, p' (since it contains p as its central factor) is a palindromic extension of p, and therefore p' is a central factor of $U^{(n)}$, which is a contradiction. Thus $\varphi(v_{\text{suf}})0^{\frac{t}{2}}$ is a suffix of v for all suffixes v_{suf} of v, therefore $v = \varphi(v)0^{\frac{t}{2}}$.

Let now s and t be odd. If v_{suf} is a suffix of the word v, then $v_{\text{suf}}\widetilde{v}_{\text{suf}}$ is a palindrome of u_β with center ε. Using Lemma 5.3.2, the following holds true.

$$\begin{aligned}
w_0 &= \varphi(v_{\text{suf}})0^{\frac{t-1}{2}} &\Longrightarrow&& w_0 0\widetilde{w}_0 &\in \mathcal{P}al(u_\beta) \\
w_1 &= \varphi(w_0)0^t &\Longrightarrow&& w_1 1\widetilde{w}_1 &\in \mathcal{P}al(u_\beta) \\
w_2 &= \varphi(w_1)0^t &\Longrightarrow&& w_2 2\widetilde{w}_2 &\in \mathcal{P}al(u_\beta) \\
& & \vdots & & & \\
w_{m-1} &= \varphi(w_{m-2})0^t &\Longrightarrow&& w_{m-1}(m-1)\widetilde{w}_{m-1} &\in \mathcal{P}al(u_\beta) \\
w_\varepsilon &= \varphi(w_{m-1})0^{\frac{s+t}{2}} &\Longrightarrow&& w_\varepsilon\widetilde{w}_\varepsilon &\in \mathcal{P}al(u_\beta)
\end{aligned}$$

Together we obtain

$$w_\varepsilon = \varphi^{m+1}(v_{\rm suf})\varphi^m(0^{\frac{t-1}{2}})\varphi^{m-1}(0^t)\cdots\varphi^2(0^t)\varphi(0^t)0^{\frac{s+t}{2}}\,.$$

Since $\varphi^m(0) = \varphi^{m-1}(0^t)\varphi^{m-2}(0^t)\cdots\varphi(0^t)0^s$, the word w_ε can be rewritten in a simpler form

$$w_\varepsilon = \varphi^{m+1}(v_{\rm suf})\varphi^m(0^{\frac{t-1}{2}})\varphi^m(0)0^{\frac{t-s}{2}} = \varphi^{m+1}(v_{\rm suf})\varphi^m(0^{\frac{t+1}{2}})0^{\frac{t-s}{2}}$$

Since w_ε is again a suffix of v, the statement of the lemma for s and t odd holds true. □

Theorem 5.6.2. *Let u_β be the fixed point of the substitution φ given by (5.2), and let $v\widetilde{v}$ be the infinite palindromic branch of u_β with center ε. Then the left-sided infinite word v is invariant under the substitution ψ defined for all letters $a \in \{0, 1, \ldots, m-1\}$ by*

$$\psi(a) = \begin{cases} w^{-1}\varphi(a)w\,, & \text{where } \ w = 0^{\frac{t}{2}}\,, \qquad\qquad \text{for } t \text{ even,} \\ w^{-1}\varphi^{m+1}(a)w\,, & \text{where } \ w = \varphi^m(0^{\frac{t+1}{2}})0^{\frac{t-s}{2}}\,, \quad \text{for } t \text{ and } s \text{ odd.} \end{cases}$$

Moreover, if t is even, then $\psi(a)$ is a palindrome for all $a \in A_\varphi$ and $v\widetilde{v}$ as a pointed sequence is invariant under the same substitution ψ.

Proof. First let us show that the substitution ψ is well defined.

- Let t be even. Since $0^{\frac{t}{2}}$ is a prefix of $\varphi(a)$ for all $a \in \{0, 1, \ldots, m-2\}$ and $\varphi(m-1) = 0^s$, therefore $0^{\frac{t}{2}}$ is a prefix of $\varphi(m-1)0^{\frac{t}{2}} = 0^{s+\frac{t}{2}}$.
- Let t and s be odd. Let us verify that w is a prefix of $\varphi^{m+1}(a)w$.
  - If $a \neq m-1$, we show that $w = \varphi^m(0^{\frac{t+1}{2}})0^{\frac{t-s}{2}}$ is a prefix of

    $$\varphi^{m+1}(a) = \varphi^m\big(0^t(a+1)\big) = \varphi^m(0^{\frac{t+1}{2}})\varphi^m(0^{\frac{t-1}{2}})\varphi^m(a+1)\,.$$

    It suffices to show that $0^{\frac{t-s}{2}}$ is a prefix of $\varphi^m(0^{\frac{t-1}{2}})$. For $t = s$ it is obvious. For $t > s \geq 1$ we obtain $t \geq 3$ and so $\varphi^m(0^{\frac{t-1}{2}}) = \varphi^m(0)\varphi^m(0^{\frac{t-3}{2}})$ and clearly $0^{\frac{t-s}{2}}$ is a prefix of $\varphi^m(0)$.
  - If $a = m-1$, then

    $$\varphi^{m+1}(m-1)w = \varphi^m(0^s)\varphi^m(0^{\frac{t+1}{2}})0^{\frac{t-s}{2}} = \varphi^m(0^{\frac{t+1}{2}})\varphi^m(0)\varphi^m(0^{s-1})0^{\frac{t-s}{2}}\,.$$

    Since $0^{\frac{t-s}{2}}$ is a prefix of $\varphi^m(0)$, the correctness of the definition of the substitution ψ is proved.

Now it is enough to prove that $\psi(v) = v$. Lemma 5.6.1 says that in the case that t is even the left infinite word $v = \cdots v_3v_2v_1$ satisfies $v = \varphi(v)w$. Thus we have

$$\begin{aligned}\psi(v) = \cdots\psi(v_3)\psi(v_2)\psi(v_1) &= \cdots w^{-1}\varphi(v_3)ww^{-1}\varphi(v_2)ww^{-1}\varphi(v_2)w = \\ &= \cdots\varphi(v_3)\varphi(v_2)\varphi(v_1)w \;=\; \varphi(v)w \;=\; v\,.\end{aligned}$$

In case that t and s are odd, the proof is the same, using φ^{m+1} instead of φ.

If t is even, it is clear from the prescription for ψ, that $\psi(a)$ is a palindrome for any letter a, which implies the invariance of the word $v\widetilde{v}$ under ψ. □

Let us mention that for t, s odd the words $\psi(a)$, $a \in A_\varphi$, may not be palindromes. In that case the right-sided word $\widetilde{v}$ is invariant under another substitution, namely $a \mapsto \widetilde{\psi(a)}$. Nevertheless even for t, s odd it may happen that $\psi(a)$ is a palindrome for all letters. Then the two-sided word $v\widetilde{v}$ is invariant under ψ. This situation is illustrated on the following example.

Example. Consider the Tribonacci word, i.e. the word u_β for $d_\beta(1) = 111$. It is the fixed point of the substitution

$$\varphi(0) = 01, \quad \varphi(1) = 02, \quad \varphi(2) = 0\,,$$

which is in the form (5.2) for $t = s = 1$ and $m = 3$. Therefore $w = \varphi^3(0) = 0102010$. The substitution ψ, under which the infinite palindromic branch $v\widetilde{v}$ of the Tribonacci word is invariant, is therefore given as

$$\begin{array}{lcl}\psi(0) & := & w^{-1}\varphi^4(0)w & = & 0102010102010,\\ \psi(1) & := & w^{-1}\varphi^4(1)w & = & 01020102010,\\ \psi(2) & := & w^{-1}\varphi^4(2)w & = & 0102010.\end{array}$$

Note that the substitution ψ has the following property: the word $\psi(a)$ is a palindrome for every $a \in A_\varphi$.

5.7 Number of palindromes in the prefixes of u_β

In [53] the authors obtain an interesting result which says that every finite word w contains at most $|w| + 1$ different palindromes. (The empty word is considered as a palindrome contained in every word.) Denote by $P(w)$ the number of palindromes contained in the finite word w. Formally, we have

$$P(w) \leq |w| + 1 \qquad \text{for every finite word } w.$$

The finite words w for which the equality is reached are called *full* (as suggested in [43]). An infinite word u is called full, if all its prefixes are full. In [53] the authors have shown that every Sturmian word is full. They have shown the same property for episturmian words.

The infinite word u_β can be full only if its language is closed under reversal, i.e. in the simple Parry case for $d_\beta(1) = tt \cdots ts$, $t \geq s \geq 1$. For $s \geq 2$ such words are not episturmian, nevertheless, we shall show that they are full. We shall use the notions and results introduced in [53].

Definition. A finite word w is said to satisfy property Ju, if there exists a palindromic suffix of w which is unioccurrent in w.

Clearly, if w satisfies Ju, then it has exactly one palindromic suffix which is unioccurrent, namely the longest palindromic suffix of w.

Proposition 5.7.1 ([53]). *Let w be a finite word. Then $P(w) = |w| + 1$ if and only if all the prefixes $\hat{w}$ of w satisfy Ju, i.e. have a palindrome suffix which is unioccurrent in $\hat{w}$.*

Theorem 5.7.2. *The infinite word u_β invariant under the substitution* (5.2) *is full.*

Proof. We show the statement using Proposition 5.7.1 by contradiction. Let w be a prefix of u_β of minimal length which does not satisfy Ju, and let $X0^k$ be a suffix of w with $X \neq 0$.

First we show that $k \in \{0, t+1\}$. For, if $1 \leq k \leq t$ or $t+2 \leq k$, then q is the maximal palindromic suffix of $w0^{-1}$ if and only if $0q0$ is the maximal palindromic suffix of w. Since $0q0$ occurs at least twice in w, then also q occurs at least twice in $w0^{-1}$, which is a contradiction with the minimality of w.

Define

$$w_1 = \begin{cases} w0^t & \text{if } w \text{ has suffix } X \neq 0\,, \\ w0^{s-1} & \text{if } w \text{ has suffix } X0^{t+1},\ X \neq 0\,. \end{cases}$$

For the maximal palindromic suffix p of w denote

$$p_1 = \begin{cases} 0^t p 0^t & \text{if } w \text{ has suffix } X \neq 0\,, \\ 0^{s-1} p 0^{s-1} & \text{if } w \text{ has suffix } X0^{t+1},\ X \neq 0\,. \end{cases}$$

Since in u_β every two non-zero letters are separated by the word 0^t or 0^{t+s}, we obtain that

(i) p_1 is the maximal palindromic suffix of w_1.

(ii) the position of centers of palindromes p and p_1 coincide in all occurrences in u_β.

Since p occurs in w at least twice, also the palindromic suffix p_1 occurs at least twice in w_1, i.e. the word w_1 is a prefix of u_β which does not satisfy Ju.

From the definition of w_1 it follows that

$$w_1 = \varphi(\hat{w})0^t$$

for some prefix $\hat{w}$ of u_β. Thus the maximal palindromic suffix p_1 of w_1 is of the form $p_1 = \varphi(\hat{p})0^t$, where $\hat{p}$ is a factor of $\hat{w}$. According to point (i) of Proposition 5.2.3, $\hat{p}$ is a palindrome, and the same proposition implies that $\hat{p}$ is the maximal palindromic suffix of $\hat{w}$. Since p_1 occurs at least twice in w_1, also $\hat{p}$ occurs at least twice in $\hat{w}$. Therefore $\hat{w}$ does not satisfy the property Ju. As

$$|\hat{w}| < |\varphi(\hat{w})| < |w|,$$

we have a contradiction with the minimality of w. □

5.8 On non-simple Parry numbers

The study of palindromic complexity of an uniformly recurrent infinite word is interesting in the case that its language is closed under reversal. Infinite words u_β associated to Parry numbers β are uniformly recurrent.

For non-simple Parry number β, the condition under which the language of the infinite word u_β is closed under reversal has been stated by Bernat [32]. He has shown that the language of u_β is closed under reversal if and only if β is a quadratic number, i.e. a root of minimal polynomial $X^2 - aX + b$, with $a \geq b+2$ and $b \geq 1$. In this case $\mathrm{d}_\beta(1) = (a-1)(a-b-1)^\omega$. The palindromic complexity of the corresponding infinite words u_β is described in [22].

Infinite words u_β for non-simple Parry numbers β are thus another example for which the equality

$$\mathcal{P}(n) + \mathcal{P}(n+1) = \Delta\mathcal{C}(n) + 2$$

is satisfied for all $n \in \mathbb{N}$. According to our knowledge, among all examples of infinite words satisfying this equality, the words u_β (for both simple and non-simple Parry number β) are exceptional in that they have the second difference $\Delta^2\mathcal{C}(n) \neq 0$.

Chapter 6

Conclusion

We have seen in course of this thesis several problems or questions which remain unsolved or opened. As a conclusion, instead of usual duplication of what have been already written in the introduction section, we will summarize these problems here in one common place.

Arithmetics. Concerning the arithmetic properties, the "most wanted" answer is a (simple) algebraic characterization of numbers β for which the Finiteness property holds, or equivalently, to characterize numbers β for which the set of β-expansions with a finite fractional part has a ring structure.

However, to give a full answer to this question is a very difficult task and it seems to be beyond out present horizon of feasibility. Hence we propose here a simpler question connected with Property (F). We gave a sufficient condition for Property (F) to hold in the terms of the minimal forbidden words (Theorem 3.1.3), but the examples illustrating this result were already known to satisfy Property (F). Therefore the first step towards a characterization of numbers satisfying Property (F) — at least from our point of view — could be to find new numbers β such that the transcription of each of its minimal forbidden words has the sum of its digits smaller than the word itself has. Note that he term "a new number β" is used in the sense that β should not belong to some known class of numbers satisfying Property (F), cf. Theorems 2.2.4, 2.2.5 and 2.2.6.

A second problem we would like to address in the future is the proof of the correctness of the algorithm performing addition in a general Pisot base.

Alpha-adic expansions. In the area of α-adic expansions the most evident open question is the unicity/multiplicity of α-adic expansions of elements of the extension ring $\mathbb{Z}[\alpha^{-1}]$ for numbers β of higher degree, or in general the problem to compute (or at least estimate) the number of α-adic expansions of a given number x.

Another possible approach to expansions in conjugates of a Pisot number β would be to assume a simultaneous expansion of a number in all the conjugates of β, that is, in the form of a $d-1$ dimensional vector in the case where β has degree d. On one hand this would be quite different from the approach we have used in this thesis, on the other hand this concept would allow us to use previous results concerning the tiling of the conjugated plane.

In the last section of this chapter we have seen (at least implicitly) another open problem, which in general could be stated as: what are the tools that transform eventual periodicity into eventual periodicity. Obvious answer are sequential functions/transducers, but it seems that this is not a necessary condition, that the concept of the sequentiality is unnecessarily strong and that even functions/machines that are not sequential can preserve eventual periodicity.

Combinatorics. In the chapter dealing with the palindromic complexity of infinite words associated with simple Pisot numbers we found a necessary condition under which u_β contains infinitely many palindromes. For words satisfying this condition we then performed computation of the palindromic complexity. In other cases the palindromic complexity eventually vanishes, i.e. $\mathcal{P}(n) = 0$ from some n_0 on, but it still may be interesting to compute these initial non-zero values of $\mathcal{P}(n)$.

Appendix A

Numerical evidences

In this appendix we supply numerical evidences to two proofs from Chapter 3. In both cases we needed to show the validity of some inequality for positive integers. However, using algebraic manipulations and estimates we succeeded for all but a few values of a given parameter. The validity of inequalities for these unsolved cases is numerically confirmed here.

A.1 Proof of Theorem 3.2.8

The aim of the proof was to show the validity of inequality (3.14), which was the following one

$$(1+m)\beta^3 + \beta^2 + (m+2)\beta + 1 > \left((m+2)\beta + m^2 + 2\right)\beta^{3/2}\,, \tag{3.14}$$

where $m \in \mathbb{N}$, $m \geq 2$ and β is a generalized golden mean, that is, an algebraic integer with minimal polynomial $x^3 - mx^2 - x - 1$, $m \geq 2$. Using algebraic manipulations we were able to show the validity of the inequality for $m \geq 10$.

Let us denote by $L(m)$ and $R(m)$ the value of the left-hand side and right-hand side of (3.14), respectively. Computed values of $L(m)$ and $R(m)$ for $m = 2, \dots, 9$ are summarized in Table A.1. Note that all numbers in the table are rounded.

Since in all lines of the table the value of $L(m)$ is strictly greater than the corresponding value of $R(m)$, Theorem 3.2.8 holds for all $m \geq 2$.

A.2 Inequalities in Section 3.2.3

The aim was to show the upper estimates $L_\oplus(\beta) \leq 1$ and $L_\otimes(\beta) \leq 2$ for β being a cubic Pisot unit with minimal polynomial of the form $x^3 - ax^2 - (a-1)x + 1$ (cf. Page 56). Using algebraic manipulations we were able to show the validity of estimates for $a \geq 11$.

To show the validity of $L_\oplus(\beta) \leq 1$ and $L_\otimes(\beta) \leq 2$ for $a = 3, \dots, 10$ it is enough, according to Theorem 2.2.8, to check the following inequalities

$$\left(\frac{1}{\beta'}\right)^{L_\oplus(\beta)} < \frac{2H}{K} \qquad \text{and} \qquad \left(\frac{1}{\beta'}\right)^{L_\otimes(\beta)} < \frac{H^2}{K} \tag{A.1}$$

m	β	$L(m)$	$R(m)$
2	2.5468	67.2317	65.7910
3	3.3830	184.2256	173.6937
4	4.2876	439.2222	388.2062
5	5.2279	922.2122	760.1705
6	6.1877	1747.1990	1346.8336
7	7.1592	3052.1854	2211.1840
8	8.1380	4999.1724	3421.4714
9	9.1216	7774.1605	5050.8299

Table A.1: Values of β, $L(m)$ and $R(m)$

or by combining terms in (A.1) into one inequality and using the fact that $K = 1$

$$\frac{1}{\beta'} < 2H \leq \left(\frac{1}{\beta'}\right)^2 < H^2 \leq \left(\frac{1}{\beta'}\right)^3 . \tag{A.2}$$

Computed values confirming the validity of (A.2) are summarized in Table A.2.

a	$\frac{1}{\beta'}$	$2H$	$\left(\frac{1}{\beta'}\right)^2$	H^2	$\left(\frac{1}{\beta'}\right)^3$
3	2.9122	8.0918	8.4811	16.3693	24.6988
4	3.9488	10.0347	15.5932	25.1738	61.5750
5	4.9663	12.0170	24.6636	36.1022	122.4858
6	5.9760	14.0096	35.7127	49.0675	213.4197
7	6.9821	16.0060	48.7491	64.0480	340.3691
8	7.9861	18.0040	63.7772	81.0359	509.3292
9	8.9889	20.0028	80.7997	100.0279	726.2970
10	9.9909	22.0020	99.8179	121.0223	997.2704

Table A.2: Values of terms in (A.2)

Appendix B

Program pisotarith

This appendix gathers two individual manuals for the program `pisotarith`. The first one is an installation manual giving a short overview of standard installation instructions. The second one is a user manual, focused on the `batch` front-end to the library, which is used for batch processing of data files.

B.1 Installation manual

An actual version of the `pisotarith` program can be downloaded from the home site of the author

`http://linux.fjfi.cvut.cz/~ampy/`

in the form of a tarball package. Usual procedure to compile should be used

```
tar -zxvf pisotarith-w.x.y-z.tar.gz
cd pisotarith-w.x.y-z/src
make
```

where `w.x.y-z` stands for the actual version of the package. Apart from `make (all)`, there are two other possible targets of `make`, namely `batch` and `cgi`. The `batch` target builds the `pisotarith-batch` front-end used for performing batches of computations (see Section B.2 of this Appendix). The `cgi` target builds the `pisotarith-cgi` front-end used as a cgi script on a web server. An example of a www-page with input form for this script can be found in the directory `pisotarith-w.x.y-z/www`.

B.2 User manual

This section provides a short help for the options of the `batch` front-end of the program. Note that if a long option shows an argument as mandatory, then it is mandatory for the equivalent short option too. If an argument, which is marked as *mandatory*, is not specified on the command line, the program asks for its value directly after the start.

`-p --operation {a,s,m}` default: `a`
: Sets the operation to be performed on expansions: `a` stands for addition, `s` for subtraction and `m` for multiplication.

`-n --notest`
: If this option is used the program does not test for the re-occurrence during the normalization. This can speed up computation and lower the amount of memory. On the other hand, it should be used with case, since in the case when the result is eventually periodic the normalization algorithm cannot stop when the re-occurrence is not being tested.

`-i --input FILE1 [FILE2]` *mandatory*
: Specifies input file(s) to be used. If only `FILE1` is specified the computation is performed for all the pairs of expansions in this file, otherwise it is performed for all the pairs such that the first member of the pair is taken from `FILE1` and the second one from `FILE2`.

`-o --output FILE` *mandatory*
: Specifies the output file into which the results will be printed.

`-d --digits N` default: 0
: By default the results of all the computations performed are printed in the output file. By using this option the user may specify the minimal number of fractional digits `N` the expansion has to have to be printed.

`-r --progressive`
: Changes the way the results are printed on the output in the following way: at the beginning of the computation sets the `--digits` parameter to 0 and then its value is progressively increased, that is, as soon as a result having d fractional digits is found (and printed on the output), the parameter `--digits` is set to $d + 1$.

`-f --finite`
: If this option is used all results having an infinite (eventually periodic) expansion are not printed on the output.

`-l --logfile FILE`
: Sets the name of the log file. The program prints into it the summary of used options and possibly also indicators of the progress of computation (according to the options `-v` and `--verbose_step`).

`-v --verbose N` default: 0
: Set the level of the verbose mode. For the value 0 no messages are printed in the output, for the value 1 the overview of values of parameters and also the progress of computation are printed on the output.

`--verbose_step N` default: 1

If `--verbose` is set to 1, the program prints on the output a message after every `N` expansions in the first file have been processed.

`-s --silent`

Program does not print on the standard output prompts for mandatory arguments, which were not specified on the command line (Rényi expansion, input file(s), output file).

`-h --help`

Prints a shortened variant of this manual on the standard output and exits.

List of Symbols

Symbols used in this work are listed in the following table. Each symbol is supplemented with a short description and with the number of the page where it is used for the first time.

$\langle x \rangle_\beta$	β-expansion of x	20
$(x)_\beta$	β-representation of x	19
$<_{\text{lex}}$	lexicographically smaller	11
$\lfloor x \rfloor$	integer part of x	20
$^\omega$	infinite iteration	11
$\sim$	reverse operator	12
$\{x\}$	fractional part of x	20
$\#$	cardinality of a set	9
$\mathcal{A}$	finite automaton	15
A	finite alphabet	11
A^*	set of finite words over A	11
$A^{\mathbb{N}}$	set of infinite words over A	11
A_β	canonical alphabet of a β-numeration system	20
$\mathbb{C}$	set of complex numbers	9
$\mathcal{C}_u$	complexity of a word u	12
$\mathrm{d}_\beta(1)$	Rényi expansion of unity	20
ε	empty word	11
$\mathrm{Fin}(\beta)$	set of numbers with finite β-expansion	22
φ	morphism or substitution	14
φ_β	canonical substitution associated with β	27
$L_\oplus(\beta)$	maximal number of fractional digits arising under addition of β-integers	26
$L_\otimes(\beta)$	maximal number of fractional digits arising under multiplication of β-integers	26
$\mathcal{L}(u)$	language of a word u	11
$\mathcal{L}_n(u)$	set of factors of length n in a word u	11
$\boldsymbol{M}_\varphi$	incidence matrix of a substitution φ	15
$\mathrm{M}_\beta(x)$	minimal polynomial of β	25
$\mathbb{N}$	set of non-negative integers	9
ν_C	normalization function	21
$\mathcal{P}_u$	palindromic complexity of a word u	13
$\mathcal{P}al(u)$	set of palindromic factors in a word u	13
π_β	β-value function	19

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