Description of Voronoi tiles in quasicrystals with octagonal symmetry

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Outline

Cut and Project scheme

$$
\begin{array}{ccc}
 & L \subset \mathbb{R}^{n+m} & \\
 {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\
M = \pi_1(L) & \xrightarrow[\ast = \pi_1^{-1} \circ \pi_2]{} & M^* = \pi_2(L)
\end{array}
$$

Cut-and-project quasicrystal with the acceptance window Ω

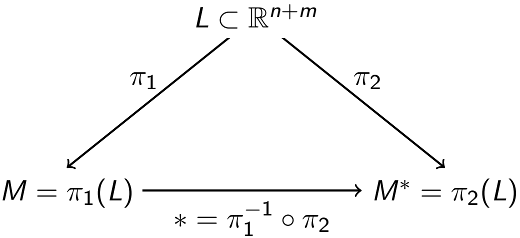
$$\Sigma(\Omega) = \{x \in M \mid x^* \in \Omega\}$$

- $\pi_1|_L$ injection
- $\pi_2(L)$ dense
- Ω bounded, $\Omega^\circ \neq \emptyset$

$\Longrightarrow$ Quasicrystal $\Sigma(\Omega)$

- is Delone set,
- has finite local complexity.

Cut and Project scheme



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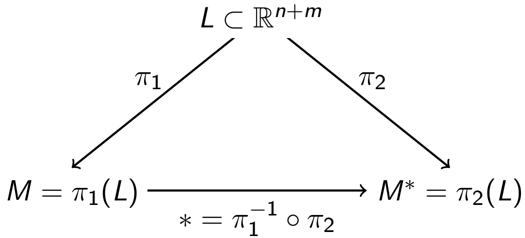

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Delone set

Definition

$P \subset \mathbb{R}^n$ is a **Delone set** if it is

1. uniformly discrete

$$\exists r, 0 < r \quad \text{such that} \quad \forall x, y \in P : r \leq ||x - y||,$$

2. relatively dense

$$\exists R, 0 < R < \infty \quad \text{such that} \quad \forall z \in \mathbb{R}^n \, \exists x \in P : ||z - x|| \leq R.$$

Covering radius

$$R_C = \min\{R > 0 \mid \forall z \in \mathbb{R}^n \, \exists x \in P : ||z - x|| \leq R\}$$

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Voronoi tiling

Definition

Voronoi tile of $x \in \Sigma(\Omega)$

$$V(x) = \{y \in \mathbb{R}^n \mid \forall z \in \Sigma(\Omega), z \neq x : ||y - x|| \leq ||y - z||\}.$$

Properties

- $\Sigma(\Omega)$ is Delone
  $\Rightarrow$ $V(x)$ are bounded polygons
- $\Sigma(\Omega)$ has finite local complexity
  $\Rightarrow$ finitely many different tiles

Remark. Importance of covering radius

$$V(x) \equiv \{y \in B_x(2R_C) \mid \forall z \in \Sigma(\Omega), z \neq x : ||y - x|| \leq ||y - z||\}.$$

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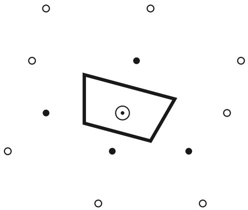

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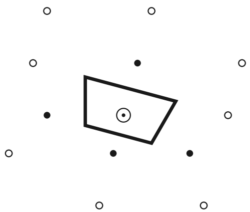

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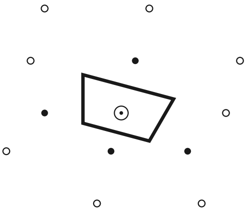

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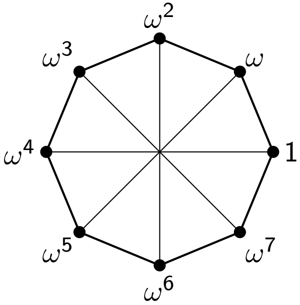
Our model



$\omega = e^{\frac{2\pi i}{8}}$ ... eigth root of unity

$\mathbb{Z}[\omega] = \{a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 \mid a_i \in \mathbb{Z}\}$

... ring of integers of $\mathbb{Q}(\omega)$

$\beta = 1 + 2\cos\frac{2\pi}{8} = 1 + \sqrt{2}$

$\beta' = 1 + 2\cos\frac{6\pi}{8} = 1 - \sqrt{2}$

... Pisot-cyclotomic number of order 8

Remark. $\mathbb{Z}[\omega] = \mathbb{Z}[\beta] + \omega\mathbb{Z}[\beta]$

C&P scheme essentials

$$M = \mathbb{Z}[\beta]\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{\mathbf{v}_1} + \mathbb{Z}[\beta]\underbrace{\begin{pmatrix}\Re(\omega)\\ \Im(\omega)\end{pmatrix}}_{\mathbf{v}_2} \qquad M^* = \mathbb{Z}[\beta]\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{\mathbf{v}_1^*} + \mathbb{Z}[\beta]\underbrace{\begin{pmatrix}\Re(\omega^3)\\ \Im(\omega^3)\end{pmatrix}}_{\mathbf{v}_2^*}$$

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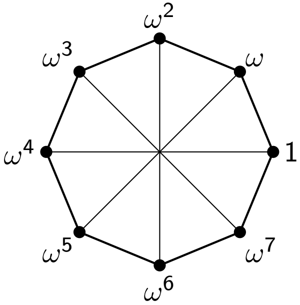

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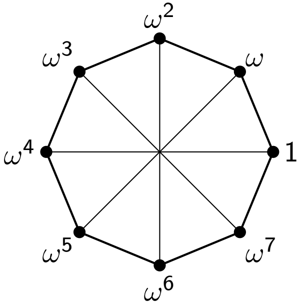

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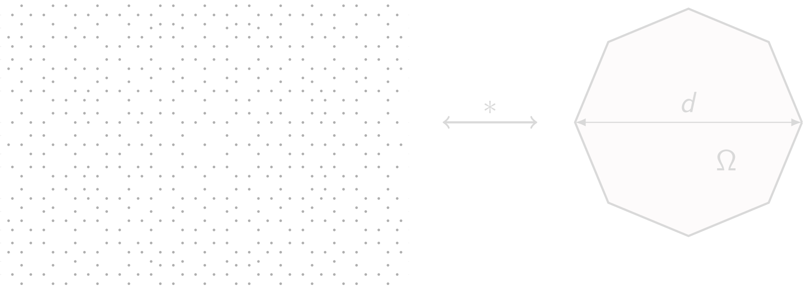

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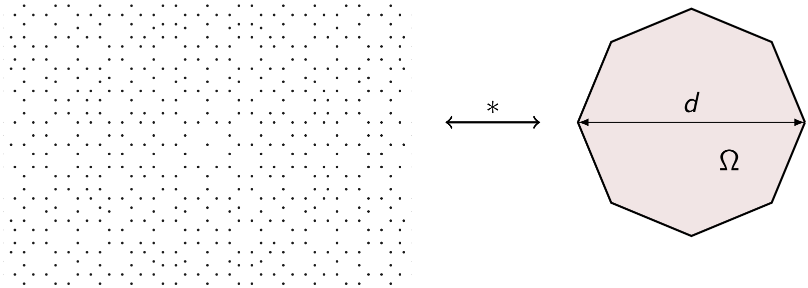

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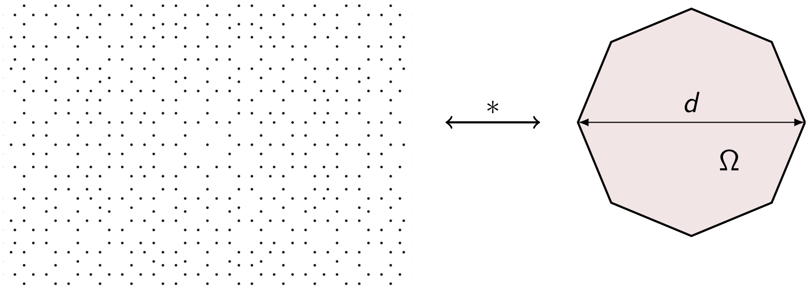

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Questions/Tasks

1. For given Ω_d describe all tiles in $\Sigma(\Omega_d)$
   - cannot be solved by observation only
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2. Inspect all $d \in (1, \beta]$, classify according to the tile set
   - $(1, \beta]$ can be covered by subintervals based on sets of tiles in corresponding quasicrystals
   - if two quasicrystals have diameters from the same subinterval then they have the same set of tiles

Remark. Both questions are completely solved for rhombic, decagonal and circular window for C&P QC with decagonal rotational symmetry.

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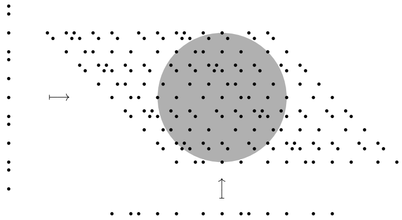
Tiles in $\Sigma(\Omega_r)$ – Rhombic window

1. $\Sigma(\Omega)$ = Cartesian product of two one-dimensional sets $\Sigma(I)$



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   … half of largest distance in $\Sigma(I)$

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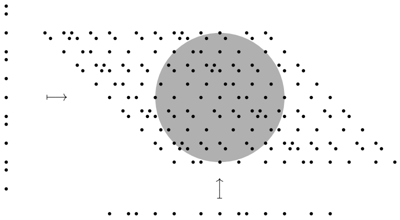

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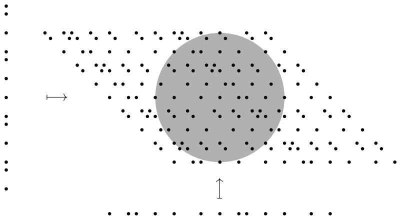

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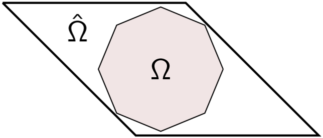

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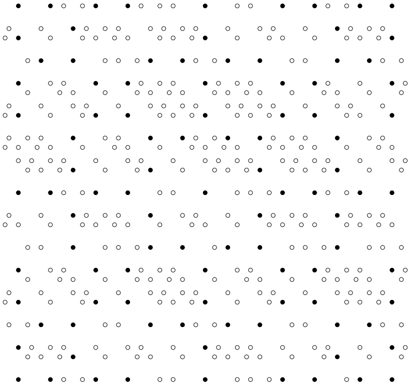

Then filter

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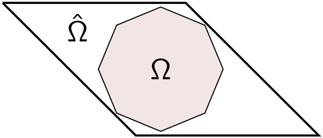

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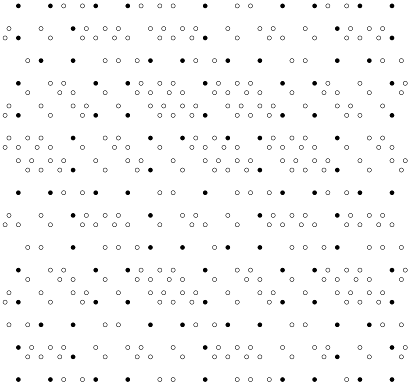

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Filter set of tiles

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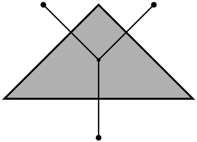

2. Compute star-images of neighbors
3. Test if they fit into the window
4. Get the region corresponding to the tile

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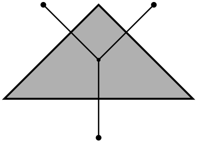

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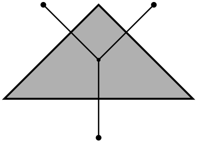

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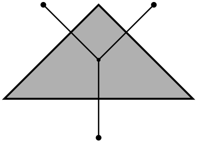

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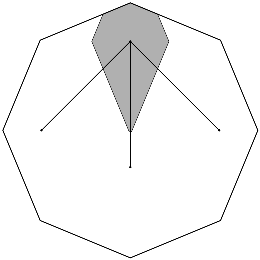
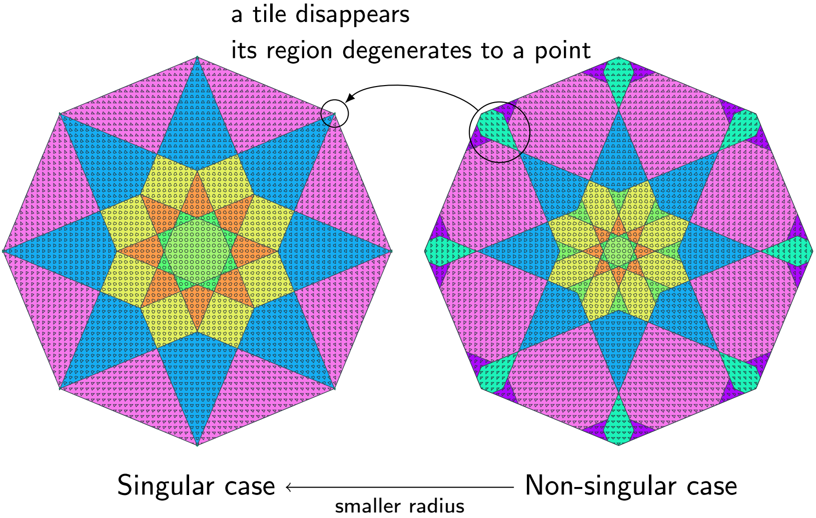


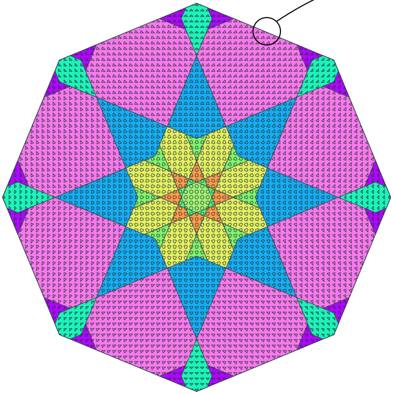
Singular cases

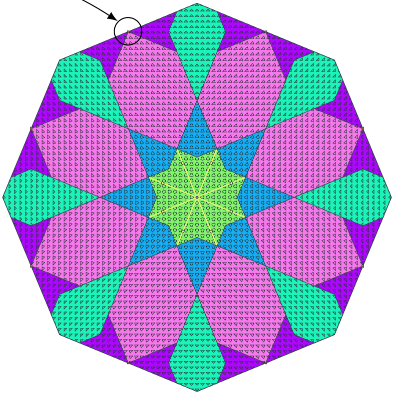


Singular cases

a new tile appears

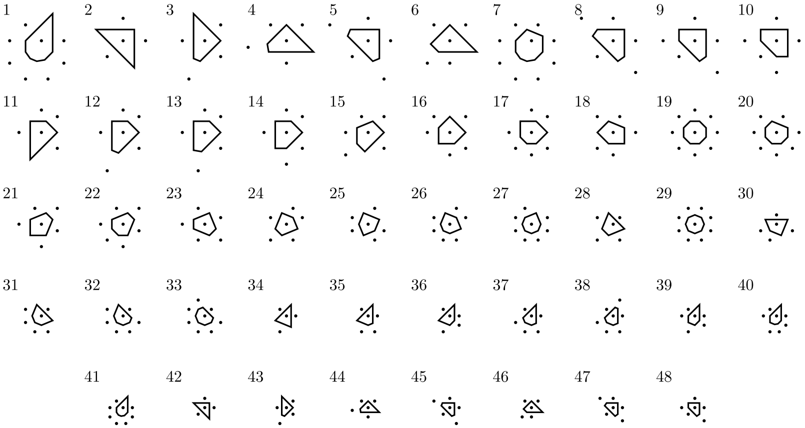
a new region is created





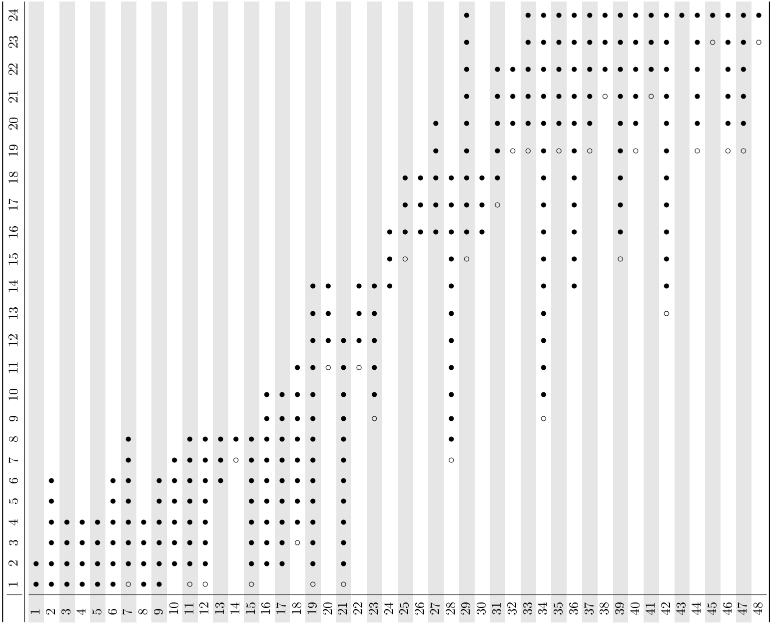
Non-Singular case $\xrightarrow[\text{larger radius}]{}$ Singular case

Results I — Voronoi tiles



r_1	r_3	r_5	r_7	r_9	r_{11}	r_{13}	r_{15}	r_{17}	r_{19}	r_{21}	r_{23}	r_{25}
1	$5\beta-11$	$\frac{3\beta-5}{2}$	$\frac{\beta}{2}$	$\beta-1$	$\frac{3}{2}$	$\frac{\beta+1}{2}$	2	$\frac{3\beta-3}{2}$	$\frac{\beta+2}{2}$	$3\beta-5$	$\frac{7-\beta}{2}$	β

Results II



Conclusion

Octagonal case

1. number of tiles in a quasicrystal

$$7 \leq \# \text{ tiles} \leq 17$$

2. number of tiles with non-zero frequency in a quasicrystal

$$5 \leq \# \text{ tiles} \leq 17$$

General case

1. Applicable to an acceptance window of arbitrary shape
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