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Aritmetika v číselných soustavách se zápornou bází

Arithmetics in number systems with a negative base

DIPLOMOVÁ PRÁCE

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Prohlášení

Prohlašuji, že jsem svou diplomovou práci vypracoval samostatně a použil jsem pouze podklady uvedené v přiloženém seznamu.

V Praze dne ....

....
Tomáš Vávra

Poděkování

Mé poděkování patří všem TIGRům; avšak lví podíl na této práci přísluší mé školitelce, doc. Zuzaně Masákové. Nejen, že zadala zajímavé téma, ale pečlivě bděla nad mou pracovní morálkou a dokázala při konzultacích vytvořit přívětivou a kreativní atmosféru. Nemalý dík patří také Vašku Potočkovi za program, který výrazně napomohl ověřování *nepravdivosti* nově vznikavších hypotéz.

Tomáš Vávra

Název práce:
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Abstrakt: Zabýváme se vlastnostmi množiny $\mathbb{Z}_{-\beta}$ $(-\beta)$-celých čísel v číselných systémech se záporným základem. Ukážeme, že pokud je β kvadratická Pisotova jednotka, pak může být množina $\mathbb{Z}_{-\beta}$ čísel vybavena takovou operací $\oplus$, že $(\mathbb{Z}_{-\beta}, \oplus)$ je grupa a navíc je operace $\oplus$ kompatibilní se sčítáním v $\mathbb{R}$. Potom budeme odhadovat počet zlomkových míst vycházejících z aritmetických operací na $(-\beta)$-celých číslech a pro jednu třídu kvadratických Pisotových bází provedeme přesné určení tohoto počtu pro sčítání.

Klíčová slova: číselná soustava, záporná báze, celá čísla, aritmetika

Title:
Arithmetics in number systems with a negative base

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Abstract: We study properties of the set of $(-\beta)$-integers in the negative-based numeration system. We show that when β is a quadratic Pisot unit, the set of $(-\beta)$-integers with a suitable operation $\oplus$ forms a group and that this operation is compatible with addition in $\mathbb{R}$. Then we estimate the number of fractional digits arising from the arithmetical operations on $(-\beta)$-integers and for one class of quadratic Pisot numbers we present the precise bound on this number for addition.

Key words: number system, negative base, integers, arithmetics

Contents

List of notations

$\lfloor x \rfloor$	floor function, $\lfloor x \rfloor = \max\{k \in \mathbb{Z} \| k \leq x\}$
$\lceil x \rceil$	ceiling function, $\lceil x \rceil = \min\{k \in \mathbb{Z} \| k \geq x\}$
$\mathbb{N}$	set of natural numbers, $\{1, 2, 3, \dots\}$
$\mathbb{N}_0$	set of natural numbers including zero, $\{0, 1, 2, \dots\}$
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	field of rational numbers
$\mathbb{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
$\mathbb{Z}[x]$	ring of polynomials with integer coefficients
$\mathbb{Q}[x]$	ring of polynomials with rational coefficients
$\mathcal{A}$	alphabet, a finite set of symbols
$\mathcal{A}^*$	set of finite words over an alphabet $\mathcal{A}$
$\mathcal{A}^{\mathbb{N}}$	set of infinite words over an alphabet $\mathcal{A}$
$\mathbb{Z}_\beta$	set of β-integers
$\mathbb{Z}_{-\beta}$	set of $(-\beta)$-integers
$\overline{A}$	abbreviated notation of an expression $-A$
τ	golden mean, number $\frac{1+\sqrt{5}}{2}$

Introduction

Nonstandard number systems have applications in many different fields of science. For example, redundant number systems, where several representations of one number are possible, allow to design parallel algorithms and thus speed the arithmetical operations up. This is useful mainly when manipulating large numbers, e.g. in cryptography. Nonstandard number systems with irrational base are particulary useful in modeling of quasicrystals. Representation of real numbers in the form

$$x = \sum_{i \leq k} x_i \beta^i \quad \text{with } x_i \in \mathbb{R}$$

for a general real $\beta > 1$ was first considered by A. Rényi in 1957 [17]. In 2009, S. Ito and T. Sadahiro [12] introduced a new number system with a negative base $-\beta < -1$. The expansion of numbers in this number system is an analogy to the positive-based number system, however, certain properties of the new number system are essentially different.

We mostly study the properties of $(-\beta)$-integers. These are the numbers that can be expressed using only non-negative powers of the base $(-\beta)$ where β is a quadratic Pisot number. In the introductory chapter, we explain the necessary notation and facts from number theory and combinatorics on words. Then we present the theory of positional number systems with a general real base α, and then recall some results obtained in both positive- and negative-based number systems. The author's original contribution is situated in Chapters 3 and 4. First we show that the set of $(-\beta)$-integers equipped with a suitable operation $\oplus$ forms a group with properties similar to rational integers. Then we study arithmetical properties of negative-based number systems and discuss the number of fractional digits arising from arithmetical operations on the set of $(-\beta)$-integers.

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Chapter 1

Preliminaries

1.1 Number theory

Let us first recall necessary notations and facts from number theory.

A complex number α is called algebraic if it is a root of a polynomial $f \in \mathbb{Q}[x]$. Among such polynomials, there is the unique one that is monic (the coefficient at the highest power of x is equal to 1) and of minimal degree. This polynomial is called the minimal polynomial of α and the degree of α is defined as the degree of its minimal polynomial. The minimal polynomial is irreducible and hence it has $n = \deg f$ distinct roots. These are called the (Galois) conjugates of α, denoted $\alpha_i,\ i \in \{2, 3, \ldots, \deg f\}$. It is also interesting that the set $\mathbb{A}$ of all algebraic numbers is an algebraically closed field, i.e. any polynomial $g \in \mathbb{A}[x]$ has roots in $\mathbb{A}$.

Let $\alpha \in \mathbb{C}$. The extension of the field $\mathbb{Q}$ by α (denoted $\mathbb{Q}(\alpha)$) is the smallest subfield (inclusion-wise) of $\mathbb{C}$ containing α, that is,

$$\mathbb{Q}(\alpha) = \bigcap \{T \mid T \text{ is a subfield of } \mathbb{C},\ \alpha \in T\}.$$

If α is an algebraic number of degree n then

$$\mathbb{Q}(\alpha) = \{a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} \mid a_i \in \mathbb{Q}\}.$$

It also means that $\mathbb{Q}(\alpha)$ as a linear space over the field $\mathbb{Q}$ has dimension n and elements $(1, \alpha, \alpha^2, \ldots, \alpha^{n-1})$ form a base of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.

For given fields F, g, a mapping $\sigma : F \to G$ that satisfies

$$\sigma(x + y) = \sigma(x) + \sigma(y) \quad \text{and} \quad \sigma(xy) = \sigma(x)\sigma(y)$$

is called an isomorphism of the fields F, G. The only subfields of $\mathbb{C}$ isomorphic to $\mathbb{Q}(\alpha)$ for $\alpha \in \mathbb{A}$ are $\mathbb{Q}(\alpha_i)$, where α_i is a Galois conjugate of α. The fields $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\alpha_i)$ are isomorphic with isomorphism

$$\sigma_i : a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} \mapsto a_0 + a_1\alpha_i + \cdots + a_{n-1}\alpha_i^{n-1}. \tag{1.1}$$

When α is a quadratic number, i.e. α is of degree 2, then $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha')$, where α' is the Galois conjugate of α. Then $\mathbb{Q}(\alpha)$ has exactly two automorphisms: the identity and (1.1). In this case, the image of $x \in \mathbb{Q}(\alpha)$ under the non-identical automorphism will be usually denoted x'.

An algebraic number $\alpha \in \mathbb{A}$ is called an algebraic integer, if it is a root of a monic polynomial $f \in \mathbb{Z}[x]$. The minimal polynomial of an algebraic integer is also a monic polynomial with integer coefficients. The set of algebraic integers is not closed under inversion, so it is not a field, but it forms a subring of $\mathbb{C}$.

A very important subclass of algebraic integers are numbers whose Galois conjugates lie in the unit circle.

Definition 1. *An algebraic integer $\beta > 1$ is called a Pisot number if all its Galois conjugates satisfy $|\beta_i| < 1$.*

If an algebraic integer satisfies $|\beta_i| \leq 1$ and it is not a Pisot number then it is called a Salem number.

Although a description of Pisot numbers using coefficients of their minimal polynomial is not known in general, it is simple in the quadratic case.

Proposition 1. *A number $\beta > 1$ is a quadratic Pisot number if and only if it is a root of one of the following polynomial for $m, n \in \mathbb{N}$*

$$\begin{aligned} &x^2 - mx - n\,, \qquad && m \geq n \geq 1\,, \\ &x^2 - mx + n\,, \qquad && m + 2 \geq n \geq 1\,. \end{aligned}$$

An algebraic integer β is called a unit if $\frac{1}{\beta}$ is also an algebraic integer. There is another equivalent characterization of algebraic units: an algebraic number β is a unit if and only if the absolute term of its minimal polynomial is ± 1. For example, quadratic Pisot units are the roots of

$$\begin{aligned} &x^2 - mx - 1\,, \qquad && m \geq 1\,, \\ &x^2 - mx + 1\,, \qquad && m \geq 3\,. \end{aligned}$$

The algebraic units also form an interesting structure – it is a multiplicative subgroup of $(\mathbb{C}, \cdot)$.

1.2 Combinatorics on words

Let $\mathcal{A}$ be a nonempty set of symbols, also called an alphabet. By $\mathcal{A}^*$ and $\mathcal{A}^{\mathbb{N}}$ we denote the set of all finite and infinite sequences over $\mathcal{A}$, respectively. The elements of these sets are called words or strings. In $\mathcal{A}^*$, the operation concatenation is defined by the natural prescription

$$(a_1 a_2 \ldots a_m) \circ (b_1 b_2 \ldots b_n) = a_1 a_2 \ldots a_m b_1 b_2 \ldots b_n .$$

This operation is associative and has a neutral element, the empty word ϵ that also belongs to $\mathcal{A}^*$. The set $(\mathcal{A}^*, \circ)$ is a monoid and the symbol $\circ$ is usually omitted.

The k-th power of $u \in \mathcal{A}^*$ is defined as

$$u^k = \underbrace{u \circ u \circ \cdots \circ u}_{k-times}$$

and the periodic repetition as

$$u^\omega = u \circ u \circ \cdots \in \mathcal{A}^{\mathbb{N}}.$$

A word u is called a factor (subword) of the word w if there exist words $a \in \mathcal{A}^*, b \in \mathcal{A}^* \cup A^{\mathbb{N}}$ such that $w = aub$. If $a, b \neq \epsilon$ then u is called a proper factor. If $a = \epsilon$ or $b = \epsilon$ then u is called a prefix or suffix of w, respectively.

The space $\mathcal{A}^{\mathbb{N}}$ can be equipped with a metric ρ, defined by the following prescription: for $a, b \in \mathcal{A}^{\mathbb{N}}$ we put

$$\rho(a,b) = 2^{-n}\,, \qquad \text{where } n = \inf\{k \geq 1 | a_k \neq b_k\}$$

with the convention $2^{-\infty} = 0$, which happens if and only if $a = b$. With the metric ρ, $\mathcal{A}^{\mathbb{N}}$ is a compact topological space. In order to extent the metric to the elements of $\mathcal{A}^*$, we "complete" them to infinite words by a symbol $\delta \notin \mathcal{A}$. Then we use the metric defined on the space $(\mathcal{A} \cup \{\delta\})^{\mathbb{N}}$.

An ordering of strings is also defined.

Definition 2. *Let $\mathcal{A}$ be a linearly ordered set. We say that $u = u_1u_2u_3 \cdots \in \mathcal{A}^{\mathbb{N}}$ is lexicographically smaller than $v = v_1v_2v_3 \cdots \in \mathcal{A}^{\mathbb{N}}$ (denoted $a \prec_{\text{lex}} b$), if there exists $k \in \mathbb{N}$, such that*

1. *$u_i = v_i$ for $i < k$*
2. *$u_k < v_k$.*

We say that u is alternately smaller than v (denoted $u \prec_{\text{alt}} v$), if there exists $k \in \mathbb{N}$, such that

1. *$u_i = v_i$ for $i < k$,*
2. *$u_k < v_k$ for k even, or $u_k > v_k$ for k odd.*

Definition 3. *Let us have a set $\mathcal{S} \subset \mathcal{A}^{\mathbb{N}}$ and let $\mathcal{F} \subset \mathcal{A}^{\mathbb{N}} \cup \mathcal{A}^*$, such that $\forall x \in \mathcal{A}^{\mathbb{N}}$ the following properties are satisfied:*

1. *A string x belongs to $\mathcal{S}$ if and only if no element $w \in \mathcal{F}$ is a factor of x.*
2. *Each proper factor of a string $w \in \mathcal{F}$ belongs to $\mathcal{S}$.*

Then $\mathcal{F}$ is called the set of forbidden strings in $\mathcal{S}$.

Chapter 2

Positional number systems

A positional number system is given by a real base α with $|\alpha| > 1$ and a finite set of digits $\mathcal{A} \subset \mathbb{R}$, also called the alphabet. If $x \in \mathbb{R}$ can be represented as

$$x = \sum_{i=-\infty}^{k} x_i \beta^i \quad \text{with } x_i \in \mathcal{A},$$

we say that x has an α-representation in $\mathcal{A}$ and we write

$$x = x_k \dots x_0 \bullet_\alpha x_{-1} \dots .$$

The indexing of the coefficients might vary in this text, but the symbol $\bullet_\alpha$ always denotes the position between positive and negative powers of the base. The representation is said to be finite, if it has the form $x_k \dots x_0 \bullet_\alpha x_{-1} \dots x_{-\ell} 0^\omega$. The repetition of zeroes might be omitted as well as α in the symbol $\bullet_\alpha$ when the base is clear from the context. It is also known that the assumption of each $x \in \mathbb{R}$ having at least one representation in $\mathcal{A}$ implies that $\#\mathcal{A} \geq |\alpha|$. The question is how to obtain a α-representation of $x \in \mathbb{R}$.

Definition 4. *Given a base $\alpha \in \mathbb{R}, |\alpha| > 1$, a finite alphabet $\mathcal{A} \subset \mathbb{R}$, and a bounded interval $\mathcal{J}$ such that $0 \in \mathcal{J}$. Let $D : \mathcal{J} \to \mathcal{A}$ be a mapping such that the transformation $T(x) := \alpha x - D(x)$ maps $\mathcal{J} \to \mathcal{J}$. Then the corresponding representation of x is*

$$d_{\alpha,\mathcal{J},D}(x) := x_1 x_2 x_3 \dots \in \mathcal{A}^{\mathbb{N}}, \quad \textit{where } x_i = D\left(T^{i-1}(x)\right). \tag{2.1}$$

Moreover, it holds that $x = \bullet_\alpha d_{\alpha,\mathcal{J},D}(x)$. This follows from the prescription for T, where for $x \in \mathcal{J}$ we have $x = \frac{D(x)}{\alpha} + \frac{T(x)}{\alpha}$. Since the value of $T(x)$ is in $\mathcal{J}$, we can use the formula for $T(x)$, i.e. $T(x) = \frac{D(T(x))}{\alpha} + \frac{T^2(x)}{\alpha}$. Then

$$x = \frac{D(x)}{\alpha} + \frac{D\left(T(x)\right)}{\alpha^2} + \frac{T^2(x)}{\alpha^2}$$

and in the limit

$$x = \frac{D(x)}{\alpha} + \frac{D\left(T(x)\right)}{\alpha^2} + \frac{D\left(T^2(x)\right)}{\alpha^3} + \dots . \tag{2.2}$$

A very important fact is that the definition determines one particular representation of $x \in \mathcal{J}$. This is crucial for defining some important sets, e.g. the set of numbers with finite α-expansion or the set of α-integers.

In order to find a α-representation of other real numbers, let us consider the set

$$\bigcup_{i\in\mathbb{Z}} \alpha^i \mathcal{J} = \begin{cases} \mathbb{R}^+ & \text{if } \mathcal{J} = [0,c) \text{ and } \alpha > 1, \\ \mathbb{R}^- & \text{if } \mathcal{J} = (-c,0] \text{ and } \alpha > 1, \\ \mathbb{R} & \text{otherwise.} \end{cases}$$

When $x \in \bigcup_{i\in\mathbb{Z}} \alpha^i \mathcal{J}$ is to be represented in the base α, one first finds an integer k such that

$$\frac{x}{\alpha^\ell} \in \mathcal{J}, \quad \forall \ell \geq k. \tag{2.3}$$

Note that such exponent k does not exist when zero is the end-point of $\mathcal{J}$ and $\alpha < -1$. Therefore we will not consider these choices of $\mathcal{J}$. If (2.3) is satisfied, the α-representation of $\frac{x}{\alpha^k} \in \mathcal{J}$ can be obtained in the form (2.2), which implies

$$x = \alpha^k \left(\frac{D\left(\frac{x}{\alpha^k}\right)}{\alpha} + \frac{D\left(T\left(\frac{x}{\alpha^k}\right)\right)}{\alpha^2} + \frac{D\left(T^2\left(\frac{x}{\alpha^k}\right)\right)}{\alpha^3} + \dots \right),$$

that is, if

$$d_{\alpha,\mathcal{J},D}\left(\frac{x}{\alpha^k}\right) = z_1 z_2 z_3 \dots \tag{2.4}$$

then

$$x = z_1 z_2 z_3 \dots z_k \bullet_\alpha z_{k+1} \dots . \tag{2.5}$$

The condition (2.3) ensures that the representation of x is independent of the chosen power k of α in (2.4) when the initial zero coefficients are omitted. An example of possible problems that may occur if $x \in \mathcal{J}$ but $\frac{x}{\alpha} \notin \mathcal{J}$ will be shown in Section 2.2. This leads to the following definition.

Definition 5. *Let $x \in \bigcup_{i\in\mathbb{Z}} \alpha^i \mathcal{J}$ and let $k \in \mathbb{N}_0$ satisfy $\frac{x}{\alpha^i} \in \mathcal{J}$, $\forall i \geq k$. If*

$$\left(\frac{x}{\alpha^k}\right) = z_1 z_2 z_3 \dots$$

then

$$z_1 z_2 z_3 \dots z_k \bullet_\alpha z_{k+1} \dots$$

is called the $(\alpha, \mathcal{J}, D)$-expansion of x.

A string $u \in \mathcal{A}^{\mathbb{N}}$ such that $u = d_{\alpha,\mathcal{J},D}(x)$ for some $x \in \mathcal{J}$ is called $(\alpha, \mathcal{J}, D)$-admissible. If J and D are clear from the context, we speak about α-expansions and α-admissibility.

As we will see, in many cases the admissibility of a string might be decided by some ordering criterion and consequently described by a set of forbidden strings. The special role in the admissibility criterion play the expansion of the boundaries of $\mathcal{J}$ (see [8] and [13]).

In "conventional" number systems (binary, decimal, ...), we are used to perform an ordering of numbers on the real line using the lexicographic ordering of their expansions. This approach, under some assumptions, is valid for every positional number system. The following fact is proved for example in [11].

Proposition 2. *Let $\alpha, \mathcal{A}, \mathcal{J}$ and D be as in Definition 4. Let numbers $x, y \in \mathcal{J}$ and let $d_{\alpha,\mathcal{J},D}(x) = x_1x_2x_3\ldots$ and $d_{\alpha,\mathcal{J},D}(y) = y_1y_2y_3\ldots$ be their α-expansions. Then*

- *if $\alpha > 1$ and $D(x)$ is non-decreasing then*

$$x < y \iff x_1x_2x_3\cdots \prec_{\text{lex}} y_1y_2y_3\ldots,$$

- *if $\alpha < -1$ and $D(x)$ is non-increasing then*

$$x < y \iff x_1x_2x_3\cdots \prec_{\text{alt}} y_1y_2y_3\ldots.$$

For $x, y \notin \mathcal{J}$ we use the so-called radix order of their $(\alpha, \mathcal{J}, D)$-expansions. It means we align them according to the powers of the base, complete the expansions by zeroes from the left side to have the same length, and use the corresponding order (lexicographical or alternate).

Example 1. *We want to order $(\alpha, \mathcal{J}, D)$-expansions $x = x_k \ldots x_0 \bullet x_{-1}\ldots$ and $y = y_\ell \ldots y_0 \bullet y_{-1}\ldots$, where $k \geq \ell$. First, y is completed with $k-\ell$ zeros to obtain the string $y' = \underbrace{0\ldots0}_{k-\ell} y_\ell \ldots y_0 \bullet y_{-1}\ldots$. Then we compare the strings*

$$x_kx_{k-1}x_{k-2}\ldots \quad \text{and} \quad \underbrace{0\ldots0}_{k-\ell}y_\ell y_{\ell-1}y_{\ell-2}\ldots$$

in case of lexicographical order or alternate order with k odd, or the strings

$$0x_kx_{k-1}x_{k-2}\ldots \quad \text{and} \quad \underbrace{0\ldots0}_{k-\ell+1}y_\ell y_{\ell-1}y_{\ell-2}\ldots$$

in case of alternate order when k is even.

2.1 The Rényi number system

In 1957, A. Rényi in [17] introduced a number system with a base $\beta > 1$. This number system can be described by specifying $\alpha, \mathcal{J}, D$ in Definition 4 to be: $\beta > 1, \mathcal{J} = [0,1), D(x) = \lfloor \beta x \rfloor$. The alphabet of this number system is $\mathcal{A}_\beta = \{0, 1, \ldots, \lceil \beta - 1 \rceil\}$. For this choice of $\beta, \mathcal{J}, D$, we will denote $d_{\beta,\mathcal{J},D}(x) = d_\beta(x)$ and the representation of x in Definition 5 will be called the β-expansion of x and denoted $\langle x \rangle_\beta$. Since $\mathcal{J} \subset \mathbb{R}^+$ and $\beta > 1$, it is impossible to extent the β-expansion to negative real numbers. Naturally, we use the symbol “minus” for describing negative numbers.

The condition for deciding admissibility of a given string $u \in \mathcal{A}_\beta^{\mathbb{N}}$ was given by W. Parry in 1960 [16]. Let us recall that, in this case, a string of digits is called admissible if it is a β-expansion of some $x \in [0,1)$.

Theorem 1 (W. Parry)**.** *A string u over integers is β-admissible if and only if*

$$0^\omega \preceq_{\text{lex}} \widetilde{u} \prec_{\text{lex}} \lim_{\varepsilon \to 0^+} d_\beta(1-\varepsilon)$$

for every suffix $\widetilde{u}$ of the string u.

Let us make some comments on the theorem. The expression $d^*_\beta(1) = \lim_{\varepsilon\to 0^+} d_\beta(1-\varepsilon)$ is called the infinite Rényi expansion of unity and is a limit with respect to the topology on $\mathcal{A}^{\mathbb{N}}$. However, it does not have to be computed as a limit. We can obtain $d^*_\beta(1)$ from the Rényi expansion of unity

$$d_\beta(1) = t_1 t_2 t_3 \ldots, \quad \textit{where } t_i = \lfloor \beta T^{i-1}(1) \rfloor$$

by the prescription

$$d^*_\beta(1) = \begin{cases} \big(t_1 \cdots t_{m-1}(t_m - 1)\big)^\omega & \text{if } d_\beta(1) = t_1 \cdots t_m 0^\omega\,,\ t_m \neq 0\,, \\ d_\beta(1) & \text{otherwise.} \end{cases}$$

From the Parry lexicographic condition it might be possible to derive the set of forbidden strings, i.e. the set of strings $\mathcal{F}$ that satisfies: "$u \in \mathcal{A}^{\mathbb{N}}_\beta$ is admissible if and only if u does not contain any element of $\mathcal{F}$ as its substring." This is possible if $d^*_\beta(1)$ is eventually periodic. Moreover, if $d^*_\beta(1)$ is purely periodic then the set of forbidden strings is finite.

Example 2. *Let $\beta = \tau = \frac{1+\sqrt{5}}{2}$ (the so called Golden mean), the greater root of $x^2 - x - 1$. Since $\lfloor \tau \rfloor = 1$, the alphabet is $\mathcal{A}_\tau = \{0, 1\}$. Now we derive the coefficients of $d_\tau(1)$*

- $t_1 = \lfloor \tau \cdot 1 \rfloor = 1$,
- $t_2 = \lfloor \tau T^1_\tau(1) \rfloor = \lfloor \tau(\tau \cdot 1 - \lfloor \tau \cdot 1 \rfloor) \rfloor = \lfloor \tau^2 - \tau \rfloor = \lfloor 1 \rfloor = 1$,
- $t_3 = \lfloor \tau T^2_\tau(1) \rfloor = \lfloor \tau T_\tau(\tau - 1) \rfloor = \lfloor \tau(\tau^2 - \tau - \lfloor \tau^2 - \tau \rfloor) \rfloor = 0$,
- $t_i = 0$ *for all* $i \geq 4$ *because* $T_\tau(0) = 0$.

We have

$$d_\tau(1) = 110^\omega\,, \qquad d^*_\tau(1) = (10)^\omega$$

and subsequently by Theorem 1 we obtain the condition of admissibility

$$0^\omega \preceq_{\text{lex}} \tilde{x} \prec_{\text{lex}} (10)^\omega.$$

The set of forbidden string is

$$\mathcal{F} = \{11, (10)^\omega\}\,.$$

Let us focus on some interesting sets related to the Rényi numeration systems. The set of numbers with finite expansion is defined as

$$\text{Fin}(\beta) = \{x \in \mathbb{R} \mid \langle |x| \rangle_\beta = x_k \ldots x_0 \bullet_\beta x_{-1} \ldots x_{-\ell} 0^\omega\}.$$

Note that $\text{Fin}(\beta)$ is symmetrical with respect to zero. It is also easy to prove that $\text{Fin}(\beta)$ is dense in $\mathbb{R}$ for any $\beta > 1$.

An interesting question is, under what conditions $\text{Fin}(\beta)$ forms a subring of $\mathbb{R}$, i.e. when $\text{Fin}(\beta)$ is closed under addition, subtraction and multiplication. In that case we say that β

satisfies the finiteness property or simply (F). A result by Ch. Frougny and B. Solomyak [9] shows that if $\mathrm{Fin}(\beta)$ is a ring then β is a Pisot number without any positive conjugate. However, the reverse implication does not hold and describing bases β satisfying (F) is still an open problem.

We also consider the set of β-integers, analogy to rational integers $\mathbb{Z}$. It is defined as

$$\mathbb{Z}_\beta = \{x \in \mathbb{R} \mid \langle|x|\rangle_\beta = x_k \dots x_0 \bullet 0^\omega\}.$$

We are mostly interested the following three questions:

1. When $\mathbb{Z}_\beta$ is a ring.
2. When $\mathrm{Fin}(\beta)$ is a ring, i.e. β satisfies (F).
3. How many fractional digits has the result of addition, subtraction or multiplication of β-integers.

The first question can be answered easily. It is a well known fact that $\mathbb{Z}_\beta$ is a ring if and only if β is an integer. For, $\lfloor\beta\rfloor + 1 \notin \mathbb{Z}_\beta$ if $\beta \notin \mathbb{Z}$. The second question is still an open problem. As we mentioned, it was proved [9] that (F) implies β is a Pisot number without positive conjugate. However, the converse does not hold and for the description of such bases only partial results are obtained for bases of small degree [2]. Since the distance between consecutive β-integers is at most 1, $\mathbb{Z}_\beta$ is always relatively dense. However, it does not have to be uniformly discrete.

Let us focus on the third question. For this purpose we define quantities denoting the number of fractional digits that may arise in a result of arithmetical operations

$$\begin{aligned} L_\oplus(\beta) &= \min\{l \in \mathbb{N}_0 \mid \forall\, x, y \in \mathbb{Z}_\beta,\ x + y \in \mathrm{Fin}(\beta) \Rightarrow x + y \in \beta^{-l}\mathbb{Z}_\beta\}\,, \\ L_\otimes(\beta) &= \min\{l \in \mathbb{N}_0 \mid \forall\, x, y \in \mathbb{Z}_\beta,\ x \cdot y \in \mathrm{Fin}(\beta) \Rightarrow x \cdot y \in \beta^{-l}\mathbb{Z}_\beta\}\,. \end{aligned}$$

As we do not consider infinite β-expansions, this definition allows $L_\oplus(\beta), L_\otimes(\beta)$ to be finite even though $\mathrm{Fin}(\beta)$ may not be a ring. Moreover, for β with (F), it might seem to be possible to find a sequence $(a_n, b_n)_{n\in\mathbb{N}}$ such that the length of the fractional part of $a_n + b_n$ or $a_n \cdot b_n$ tends to infinity. However, this is not possible since (F) implies β is a Pisot number and that implies finite values of $L_\oplus(\beta), L_\otimes(\beta)$ (see [4]).

Derivation of $L_\oplus(\beta)$ and $L_\otimes(\beta)$ is a difficult problem which has been (almost) solved only for quadratic Pisot numbers and a small class of cubic Pisot numbers. For example, for quadratic Pisot units, these values have been derived in [5].

Theorem 2.

- *For $\beta > 1$ root of $x^2 - mx - 1$ $m \geq 1$ is*

$$L_\oplus(\beta) = 2 = L_\otimes(\beta)\,.$$

- *For $\beta > 1$ root of $x^2 - mx + 1$ $m \geq 3$ is*

$$L_\oplus(\beta) = 1 = L_\otimes(\beta)\,.$$

The values of $L_\oplus(\beta)$ and $L_\otimes(\beta)$ for other quadratic Pisot numbers are also studied in [3], [10] and finally [7].

Theorem 3. *Let $\beta > 1$ be a root of $x^2 = mx + n$ with $m \geq n \geq 1$. Then*

$$L_\oplus(\beta) = 2m \quad \text{if} \quad m = n$$

and

$$L_\oplus(\beta) = 2\left\lfloor \frac{m+1}{m-n+1} \right\rfloor \quad \text{if} \quad m > n.$$

Theorem 4. *Let $\beta > 1$ be a root of $x^2 = mx - n$ with $m - 2 \geq n \geq 1$. Then*

$$L_\oplus(\beta) = 1 \quad \text{if} \quad n = 1 \text{ or } m \geq 3n + 1$$

and

$$\left\lfloor \frac{m-2}{m-n-1} \right\rfloor \leq L_\oplus(\beta) \leq \left\lceil \frac{m-1}{m-n-1} \right\rceil \quad \text{otherwise.}$$

2.2 The Ito-Sadahiro number system

In 2009, S. Ito and T. Sadahiro introduced a number system with negative base $-\beta < -1$ having similar properties as the Rényi number systems. The analogies are that the digits belong to a non-negative and minimal alphabet, the interval $\mathcal{J}$ is of unit length, and the prescription for the digit function D is similar, precisely

$$\mathcal{A}_{-\beta} = \{0, 1, \ldots, \lfloor\beta\rfloor\}, \quad \mathcal{J} = \left[-\frac{\beta}{\beta+1}, \frac{1}{\beta+1}\right), \quad \text{and} \quad D(x) = \left\lfloor -\beta x + \frac{\beta}{\beta+1} \right\rfloor.$$

In this case, we denote $d_{-\beta,\mathcal{J},D}(x) = d_{-\beta}(x)$ and the representation of x in Definition 5 is called the $(-\beta)$-representation of x and denoted $\langle x\rangle_{-\beta}$. We use ℓ_β and r_β to denote the left- and the right-end of the interval $\mathcal{J}$, respectively.

Since $\bigcup_{n\geq 0} \mathcal{J}^n = \mathbb{R}$, any real number can be expanded without using the minus sign. Let us justify the requirement (2.3). If the $(-\beta)$-expansion of ℓ_β is

$$d_{-\beta}(\ell_\beta) = d_1 d_2 d_3 \ldots$$

then

$$d_{-\beta}\left(\frac{\ell_\beta}{(-\beta)^2}\right) = 1 d_1 d_2 d_3 \ldots$$

instead of the expected string

$$00 d_1 d_2 d_3 \ldots.$$

This is caused by $\frac{x}{-\beta} = r_\beta$ not being in $\mathcal{J}$. In fact, we would have countably many real numbers with two admissible $(-\beta)$-representations,

$$d_1 d_2 \ldots d_n \bullet_{-\beta} d_{n+1} \cdots \quad \text{and} \quad 1 d_1 d_2 \ldots d_{n+1} \bullet_{-\beta} d_{n+2} \ldots.$$

Therefore, we choose only one of them by requiring (2.3). Consequently,

$$d_{-\beta}(\ell_\beta) = d_1 d_2 d_3 \ldots \quad \text{but} \quad \langle \ell_\beta\rangle_{-\beta} = 1 d_1 \bullet d_2 \ldots. \tag{2.6}$$

Example 3. *For quadratic Pisot numbers, we have the following expansions of* ℓ_β.

1. *For* $\beta > 1$ *satisfying* $\beta^2 = m\beta - n$, $m-2 \geq n \geq 1$ *is* $d_{-\beta}(\ell_\beta) = [(m-1)\,n]^\omega$.
2. *For* $\beta > 1$ *satisfying* $\beta^2 = m\beta + n$, $m \geq n \geq 1$ *is* $d_{-\beta}(\ell_\beta) = m\,(m-n)^\omega$.

The analogy of Theorem 1 describing admissible $(-\beta)$-expansions was derived in [12] and uses the alternate order.

Theorem 5. *A string* u *over integers is* $(-\beta)$*-admissible if and only if*

$$d_{-\beta}(\ell_\beta) \preceq_{\text{alt}} \widetilde{u} \prec_{\text{alt}} \lim_{\varepsilon \to 0^+} d_{-\beta}(r_\beta - \varepsilon)$$

for every suffix $\widetilde{u}$ *of the string* u.

In [12] it was also shown that we can obtain $\lim_{\varepsilon\to 0^+} d_{-\beta}(r_\beta - \varepsilon)$ from $d_{-\beta}(\ell_\beta)$ by an explicit prescription, namely

$$d^*_{-\beta}(r_\beta) = \begin{cases} (0d_1d_2\dots d_{m-1}(d_m-1))^\omega & \text{if } d_{-\beta}(\ell_\beta) = (d_1\dots d_m)^\omega\,,\ d_1 \neq d_m\,, m \text{ odd,} \\ 0d_1d_2d_3\dots & \text{otherwise.} \end{cases}$$

According to Proposition 2, for the Ito-Sadahiro number system we obtain

$$x \leq y \quad \Longleftrightarrow \quad \langle x\rangle_{-\beta} \preceq_{\text{alt}} \langle y\rangle_{-\beta}.$$

It means that the ordering of real numbers corresponds to the alternate order of their $(-\beta)$-representations.

The set of numbers with finite $(-\beta)$-expansion is defined as

$$\text{Fin}(-\beta) = \{x \in \mathbb{R} \mid \langle x\rangle_{-\beta} = x_n \dots x_0 \bullet_{-\beta} x_{-1}\dots x_{-\ell}0^\omega\}.$$

Note that unlike in positive base number system, $\text{Fin}(-\beta)$ is defined on the real line without using the absolute value. Hence $\text{Fin}(-\beta)$ is not symmetrical with respect to zero. Although it is even not known whether $\text{Fin}(-\beta)$ is dense in $\mathbb{R}$ for general $\beta > 1$, the following properties of $\text{Fin}(-\beta)$ have been shown in [14] and [15].

Proposition 3. $\text{Fin}(-\beta) = \{0\}$ *if and only if* $\beta < \frac{1+\sqrt{5}}{2}$.

Theorem 6. *Let* $\beta > 1$ *be the root of* $x^2 - mx + n$ *for* $m - 2 \geq n \geq 1$. *Then* $\text{Fin}(-\beta)$ *is a ring.*

Theorem 7. *Let* $\beta > 1$ *be the root of* $x^2 - mx - n$ *for* $m \geq n \geq 1$. *Then* $\text{Fin}(-\beta)$ *is closed under addition but not under subtraction.*

Of course, the set of $(-\beta)$-integers can be also defined by the natural way

$$\mathbb{Z}_{-\beta} = \{x \in \mathbb{R} \mid \langle x\rangle_{-\beta} = x_n \dots x_0 \bullet_{-\beta} 0^\omega\}.$$

As in the Rényi system, $\mathbb{Z}_{-\beta}$ is a ring if and only if $\beta \in \mathbb{N}$, unless $\beta \in \left(1, \frac{1+\sqrt{5}}{2}\right)$, when $\mathbb{Z}_{-\beta} = \{0\}$ is the trivial ring according to Proposition 3. The set of $(-\beta)$-integers is not relatively dense or uniformly discrete in general (see [18]).

In many cases, the lengths of distances between consecutive elements of $\mathbb{Z}_{-\beta}$ can be computed from $d_\beta(\ell_\beta)$ and their ordering can be described as a fixed point of a morphism on the alphabet coding the lengths (see [1]). For quadratic Pisot numbers, we have two lengths of distances in $\mathbb{Z}_{-\beta}$.

Proposition 4. *Let Δ_0, Δ_1 denote the lengths of distances between consecutive $(-\beta)$-integers. Then*

$$\{\Delta_0, \Delta_1\} = \begin{cases} \{1, \frac{m}{\beta}\} & \text{for } \beta \text{ root of } x^2 - mx - m, m \geq 1, \\ \{1, 1 + \frac{n}{\beta}\} & \text{for } \beta \text{ root of } x^2 - mx - n, m > n \geq 1, \\ \{1, 2 - \frac{n}{\beta}\} & \text{for } \beta \text{ root of } x^2 - mx + n, m - 2 \geq n \geq 1. \end{cases}$$

One can also observe the number of fractional digits arising from the arithmetical operations. In analogy with the Rényi number system we define

$$L_\oplus(-\beta) = \min\{l \in \mathbb{N}_0 \mid \forall x, y \in \mathbb{Z}_{-\beta},\ x \pm y \in \mathrm{Fin}(-\beta) \Rightarrow x \pm y \in (-\beta)^{-l}\mathbb{Z}_{-\beta}\}, \quad (2.7)$$
$$L_\otimes(-\beta) = \min\{l \in \mathbb{N}_0 \mid \forall x, y \in \mathbb{Z}_{-\beta},\ x \cdot y \in \mathrm{Fin}(-\beta) \Rightarrow x \cdot y \in (-\beta)^{-l}\mathbb{Z}_{-\beta}\}. \quad (2.8)$$

The following theorem, proven in [14], might be helpful for an estimation of the values $L_\oplus(-\beta), L_\otimes(-\beta)$.

Theorem 8. *Let β be an algebraic number, and let β' be one of its conjugates satisfying $|\beta'| < 1$. Denote*

$$\begin{aligned} H &:= \sup\{|z'| \mid z \in \mathbb{Z}_{-\beta}\}, \\ K &:= \inf\{|z'| \mid z \in \mathbb{Z}_{-\beta} \setminus (-\beta)\mathbb{Z}_{-\beta}\}. \end{aligned} \quad (2.9)$$

If $K > 0$, then

$$\frac{1}{|\beta'|^{L_\oplus}} \leq \frac{2H}{K} \quad \text{and} \quad \frac{1}{|\beta'|^{L_\otimes}} \leq \frac{H^2}{K}. \quad (2.10)$$

Moreover, if the supremum or infimum in (2.9) *is not reached, then strict inequality holds in both of* (2.10).

Using Theorem 8, the following values have been derived for quadratic Pisot units in [14] and [15].

Theorem 9. *1. Let $\beta > 1$ be a root of $x^2 - x - 1$. Then $L_\oplus(-\beta) = 2 = L_\otimes(-\beta)$.*

2. Let $\beta > 1$ be a root of $x^2 - mx - 1$ for $m \geq 2$. Then $L_\oplus(-\beta) = 1 = L_\otimes(-\beta)$.

3. Let $\beta > 1$ be a root of $x^2 - mx + 1$ for $m \geq 3$. Then $L_\oplus(-\beta) = 2 = L_\otimes(-\beta)$.

An estimation of $L_\oplus$ and $L_\otimes$ for the remaining quadratic Pisot numbers is the aim of Chapter 4 in this work.

Example 4. *Let us take as a base the negative value of the so-called golden mean, i.e.* $\beta = \frac{1+\sqrt{5}}{2} = \tau$*, the greater root of* $x^2 - x - 1$*. Then*

$$\mathcal{J} = \left[-\frac{\tau}{\tau+1}, \frac{1}{\tau+1}\right) = \left[-\frac{1}{\tau}, \frac{1}{\tau^2}\right) \quad \text{and} \quad D(x) = \left\lfloor -\tau x + \frac{1}{\tau} \right\rfloor$$

and the coefficients of the expansion of ℓ_β *are computed as follows. We have*

$$d_1 = D(T^0(\ell_\beta)) = D(\ell_\beta) = \left\lfloor -\tau\left(-\frac{1}{\tau}\right) + \frac{1}{\tau} \right\rfloor = \left\lfloor 1 + \frac{1}{\tau} \right\rfloor = \lfloor \tau \rfloor = 1.$$

Since

$$T(\ell_\beta) = -\tau\left(-\frac{1}{\tau}\right) - D(\ell_\beta) = 0$$

and $T(0) = D(0) = 0$*, the rest of digits are zeros, i.e.* $d_{-\beta}(\ell_\beta) = 10^\omega$*. According to Theorem 5, the limit expansion of* r_β *is* $d^*(r_\beta) = 010^\omega$ *and the condition for a string* $u \in \mathcal{A}^\mathbb{N}$ *to be admissible is that*

$$10^\omega \preceq_{\text{alt}} \widetilde{u} \prec_{\text{alt}} 010^\omega$$

for all suffixes $\widetilde{u}$ *of the string* u*.*

The set of forbidden strings is

$$\mathcal{F} = \{1\,0^{2k-1}1 \mid k \in \mathbb{N}\} \cup \{0\,1\,0^\omega\}.$$

Note that the string 10^ω *is admissible while the string* 010^ω *is not.*

Let us derive the values $L_\oplus(-\tau)$ *and* $L_\otimes(-\tau)$ *using Theorem 8. The Galois conjugate of* τ *is* $-\frac{1}{\tau}$*. For a number* $z = \sum_{i=0}^{k} z_i\,(-\tau)^i \in \mathbb{Z}_{-\beta}$ *we have*

$$0 \le z' = \sum_{i=0}^{k} z_i \left(\frac{1}{\tau}\right)^i < \sum_{i=0}^{+\infty} 1 \cdot \left(\frac{1}{\tau}\right)^i = \tau^2.$$

Hence $H = \tau^2$ *and the supremum is not reached.*

One might think that all elements of the set $\mathbb{Z}_{-\tau} \setminus (-\tau)\mathbb{Z}_{-\tau}$ *are in the form*

$$z = a_k a_{k-1} \dots a_1 a_0 \bullet 0^\omega, \quad a_0 \neq 0,$$

so we can use $1\bullet$ *to estimate the value* K*. However,* $1\bullet \in (-\tau)\mathbb{Z}_{-\tau}$ *since* $1\bullet = 110\bullet$ *and therefore*

$$K = \inf\{|z'| \mid z \in \mathbb{Z}_{-\tau} \setminus (-\tau)\mathbb{Z}_{-\tau}\} \neq 1\,.$$

Here the infimum is reached for $z = 11\bullet = 0\bullet 1$ *and hence*

$$K = 1 + \frac{1}{\tau} = \tau.$$

As a result we have

$$\tau^{L_\oplus} < \frac{2H}{K} = \frac{2\tau^2}{\tau} \le \frac{\tau^4}{\tau} = \tau^3$$

which implies $L_{\oplus}(-\tau) \leq 2$. *For the multiplication we have*

$$\tau^{L_{\otimes}} < \frac{H^2}{K} = \frac{\tau^4}{\tau} = \tau^3$$

and hence $L_{\otimes}(-\tau) \leq 2$.

By using the examples

$$1111 \bullet +1111\bullet = 110000 \bullet 11$$

and

$$1111 \bullet \times 1111\bullet = 11100 \bullet 11,$$

we get $L_{\oplus}(-\tau) \geq 2$ *and* $L_{\otimes}(-\tau) \geq 2$, *respectively.*

Altogether we obtain the result

$$L_{\oplus}(-\tau) = 2 = L_{\otimes}(-\tau).$$

Chapter 3

$(-\beta)$-integers as a group

The aim of this chapter is to describe the set of $(-\beta)$-integers as a group with a suitable operation $\oplus$. Since $\mathbb{Z}_{-\beta}$ is countable, one can find a mapping $\varphi : \mathbb{Z}_{-\beta} \longrightarrow \mathbb{Z}$ and define the operation $\oplus$ as

$$b_m \oplus b_n = b_{m+n}, \quad \forall m, n \in \mathbb{Z}.$$

Then the group $(\mathbb{Z}_{-\beta}, \oplus)$ is isomorphic to $\mathbb{Z}$. We will show that when β is a quadratic Pisot unit, it is possible to use this approach to obtain a group such that operation $\oplus$ is compatible with standard addition of real numbers, i.e. there exists a constant $C < +\infty$ such that

$$|(x \oplus y) - (x + y)| \leq C, \qquad \text{for any } x, y \in \mathbb{Z}_{-\beta}$$

and we will determine the value of C. Moreover, when an operation $\ominus : \mathbb{Z}_{-\beta} \to \mathbb{Z}_{-\beta}$ is defined as $a \ominus b = a \oplus (\ominus b)$, where $\ominus b$ is the inverse element of b in the group $(\mathbb{Z}_{-\beta}, \oplus)$, the operation $\ominus$ is compatible with subtraction in $\mathbb{R}$.

This result is in analogy with [6] where compatibility of addition is shown for $\mathbb{Z}_\beta$ with $\beta > 1$ being a quadratic Pisot unit. We consider here $\mathbb{Z}_{-\beta}$ for the same class of numbers β.

Definition 6. *Let $\boldsymbol{u} = (u_i)_{i \in \mathbb{Z}}$ be an integer sequence. Let*

$$\mathcal{M} := \left\{ \sum_{i=-\ell}^{k} x_i (-\beta)^i \;\middle|\; x_i \in \mathbb{Z} \right\}$$

denote the set of numbers with a finite $(-\beta)$-representation. Then we define the mapping $\phi : \mathcal{M} \to \mathbb{Z}$ by the prescription

$$\phi : \sum_{i=-\ell}^{k} x_i (-\beta)^i \mapsto \sum_{i=-\ell}^{k} x_i u_i.$$

The definition is correct when the mapping ϕ is independent on chosen representation of x. This can be ensured by choosing β an algebraic integer and $\boldsymbol{u}$ as a solution of a linear

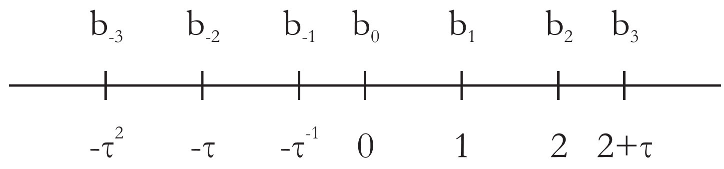

Figure 3.1: The enumeration of $\mathbb{Z}_{-\tau}$ integers

recurrence having the minimal polynomial of $(-\beta)$ as its characteristic polynomial, that is, if β is a root of

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

then $\boldsymbol{u}$ is a solution of

$$(-1)^n u_n + (-1)^{n-1} a_{n-1} u_{n-1} + \cdots + (-1) a_1 u_1 + u_0 = 0.$$

When $\boldsymbol{u}$ is selected to have this property, the mapping ϕ is independent on the chosen representation of x in the base $(-\beta)$.

In the following text, we denote the $(-\beta)$-integers by elements of an increasing sequence $(b_n)_{n\in\mathbb{Z}}$ such that

$$\mathbb{Z}_{-\beta} = \{b_j \mid j \in \mathbb{Z}\}, \quad \text{where } b_0 = 0 \text{ and } b_j < b_{j+1}. \tag{3.1}$$

We will present several lemmas and propositions for three classes of quadratic Pisot numbers first. The summary is situated to the very end of this chapter.

3.1 Case $\beta^2 = \beta + 1$

In this case we choose the sequence $\boldsymbol{u} = (u_i)_{i\in\mathbb{Z}}$ by the prescription

1. $u_{k+2} = -u_{k+1} + u_k$;
2. $u_0 = 1, u_{-1} = -1$.

To illustrate the sequence $\boldsymbol{u}$, let us write down first few members:

$$\ldots u_{-2} = 0, u_{-1} = 1, u_0 = 1, u_1 = -2, u_2 = 3 \ldots.$$

The following lemma proven in [19] describes the lengths of gaps between consecutive $(-\beta)$-integers.

Lemma 5. *Let $x < y$ be consecutive $(-\beta)$-integers. Then either $y = x + 1$ or $y = x + \frac{1}{\beta}$.*

Lemma 6. *The mapping ϕ from Definition 6 has the following properties:*

1. $\phi(x + y) = \phi(x) + \phi(y), \phi(-x) = -\phi(x)$;
2. $\phi(b_j) = j, \forall j \in \mathbb{Z}$;

3. *ϕ restricted to $\mathbb{Z}_{-\beta}$ is a bijection.*

Proof. 1. This property follows from

$$\phi(x+y) = \sum_{-\ell}^{k}(x_i+y_i)u_i = \sum_{-\ell}^{k} x_i u_i + \sum_{-\ell}^{k} y_i u_i = \phi(x)+\phi(y).$$

2. • $\phi(b_0) = \phi(0) = 0$ and $\phi(b_1) = \phi(1) = 1$ directly from the definition.

   • Assume that the statement is true for b_j. According to Proposition 5, it holds that

$$b_{j+1} = \begin{cases} b_j + 1\bullet \\ b_j + \overline{1}\,\overline{1}\bullet \end{cases} \quad \text{and} \quad b_{j-1} = \begin{cases} b_j - \overline{1}\bullet \\ b_j - \overline{1}\,\overline{1}\bullet \end{cases}$$

   where we write numbers 1 and $\frac{1}{\beta} = \beta - 1$ as their $(-\beta)$-representations. We have

$$\begin{aligned}
\phi(b_{j+1}) &= \phi(b_j + 1\bullet) = \phi(b_j) + \phi(1\bullet) = \phi(b_j) + \phi(b_1) = j+1, \\
\phi(b_{j-1}) &= \phi(b_j - 1\bullet) = \phi(b_j) - \phi(1\bullet) = \phi(b_j) - \phi(b_1) = j-1, \\
\phi(b_{j+1}) &= \phi(b_j) - \phi(10\bullet) - \phi(1\bullet) = j - (-2) - 1 = j+1, \\
\phi(b_{j-1}) &= \phi(b_j) + \phi(10\bullet) + \phi(1\bullet) = j + (-2) + 1 = j-1.
\end{aligned}$$

   We used $\phi(10\bullet) = u_1 = -2$ and $\phi(1\bullet) = u_0 = 1$.

3. This is a consequence of the step 2.

□

Note that obviously, the map ϕ is not injective on $\mathrm{Fin}(-\beta)$, because every integer is the image of some element of $\mathbb{Z}_{-\beta} \subset \mathrm{Fin}(-\beta)$.

For quadratic numbers, let us denote $\mathbb{Z}[\beta] = \{a\beta + b \mid a, b \in \mathbb{Z}\}$. For quadratic Pisot units we also have $\mathbb{Z}[\beta] = \mathbb{Z}[\frac{1}{\beta}] = \{\frac{a}{\beta} + b \mid a, b \in \mathbb{Z}\}$ since $\frac{1}{\beta} = m - \beta$ or $\frac{1}{\beta} = -m + \beta$. That also means that when $x \in \mathbb{Z}[\beta]$ then $x' \in \mathbb{Z}[\beta]$. That is because $x = a\beta + b \Rightarrow x' = \pm\frac{a}{\beta} + b \in \mathbb{Z}[\frac{1}{\beta}] = \mathbb{Z}[\beta]$. We will use this fact later.

Let us recall the following statement from [19] which uses the Galois image x' of elements x of the field $\mathbb{Q}(\beta)$.

Lemma 7. *Let β satisfy $\beta^2 = \beta + 1$. Then*

$$\mathbb{Z}_{-\beta} = \{x \in \mathbb{Z}[\beta] \mid x' \in [0, \beta^2)\}.$$

Lemma 8. *Let $x, y \in \mathbb{Z}_{-\beta}$, where β satisfies $\beta^2 = \beta + 1$. Then there exist unique $g, h \in \mathbb{Z}_{-\beta}$ and $\eta, \mu \in \{0, 1\}$, such that*

$$x + y = g + \frac{\eta}{\beta^2} \quad \text{and} \quad x - y = h - \frac{\mu}{\beta^2}.$$

Proof. For $x, y \in \mathbb{Z}_{-\beta}$ we put $z = x + y$. According to Lemma 7, $0 \leq z' = x' + y' < 2\beta^2$. We will distinguish two cases:

- If $0 \leq z' < \beta^2$ then $z = x + y \in \mathbb{Z}_{-\beta}$, so the statement is true for $\eta = 0$.
- If $\beta^2 \leq z' < 2\beta^2$, we put $g := z - \beta^{-2}$, i.e. $g' = z' - \beta^2$. We used that $\beta' = -\frac{1}{\beta}$. Then we have
$$0 \leq g' < \beta^2,$$
which means, by Lemma 7, $g \in \mathbb{Z}_{-\beta}$ and
$$x + y = z = g + \frac{1}{\beta^2}.$$

The proof for the subtraction is similar. We put $w = x - y$. According to Lemma 7, we have $-\beta^2 < w' < \beta^2$. Then

1. If $0 \leq w' < \beta^2$, then $w = x - y \in \mathbb{Z}_{-\beta}$, or
2. If $-\beta^2 < w' < 0$, then we put $h := w + \beta^{-2}$. Then $h' = w' + \beta^2 \in \left(0, \beta^2\right)$, i.e. $h \in \mathbb{Z}_{-\beta}$ and $w = x - y = h - \beta^{-2}$.

To prove the uniqueness, we assume $p \pm \frac{\eta}{\beta^2} = q + \pm\frac{\mu}{\beta^2}$, i.e. $(p - q) \pm (\eta - \mu)\beta^{-2} = 0$. If $p = q$ then necessarily $\eta = \mu$ and vice versa. The case $p \neq q$ and $\eta - \mu \in \{-1, 1\}$ leads to a contradiction since one cannot have
$$\frac{1}{\beta} \leq |p - q| = |(\mu - \eta)\beta^{-2}| = \frac{1}{\beta^2},$$
where we use that distances between consecutive $(-\beta)$-integers are 1 or $\frac{1}{\beta}$. □

Definition 7. *For $\beta > 1$ satisfying $\beta^2 = \beta + 1$ we define the operations $\oplus, \ominus : \mathbb{Z}_{-\beta} \to \mathbb{Z}_{-\beta}$ as*
$$x \oplus y = g \quad \text{and} \quad x \ominus y = h$$
with $g, h \in \mathbb{Z}_{-\beta}$ being as in Lemma 8.

Proposition 9. *Let $(b_j)_{j\in\mathbb{Z}}$ be the sequence of $(-\beta)$-integers satisfying* (3.1)*. Then*

1. $b_n \oplus b_m = b_{n+m}$,
2. $b_n \ominus b_m = b_{m-n}$.

Proof. By Lemma 7, we have $b_m + b_n = g + \eta\beta^{-2}$ and hence $b_m \oplus b_n = g$ with $g \in \mathbb{Z}_{-\beta}$. Then
$$\begin{aligned}\phi(b_m \oplus b_n) = \phi(g) &= \phi(g) + \underbrace{\eta u_{-2}}_{=0} \\ &= \phi(g) + \phi(\eta(-\beta)^{-2}) = \phi(g + \eta\beta^{-2}) = \phi(b_m + b_n) = m + n = \phi(b_{m+n}).\end{aligned}$$
We used $u_{-2} = 0$ and the additivity of ϕ shown in Lemma 6. Hence the expressions $b_m \oplus b_n = g \in \mathbb{Z}_{-\beta}$ and $b_{m+n} \in \mathbb{Z}_{-\beta}$ must be equal, since ϕ is a bijection on $\mathbb{Z}_{-\beta}$. The proof for the subtraction is similar. □

3.2 Case $\beta^2 = m\beta + 1$, $m \geq 2$

Here we choose a similar recurrent prescription for $\boldsymbol{u}$, but with different boundary conditions. As one can see below, this is because of different length of the distances between two consecutive elements of $\mathbb{Z}_{-\beta}$. We define $\boldsymbol{u}$ to satisfy

1. $u_{k+2} = -mu_{k+1} + u_k$;
2. $u_0 = 1, u_{-1} = 0$.

First, we need to prove that the analogy of Lemma 6 is true for β roots of $x^2 - mx - 1$, where $m \geq 2$. Proof of statements 1 and 3 is a straightforward analogy. To prove the second statement of Lemma 6, we will use the fact of having different boundary conditions. The gaps between the elements of $\mathbb{Z}_{-\beta}$ are now $1 = 1\bullet$ and $1 + \frac{1}{\beta} = 1 \bullet \overline{1}$. Since $\phi(1\bullet) = 1$ and $\phi(1 \bullet \overline{1}) = \phi(1\bullet) + \underbrace{\phi(0 \bullet \overline{1})}_{=-u_{-1}=0} = 1$, the analogy of Lemma 6 also holds for β satisfying $\beta^2 = m\beta + 1$.

In this case, the element of $\mathbb{Z}_{-\beta}$ can be also characterized by its image under the non-identical automorphism of $\mathbb{Q}(\beta)$. The following lemma was proved in [19].

Lemma 10. *For β satisfying $\beta^2 = m\beta + 1$, where $m \geq 2$, is*

$$\mathbb{Z}_{-\beta} = \{x \in \mathbb{Z}[\beta] \mid x' \in [0, \beta)\}.$$

Lemma 11. *Let $\beta > 1$ satisfy $\beta^2 = m\beta + 1$ for $m \geq 3$. Then for any $x, y \in \mathbb{Z}_{-\beta}$ there exist the unique $g, h \in \mathbb{Z}_{-\beta}$ and $\eta, \mu \in \{0, 1\}$ such that*

1. $x + y = g - \frac{\eta}{\beta}$,
2. $x - y = h + \frac{\mu}{\beta}$.

Proof. For $x, y \in \mathbb{Z}_{-\beta}$ we put $z = x + y$ and $w = x - y$. According to the Lemma 10 is

$$0 \leq z' < 2\beta \quad \text{and} \quad -\beta < w' < \beta.$$

We will distinguish three cases.

1. If $0 \leq z' < \beta$ (or $0 \leq w' < \beta$) then $z \in \mathbb{Z}_{-\beta}$ (or $w \in \mathbb{Z}_{-\beta}$) and the statement is true for $\eta = 0$ (or $\mu = 0$).
2. If $\beta \leq z' < 2\beta$, we put $g := z + \beta^{-1}$, i.e. $g' = z' - \beta$. Then we have $0 \leq g' < \beta$, which means $g \in \mathbb{Z}_{-\beta}$ and $x + y = z = g - \frac{1}{\beta}$.
3. If $-\beta < w' < 0$, we put $h := w - \beta^{-1}$, i.e. $h' = w' + \beta \in (0, \beta)$ and hence $x - y = w = h + \frac{1}{\beta}$ with $h \in \mathbb{Z}_{-\beta}$.

To prove the uniqueness, let us assume that

$$h \pm \frac{\alpha}{\beta} = g \pm \frac{\gamma}{\beta}$$

for $h, g \in \mathbb{Z}_{-\beta}$ and $\alpha, \gamma \in \{0, 1\}$. Clearly, if $h = g$ then $\alpha = \gamma$ and vice versa. Otherwise, $h - g = \pm\frac{\gamma}{\beta}$ which is not possible since according to Proposition 4, it holds that $|h - g| \geq 1$. $\square$

Definition 8. *For $\beta > 1$ satisfying $\beta^2 = m\beta + 1, m \geq 2$ we define the operations $\oplus, \ominus : \mathbb{Z}_{-\beta} \to \mathbb{Z}_{-\beta}$ as*

$$x \oplus y = g \quad \text{and} \quad x \ominus y = h$$

where $g, h \in \mathbb{Z}_{-\beta}$ are as in Lemma 11.

Proposition 12. *Let $\beta > 1$ satisfy $\beta^2 = m\beta + 1$ for $m \geq 2$ and let $(b_j)_{j\in\mathbb{Z}}$ be the sequence of $(-\beta)$-integers satisfying* (3.1)*. Then*

$$b_n \oplus b_m = b_{n+m} \quad \text{and} \quad b_n \ominus b_m = b_{m-n}.$$

Proof. Let $b_m + b_n = g - \eta\beta^{-1}$ and $b_m \oplus b_n = g$ with $g \in \mathbb{Z}_{-\beta}$. Then

$$\phi(b_m \oplus b_n) = \phi(g) + \eta \underbrace{\phi(0 \bullet 1)}_{=u_{-1}=0} = \phi(g - \eta\beta^{-1}) = \phi(b_m + b_n) = m + n = \phi(b_{m+n}).$$

The proof for subtraction is analogical. $\square$

3.3 Case $\beta^2 = m\beta - 1$

In this case the sequence $\boldsymbol{u}$ is chosen as the solution of

1. $u_{k+2} = -mu_{k+1} - u_k$,
2. $u_0 = 1, u_{-1} = -1$.

Thus we have

$$\ldots, u_{-2} = m - 1, u_{-1} = -1, u_0 = 1, u_1 = 1 - m, u_2 = m^2 - m - 1, \ldots.$$

The distances between consecutive elements of $\mathbb{Z}_{-\beta}$ are $1 = 1\bullet$ and $2 - \frac{1}{\beta} = 2 \bullet 1$, thus their images under the mapping ϕ from Definition 6 are

$$\phi(1\bullet) = u_0 = 1 = \phi(2 \bullet 1) = 2u_0 + u_{-1}.$$

Hence the analogy of Lemma 6 still holds for this class of bases.

The characterization of elements of $\mathbb{Z}_{-\beta}$ using nonidentical automorphism in $\mathbb{Q}(\beta)$ (also proved in [19]) is:

Proposition 13.

$$\mathbb{Z}_{-\beta} = \left\{ x \in \mathbb{Z}[\beta] \mid x' \in \left(-\frac{\beta-1}{\beta+1}, \beta\frac{\beta-1}{\beta+1} \right) \right\}.$$

Lemma 14. *Let $x, y \in \mathbb{Z}_{-\beta}$. Then there exist unique $\eta, \mu \in \{-1, 0, 1\}$ and $g, h \in \mathbb{Z}_{-\beta}$ such that*

$$x + y = g + \eta \left(\frac{m-1}{-\beta} + \frac{1}{(-\beta)^2} \right),$$
$$x - y = h + \mu \left(\frac{m-1}{-\beta} + \frac{1}{(-\beta)^2} \right).$$

Proof. First we show that the numbers $-\frac{\beta-1}{\beta+1}$ and $\beta\frac{\beta-1}{\beta+1}$ have infinite $(-\beta)$-expansion. That is because

$$-\frac{\beta-1}{\beta+1} = -\frac{\beta}{\beta+1} - \frac{1}{(-\beta)} \frac{1}{1 - \frac{1}{(-\beta)}}, \tag{3.2}$$

where we have a $(-\beta)$-representation

$$-\frac{\beta}{\beta+1} = \ell_\beta = \bullet\, d_1 d_2 d_3 \cdots = \bullet\, [(m-1)\, n]^\omega \qquad \text{(see Example 3)}$$

and the second term in 3.2 is the series

$$\sum_{i=-1}^{+\infty} (-\beta)^i = \bullet 1^\omega.$$

Thus we have the $(-\beta)$-expansion

$$\left\langle -\frac{\beta-1}{\beta+1} \right\rangle_{-\beta} = \bullet(d_1-1)(d_2-1)(d_3-1)\cdots = \bullet(m-2)\,0\,(m-2)\,0\ldots$$

which implies $-\frac{\beta-1}{\beta+1} \notin \mathrm{Fin}(-\beta)$. Also $\beta\frac{\beta-1}{\beta+1} \notin \mathrm{Fin}(-\beta)$ since it is just the same number multiplied by the base and therefore

$$\beta\frac{\beta-1}{\beta+1} = (m-2) \bullet 0\,(m-2)\,0\ldots.$$

For $x, y \in \mathbb{Z}_{-\beta}$ is $x \pm y \in \mathrm{Fin}(-\beta) \subset \mathbb{Z}[\beta]$ since β is a quadratic Pisot unit and $\mathrm{Fin}(-\beta)$ is a ring. Then also $(x \pm y)' \in \mathbb{Z}[\beta]$ and since it can be written as a finite addition or subtraction of β's and 1's, then $(x \pm y)' \in \mathrm{Fin}(-\beta)$. Therefore it is not possible that $(x \pm y)' = -\frac{\beta-1}{\beta+1} \notin \mathrm{Fin}(-\beta)$. Hence we do not consider the values $-\frac{\beta-1}{\beta+1}$ and $\beta\frac{\beta-1}{\beta+1}$ as the ends of intervals in the rest of the proof.

Let us prove the statement for addition. According to Proposition 13, for $x, y \in \mathbb{Z}_{-\beta}$ it holds $(x+y)' \in \left(-2\frac{\beta-1}{\beta+1}, 2\beta\frac{\beta-1}{\beta+1}\right)$. We distinguish several cases:

1. If $(x+y)' \in \left(-\frac{\beta-1}{\beta+1}, \beta\frac{\beta-1}{\beta+1}\right) =: \Omega$ then clearly $x + y \in \mathbb{Z}_{-\beta}$.
2. The case $(x+y)' \in \left(-2\frac{\beta-1}{\beta+1}, -\frac{\beta-1}{\beta+1}\right) =: \Omega_1$ will be solved as a subcase of Case 5., since $\Omega_1 \subset \Omega_1^*$.

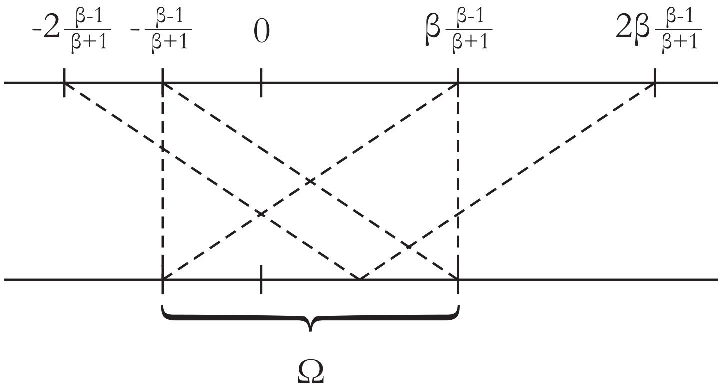

Figure 3.2: Transformations of the intervals for addition

3. If $(x+y)' \in \left(\beta\frac{\beta-1}{\beta+1}, 2\beta\frac{\beta-1}{\beta+1}\right) = \Omega_2$, we put
$$g := x + y - \left(\frac{m-1}{-\beta} + \frac{1}{(-\beta)^2}\right).$$
The left-end of Ω_2 will transform as
$$\beta\frac{\beta-1}{\beta+1} + (m-1)\beta - \beta^2 = -\frac{\beta-1}{\beta+1}$$
which is the left-end of Ω. Again, the length of Ω_2 is smaller than the length of Ω, so $z' \in \Omega$ and hence
$$x + y = g + \left(\frac{m-1}{-\beta} + \frac{1}{(-\beta)^2}\right),$$
where $g \in \mathbb{Z}_{-\beta}$.

For subtraction we have
$$(x-y)' \in (-(\beta-1), \beta-1).$$
We distinguish three more cases.

4. If $(x-y)' \in \Omega$ then clearly $x - y \in \mathbb{Z}_{-\beta}$.
5. If $(x-y)' \in \left(-(\beta-1), -\frac{\beta-1}{\beta+1}\right) =: \Omega_1^*$, we put $h = x - y + \frac{m-1}{-\beta} + \frac{1}{(-\beta)^2}$, i.e. $h' = (x-y)' - (m-1)\beta + \beta^2$. Since the left-end of Ω_1^* is transformed as
$$-\beta + 1 - (m-1)\beta + \beta^2 = 0$$
and the length of Ω_1^* is the same as the length of $\left(0, \beta\frac{\beta-1}{\beta+1}\right)$, we have
$$\Omega_1^* - (m-1)\beta + \beta^2 = \left(0, \beta\frac{\beta-1}{\beta+1}\right) \subset \Omega.$$
Therefore $h' \in \Omega$ which implies
$$h = (x-y) + \frac{m-1}{-\beta} + \frac{1}{(-\beta)^2} \in \mathbb{Z}_{-\beta}$$
and
$$x - y = h - \left(\frac{m-1}{-\beta} + \frac{1}{(-\beta)^2}\right).$$

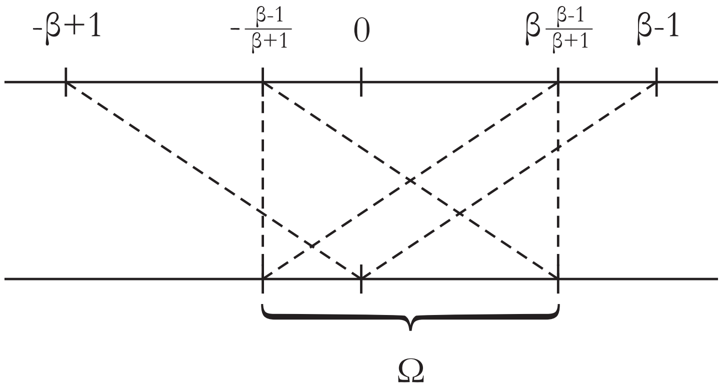

Figure 3.3: Transformations of the intervals for subtraction

6. The case $(x-y)' \in \left(\beta\frac{\beta-1}{\beta+1}, \beta-1\right) =: \Omega_2^*$ is in fact solved in Case 3. of addition because $\Omega_2^* \subset \Omega_2$.

   For $x+y \notin \mathbb{Z}_{-\beta}$, we have presented the unique way of shifting $(x+y)' \notin \mathbb{Z}_{-\beta}$ inside of the interval Ω (see Figure 3.2). For $x+y \in \mathbb{Z}_{-\beta}$ is

$$x+y \pm \left(\frac{m-1}{-\beta} + \frac{1}{(-\beta)^2}\right) \notin \mathbb{Z}_{-\beta},$$

   since z' would be transformed outside of Ω (see Figure 3.2 and 3.3). This proves the uniqueness of η. The proof of the uniqueness for subtraction is analogical.

□

Definition 9. *Let $\beta > 1$ satisfy $\beta^2 = m\beta - 1$ for $m \geq 3$. Then we define the operations $\oplus, \ominus : \mathbb{Z}_{-\beta} \to \mathbb{Z}_{-\beta}$ as*

$$x \oplus y = g \quad \text{and} \quad x \ominus y = h$$

with $g, h \in \mathbb{Z}_{-\beta}$ as in Lemma 14.

Theorem 10. *Let $\beta > 1$ satisfy $\beta^2 = m\beta - 1$ for $m \geq 3$ and let $(b_j)_{j\in\mathbb{Z}}$ be the sequence of $(-\beta)$-integers satisfying* (3.1). *Then*

1. $b_m \oplus b_n = b_{m+n}$,
2. $b_m \ominus b_n = b_{m-n}$.

Proof. Let $b_m + b_n = g + \eta\left(\frac{m-1}{-\beta} + \frac{1}{(-\beta)^2}\right)$. Then we have

$$\phi(b_m \oplus b_n) = \phi(g) = \phi(g) + \underbrace{\eta[(m-1)u_{-1} + u_{-2}]}_{=0} = \phi\left(g + \eta\left(-(m-1)\beta^{-1} + \beta^{-2}\right)\right)$$
$$= \phi(b_m + b_n) = m + n = \phi(b_{m+n}).$$

The proof for subtraction is analogical. □

The compatibility of $\oplus$ and $\ominus$ with the operations $+, -$ in $\mathbb{R}$ is in fact the corollary of Propositions 9, 12 and 3.3. This fact follows from

$$b_m + b_n = g + \xi = b_m \oplus b_n + \xi = b_{m+n} + \xi$$

which means not only that the operations $\oplus$ are compatible with $C = \xi$ but that the $(m+n)$-th integer is a good approximation of the summation of m- and n-th integer.

The neutral element in the group $(\mathbb{Z}_{-\beta}, \oplus)$ is $b_0 = 0$ since $b_j + b_0 = b_{j+0} = b_j$, $\forall j \in \mathbb{Z}$ and hence the inverse element of b_j is b_{-j}. Then

$$b_m - b_n = h + \xi = b_m \ominus b_n + \xi = b_{m-n} + \xi = b_m \oplus (\ominus\, b_n) + \xi.$$

This proves that the operation $\ominus$ is the addition of an inverse element and is compatible with subtraction in $\mathbb{R}$ with $C = \xi$.

Chapter 4

Estimations of $L_{\oplus}(-\beta)$ and $L_{\otimes}(-\beta)$ for quadratic Pisot numbers

In this chapter we try to estimate the values $L_{\oplus}(-\beta)$ and $L_{\otimes}(-\beta)$ defined as (2.7) and (2.8) for quadratic Pisot numbers. These values for quadratic Pisot units have been already determined in [14] and [15].

Usually the easiest and the most straightforward approach follows from Theorem 8. However, there is a class of quadratic Pisot numbers for which the value K equals to zero. That makes Theorem 8 impossible to use. Still, we will be able to obtain a partial result on $L_{\oplus}(-\beta)$ for this class. For the class of quadratic Pisot number with negative Galois conjugate, we do the estimation on $L_{\oplus}(-\beta)$, $L_{\otimes}(-\beta)$ and derive the exact bound on the number of fractional digits arising from addition of $(-\beta)$-integers.

Let us first present the sets of forbidden strings for quadratic Pisot numbers.

Proposition 15. *1. Let $\beta > 1$ satisfy $\beta^2 = m\beta - n$ for $m - 2 \geq n \geq 1$. Then*

$$\begin{aligned} \mathcal{F} = &\{(m-1)\,A \mid A \leq n-1\} \\ &\cup \{0\,[(m-1)\,n]^{\omega}\}\,. \end{aligned}$$

2. Let $\beta > 1$ satisfy $\beta^2 = m\beta + n$ for $m \geq n \geq 1$. Then

$$\begin{aligned} \mathcal{F} = &\{m\,(m-n)^{2k}C \mid C \leq m-n-1\,,\ k \in \mathbb{N}_0\} && (4.1)\\ &\cup \{m\,(m-n)^{2k+1}D \mid D \geq m-n+1\,,\ k \in \mathbb{N}_0\} && (4.2)\\ &\cup \{0\,m\,(m-n)^{\omega}\}. && (4.3) \end{aligned}$$

4.1 Case $\beta^2 = m\beta - n$

First, we present a class of numbers, including a subset of quadratic Pisot numbers, for which is using Theorem 8 not possible since $K = 0$.

Proposition 16. *Let $\beta > 1$ be an algebraic number with a Galois conjugate $\beta' \in (0,1)$ and let $\left\lfloor \frac{1}{(\beta')^2} \right\rfloor \leq \lfloor \beta \rfloor - 1$. Then the value $K = \inf\{|z'| \mid z \in \mathbb{Z}_{-\beta} \setminus (-\beta)\mathbb{Z}_{-\beta}\}$ equals to zero.*

Proof. We shall find the sequence $(z_n)_{n\in\mathbb{N}}$ of $(-\beta)$-integers in $\mathbb{Z}_{-\beta} \setminus (-\beta)\mathbb{Z}_{-\beta}$ such that $\lim_{n\to+\infty} z_n' = 0$. Here z_n' stands for the image of z_n under the field isomorphism induced by the Galois conjugate $\beta' \in (0,1)$.

Let us denote $\gamma = \frac{1}{\beta'} > 1$. Since $\frac{1}{\gamma} = \beta' \in (0,1)$, we are able to obtain its expansion in the positive base γ^2 in the form

$$\frac{1}{\gamma} = \sum_{i=1}^{+\infty} \frac{b_i}{(\gamma^2)^i}.$$

We put

$$z_n := b_n\, 0\, b_{n-1}\, 0 \ldots b_2\, 0\, b_1\, 1 \bullet_{-\beta}\, .$$

This $(-\beta)$-representation is a $(-\beta)$-expansion since the assumption $\left\lfloor \frac{1}{(\beta')^2} \right\rfloor = \lfloor \gamma^2 \rfloor \leq \lfloor \beta \rfloor - 1$ implies that the maximal digit $\lfloor \gamma^2 \rfloor$ of the alphabet of γ^2-expansions is strictly smaller than the maximal digit $\lfloor \beta \rfloor$ of $(-\beta)$-expansion. Since the $(-\beta)$-expansion of z_n does not contain the maximal digit $\lfloor \beta \rfloor - 1$, it has to be admissible, as can be seen from Theorem5 realizing that the first digit of $d_{-\beta}(\ell_\beta)$ is equal $\lfloor \beta \rfloor$.

Then we have

$$\begin{aligned} z_n' &= 1 + b_1(-\beta') + b_2(-\beta')^3 + \cdots + b_n(-\beta')^{2n-1} \\ &= 1 + \frac{1}{(-\beta')}\left(b_1(-\beta')^2 + b_2(-\beta')^4 + \cdots + b_n(-\beta')^{2n}\right) \\ &= 1 - \gamma\, \underbrace{\left(b_1\gamma^2 + b_2\gamma^4 + \cdots + b_n\gamma^{2n}\right)}_{\to\gamma^{-1}} \longrightarrow 0. \end{aligned}$$

∎

We show that for a large class of quadratic Pisot numbers with positive conjugate, the value K equals to zero. For $\beta > 1$ root of $x^2 - mx + n$, $n^2 \geq \frac{m^2}{m-2}$ we can do an estimation

$$\left\lfloor \frac{1}{(\beta')^2} \right\rfloor = \left\lfloor \frac{\beta^2}{n^2} \right\rfloor \leq \left\lfloor \frac{m^2}{n^2} \right\rfloor \leq m - 2 \leq \lfloor \beta \rfloor - 1.$$

Hence we have to use other methods to estimate $L_\oplus(-\beta)$ or $L_\otimes(-\beta)$. Estimation of the value $L_\otimes(-\beta)$ is a very difficult problem and we will focus only on $L_\oplus(-\beta)$.

Lemma 17. *Let $\beta > 1$ satisfy $\beta^2 = m\beta - n$ for $m-2 \geq n \geq 1$. Then $x + y \in \mathrm{Fin}(-\beta)$ for every $x \in \mathrm{Fin}(-\beta)$, $y \in \{0, 1, \dots, m-n-1\}$. Moreover, let*

$$\langle x \rangle_{-\beta} = x_k \dots x_0 \bullet x_{-1} \dots x_{-r}\, 0^\omega \qquad \textit{where } x_{-r} \neq 0.$$

Then one of the following statements is true.

1. $\langle x+y \rangle_{-\beta} = z_j \dots z_0 \bullet z_{-1} \dots z_{-r}\, 0^\omega$,
2. $\langle x+y \rangle_{-\beta} = z_j \dots z_0 \bullet X_1 Y_1 \dots X_l Y_l\, (x_{-r} + m - n)\, n\, 0^\omega$,
3. $\langle x+y \rangle_{-\beta} = z_j \dots z_0 \bullet X_1 Y_1 \dots X_l [x_{-r} - (m-n-1)]\, (m-n)\, n\, 0^\omega$,

where $X_i \in \{m-n-1, \dots, m-2\}$ and $Y_i \in \{0, 1, \dots, n-1\}$.

Proof. Let us first realize that since β is the root of $x^2 - mx + n$, we have the following representations of 0,

$$1\ m\ n \bullet = \overline{1}\ \overline{m}\ \overline{n} \bullet = 0\,,$$

where $\overline{A}$ is a compact form of $-A$. By repeated application of this relation, we obtain for every $k \in \mathbb{N}$,

$$\begin{array}{cccccccc} & 1 & (m-1) & \overline{[(m-n-1)} & \overline{(m-n-1)]^k} & \overline{(m-n)} & \overline{n} & \bullet & = 0 \\ 1 & (m-1) & \overline{(m-n-1)} & [(m-n-1) & \overline{(m-n-1)]^k} & (m-n) & n & \bullet & = 0 \end{array} \tag{4.4}$$

Adding $y \in \{0, 1, \dots, m-n-1\}$ to a number x written as an admissible digit string may result in a non-admissible digit string, which, nevertheless, represents the number $x + y$. We show that $x+y$ belongs to $\mathrm{Fin}(-\beta)$ by providing its finite $(-\beta)$-expansion. In order to see that the two strings represent the same number, one can verify that the second one is obtained from the first one by adding digit-wise a zero which is in the form (4.4). We give a list of cases. One verifies by inspection that the list contains all cases of non-admissible strings that arise from admissible ones by adding y.

According to Proposition 15, a digit string may be non-admissible by breaking one of the two conditions, namely, either it is not over the alphabet $\{0, 1, \dots, m-1\}$, or it contains the subsequence $(m-1)A$, where $A \leq n-1$.

Case 1. Consider an $x \in \mathrm{Fin}(\beta)$ such that its $(-\beta)$-expansion has digit $x_0 \geq (m-y)$ at $(-\beta)^0$, and the digit at position $(-\beta)^{-1}$, denote it by C, is at least n. Find $k \in \{0, 1, 2, \dots\}$ such that we have a representation of $x + y$ in the form

$$x{+}y \ = \ \cdots\ A\, B\, [(m{-}1)\, n]^k\, (x_0{+}y) \bullet C\ \cdots \qquad \text{where the string } A\,B \neq (m{-}1)\, n\,. \tag{4.5}$$

Case 1.1. First take $B = 0$. To the representation (4.5) of the number $x + y$ we add digit-wise a representation of 0,

$$\begin{array}{cccccccccc} x+y & = & \cdots & A & 0 & [(m{-}1) & n]^k & (x_0+y) & \bullet & C & \cdots \\ 0 & = & & 1 & (m{-}1) & \overline{[(m{-}n{-}1)} & (m{-}n{-}1)]^k & \overline{(m{-}n)} & \bullet & \overline{n} & \\ \hline x+1 & = & \cdots & (A{+}1) & (m{-}1) & [n & (m{-}1)]^k & z_0 & \bullet & C{-}n & \cdots \end{array}$$

Here we have

$$z_0 = x_0 + y - m + n \leq x_0 + m - n - 1 - m + n = x_0 - 1 \leq m - 2.$$

Since $B = 0$ in the $(-\beta)$-expansion of x, we necessarily have $A \leq m - 2$. Therefore also the resulting representation of $x + y$ is admissible as $(-\beta)$-expansion of $x + y$.

The case $B \geq 1$ is divided into two subcases.

Case 1.2. Let $B \geq 1$ and $k = 0$. Again, we add to the non-admissible representation of $x + y$ in the form (4.5) a suitable representation of 0,

$$\begin{array}{ccccccccc} x+y & = & \cdots & A & B & (x_0+y) & \bullet & C & \cdots \\ 0 & = & & & \overline{1} & \overline{m} & \bullet & \overline{n} & \\ \hline x+y & = & \cdots & A & (B-1) & z_0 & \bullet & C-n & \cdots \end{array}$$

Here we have, by the assumption $x_0 \geq m - y$, that $z_0 = x_0 + y - m \geq 0$. Since $A\ B \neq (m-1)\ n$, the resulting representation of $x + y$ is the $(-\beta)$-expansion of $x + y$.

Case 1.3. Let $B \geq 1$ and $k \geq 1$. In this case we rewrite

$$\begin{array}{ccccccccccccc} x+y & = & \cdots & A & B & (m-1) & n & [(m-1) & n]^{k-1} & (x_0+y) & \bullet & C & \cdots \\ 0 & = & & & \overline{1} & \overline{(m-1)} & (m-n-1) & [\overline{(m-n-1)} & (m-n-1)]^{k-1} & \overline{(m-n)} & \bullet & \overline{n} & \\ \hline x+y & = & \cdots & A & (B-1) & 0 & (m-1) & [n & (m-1)]^{k-1} & z_0 & \bullet & C-n & \cdots \end{array}$$

Here $z_0 = x_0 + y - m + n \leq m - 2$ and $A\,B \neq (m-1)\,n$, therefore the result is $(-\beta)$-admissible.

Case 2. Consider an $x \in \mathrm{Fin}(\beta)$ such that its $(-\beta)$-expansion has digit $x_0 \leq m - 1 - y$ at the position $(-\beta)^0$. In order that after adding y one obtains a non-admissible string different from Case 1, necessarily the digit C at position $(-\beta)^{-1}$ satisfies $C \leq n - 1$. Denote by M the set of pairs of digits

$$M := \big\{\, X\ Y \;\big|\; X \in \{m-n-1, \ldots, m-2\},\ Y \in \{0, 1, \ldots, n-1\} \,\big\}\,. \tag{4.6}$$

Then we can find $k, l \in \{0, 1, 2, \ldots\}$ such that

$$x + y \;=\; \cdots\ A\ B\ [(m-1)\ n]^k\ (x_0{+}y) \bullet C\ X_1\ Y_1 \cdots X_l\ Y_l\ D\ E\ \cdots \tag{4.7}$$

where the string $A\ B \neq (m-1)\ n$ and the string $D\ E$ does not belong to M. Denote by p_1, p_2 the $(-\beta)$-integers

$$\begin{array}{cccccccc} p_1 & = & & \overline{1} & \overline{(m-1)} & [(m-n-1) & \overline{(m-n-1)}]^k & \bullet \\ p_2 & = & 1 & (m-1) & \overline{(m-n-1)} & [(m-n-1) & \overline{(m-n-1)}]^k & \bullet \end{array}$$

and by z_1, z_2 the following numbers with only $(-\beta)$-fractional part,

$$\begin{array}{ccccccc} z_1 & = & \bullet & [(m-n-1) & \overline{(m-n-1)}]^l & (m-n) & n \\ z_2 & = & \bullet & [(m-n-1) & \overline{(m-n-1)}]^l & (m-n-1) & \overline{(m-n)}\ \ \overline{n} \end{array}$$

Using (4.4) one can easily see that $p_i + z_j = 0$ for $i, j \in \{1, 2\}$. We shall work separately with the $(-\beta)$-integer and $(-\beta)$-fractional part of $x + y$.

First consider the $(-\beta)$-fractional part of $x + y$. Recall that $C \leq n - 1$ and the string $D\ E \notin M$. If $D \leq m - n - 2$ or $D = m - n - 1$ (the latter implies $E \geq n$), then we add z_1 to the $(-\beta)$-fractional part of $x + y$. We obtain

$$\begin{array}{lcccccccc}
\bullet & C & X_1 & Y_1 & \cdots & X_l & Y_l & D & E\ \cdots \\
\bullet & (m-n-1) & \overline{(m-n-1)} & (m-n-1) & \cdots & \overline{(m-n-1)} & (m-n) & n & \\
\hline
\bullet & C+m-n-1 & X_1-(m-n-1) & Y_1+m-n-1 & \cdots & X_l-(m-n-1) & Y_l+m-n & D+n & E\ \cdots
\end{array} \tag{4.8}$$

The resulting fractional part is an admissible digit string. If $D \geq m - n$ (which implies $E \geq n$), then we add z_2 to the $(-\beta)$-fractional part of $x + y$. We obtain

$$\begin{array}{lcccccccc}
\bullet & C & X_1 & Y_1 & \cdots & X_l & Y_l & D & E\ \cdots \\
\bullet & (m-n-1) & \overline{(m-n-1)} & (m-n-1) & \cdots & \overline{(m-n-1)} & (m-n-1) & \overline{(m-n)} & \overline{n} \\
\hline
\bullet & C+m-n-1 & X_1-(m-n-1) & Y_1+m-n-1 & \cdots & X_l-(m-n-1) & Y_l+m-n-1 & D-m+n & E-n\ \cdots
\end{array} \tag{4.9}$$

Again, the resulting string is admissible.

Let us now take the $(-\beta)$-integer part of $x + y$. Recall that $A\ B \neq (m-1)\ n$. If $B = 0$, then to the $(-\beta)$-integer part of $x + y$ we add p_2, if $B \geq 1$, we add p_1. For $B = 0$ we have

$$\begin{array}{ccccccc}
\cdots & A & 0 & [(m-1) & n]^k & (x_0+y) & \bullet \\
 & 1 & (m-1) & [\overline{(m-n-1)} & (m-n-1)]^k & \overline{(m-n-1)} & \bullet \\
\hline
\cdots & (A+1) & (m-1) & [n & (m-1)]^k & z_0 & \bullet
\end{array}$$

For $B \geq 1$ we have

$$\begin{array}{ccccccccc}
\cdots & A & B & [(m-1) & n]^{k-1} & (m-1) & n & (x_0+y) & \bullet \\
 & & \overline{1} & \overline{(m-1)} & [(m-n-1) & \overline{(m-n-1)}]^{k-1} & (m-n-1) & \overline{(m-n-1)} & \bullet \\
\hline
\cdots & A & (B-1) & 0 & [(m-1) & n]^{k-1} & (m-1) & z_0 & \bullet
\end{array}$$

For $B \geq 1$ and $k = 0$ we have

$$\begin{array}{ccccc}
\cdots & A & B & (x_0+y) & \bullet \\
 & & \overline{1} & \overline{(m-1)} & \bullet \\
\hline
\cdots & A & (B-1) & z_0 & \bullet
\end{array}$$

In all cases, the result is an admissible string with the last digit z_0 satisfying $n \leq z_0 \leq m - 2$. Concatenating such a string with an admissible digit string resulting from the $(-\beta)$-fractional part, we obtain an admissible digit string. Therefore, we have provided a prescription to rewrite the original non-admissible representation of $x+y$ of the form (4.7) by adding 0 in the form $p_i + z_j$, into the $(-\beta)$-expansion of $x+y$.

The sum of numbers $x = x_k \dots x_0 \bullet x_{-1} \dots x_{-r}$ and $y \in \{0, 1, \dots, m-n-1\}$ is in the form $x + y = z_\ell \dots z_0 \bullet z_{-1} \dots z_{-r}$ when Case 1. or (4.8) happens. When $x_{-r}\, 0 = Y_l\, D$ as in (4.8), the sum is clearly in the form

$$\langle x+y\rangle_{-\beta} = z_\ell \dots z_0 \bullet (x_{-1}+\gamma)\,(x_{-2}-\gamma)(x_{-3}+\gamma)\dots(x_{-r+1}-\gamma)(x_{-r}+m-n)\,n$$

where $\gamma = m - n - 1$.

When $x_{-r}\, 0\, 0 = X_l\, Y_l\, D$ as in (4.9) then

$$\langle x+y\rangle_{-\beta} = z_\ell \dots z_0 \bullet (x_{-1}+\gamma)\,(x_{-2}-\gamma)(x_{-3}+\gamma)\dots(x_{-r+1}+\gamma)(x_{-r}-\gamma)\,(m-n)\,n$$

where $\gamma = m - n - 1$. This completes the proof. $\square$

Lemma 18. *Let* $x \in \mathbb{Z}_{-\beta}$ *and* $y = \sum_{i=0}^{k} y_i(-\beta)^i$, $y_i \in \{0, 1, \dots, m-n-1\}$. *Then* $x + y \in \frac{1}{(-\beta)^2}\mathbb{Z}_{-\beta}$.

Proof. We add the number y digit by digit from the "most-left" one, that is, first we add $y_i(-\beta)^i$ where $i = \max\{j \mid y_j \neq 0\}$ and we repeat this procedure until the whole y is added to x.

Let us consider the case when we added $y_i(-\beta)^i$ and new fractional digits appeared for the first time. Then according to Lemma 17 we have $x + y_i(-\beta)^i$ in the form

$$\langle x + y_i(-\beta)^i\rangle_{-\beta} = \cdots z_i X_1 Y_1 \cdots X_l Y_l\,(x_0+m-n) \bullet\ n\,0^\omega$$

or

$$\langle x + y_i(-\beta)^i\rangle_{-\beta} = \cdots z_i X_1 Y_1 \cdots X_l [x_0-(m-n-1)] \bullet\ (m-n)\,n\,0^\omega$$

where $X_j \in \{m-n+1, \dots, m-2\}$, $Y_j \in \{0, 1, \dots, n-1\}$, and z_i denotes a position of $(-\beta)^i$. We distinguish two cases of what can happen when the next digit after y_i is added.

1. If the next digit is to be added is at the position $(-\beta)^{i-2t}$ (i.e. to the position where Y_t is), then we have

   $$z_i = Y_t + y_{i-2t} \le (n-1) + (m-n-1) \le m-2.$$

   Therefore we can add $y_i(-\beta)^i$ digit-wise without making a forbidden string. No new fractional digit was made and the result is in the form

   $$\langle x + y_i(-\beta)^i + y_{i-2t}\rangle_{-\beta} = \dots z_{i-2t} X_{t+1} Y_{t+1} \dots X_l Y_l \bullet (m-n)\,n.$$

   That means the addition of next digit will happen in the same manner.

2. If the next digit is to be added is at the position $(-\beta)^{i-(2t-1)}$, where X_t is, and $x_{i-(2t-1)} + y_{i-(2t-1)} \geq m-1$, notice that the string beginning with x_{i-2t} is in the form
$$\widetilde{X_1}\,\widetilde{Y_1}\ldots\widetilde{X_s}\,\widetilde{Y_s}\,D\,E$$
with $\widetilde{X_j} \in \{m-n-1,\ldots,m-2\}$, $Y_j \in \{0,\ldots,n-1\}$ and $D \geq (m-n)$, $E \geq n$. In that case we use the rewriting rule (4.9). In fact, since we rewrite the suffix $(Y_l{+}m{-}n)\,n\,0^\omega \mapsto Y_l\,0\,0^\omega$, the fractional digits created when adding $y_i(-\beta)^i$ vanished in this step.

We showed that when fractional digits appear (at most two of them), we can either add the next digit directly (without rewriting) or the fractional digits disappear during the next rewriting. This completes the proof. ☐

Lemma 19. *Let $\beta > 1$ satisfy $\beta^2 = m\beta - n$ for $m-2 \geq n \geq 1$. Then every $x \in \mathbb{Z}_{-\beta}$ has a $(-\beta)$-representation in the form $-x = \sum_{j\geq -1} x_j(-\beta)^j$ with $j \in \{0,1,\ldots,m\}$.*

Proof. Let $-x = x_k \ldots x_0\bullet$ with $x_i \in \{0,-1,-2,\cdots-(m-1)\}$. We will apply the rule
$$x_{i+1}x_ix_{i-1} \mapsto (x_{i+1}+1)(x_i+m)(x_{i-1}+n)$$
at every position where $x_i < 0$ starting from $i = k$ up to $i = 0$. Notice that since x_{i-1} can change from negative to positive when $x_i < 0$, it matters that we apply the rule from $i = k$ to $i = 0$ and not in the opposite order. We show that using this algorithm we obtain a $(-\beta)$-representation of $-x$ over the alphabet $\{1,\ldots,m\}$ in the form
$$-x = y_{k+1}\ldots y_0 \bullet y_{-1}0^\omega. \tag{4.10}$$

Let us assume we the algorithms is rewriting the first position $x_i < 0$. We have
$$x_{i+1}x_ix_{i-1} \mapsto (x_{i+1}+1)(x_i+m)(x_{i-1}+n) \quad \text{where } x_{i+1} = 0,\ x_i < 0.$$

Here x_i will be rewritten either to $(x_i+m) \leq m-1$ or $(x_i+m+1) \leq m$ (in the next step), depending on the digit x_{i-1}. If $-n \leq x_{i-1} \leq 0$, then algorithm will not put $+1$ to the i-th position. If $-(m-1) \leq x_{i-1} < -n$ then $x_{i-1}+n < 0$ and the algorithm also rewrites
$$(x_i+m)(x_{i-1}+n)x_{i-2} \mapsto (x_i+m+1)(x_{i-1}+n+m)(x_{i-2}+n).$$

Here $(x_i+m+1) \leq -1+m+1 = m$ and $(x_{i-1}+n+m) \leq -(n+1)+n+m = m-1$. At most $+1$ can be added to the $(i-1)$-th position and therefore the resulting string is over the alphabet $\{0,1,\ldots,m\}$.

The fractional digits appears for example when rewriting $-1\bullet \mapsto 1\,(m-1)\bullet n$.

☐

Lemma 18 together with Lemma 19 give us the following theorem.

Theorem 11. *Let β satisfy $\beta^2 = m\beta - n$. Then $L_{\oplus}(-\beta) \leq 2\left\lceil \frac{m}{m-n-1} \right\rceil + 1$.*

Proof. When adding $x, y \in \mathbb{Z}_{-\beta}$, one can write $x + y = x + y^{(1)} + y^{(2)} + \cdots + y^{(N)}$, where $N = \left\lceil \frac{m-1}{m-n-1} \right\rceil$ and

$$y^{(i)} = \sum_{j \geq 0} y_j^{(i)}, \qquad \text{with} \;\; y_j^{(i)} \in \{0, 1, \ldots, m-n-1\}.$$

During each addition, according to Lemma 18, at most two new fractional can be created, which means $x + y \in \frac{1}{(-\beta)^{2N}} \mathbb{Z}_{-\beta}$.

For $y \in \mathbb{Z}_{-\beta}$ is $-y \in \frac{1}{-\beta}\mathbb{Z}_{-\beta}$ and hence we consider $(-\beta)(-y) \in \mathbb{Z}_{-\beta}$ first. Then we follow the same principle. According to Lemma 19, we can write $(-\beta)(-y) = y^{(1)} + y^{(2)} + \cdots + y^{(M)}$, where $M = \left\lceil \frac{m}{m-n-1} \right\rceil$ and

$$y^{(i)} = \sum_{j \geq 0} y_j^{(i)}, \qquad \text{with} \;\; y_j^{(i)} \in \{0, 1, \ldots, m-n-1\}.$$

When subtracting $(-\beta)x - (-\beta)y$, only $2M$ fractional digits can appear, which means $(-\beta)(x-y) \in \frac{1}{(-\beta)^{2M}}\mathbb{Z}_{-\beta}$ or in other words $x - y \in \frac{1}{(-\beta)^{2M+1}}\mathbb{Z}_{-\beta}$. This completes the proof. □

Numerical experiments showed that this estimation of $L_{\oplus}(-\beta)$ should be close to the actual value, however, no example for which $L_{\oplus}(-\beta) = \frac{1}{(-\beta)^{2M+1}}\mathbb{Z}_{-\beta}$ was found so far.

4.2 Case $\beta^2 = m\beta + n$

Let us first try to derive $L_{\oplus}(-\beta)$ using Theorem 8. When $m = n$, the string $mmm\ldots m0^{\omega}$ is $(-\beta)$-admissible. Hence we can do an estimation of H as

$$0 \leq z' = \sum_{i=0}^{k} z_i(-\beta')^i = \sum_{i=0}^{k} z_i \left(\frac{n}{\beta}\right)^i < \sum_{i=0}^{+\infty} m \left(\frac{n}{\beta}\right)^i = \frac{m\beta}{\beta - n} = H.$$

The case $m = n = 1$ has been already solved in [15] and therefore we can take $m \geq 2$. Then it always holds $1\bullet \in \mathbb{Z}_{-\beta} \backslash (-\beta)\mathbb{Z}_{-\beta}$ (does not hold in the case $m = n = 1$, see Example 4) and hence $K = 1$.

Then we obtain

$$\left(\frac{\beta}{m}\right)^{L_{\oplus}} = \left(\frac{1}{\beta'}\right)^{L_{\oplus}} < 2\frac{m\beta}{\beta - m} = 2\beta^2$$

and consequently $L_{\oplus} < \frac{\ln 2\beta^2}{\ln \frac{\beta}{m}}$. Using the estimations $\beta < m+1$ and

$$\frac{\beta}{m} = 1 + \frac{1}{\beta} > 1 + \frac{1}{m+1} > e^{\frac{1}{m+2}}$$

we finally get

$$L_{\oplus}(-\beta) < \frac{\ln 2(m+1)^2}{\ln e^{\frac{1}{m+2}}} = (m+2)\ln\left[2(m+1)^2\right]. \tag{4.11}$$

The analogical approach can be used to obtain an estimation of $L_\otimes$ in the form

$$L_\otimes < \frac{\ln(m+1)^4}{\ln e^{\frac{1}{m+2}}} < 4(m+2)\ln(m+1).$$

When $m > n$, the string $mmm\dots m(m-1)0^\omega$ is admissible (unlike $mmm\dots m0^\omega$). Then $H = \frac{m\beta}{\beta-n} - 1$ and $K = 1$ which follows from the same approach as in the case $m = n$. Substituting these values into Theorem 8 we obtain

$$\left(\frac{m}{n}\right)^{L_\oplus} < \left(\frac{\beta}{n}\right)^{L_\oplus} < \frac{2H}{K} = 2\left(\frac{m\beta}{\beta-n} - 1\right) = 2\beta\frac{\beta-1}{\beta-n} < 2\frac{(m+1)^2}{m-n}$$

and hence

$$L_\oplus(-\beta) < \log_{\frac{m}{n}} 2\frac{(m+1)^2}{m-n} = \frac{\ln 2 + 2\ln(m+1) - \ln(m-n)}{\ln\frac{m}{n}}.$$

Analogically we obtain

$$L_\otimes(-\beta) < 2\log_{\frac{m}{n}} \frac{(m+1)^2}{m-n} = \frac{4\ln(m+1) - 2\ln(m-n)}{\ln\frac{m}{n}}.$$

The results above lead to the following proposition.

Proposition 20. *Let $\beta > 1$ satisfy $\beta^2 = m\beta + n$ for $m \geq n \geq 1$. Then*

1. *for $m > n$ are*
$$L_\oplus(-\beta) < \frac{\ln 2 + 2\ln(m+1) - \ln(m-n)}{\ln\frac{m}{n}}$$
and
$$L_\otimes(-\beta) < \frac{4\ln(m+1) - 2\ln(m-n)}{\ln\frac{m}{n}}.$$

2. *for $m = n \geq 2$ are*
$$L_\oplus < (m+2)\ln\left[2(m+1)^2\right] \qquad \textit{and} \qquad L_\otimes < 4(m+2)\ln(m+1).$$

This estimation can be useful when $m \gg n$. For example, assumption $m-n \geq 2\left(\frac{m+1}{m}\right)^2 n^2$ gives us $L_\oplus \leq 1$. That means, in fact, that $L_\oplus = 1$, for $1\,(m-1)\bullet +1\bullet = 0\bullet n$.

For a large subclass of quadratic Pisot numbers with negative conjugate, we have been able to obtain precise number of fractional digits arising from addition of $(-\beta)$-integers.

For this purpose, we define the quantity denoting the number of fractional digits that may arise from addition as

$$L_\oplus^+(-\beta) = \min\{l \in \mathbb{N}_0 \mid \forall\, x, y \in \mathbb{Z}_{-\beta},\ x + y \in \mathrm{Fin}(-\beta) \Rightarrow x + y \in (-\beta)^{-l}\mathbb{Z}_{-\beta}\}.$$

First, let us present a lemma whose part was given in [15].

Lemma 21. Let $\beta > 1$ be root of $x^2 - mx - n$, $m \geq n \geq 1$. Then

$$x := \sum_{i=0}^{N} a_i(-\beta)^i \in \mathrm{Fin}(-\beta)$$

for arbitrary $a_i \in \{0, 1, \ldots, m\}$. Moreover, $x \in \mathbb{Z}_{-\beta}$ except when both $m > n$ and $a_0 = m$, in which case $x \in \frac{1}{-\beta}\mathbb{Z}_{-\beta}$. Also, when $m > n$ and $x = a_N \ldots a_1 a_0 \bullet$ with a_0, $a_1 \neq m$, then $\langle x \rangle_{-\beta} = x_k \ldots x_1\, a_0 \bullet$.

Proof. Consider the $(-\beta)$-representation $a_N a_{N-1} \ldots a_0 \bullet$ of x. If it is not the $(-\beta)$-expansion of x, then $a_N a_{N-1} \ldots a_0 0^\omega$ contains one of forbidden strings listed in Proposition 15. We shall rewrite the left-most forbidden string in $a_N a_{N-1} \ldots a_0 0^\omega$ by adding a suitable $(-\beta)$-representation of 0. The new $(-\beta)$-representation of x is 'better' than $a_N a_{N-1} \ldots a_0 0^\omega$ in the way that the left-most forbidden string starts at a lower power of $(-\beta)$. Such rewriting does not add non-zero digits to the right, (unless we deal with the last occurring forbidden string). Therefore, by repeating such rewriting rules, we finish in finitely many steps with a $(-\beta)$-representation which does not contain any forbidden strings, i. e. it is the $(-\beta)$-expansion of x.

Since β is a root of $x^2 - mx - n$, we have

$$1\ m\ \overline{n} \bullet = \overline{1}\ \overline{m}\ n \bullet = 0\,. \tag{4.12}$$

(Here for a digit d we write $\overline{d}$ instead of $-d$.)

We distinguish several cases, according to the type of the left-most forbidden string (cf. Proposition 15).

Case 1. Consider first that $m > n$ and take the forbidden string (4.1), together with two digits A, B in the $(-\beta)$-representation of x at higher powers of $(-\beta)$,

$$\ldots\ A\ B\ m\ (m-n)^{2k}\ C \ldots \qquad k \in \mathbb{N}_0,\ C \leq m - n - 1\,. \tag{4.13}$$

The way to rewrite the forbidden string depends on the digits A, B.

Subcase 1.1. Let $B = 0$, and consequently $A \in \{0, 1, \ldots, m-1\}$. (Otherwise $A0m$ is also forbidden, which contradicts the fact that we take the left-most forbidden string.) We rewrite

$$\begin{array}{ccccccc} \ldots & A & 0 & m & (m-n)^{2k} & C & \ldots \\ \ldots & A+1 & m & (m-n) & (m-n)^{2k} & C & \ldots \end{array}$$

It is easy to verify that now no forbidden string occurs left from the digit C, which was our aim.

Subcase 1.2. Let in (4.13) be $B \neq 0$ and $k \geq 1$. Then

$$\begin{array}{ccccccc} \ldots & A & B & m & (m-n) & (m-n)^{2k-1} & C & \ldots \\ \ldots & A & B-1 & 0 & m & (m-n)^{2k-1} & C & \ldots \end{array}$$

Again, the latter may contain a forbidden string only starting from the digit C.

Subcase 1.3. Let $B \neq 0$ and $k = 0$. We write

$$\begin{array}{cccccc} \dots & A & B & m & C & \dots \\ \dots & A & B-1 & 0 & C+n & \dots \end{array}$$

where the latter has no forbidden strings up to the digit $C + n$.

Case 2. Take the forbidden string (4.2) which occurs for both $m > n$ and $m = n$,

$$\dots\ A\ B\ m\ (m-n)^{2k+1}\ D \dots \qquad k \in \mathbb{N}_0,\ D \geq m - n + 1\,. \tag{4.14}$$

The rewriting is analogous to subcases 1.1. and 1.2., subcase 1.3. now has no analogue.

Subcase 2.1. Let $B = 0$, and consequently $A \in \{0, 1, \dots, m-1\}$. We rewrite

$$\begin{array}{ccccccc} \dots & A & 0 & m & (m-n)^{2k+1} & D & \dots \\ \dots & A+1 & m & (m-n) & (m-n)^{2k+1} & D & \dots \end{array}$$

where the latter has no forbidden strings up to the digit D.

Subcase 2.2. Let in (4.14) be $B \neq 0$. Then

$$\begin{array}{cccccccc} \dots & A & B & m & (m-n) & (m-n)^{2k} & D & \dots \\ \dots & A & B-1 & 0 & m & (m-n)^{2k} & D & \dots \end{array}$$

where the latter has no forbidden strings up to the digit D.

For $m \neq n$, the situation when the last nonzero digit can be affected is **Case 3.** Consider $m = n$. According to Proposition 15 it remains to solve the case that the only forbidden string in the $(-\beta)$-representation of x is $0m$ at the end. Necessarily, the $(-\beta)$-representation ends with $A0m$, where $A \leq m - 1$. We rewrite

$$\begin{array}{cccc} \dots & A & 0 & m \\ \dots & A+1 & m & 0 \end{array}$$

By that, we have shown that $x \in \mathrm{Fin}(-\beta)$. In order to show $x \in \mathbb{Z}_{-\beta}$, note that in all cases except subcase 1.3, the rewriting of the forbidden string did not influence the digits starting from C (resp. D) to the right. Thus, if the original $(-\beta)$-representation of x had vanishing digits at negative powers of $(-\beta)$, then the same is valid for the rewritten $(-\beta)$-representation of x. The only case where new non-zero digits at negative powers of $(-\beta)$ may arise, is 1.3 for $m > n$, and that only if $x = a_N a_{N-1} \dots a_0 \bullet = \dots ABm\bullet$, i.e. $a_0 = m$.

For $m \neq n$, when the digit $a_0 \neq m$ is to be rewritten, only Subcase 1.3. when $a_0 = C$ has to be considered. However, Subcase 1.3 requires that $a_1 = m$. This completes the proof. $\square$

Theorem 12. *Let $\beta > 1$ satisfy $\beta^2 = m\beta + n$ for $m \geq 2n$ and let $x, y \in \mathbb{Z}_{-\beta}$. Then*

$$L_{\oplus}^{+}(-\beta) \leq \begin{cases} 1 & \textit{for } m \geq 2n+1 \textit{ or } m = 2, n = 1, \\ 2 & \textit{for } m = 2n,\ n \neq 1. \end{cases}$$

Proof. The assumption $m \geq 2n$ in both cases implies $2(m-n) \geq m$. Therefore $y \in \mathbb{Z}_{-\beta}$ can be written as

$$y = y^{(1)} + y^{(2)} = \sum_{j \geq 0} y_j(-\beta)^j + \sum_{j \geq 0} y_j^*(-\beta)^j$$

with $y_j, y_j^* \in \{0, 1, \ldots, m-n\}$ and with the coefficient y_0 satisfying $y_0 \leq m-n-1$. This is possible since necessarily the digit at the power $(-\beta)^0$ of y is $y_0 \leq m-1$ since the string $m\,0^\omega$ is forbidden in $y \in \mathbb{Z}_{-\beta}$.

Let us add the number $y \in \mathbb{Z}_{-\beta}$ from the "most-left" position, i.e. first we add $y_i(-\beta)^i$ where $i = \max\{j \mid y_j \neq 0\}$. If $x_i + y_i \leq m$ then we sum the digits and rewrite the result $x + y_1(-\beta)^i$ into its $(-\beta)$-expansion. According to Lemma 21 we have $x + y_i(-\beta)^i \in \mathbb{Z}_{-\beta}$ or $x + y_i(-\beta)^i \in \frac{1}{-\beta}\mathbb{Z}_{-\beta}$ when $i = 0$ and $x_0 + y_0 = m$. Then we move to the next digit which we add a $(-\beta)$-expansion. In the latter case we added the whole $y^{(1)}$.

Let us now consider the case when $x_i + y_i \geq m+1$. By iterating the rewriting rule $0 = 1\,m\,\overline{n} = \overline{1}\,\overline{m}, n$, we obtain the following representations of zero for $k \in \mathbb{N}_0$.

$$\begin{array}{rcccccc}
0 = & \overline{1} & \overline{m+1} & \overline{(m-n+1)}^k & \overline{m-n} & n & \\
= & 1 & (m-1) & \overline{(m+n)} & n & & \\
= & 1 & (m-1) & \overline{(m+n+1)} & \overline{(m-n+1)}^k & \overline{(m-n)} & n
\end{array}$$

When $x_i + y_i \geq m+1$, we distinguish several cases.

Case 1. Let $x_{i+2}x_{i+1}x_i = A\,0\,x_i$ where necessarily $A \leq m-1$ since $m\,0$ is a forbidden string.

Case 1a. When $x_i + y_i \leq m+n$. Then we rewrite

$$\begin{array}{ccccccc}
x + y_i(-\beta)^i & = & \cdots & A & 0 & (x_i + y_i) & \cdots \\
0 & = & & 1 & m & \overline{n} & \\
\hline
x + y_i(-\beta)^i & = & \cdots & (A{+}1) & m & (x_i + y_i - n) & \cdots
\end{array}$$

Here $x_i + y_i - n \in \{m-n+1, \ldots, m\}$.

Case 1b. If $x_i + y_i \geq m+n+1$ and there is an integer $l \in \mathbb{N}$ such that the $(-\beta)$-representation of $x + y_i(-\beta)^i$ is in the following form with $X_i \geq m-n+1$ and $Y_i \leq m-n$, we rewrite

$$\begin{array}{ccccccccc}
x + y_i(-\beta)^i = \cdots & A & 0 & (x_i + y_i) & X_1 & \cdots & X_{l-1} & X_l & Y \\
0 \quad = & 1 & (m-1) & \overline{(m+n+1)} & \overline{\gamma} & \ldots & \overline{\gamma} & \overline{(m-n)} & n \\
\hline
x + y_i(-\beta)^i = \cdots & (A{+}1) & (m{-}1) & z_i & (X_1{-}\gamma) & \cdots & (X_{l-1}{-}\gamma) & X_l{-}(m-n) & Y{+}n
\end{array}$$

where $\gamma = m-n+1$. The latter $(-\beta)$-representation of $x + y_i(-\beta)^i$ is over the alphabet $\mathcal{A} = \{0, 1, \ldots m\}$ and uses only non-negative powers of $(-\beta)$, except if $Y = 0$ is at the position of $(-\beta)^{-1}$.

Case 1c. When $x_i + y_i \geq m+n+1$ and $l = 0$ we rewrite

$$\begin{array}{cccccccc}
x + y_i(-\beta)^i & = & \cdots & A & 0 & (x_i + y_i) & Y & \cdots \\
0 & = & & 1 & (m-1) & \overline{(m+n)} & n & \\
\hline
x + y_i(-\beta)^i & = & \cdots & (A{+}1) & m & (x_i + y_i - (m+n)) & (Y+n) & \cdots
\end{array}$$

Case 2a. Let $x_{i+1}x_i = B\,x_i$ where $B \geq 1$. Again, we find an integer $l \in \mathbb{N}$, such that the $(-\beta)$-representation of $x + y_i(-\beta)^i$ is in the following form with $X_i \geq m-n+1$ and $Y_i \leq m-n$.

$$\begin{array}{cccccccc} x+y_i(-\beta)^i = \cdots & B & (x_i+y_i) & X_1 & \cdots & X_{l-1} & X_l & Y \\ 0 \quad = & \overline{1} & \overline{(m+1)} & \overline{\gamma} & \cdots & \overline{\gamma} & \overline{(m-n)} & n \\ \hline x+y_i(-\beta)^i = \cdots & (B-1) & z_i & (X_1-\gamma) & \cdots & (X_{l-1}-\gamma) & X_l-(m-n) & Y+n \end{array}$$

where $\gamma = m-n+1$. Again, the latter $(-\beta)$-representation of $x + y_i(-\beta)^i$ is over the alphabet $\mathcal{A} = \{0, 1, \ldots, m\}$.

Case 2b. When $x_{i+1}x_i = B\,x_i$ with $B \geq 1$ and $l = 0$ we rewrite

$$\begin{array}{cccccccc} x+y_i(-\beta)^i & = & \cdots & B & (x_i+y_i) & Y & \cdots \\ 0 & = & & \overline{1} & \overline{m} & n & \\ \hline x+y_i(-\beta)^i & = & \cdots & (B-1) & (x_i+y_i-m) & (Y+n) & \cdots \end{array}$$

After adding $y_i(-\beta)^i$ we rewrite the result into its $(-\beta)$-expansion and according to Lemma 21 either $x + y_i(-\beta)^i \in \mathbb{Z}_{-\beta}$ or $x + y_i(-\beta)^i \frac{1}{-\beta} \in \mathbb{Z}_{-\beta}$. The latter can be happen in two cases and we will show that when $x + y_i(-\beta)^i \in \frac{1}{-\beta}\mathbb{Z}_{-\beta}$, the rest of $y^{(1)}$ can be added and $\langle x + y^{(1)}\rangle_{-\beta} = z_N \ldots z_0 \bullet n$.

The first case is when $x_0 = Y = m-n$. In that case, it is possible to add all the digits $y_j,\ 1 \leq j < i$ since $X_j - \gamma + y_j \leq m$. Then

$$x + \sum_{1\leq j<i} y_j(-\beta)^j = \cdots C\, m \bullet 0^\omega = \cdots (C-1)\, 0 \bullet n,$$

where we used $\overline{1}\,\overline{m} \bullet n = 0$ and the fact that $C \neq 0$, since $C = X_l - (m-n) \geq 1$ (in Cases 1b. and 2a.) or $C = x_i + y_i - (m-n) \geq 1$ (in Case 1c.) and finally $C = x_i + y_i - m \geq 1$ in Case 2b. Now the digit y_0 can be added and $x + y^{(1)} = \cdots (C-1)\, y_0 \bullet n$.

In the second case, we have $X_l\, Y = x_0\, 0$. Then all the digits $y_j,\ 0 \leq j < i$ can be added for the same reason as above, and we have $x + y^{(1)} = \cdots z_0 \bullet n$. Moreover, the coefficient $z_0 = X_l - (m-n) + y_0$ satisfies $z_0 \leq m-1$ (recall that $y_0 \leq m-n-1$).

In both cases we have $x + y^{(1)} = \cdots Z \bullet n$ with $Z \leq m-1$ and therefore according to Lemma 21, the $(-\beta)$-expansion of $x + y^{(1)}$ is in the form

$$\langle x + y^{(1)}\rangle_{-\beta} = z_k \ldots z_0 \bullet n.$$

The addition of $y^{(2)}$ is similar to adding $y^{(1)}$. When $x + y^{(1)} \in \mathbb{Z}_{-\beta}$, it has been already shown that one can obtain at most one fractional digit when adding $y^{(2)}$. Note that now we can have $y_0^* = m-n$ while we required $y_0 \leq m-n-1$. However, this requirement was important only for showing that if $x + y^{(1)} \notin \mathbb{Z}_{-\beta}$ then $\langle x\rangle_{-\beta} = \cdots \bullet n$. When we have $x + y^{(1)} = \cdots \bullet n$, then either $x + y^{(1)} + y^{(2)} = \cdots \bullet n$ or $x + y^{(1)} + y^{(2)} = \cdots \bullet (2n)$ (if cases 1b., 1c., 2a or 2b happen, in that case we have $\bullet n = \bullet Y$). Obviously, for $2n+1 \leq m$ is $x + y = \cdots \bullet (2n) \in \frac{1}{-\beta}\mathbb{Z}_{-\beta}$ and for $2n = m$ is $x + y = \cdots \bullet (2n) = \cdots \bullet m \in \frac{1}{(-\beta)^2}\mathbb{Z}_{-\beta}$ according to Lemma 21.

The fact that for $n = 1$ and $m = 2$ is $L_\oplus^+(-\beta) = 1$ gives Theorem 2. $\square$

To show that the bound on $L_\oplus^+(-\beta)$ is precise, we present two examples.

Example 5. *For $2n < m$ we have $1\,(m-1) \bullet + 1\bullet = 1\,m\bullet\ = 0 \bullet n \notin \mathbb{Z}_{-\beta}$.*

Example 6. *For $2n = m$, $n \geq 2$ we consider the following representation of zero.*

$$\begin{aligned} 0 &= 1\,m\,\overline{n}\,0\,0 \bullet + \overline{1}\,\overline{m}\,n\,0 \bullet + \overline{2}\,\overline{(2m)}\,\overline{(2n)} \bullet + \overline{2}\,\overline{(2m)} \bullet\ \overline{(2n)} + \overline{1} \bullet\ \overline{m}\,n \\ &= 1\,(m-1)\,\overline{(m+n+2)}\,\overline{(2m-n+2)}\,\overline{(2m-2n+1)} \bullet\ 0\,n \end{aligned}$$

Clearly, the number $x := m\,m\,(m-1)\bullet_{-\beta} \in \mathbb{Z}_{-\beta}$. Then for the sum $x + x$ we have

$$\begin{array}{rllllll} 2x = & 0 & 0 & (2m) & (2m) & (2m-2) & \bullet\, 0\ 0 \\ 0\ = & 1 & (m-1) & \overline{(m+n+2)} & \overline{(2m-n+2)} & \overline{(2m-2n+1)} & \bullet\, 0\ n \\ \hline 2x = & 1 & (m-1) & (m-n-2) & (n-2) & (2n-3) & \bullet\, 0\ n \end{array}$$

The latter $(-\beta)$-representation of $2x$ is admissible for $n \geq 2$ and therefore the bound given by Theorem 12 can be reached.

From now on we will focus on estimating $L_\oplus^+(-\beta)$ for β root of $x^2 - mx - n$ for $2n > m$.

Lemma 22. *Let $\beta > 1$ satisfy $\beta^2 = m\beta + n$ for $2n \geq m+1$. Let $x \in \mathrm{Fin}(-\beta)$, $\langle x\rangle_{-\beta} = x_k \dots x_0 \bullet x_{-1} \dots x_{-r} 0^\omega$ with $x_{-r} \neq 0$ and $y = \sum_{i \geq 0} y_i(-\beta)^i, y_i \in \{0, 1, \dots, m-n+1\}$. Then it holds that*

1. *If $r \geq 1$ then $x + y \in \frac{1}{(-\beta)^{r+1}}\mathbb{Z}_{-\beta}$.*
2. *If $r \leq 0$ and $y_0 \leq m-n$ then $x + y \in \frac{1}{-\beta}\mathbb{Z}_{-\beta}$.*
3. *If $r \leq 0$ and $y_0 = m-n+1$ then $x + y \in \frac{1}{(-\beta)^2}\mathbb{Z}_{-\beta}$.*

Proof. We will add the number $y = \sum_{j\geq 0} y_j(-\beta)^j$ from the most-left digit y_i where $i = \max\{j \mid y_j \neq 0\}$ and then we add to the result $y_{i-1}(-\beta)^{i-1}$ etc. until we add the whole number y.

First, let us realize that when $x_i + y_i \leq m$ for $i \neq 0$, we have a $(-\beta)$-representation of $x + y_i(-\beta)^i$ over the alphabet $\mathcal{A} = \{0, 1, \dots, m\}$, and according to Lemma 21 is $x + y_i(-\beta)^i \in \mathbb{Z}_{-\beta}$ or $x + y_i(-\beta)^i \in \frac{1}{-\beta}\mathbb{Z}_{-\beta}$ when $y_i = y_0$, $x_0 + y_0 = m$ and $m > n$.

When $x_i + y_i \geq m+1$, we distinguish several cases. Note that $x_i + y_i \leq m+n$ since $y_i \leq m-n+1 \leq n$. We show, how $x + y_i(-\beta)^i$ can be rewritten to a representation over the "right" alphabet $\mathcal{A}_{-\beta} = \{0, 1, \dots, m\}$.

Case 1. If $x + y_i(-\beta)^i$ is in the following form with $A \leq m$ (which is necessary since $m\,0$ is forbidden), we shall rewrite.

$$\begin{array}{ccccccc} x + y_i(-\beta)^i & = & \cdots & A & 0 & (x_i + y_i) & \cdots \\ 0 & = & & 1 & m & \overline{n} & \\ \hline x + y_i(-\beta)^i & = & \cdots & A+1 & m & z_i & \cdots \end{array}$$

the latter $(-\beta)$-representation is over the alphabet $\mathcal{A}_{-\beta}$.

Case 2. Here is $B \in \{1, 2, \dots, m\}$ and $Y \le m - n$. In this case we rewrite $x + y_i(-\beta)^i$ as

$$\begin{array}{rcccccc}
x + y_i(-\beta)^i & = & \cdots & B & (x_i + y_i) & Y & \cdots \\
0 & = & & \overline{1} & \overline{m} & n & \\
\hline
x + y_i(-\beta)^i & = & \cdots & B-1 & z_i & (Y+n) & \cdots
\end{array}$$

The result is over the right alphabet.

Case 3. We find an integer $l \in \mathbb{N}$ and digits $X_i \in \{m-n+1, \dots, m\}$, $Y \le m - n$, such that we have $x + y_i(-\beta)^i$ in the form

$$\begin{array}{rccccccccc}
x + y_i(-\beta)^i & = \cdots & B & (x_i + y_i) & X_1 & \cdots & X_{l-1} & X_l & Y & \cdots \\
0 & = & \overline{1} & \overline{m+1} & \overline{(m-n+1)} & \cdots & \overline{(m-n+1)} & \overline{(m-n)} & n & \\
\hline
x + y_i(-\beta)^i & = \cdots & B-1 & z_0 & (X_1 - \gamma) & \cdots & (X_k - \gamma) & (X_l - m + n) & Y + n & \cdots
\end{array}$$

where we denote $\gamma = m - n + 1$. Of course, the latter representation is over the alphabet $\mathcal{A}_{-\beta}$.

Let us discuss when new fractional digits arise in the $(-\beta)$-expansion of $x + y_i(-\beta)^i$.

If a new fractional digits appears when **Case 2.** happens then necessarily $Y = x_{-1}$ and $y_i = y_0$. That means we added the whole number y and we have $x + y = \cdots \bullet n$. Then according to Lemma 21 is $x + y \in \frac{1}{-\beta}\mathbb{Z}_{-\beta}$.

Let us study the case when **Case 3.** causes a direct (before doing an expansion) appearing of a new fractional digit. We distinguish three cases:

1. If $-r > -1$ and $Y = 0 = a_{-r-1}$, one can directly add all y_j, $0 \le j < i$ since $\gamma = m - n + 1 \ge y_i$ was subtracted from these positions. Then we have $x + y \in \frac{1}{(-\beta)^{r+1}}\mathbb{Z}_{-\beta}$.
2. Let $r \le 0$, i.e. $x \in \mathbb{Z}_{-\beta}$, and $y_0 \le m - n$. In order to obtain fractional digits caused by **Case 3.**, we have $Y = x_{-1} = 0$ and consequently we can add all the digits $y_j, 1 \le j \le i$.
   (a) When $y_0 \le m - n$, it can be also directly added. Since the last digit is $x_{-1} = n$, it holds $x + y \in \frac{1}{-\beta}\mathbb{Z}_{-\beta}$.
   (b) When $y_0 = m - n + 1$, we can add $y_j, 1 \le j \le i$ and $(y_0 - 1)(-\beta)^0 = m - n$ to obtain $x + y - 1 \in \frac{1}{-\beta}\mathbb{Z}_{-\beta}$ as was proven above. Then for $m > n$ we already have $(x + y - 1) + 1 \in \frac{1}{(-\beta)^2}\mathbb{Z}_{-\beta}$ since $1 \le m - n$.
   (c) When $m = n$, we can either add $y_0 = m - n + 1 = 1$ directly (i.e. $x_0 \le m - 1$) or we can first rewrite $x + y - y_0 = \dots m \bullet m \mapsto \dots (m-1) \bullet 0\, m$. Then we can add $y_0 = 1$ to obtain $x + y = \dots m \bullet\ 0\, m \in \frac{1}{(-\beta)^2}\mathbb{Z}_{-\beta}$.
3. The last problem to consider is when $x_0 = m - n = Y$ in Cases 2. and 3. Then we rewrite

$$x + y_i(-\beta)^i = \dots z_i\, m \bullet \mapsto (z_i - 1)\, 0 \bullet\ n,$$

(Case 2.) or $(X_l - m + n)\, m \bullet \mapsto (X_l - m + n - 1)\, 0 \bullet n$ (Case 3.) and y_0 can be added. In this case, the result belongs to $\frac{1}{-\beta}\mathbb{Z}_{-\beta}$.

According to Lemma 21, no new fractional digit appears when doing the $(-\beta)$-expansion in all cases. $\square$

Theorem 13. *Let $\beta > 1$ satisfy $\beta^2 = m\beta + n$ for $m \le 2n+1$. Then*

$$L_{\oplus}^{+}(-\beta) \le \begin{cases} \left\lceil \frac{m}{m-n+1} \right\rceil & \text{for } \frac{m}{2} < n \le m-1, \\ m+1 & \text{for } m = n. \end{cases}$$

Proof. We can write $x + y = x + \sum_{i=1}^{N} y^{(i)}$, where

$$y^{(i)} = \sum_{j \ge 0} y_j^{(i)} (-\beta)^j \quad \text{with } y_j^{(i)} \in \{0, 1, \ldots, m-n+1\}$$

and $y_0^{(1)} \le m - n$ for $m > n$. The statement of the theorem is then a straightforward application of Lemma 22. $\square$

In order to show that the estimation of $L_{\oplus}^{+}(-\beta)$ given in Theorem 13 is precise for $m = n$, we give an example which show that the bound can be reached.

Example 7. *For $\beta > 1$ satisfying $\beta^2 = m\beta + m$ is $x := m \sum_{i=0}^{m+3} (-\beta)^i \in \mathbb{Z}_{-\beta}$. We consider the following $(-\beta)$-representation of zero.*

$$\begin{array}{ccccccccccccccccc}
1 & m & \overline{(m+1)} & \overline{(m+1)} & \overline{1} & \overline{1} & \overline{1} & \ldots & \overline{1} & \overline{2} & \overline{(m+1)} & \bullet & m & m & 0 & & \\
 & & & \overline{1} & \overline{(m+1)} & \overline{1} & \overline{1} & \ldots & \overline{1} & \overline{1} & \overline{1} & \bullet & \overline{1} & 0 & m & & \\
 & & & & \overline{1} & \overline{(m+1)} & \overline{1} & \ldots & \overline{1} & \overline{1} & \overline{1} & \bullet & \overline{1} & \overline{1} & 0 & m & \\
 & & & & & & \ddots & & \vdots & \vdots & \vdots & & & & & & \ddots \\
 & & & & & & & \overline{1} & \overline{(m+1)} & \overline{1} & \overline{1} & \bullet & \overline{1} & \overline{1} & \ldots & \overline{1} & 0 \ m \\
\hline
1 & m & \overline{(m+1)} & \overline{(m+2)} & \overline{(m+3)} & \overline{(m+4)} & & \ldots & \overline{(2m)} & \overline{(m+1)} & \overline{(2m)} & \bullet & 1 & 2 & \ldots & (m-1) & m \ m
\end{array}$$

We will sum $x + x$ and rewrite the digit-wise addition using the $(-\beta)$-representation of zero mentioned above.

$$\begin{array}{cccccccccccccccccc}
2x & = & & & (2m) & (2m) & (2m) & \ldots & (2m) & (2m) & (2m) & \bullet & & & & & \\
0 & = & 1 & m & \overline{(m+1)} & \overline{(m+2)} & \overline{(m+3)} & \ldots & \overline{(2m)} & \overline{(m+1)} & \overline{(2m)} & \bullet & 1 & 2 & \ldots & m & m \\
\hline
2x & = & 1 & m & (m-1) & (m-2) & (m-3) & \ldots & 0 & (m-1) & 0 & \bullet & 1 & 2 & \ldots & m & m
\end{array}$$

The result does not contain any forbidden string and thus it is a $(-\beta)$-expansion with $m+1$ fractional digits. Hence $L_{\oplus}^{+}(-\beta) = m+1$.

In order to obtain the value $L_{\oplus}(-\beta)$ one must show analogical statements also for subtraction. Some steps to prove such statements have been already made and we conjecture that the number of fractional digits arising from subtraction will not exceed the values derived in Theorems 12 and 13.

Conclusions

This work was devoted to the study of the arithmetical properties of the set of $(-\beta)$-integers when β is a quadratic Pisot number. In Chapter 3 we have shown that when β is a quadratic Pisot unit, the set $\mathbb{Z}_{-\beta}$ with a suitable operation $\oplus$ forms a group and that this operation is compatible with addition in $\mathbb{R}$, that is, there exists $C > 0$, such that

$$|(x+y) - (x \oplus y)| \leq C,$$

and we determined the value of C. Moreover, if an operation $\ominus$ is defined as $x \ominus y = x \oplus (\ominus y)$, where $\ominus y$ denotes an inverse element of y in $(\mathbb{Z}_{-\beta}, \oplus)$, then $\ominus$ is compatible with subtraction in $\mathbb{R}$.

In Chapter 4 we studied the bound on the number of fractional digits arising from arithmetical operations on $\mathbb{Z}_{-\beta}$. We derived a bound on the addition, subtraction and multiplication of $(-\beta)$-integers in the case when $\beta > 1$ satisfies $\beta^2 = m\beta + n$ for $m \geq n \geq 1$. When $2n \leq m$ or $m = n$, we were able to derive a precise value of $L_{\oplus}^{+}(-\beta)$, i.e. the maximal number of fractional digits arising from a summation of $(-\beta)$-integers. Determination of this value also for the case $2n - 1 \geq m$ remains an open problem as well as determination of the exact values for the subtraction and multiplication for this class of numbers.

For a class of quadratic Pisot numbers β satisfying $\beta^2 = m\beta - n$ for $m - 2 \geq n \geq 1$ we derived a bound on $L_{\oplus}(-\beta)$. However, this bound seems to be rough and therefore a better estimation remains as an open problem. Moreover, the bound on the number of fractional digits appearing after a multiplication of $(-\beta)$-integers is unknown at all.

A recently discovered problem is related to describing bases β for which is

$$\mathbb{Z}_{-\beta} = \left\{ \sum_{i \geq 0} a_i (-\beta)^i \mid a_i \in \{0, 1, \ldots, \lfloor \beta \rfloor\} \right\}.$$

It has been shown that the bases β with $d_{-\beta}(\ell_\beta) = (m\,0)^k\,0^\omega$ and $d_{-\beta}(\ell_\beta) = (m\,0)^k\,m^\omega$ have this property. This might be useful for showing the relation between arithmetical operations in the $(-\beta)$-number system and finite automatons.

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