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On the Eigenvalue Problem for a Particular Class of Jacobi Matrices

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Contents

Abstract

A function $\mathfrak{F}$ with simple and nice algebraic properties is defined on a subset of the space of complex sequences. Some special functions are expressible in terms of $\mathfrak{F}$, first of all the Bessel functions of the first kind. A compact formula in terms of the function $\mathfrak{F}$ is given for the determinant of a Jacobi matrix. Further we focus on the particular class of Jacobi matrices whose parallels to the diagonal are constant and whose diagonal depends linearly on the index. A formula in terms of the function $\mathfrak{F}$ is derived for the characteristic function. A special basis is constructed in which the Jacobi matrix becomes a sum of a diagonal matrix and a rank-one matrix operator. A vector-valued function on the complex plain is constructed having the property that its values on spectral points of the Jacobi matrix are equal to corresponding eigenvectors. It is shown the spectrum of the infinite Jacobi matrix with linear diagonal and constant parallels coincides with zeros of the Bessel function of the first kind as function of its order.

Keywords: tridiagonal matrix, finite Jacobi matrix, eigenvalue problem, characteristic function

1 Introduction

The results of the current paper are mostly related to the eigenvalue problem for finite-dimensional symmetric tridiagonal (Jacobi) matrices. Notably, the eigenvalue problem for finite Jacobi matrices is solvable explicitly in terms of generalized hypergeometric series [7]. Here we focus on a very particular class of Jacobi matrices which makes it possible to derive some expressions in a comparatively simple and compact form. We do not aim at all, however, at a complete solution of the eigenvalue problem. We restrict ourselves to derivation of several explicit formulas, first of all that for the characteristic function, as explained in more detail below. We also develop some auxiliary notions which may be, to our opinion, of independent interest.

First, we introduce a function, called $\mathfrak{F}$, defined on a subset of the space of complex sequences. In the remainder of the paper it is intensively used in various formulas. The function $\mathfrak{F}$ has remarkably simple and nice algebraic properties. Among others, with the aid of $\mathfrak{F}$ one can relate an infinite continued fraction to any sequence from the definition domain on which $\mathfrak{F}$ takes a nonzero value. This may be compared to the fact that there exists a correspondence between infinite Jacobi matrices and infinite continued fractions, as explained in [2, Chp. 1]. Let us also note that some special functions are expressible in terms of $\mathfrak{F}$. First of all this concerns the Bessel functions of first kind. We examine the relationship between $\mathfrak{F}$ and the Bessel functions and provide some supplementary details on it.

Further we introduce an infinite antisymmetric matrix, with entries indexed by integers, such that its every row or column obeys a second-order difference equation which is very well known from the theory of Bessel functions. With the aid of function

$\mathfrak{F}$ one derives a general formula for entries of this matrix. The matrix also plays an essential role in the remainder of the paper.

As an application we present a comparatively simple formula for the determinant of a Jacobi matrix under the assumption that the neighboring parallels to the diagonal are constant. As far as the determinant is concerned this condition is not very restrictive since a Jacobi matrix can be written as a product of another Jacobi matrix with all units on the neighboring parallels which is sandwiched with two diagonal matrices. Yet another formula for the determinant of a Jacobi matrix with the antisymmetric diagonal (with respect to its center) is presented. In that case zero is always an eigenvalue and we give an explicit formula for the corresponding eigenvector.

Next we focus on the rather particular class of Jacobi matrices whose parallels to the diagonal are constant and whose diagonal depends linearly on the index. In this case we derive a formula for the characteristic function. Moreover, we construct a basis in which the Jacobi matrix becomes a sum of a diagonal matrix and a rank-one matrix operator. This form is rather suitable for various computations. Particularly, one can readily derive a formula for the resolvent. In addition, a vector-valued function on the complex plain is constructed having the property that its values on spectral points of the Jacobi matrix are equal to corresponding eigenvectors.

Finally we focus on the spectrum of the infinite Jacobi matrix whose parallels to the diagonal are positive constant and whose diagonal depends linearly on the index. By using the knowledge of correspondence between function $\mathfrak{F}$ and the Bessel function of the first kind, we prove the spectrum of such matrices coincides with zeros of the Bessel function of the first kind as function of its order.

2 The function $\mathfrak{F}$

We introduce a function $\mathfrak{F}$ defined on a subset of the linear space formed by all complex sequences $x = \{x_k\}_{k=1}^{\infty}$.

Definition 1. Define $\mathfrak{F} : D \to \mathbb{C}$,

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \dots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \dots x_{k_m} x_{k_m+1}$$

where

$$D = \left\{ \{x_k\}_{k=1}^{\infty};\ \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}.$$

For a finite number of complex variables we identify $\mathfrak{F}(x_1, x_2, \dots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$. By convention, we also put $\mathfrak{F}(\emptyset) = 1$ where $\emptyset$ is the empty sequence.

Remark 2. Note that the domain D is not a linear space. One has, however, $\ell^2(\mathbb{N}) \subset D$.

Obviously, if all but finitely many elements of a sequence x are zeroes then $\mathfrak{F}(x)$ reduces to a finite sum. Thus one can introduce some simple examples.

Example 3.

$$\mathfrak{F}(x_1) = 1,\ \mathfrak{F}(x_1, x_2) = 1 - x_1x_2,\ \mathfrak{F}(x_1, x_2, x_3) = 1 - x_1x_2 - x_2x_3,$$
$$\mathfrak{F}(x_1, x_2, x_3, x_4) = 1 - x_1x_2 - x_2x_3 - x_3x_4 + x_1x_2x_3x_4,\ \text{etc.}$$

Let T denote the truncation operator from the left defined on the space of all sequences:

$$T(\{x_k\}_{k=1}^{\infty}) = \{x_{k+1}\}_{k=1}^{\infty}.$$

T^n, $n = 0, 1, 2, \ldots$, stands for a power of T. Hence $T^n(\{x_k\}_{k=1}^{\infty}) = \{x_{k+n}\}_{k=1}^{\infty}$.

Proposition 4. *For all $x \in D$ one has*

$$\mathfrak{F}(x) = \mathfrak{F}(Tx) - x_1x_2\,\mathfrak{F}(T^2x). \tag{1}$$

Particularly, if $n \geq 2$ then

$$\mathfrak{F}(x_1, x_2, x_3, \ldots, x_n) = \mathfrak{F}(x_2, x_3, \ldots, x_n) - x_1x_2\,\mathfrak{F}(x_3, \ldots, x_n). \tag{2}$$

Proof. Considering that

$$\mathfrak{F}(T^nx) = 1 + \sum_{m=1}^{\infty}(-1)^m \sum_{k_1=n+1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1}x_{k_1+1}x_{k_2}x_{k_2+1}\ldots x_{k_m}x_{k_m+1}$$

one has

$$\mathfrak{F}(x) - \mathfrak{F}(T^1x) = -x_1x_2 + \sum_{m=2}^{\infty}(-1)^m \sum_{k_2=3}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_1x_2x_{k_2}x_{k_2+1}\ldots x_{k_m}x_{k_m+1}$$
$$= -x_1x_2\mathfrak{F}(T^2x).$$

□

Remark 5. Clearly, given that $\mathfrak{F}(\emptyset) = \mathfrak{F}(x_1) = 1$, relation (2) determines recursively and unambiguously $\mathfrak{F}(x_1, \ldots, x_n)$ for any finite number of variables $n \in \mathbb{Z}_+$ (including $n = 0$).

Remark 6. One readily verifies that

$$\mathfrak{F}(x_1, x_2, \ldots, x_n) = \mathfrak{F}(x_n, \ldots, x_2, x_1). \tag{3}$$

Hence equality (2) implies, again for $n \geq 2$,

$$\mathfrak{F}(x_1, \ldots, x_{n-2}, x_{n-1}, x_n) = \mathfrak{F}(x_1, \ldots, x_{n-2}, x_{n-1}) - x_{n-1}x_n\,\mathfrak{F}(x_1, \ldots, x_{n-2}). \tag{4}$$

Remark 7. For a given $x \in D$ such that $\mathfrak{F}(x) \neq 0$ let us introduce sequences $\{P_k\}_{k=0}^{\infty}$ and $\{Q_k\}_{k=0}^{\infty}$ by $P_0 = 0$ and $P_k = \mathfrak{F}(x_2, \ldots, x_k)$ for $k \geq 1$, $Q_k = \mathfrak{F}(x_1, \ldots, x_k)$ for $k \geq 0$. According to (4), the both sequences obey the difference equation

$$Y_{k+1} = Y_k - x_kx_{k+1}Y_{k-1},\ k = 1, 2, 3, \ldots,$$

with the initial conditions $P_0 = 0$, $P_1 = 1$, $Q_0 = Q_1 = 1$, and define the infinite continued fraction

$$\frac{\mathfrak{F}(Tx)}{\mathfrak{F}(x)} = \lim_{k\to\infty} \frac{P_k}{Q_k} = \cfrac{1}{1 - \cfrac{x_1x_2}{1 - \cfrac{x_2x_3}{1 - \cfrac{x_3x_4}{1 - \dots}}}}.$$

Proposition 4 admits a generalization.

Proposition 8. *For every $x \in D$ and $k \in \mathbb{N}$ one has*

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \dots, x_k)\, \mathfrak{F}(T^k x) - \mathfrak{F}(x_1, \dots, x_{k-1}) x_k x_{k+1}\, \mathfrak{F}(T^{k+1}x). \tag{5}$$

Proof. Let us proceed by induction in k. For $k = 1$, equality (5) coincides with (1). Suppose (5) is true for $k \in \mathbb{N}$. Applying Proposition 4 to the sequence $T^k x$ and using (4) one finds that the RHS of (5) equals

$$\begin{aligned}
&\mathfrak{F}(x_1, \dots, x_k)\, \mathfrak{F}(T^{k+1}x) - \mathfrak{F}(x_1, \dots, x_k) x_{k+1}x_{k+2}\, \mathfrak{F}(T^{k+2}x) \\
&\quad - \mathfrak{F}(x_1, \dots, x_{k-1}) x_k x_{k+1}\, \mathfrak{F}(T^{k+1}x) \\
&= \mathfrak{F}(x_1, \dots, x_k, x_{k+1})\, \mathfrak{F}(T^{k+1}x) - \mathfrak{F}(x_1, \dots, x_k) x_{k+1}x_{k+2}\, \mathfrak{F}(T^{k+2}x).
\end{aligned}$$

This concludes the verification. □

Remark 9. With the aid of Proposition 4 one can rewrite equality (5) as follows

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \dots, x_k)\, \mathfrak{F}\left(\frac{\mathfrak{F}(x_1, \dots, x_{k-1})}{\mathfrak{F}(x_1, \dots, x_k)}\, x_k, x_{k+1}, x_{k+2}, x_{k+3}, \dots \right). \tag{6}$$

Proposition 10. *The function $\mathfrak{F}$ is a continuous functional on $\ell^2(\mathbb{N})$.*

Proof. Let $x_n \equiv \{x_k^n\}_{k=1}^\infty$, $x \equiv \{x_k\}_{k=1}^\infty \in \ell^2(\mathbb{N})$ such that

$$\|x - x_n\|_2^2 \equiv \sum_{k=1}^\infty |x_k - x_k^n|^2 \to 0$$

with $n \to \infty$. Since $x_n \to x$ in $\ell^2(\mathbb{N})$ and $\sum_{k=1}^\infty |x_k x_{k+1}| \le \|x\|_2^2$ there exist constants $N, n_0 \in \mathbb{N}$ and $0 < L < 1$ such that

$$\sum_{k=N}^\infty |x_k x_{k+1}| \le L$$

and

$$\sum_{k=N}^\infty |x_k^n x_{k+1}^n| \le L$$

for all $n \geq n_0$. Let us examine the difference $\left|\mathfrak{F}(T^N x) - \mathfrak{F}(T^N x_n)\right|$ which is equal to

$$
\begin{aligned}
&\Bigg|\sum_{m=1}^{\infty}(-1)^m\Bigg[\sum_{k_1=N+1}^{\infty}\sum_{k_2=k_1+2}^{\infty}\cdots\sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1}x_{k_1+1}x_{k_2}x_{k_2+1}\dots x_{k_m}x_{k_m+1} \\
&\qquad - \sum_{k_1=N+1}^{\infty}\sum_{k_2=k_1+2}^{\infty}\cdots\sum_{k_m=k_{m-1}+2}^{\infty} x^n_{k_1}x^n_{k_1+1}x^n_{k_2}x^n_{k_2+1}\dots x^n_{k_m}x^n_{k_m+1}\Bigg]\Bigg| \\
&\leq \sum_{m=1}^{\infty}\Bigg[\sum_{k_1=N+1}^{\infty}|x_{k_1}x_{k_1+1} - x^n_{k_1}x^n_{k_1+1}|\sum_{k_2=k_1+2}^{\infty}|x_{k_2}x_{k_2+1}|\cdots\sum_{k_m=k_{m-1}+2}^{\infty}|x_{k_m}x_{k_m+1}| \\
&+ \sum_{k_1=N+1}^{\infty}|x^n_{k_1}x^n_{k_1+1}|\sum_{k_2=k_1+2}^{\infty}|x_{k_2}x_{k_2+1} - x^n_{k_2}x^n_{k_2+1}|\cdots\sum_{k_m=k_{m-1}+2}^{\infty}|x_{k_m}x_{k_m+1}| \\
&+ \dots + \sum_{k_1=N+1}^{\infty}|x^n_{k_1}x^n_{k_1+1}|\sum_{k_2=k_1+2}^{\infty}|x^n_{k_2}x^n_{k_2+1}|\cdots\sum_{k_m=k_{m-1}+2}^{\infty}|x_{k_m}x_{k_m+1} - x^n_{k_m}x^n_{k_m+1}|\Bigg].
\end{aligned}
$$

Consequently, one has, for $n \geq n_0$,

$$\left|\mathfrak{F}(T^N x) - \mathfrak{F}(T^N x_n)\right| \leq \sum_{m=1}^{\infty} mL^{m-1}\sum_{k=1}^{\infty}|x_k x_{k+1} - x^n_k x^n_{k+1}|.$$

Next, by considering the inequality

$$\sum_{k=1}^{\infty}|x_k x_{k+1} - x^n_k x^n_{k+1}| \leq \|x\|_2\|x - x_n\|_2 + \|x_n\|_2\|x - x_n\|_2$$

one finds out

$$\lim_{n\to\infty}\mathfrak{F}(T^N x_n) = \mathfrak{F}(T^N x).$$

Finally, by (5), one gets

$$\mathfrak{F}(x_n) = \mathfrak{F}(x^n_1,\dots,x^n_N)\,\mathfrak{F}(T^N x_n) - \mathfrak{F}(x^n_1,\dots,x^n_{N-1})x^n_N x^n_{N+1}\,\mathfrak{F}(T^{N+1}x_n).$$

To conclude the proof it is sufficient to send n to infinity and use (5) again. □

Later on, we shall also need the following identity.

Lemma 11. *For any $n \in \mathbb{N}$ one has*

$$
\begin{aligned}
&u_1\mathfrak{F}(u_2,u_3,\dots,u_n)\mathfrak{F}(v_1,v_2,v_3,\dots,v_n) - v_1\mathfrak{F}(u_1,u_2,u_3,\dots,u_n)\mathfrak{F}(v_2,v_3,\dots,v_n) \\
&= \sum_{j=1}^{n}\left(\prod_{k=1}^{j-1}u_k v_k\right)(u_j - v_j)\,\mathfrak{F}(u_{j+1},u_{j+2},\dots,u_n)\mathfrak{F}(v_{j+1},v_{j+2},\dots,v_n). \qquad (7)
\end{aligned}
$$

Proof. The equality can be readily proved by induction in n with the aid of (2). □

Example 12. Let $t, w \in \mathbb{C}$, $|t| < 1$. By using the identity

$$\sum_{k_1=n}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} t^{2k_1-1} t^{2k_2-1} \ldots t^{2k_m-1} = \frac{t^{m(2m-3)} t^{2mn}}{(1-t^2)(1-t^4)\ldots(1-t^{2m})}$$

which can be verified by induction in m, one arrives at the equality

$$\mathfrak{F}\big(\{t^{k-1}w\}_{k=1}^{\infty}\big) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{t^{m(2m-1)} w^{2m}}{(1-t^2)(1-t^4)\cdots(1-t^{2m})} . \tag{8}$$

This function can be identified with a basic hypergeometric series (also called q-hypergeometric series) defined by

$${}_r\phi_s(a; b; q, z) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \ldots (a_r; q)_k}{(b_1; q)_k \ldots (b_s; q)_k} \left((-1)^k q^{\frac{1}{2}k(k-1)}\right)^{1+s-r} \frac{z^k}{(q; q)_k}$$

where $r, s \in \mathbb{Z}_+$ (nonnegative integers) and

$$(\alpha; q)_k = \prod_{j=0}^{k-1} \left(1 - \alpha q^j\right), \; k = 0, 1, 2, \ldots,$$

see [5]. In fact, the RHS in (8) equals ${}_0\phi_1(; 0; t^2, -tw^2)$ where

$${}_0\phi_1(; 0; q, z) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)}}{(q; q)_k} z^k = \sum_{k=0}^{\infty} \frac{q^{k(k-1)}}{(1-q)(1-q^2)\ldots(1-q^k)} z^k,$$

with $q, z \in \mathbb{C}$, $|q| < 1$, and the recursive rule (1) takes the form

$${}_0\phi_1(; 0; q, z) = {}_0\phi_1(; 0; q, qz) + z\, {}_0\phi_1(; 0; q, q^2 z). \tag{9}$$

Put $e(q; z) = {}_0\phi_1(; 0; q, (1-q)z)$. Then $\lim_{q\uparrow 1} e(q; z) = \exp(z)$. Hence $e(q; z)$ can be regarded as a q-deformed exponential function though this is not the standard choice (compare with [5] or [6] and references therein). Equality (9) can be interpreted as the discrete derivative

$$\frac{e(q; z) - e(q; qz)}{(1-q)z} = e(q; q^2 z).$$

Moreover, in view of Remark 7, one has

$$\cfrac{1}{1 + \cfrac{z}{1 + \cfrac{qz}{1 + \cfrac{q^2 z}{1 + \ldots}}}} = \frac{{}_0\phi_1(; 0; q, qz)}{{}_0\phi_1(; 0; q, z)} .$$

This equality is related to the Rogers-Ramanujan identities, see the discussion in [3, Chp. 7].

Example 13. The Bessel functions of the first kind can be expressed in terms of function $\mathfrak{F}$. More precisely, for $\nu \notin -\mathbb{N}$, one has

$$J_\nu(2w) = \frac{w^\nu}{\Gamma(\nu+1)} \mathfrak{F}\left(\left\{\frac{w}{\nu+k}\right\}_{k=1}^{\infty}\right). \tag{10}$$

The recurrence relation (1) transforms to the well known identity

$$zJ_\nu(z) - 2(\nu+1)J_{\nu+1}(z) + zJ_{\nu+2}(z) = 0.$$

To prove (10) one has to show that

$$\begin{aligned}
&\sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} \\
&\quad\times \frac{1}{(\nu+k_1)(\nu+k_1+1)(\nu+k_2)(\nu+k_2+1)\ldots(\nu+k_m)(\nu+k_m+1)} \\
&= \frac{1}{m!\,(\nu+1)(\nu+2)\ldots(\nu+m)} = \frac{\Gamma(\nu+1)}{m!\,\Gamma(\nu+m+1)}.
\end{aligned}$$

The first equality is verified in the proof of Proposition 14 (it suffices to send n to infinity in (14)). And so

$$\frac{w^\nu}{\Gamma(\nu+1)} \mathfrak{F}\left(\left\{\frac{w}{n+k}\right\}_{k=1}^{\infty}\right) = \sum_{m=0}^{\infty} (-1)^m \frac{w^{2m+\nu}}{m!\,\Gamma(\nu+m+1)},$$

as claimed. Furthermore, Remark 7 provides us with the infinite fraction

$$\frac{\nu+1}{w} \frac{J_{\nu+1}(2w)}{J_\nu(2w)} = \cfrac{1}{1 - \cfrac{\cfrac{w^2}{(\nu+1)(\nu+2)}}{1 - \cfrac{\cfrac{w^2}{(\nu+2)(\nu+3)}}{1 - \cfrac{\cfrac{w^2}{(\nu+3)(\nu+4)}}{1 - \ldots}}}}.$$

This can be rewritten as

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = \cfrac{z}{2(\nu+1) - \cfrac{z^2}{2(\nu+2) - \cfrac{z^2}{2(\nu+3) - \cfrac{z^2}{2(\nu+4) - \ldots}}}}.$$

Comparing to Example 13, one can also find the value of $\mathfrak{F}$ on the truncated sequence $\{w/(\nu+k)\}_{k=1}^{n}$.

Proposition 14. *For $n \in \mathbb{Z}_+$ and $\nu \in \mathbb{C} \setminus \{-n, -n+1, \dots, -1\}$ one has*

$$\mathfrak{F}\left(\frac{w}{\nu+1}, \frac{w}{\nu+2}, \dots, \frac{w}{\nu+n}\right) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+n+1)} \sum_{s=0}^{[n/2]} (-1)^s \frac{(n-s)!}{s!\,(n-2s)!}\, w^{2s} \prod_{j=s}^{n-1-s} (\nu+n-j). \tag{11}$$

In particular, for $m, n \in \mathbb{Z}_+$, $m \leq n$, one has

$$\mathfrak{F}\left(\frac{w}{m+1}, \frac{w}{m+2}, \dots, \frac{w}{n}\right) = \frac{m!}{n!} \sum_{s=0}^{[(n-m)/2]} (-1)^s \frac{(n-s)!\,(n-m-s)!}{s!\,(m+s)!\,(n-m-2s)!}\, w^{2s}. \tag{12}$$

Proof. Firstly, the equality

$$\begin{aligned}\sum_{k=1}^{n} &\frac{(n+1-k)(n+2-k)\dots(n+s-1-k)}{(\nu+k)(\nu+k+1)\dots(\nu+k+s)}\\ &= \frac{n\,(n+1)\dots(n+s-1)}{s\,(\nu+n+s)\,(\nu+1)(\nu+2)\dots(\nu+s)}\end{aligned} \tag{13}$$

holds for all $n \in \mathbb{Z}_+$, $\nu \in \mathbb{C}$, $\nu \notin -\mathbb{N}$, and $s \in \mathbb{N}$. To show (13) one can proceed by induction in s. The case $s = 1$ is easy to verify. For the induction step from $s-1$ to s, with $s > 1$, let us denote the LHS of (13) by $Y_s(\nu, n)$. One observes that

$$Y_s(\nu, n) = \frac{\nu+n+s-1}{s}\, Y_{s-1}(\nu, n) - \frac{\nu+n+2s-1}{s}\, Y_{s-1}(\nu+1, n).$$

Applying the induction hypothesis the equality readily follows.

Next one shows that

$$\begin{aligned}&\sum_{k_1=1}^{n-2s+2} \sum_{k_2=k_1+2}^{n-2s+4} \cdots \sum_{k_s=k_{s-1}+2}^{n}\\ &\quad\times \frac{1}{(\nu+k_1)(\nu+k_1+1)(\nu+k_2)(\nu+k_2+1)\dots(\nu+k_s)(\nu+k_s+1)}\\ &= \frac{(n-2s+2)(n-2s+3)\dots(n-s+1)}{s!\,(\nu+1)(\nu+2)\dots(\nu+s)\,(\nu+n-s+2)(\nu+n-s+3)\dots(\nu+n+1)}\end{aligned} \tag{14}$$

holds for all $n \in \mathbb{Z}_+$, $s \in \mathbb{N}$, $2s \leq n+2$. To this end, we again proceed by induction in s. The case $s = 1$ is easy to verify. In the induction step from $s-1$ to s, with $s > 1$, one applies the induction hypothesis to the LHS of (14) and arrives at the expression

$$\begin{aligned}&\sum_{k=1}^{n-2s+2} \frac{1}{(\nu+k)(\nu+k+1)\,(s-1)!}\\ &\times \frac{(n-k-2s+3)(n-k-2s+4)\dots(n-k-s+1)}{(\nu+k+2)(\nu+k+3)\dots(\nu+k+s)\,(\nu+n-s+3)(\nu+n-s+4)\dots(\nu+n+1)}.\end{aligned}$$

Using (13) one obtains the RHS of (14), as claimed.

Finally, to conclude the proof, it suffices to notice that

$$\mathfrak{F}\left(\frac{w}{\nu+1},\frac{w}{\nu+2},\dots,\frac{w}{\nu+n}\right) = 1+\sum_{s=1}^{[n/2]}(-1)^s\sum_{k_1=1}^{n-2s+1}\sum_{k_2=k_1+2}^{n-2s+3}\cdots\sum_{k_s=k_{s-1}+2}^{n-1}$$
$$\times\frac{w^{2s}}{(\nu+k_1)(\nu+k_1+1)(\nu+k_2)(\nu+k_2+1)\dots(\nu+k_s)(\nu+k_s+1)}$$

and to use equality (14). ☐

One can complete Proposition 14 with another relation to Bessel functions.

Proposition 15. *For $m,n\in\mathbb{Z}_+$, $m\le n$, one has*

$$\pi J_m(2w)Y_{n+1}(2w) = -\frac{n!}{m!}\,w^{m-n-1}\,\mathfrak{F}\left(\frac{w}{m+1},\frac{w}{m+2},\dots,\frac{w}{n}\right) \tag{15}$$
$$-\sum_{s=0}^{m-1}\frac{(m-s-1)!\,(n-m+2s+1)!}{s!\,(n+s+1)!\,(n-m+s+1)!}\,w^{n-m+2s+1}+O\big(w^{m+n+1}\log(w)\big).$$

Proof. Recall the following two facts from the theory of Bessel functions (see, for instance, [4, Chapter VII]). Firstly, for $\mu,\nu\notin-\mathbb{N}$, one has

$$J_\mu(z)J_\nu(z)=\sum_{s=0}^{\infty}(-1)^s\,\frac{(s+\mu+\nu+1)_s}{s!\,\Gamma(\mu+s+1)\Gamma(\nu+s+1)}\left(\frac{z}{2}\right)^{\mu+\nu+2s}$$

where $(a)_s=a(a+1)\dots(a+s-1)$ is the Pochhammer symbol. Secondly, for $n\in\mathbb{Z}_+$,

$$\pi Y_n(z)=\frac{\partial}{\partial\nu}\left(J_\nu(z)-(-1)^nJ_{-\nu}(z)\right)\Bigg|_{\nu=n}.$$

For $m,n\in\mathbb{Z}_+$, $m\le n$, a straightforward computation based on these facts yields

$$\pi J_m(z)Y_n(z) = -\sum_{s=0}^{[(n-m-1)/2]}(-1)^s\,\frac{(n-s-1)!\,(n-m-s-1)!}{s!\,(m+s)!\,(n-m-2s-1)!}\left(\frac{z}{2}\right)^{m-n+2s}$$
$$-\sum_{s=0}^{m-1}\frac{(m-s-1)!\,(n-m+2s)!}{s!\,(n+s)!\,(n-m+s)!}\left(\frac{z}{2}\right)^{n-m+2s}+2J_m(z)J_n(z)\log\left(\frac{z}{2}\right) \tag{16}$$
$$+\sum_{s=0}^{\infty}(-1)^s\,\frac{(m+n+2s)!}{s!\,(m+s)!\,(n+s)!\,(m+n+s)!}\left(\frac{z}{2}\right)^{m+n+2s}\Big(2\psi(m+n+2s+1)$$
$$-\psi(m+s+1)-\psi(n+s+1)-\psi(m+n+s+1)-\psi(s+1)\Big)$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$ is the digamma function. The proposition follows from (16) and (12). ☐

Remark 16. Note that the first term on the RHS of (15) contains only negative powers of w. One can extend (15) to the case $n=m-1$. Then

$$\pi J_m(2w)Y_m(2w)=-\sum_{s=0}^{m-1}\frac{(m-s-1)!\,(2s)!}{(s!)^2\,(m+s)!}\,w^{2s}+O\big(w^{2m}\log(w)\big).$$

3 The matrix $\mathfrak{J}$

In this section we introduce an infinite matrix $\mathfrak{J}$ that is basically determined by two simple properties – it is antisymmetric and its every row satisfies a second-order difference equation known from the theory of Bessel functions. Of course, in that case every column of the matrix satisfies the difference equation as well.

Lemma 17. *Suppose $w \in \mathbb{C} \setminus \{0\}$. The dimension of the vector space formed by infinite-dimensional matrices $A = \{A(m,n)\}_{m,n\in\mathbb{Z}}$ satisfying, for all $m, n \in \mathbb{Z}$,*

$$wA(m, n-1) - nA(m, n) + wA(m, n+1) = 0 \tag{17}$$

and

$$A(n, m) = -A(m, n), \tag{18}$$

equals 1*. Every such a matrix is unambiguously determined by the value $A(0,1)$, and one has*

$$\forall n \in \mathbb{Z},\ A(n, n+1) = A(0, 1). \tag{19}$$

Proof. Suppose A solves (17) and (18). Then $A(m,m) = 0$. Equating $m = n$ in (17) and using (18) one finds that $A(n, n+1) = -A(n, n-1) = A(n-1, n)$. Hence (19) is fulfilled. Clearly, the matrix A is unambiguously determined by the second-order difference equation (17) in n and by the initial conditions $A(m,m) = 0$, $A(m, m+1) = A(0,1)$, when m runs through $\mathbb{Z}$.

Conversely, choose $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Let A be the unique matrix determined by (17) and the initial conditions $A(m,m) = 0$, $A(m, m+1) = \lambda$. It suffices to show that A satisfies (18) as well. Note that $A(m, m-1) = -\lambda$. Furthermore,

$$\begin{aligned}&wA(m-1, m+1) - mA(m, m+1) + wA(m+1, m+1)\\ &= wA(m-1, m+1) - mA(m-1, m) + wA(m-1, m-1)\\ &= 0.\end{aligned}$$

From (17) and the initial conditions it follows that $A(m, m+2) = (m+1)\lambda/w$, and so $mA(m, m+2) = (m+1)A(m-1, m+1)$. Consequently,

$$\begin{aligned}&wA(m-1, m+2) - mA(m, m+2) + wA(m+1, m+2)\\ &= wA(m-1, m+2) - (m+1)A(m-1, m+1) + wA(m-1, m)\\ &= 0.\end{aligned}$$

One observes that, for a given $m \in \mathbb{Z}$, the sequence

$$x_n = -A(m-1, n) + \frac{m}{w}\, A(m, n),\ n \in \mathbb{Z},$$

solves the difference equation

$$wx_{n-1} - nx_n + wx_{n+1} = 0 \tag{20}$$

with the initial conditions $x_{m+1} = A(m+1, m+1)$, $x_{m+2} = A(m+1, m+2)$. By the uniqueness, $x_n = A(m+1, n)$. This means that, for all $m, n \in \mathbb{Z}$,

$$wA(m-1, n) - mA(m, n) + wA(m+1, n) = 0.$$

Put $B(m, n) = -A(n, m)$. Then B fulfills (17) and $B(m, m) = 0$, $B(m, m+1) = \lambda$. Whence $B = A$. □

Lemma 18. *Suppose $w \in \mathbb{C} \setminus \{0\}$. If a matrix $A = \{A(m, n)\}_{m,n\in\mathbb{Z}}$ satisfies (17) and (18) then*

$$\forall m, n \in \mathbb{Z},\ A(m, -n) = (-1)^n A(m, n),\ A(-m, n) = (-1)^m A(m, n). \tag{21}$$

Proof. For any sequence $\{x_n\}_{n\in\mathbb{Z}}$ satisfying the difference equation (20) one can verify, by mathematical induction, that $x_{-n} = (-1)^n x_n$, $n = 0, 1, 2, \ldots$. The second equality in (21) follows directly from the first one and property (18). □

Definition 19. For a given parameter $w \in \mathbb{C} \setminus \{0\}$ let $\mathfrak{J} = \{\mathfrak{J}(m, n)\}_{m,n\in\mathbb{Z}}$ denote the unique matrix satisfying (17), (18) with $\mathfrak{J}(0, 1) = 1$.

Remark 20. Here are several particular entries of the matrix $\mathfrak{J}$,

$$\mathfrak{J}(m, m) = 0,\ \mathfrak{J}(m, m+1) = 1,\ \mathfrak{J}(m, m+2) = \frac{m+1}{w},\ \mathfrak{J}(m, m+3) = \frac{(m+1)(m+2)}{w^2} - 1,$$

with $m \in \mathbb{Z}$. Some other particular values follow from (18) and (21). Below, in Proposition 24, we derive a general formula for $\mathfrak{J}(m, n)$.

Lemma 21. *For $0 \le m < n$ one has (with the convention $\mathfrak{F}(\emptyset)=1$)*

$$\mathfrak{J}(m, n) = \frac{(n-1)!}{m!}\, w^{m-n+1}\, \mathfrak{F}\!\left(\frac{w}{m+1}, \frac{w}{m+2}, \ldots, \frac{w}{n-1}\right). \tag{22}$$

Proof. The RHS of (22) equals 1 for $n = m+1$, and $(m+1)/w$ for $n = m+2$. Moreover, in view of (4), the RHS satisfies the difference equation (20) in the index n. □

Remark 22. From (22) and (10) it follows that

$$\forall m \in \mathbb{Z},\ \lim_{n\to\infty} \frac{w^{n-1}}{(n-1)!}\, \mathfrak{J}(m, n) = J_m(2w).$$

This is in agreement with the well known fact that, for any $w \in \mathbb{C}$, the sequence $\{J_n(2w)\}_{n\in\mathbb{Z}}$ fulfills the second-order difference equation (20).

Remark 23. Rephrasing Proposition 15 and Remark 16 one has, for $m, n \in \mathbb{Z}_+$, $m \le n$,

$$\begin{aligned}\pi J_m(2w) Y_n(2w) &= -w^{-1}\mathfrak{J}(m, n) - \sum_{s=0}^{m-1} \frac{(m-s-1)!\,(n-m+2s)!}{s!\,(n+s)!\,(n-m+s)!}\, w^{n-m+2s} \\ &\quad + O\big(w^{m+n}\log(w)\big).\end{aligned}$$

Since, by definition, the matrix $\mathfrak{J}$ is antisymmetric it suffices to determine the values $\mathfrak{J}(m,n)$ for $m \le n$, $m,n \in \mathbb{Z}$.

Proposition 24. *For $m,n \in \mathbb{Z}$, $m \le n$, one has*

$$\mathfrak{J}(m,n) = \sum_{s=0}^{[(n-m-1)/2]} (-1)^s \binom{n-s-1}{n-m-2s-1} \frac{(n-m-s-1)!}{s!}\, w^{m-n+2s+1}. \tag{23}$$

Proof. We distinguish several cases. First, consider the case $0 \le m < n$. Then (23) follows from (22) and (12). Observe also that for $m = n$, $m,n \in \mathbb{Z}$, the RHS of (23) is an empty sum and so the both sides in (23) are equal to 0.

Second, consider the case $m \le 0 \le n$. Put $m = -k$, $k \in \mathbb{Z}_+$. The RHS of (23) becomes

$$\sum_{s=0}^{[(n+k-1)/2]} (-1)^s \binom{n-s-1}{n+k-2s-1} \frac{(n+k-s-1)!}{s!}\, w^{-k-n+2s+1}. \tag{24}$$

Suppose $k \le n$. Then the summands in (24) vanish for $s = 0,1,\ldots,k-1$, and so the sum equals

$$\sum_{s=0}^{[(n-k-1)/2]} (-1)^{s+k} \frac{(n-k-s-1)!}{(n-k-2s-1)!\,s!} \frac{(n-s-1)!}{(s+k)!}\, w^{k-n+2s+1}.$$

By the first step, this expression is equal to $(-1)^k\mathfrak{J}(k,n) = \mathfrak{J}(-k,n)$ (see Lemma 18). Further, suppose $k \ge n$. Then the summands in (24) vanish for $s = 0,1,\ldots,n-1$, and so the sum equals

$$\sum_{s=0}^{[(k-n-1)/2]} (-1)^{n+s} \binom{-s-1}{k-n-2s-1} \frac{(k-s-1)!}{(n+s)!}\, w^{n-k+2s+1}.$$

Using once more the first step, this expression is readily seen to be equal to $(-1)^{k+1}\mathfrak{J}(n,k) = \mathfrak{J}(-k,n)$.

Finally, consider the case $m \le n \le 0$. Put $m = -k$, $n = -\ell$, $k,\ell \in \mathbb{Z}_+$. Hence $0 \le \ell \le k$. The RHS of (23) becomes

$$\sum_{s=0}^{[(k-\ell-1)/2]} (-1)^s \binom{-\ell-s-1}{k-\ell-2s-1} \frac{(k-\ell-s-1)!}{s!}\, w^{\ell-k+2s+1}.$$

Using again the first step, this expression is readily seen to be equal to $(-1)^{k+\ell+1}\mathfrak{J}(\ell,k)$ $= \mathfrak{J}(-k,-\ell)$. □

4 The characteristic function for the antisymmetric diagonal

First a simple relationship between the function $\mathfrak{F}$ applied to a finite sequence and the determinant of a Jacobi matrix is introduced.

Proposition 25. *For $d \in \mathbb{N}$ and $\{a_j\}_{j=1}^d \subset \mathbb{C}$, one has*

$$\mathfrak{F}(a_1, a_2, \ldots, a_d) = \left| \begin{pmatrix} 1 & a_1 & & & & \\ a_2 & 1 & a_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{d-2} & 1 & a_{d-1} \\ & & & & a_{d-1} & 1 \end{pmatrix} \right|. \tag{25}$$

Proof. The case $d = 1, 2$ is easy to verify. Denote the RHS of (25) by $D(a_1, a_2, \ldots, a_d)$. By expanding $D(a_1, a_2, \ldots, a_d)$ along the first row one finds out the recurrence rule

$$D(a_1, a_2, \ldots, a_d) = D(a_2, a_3, \ldots, a_d) - a_1 a_2 D(a_3, a_4, \ldots, a_d)$$

holds which, according to Remark 5, concludes the proof. □

Corollary 26. *For $d \in \mathbb{N}$, $w \in \mathbb{C}$ and $\{\lambda_j\}_{j=1}^d \subset \mathbb{C} \setminus \{0\}$, one has*

$$\left(\prod_{j=1}^{d} \lambda_j \right) \mathfrak{F}\left(\frac{w}{\lambda_1}, \frac{w}{\lambda_2}, \ldots, \frac{w}{\lambda_d} \right) = \left| \begin{pmatrix} \lambda_1 & w & & & & \\ w & \lambda_2 & w & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & w & \lambda_{d-1} & w \\ & & & & w & \lambda_d \end{pmatrix} \right|. \tag{26}$$

Proof. It suffices to put $a_j = \frac{w}{\lambda_j}$ in (25) and adjust the determinant. □

For a given $d \in \mathbb{Z}_+$ let $E_\pm$ denote the $(2d+1) \times (2d+1)$ matrix with units on the upper (lower) parallel to the diagonal and with all other entries equal to zero. Hence

$$(E_+)_{j,k} = \delta_{j+1,k}, \ (E_-)_{j,k} = \delta_{j,k+1}, \ j, k = -d, -d+1, -d+2, \ldots, d.$$

For $y = (y_{-d}, y_{-d+1}, y_{-d+2}, \ldots, y_d) \in \mathbb{C}^{2d+1}$ let $\operatorname{diag}(y)$ denote the diagonal $(2d+1) \times (2d+1)$ matrix with the sequence y on the diagonal. Everywhere in what follows, I stands for a unit matrix.

Next a more complicated formula, which is needed later, is presented for the determinant of a Jacobi matrix with a general diagonal but with constant neighboring parallels to the diagonal. As explained in the subsequent remark, however, this formula can be extended to the general case with the aid of a simple decomposition of the Jacobi matrix in question.

Proposition 27. *For $d \in \mathbb{N}$, $w \in \mathbb{C}$ and $y = (y_{-d}, y_{-d+1}, y_{-d+2}, \ldots, y_d) \in \mathbb{C}^{2d+1}$, $\prod_{k=1}^d y_k y_{-k} \neq 0$, one has*

$$\det\big(\operatorname{diag}(y) + wE_+ + wE_-\big) = \left(\prod_{k=1}^{d} y_k y_{-k} \right) \bigg[y_0 \, \mathfrak{F}\left(\frac{w}{y_1}, \ldots, \frac{w}{y_d} \right) \mathfrak{F}\left(\frac{w}{y_{-1}}, \ldots, \frac{w}{y_{-d}} \right)$$
$$- \frac{w^2}{y_1} \mathfrak{F}\left(\frac{w}{y_2}, \ldots, \frac{w}{y_d} \right) \mathfrak{F}\left(\frac{w}{y_{-1}}, \ldots, \frac{w}{y_{-d}} \right) - \frac{w^2}{y_{-1}} \mathfrak{F}\left(\frac{w}{y_1}, \ldots, \frac{w}{y_d} \right) \mathfrak{F}\left(\frac{w}{y_{-2}}, \ldots, \frac{w}{y_{-d}} \right) \bigg]. \tag{27}$$

Proof. Let us proceed by induction in d. The case $d = 1$ is easy to verify. Put $\mathcal{N}_d(w; y) = \det\big(\operatorname{diag}(y) + wE_+ + wE_-\big)$. Suppose (27) is true for some $d \geq 1$. For given $w \in \mathbb{C}$ and $y \in \mathbb{C}^{2d+3}$ consider the quantity $\mathcal{N}_{d+1}(w; y)$. Let us split the corresponding $(2d + 3) \times (2d + 3)$ Jacobi matrix into four blocks by splitting the set of indices into two disjoint sets $\{-d - 1, d + 1\}$ and $\{-d, -d + 1, -d + 2, \ldots, d\}$. Applying the rule

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B)$$

(see Appendix) one derives the recurrence relation

$$\mathcal{N}_{d+1}(w; y_{-d-1}, y_{-d}, y_{-d+1}, \ldots, y_d, y_{d+1}) = y_{d+1}y_{-d-1}\mathcal{N}_d(w; y'_{-d}, y_{-d+1}, y_{-d+2}, \ldots, y_{d-1}, y'_d)$$

where

$$y'_d = \left(1 - \frac{w^2}{y_d y_{d+1}}\right) y_d,\ y'_{-d} = \left(1 - \frac{w^2}{y_{-d} y_{-d-1}}\right) y_{-d}.$$

Now it is sufficient to use the induction hypothesis jointly with the equality

$$(1 - x_{n-1}x_n)\,\mathfrak{F}\left(x_1, x_2, \ldots, x_{n-2}, \frac{x_{n-1}}{1 - x_{n-1}x_n}\right) = \mathfrak{F}(x_1, x_2, \ldots, x_{n-1}, x_n)$$

which is valid for $n \geq 2$ and which follows from relations (3) and (6), with $k = 2$. □

Remark 28. Let us consider a general finite symmetric Jacobi matrix J of the form

$$J = \begin{pmatrix} \lambda_1 & w_1 & & & & \\ w_1 & \lambda_2 & w_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & w_{n-2} & \lambda_{n-1} & w_{n-1} \\ & & & & w_{n-1} & \lambda_n \end{pmatrix}$$

such that $\prod_{k=1}^{n-1} w_k \neq 0$. The Jacobi matrix can be decomposed into the product

$$J = G\tilde{J}G \tag{28}$$

where $G = \operatorname{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ is a diagonal matrix and $\tilde{J}$ is a Jacobi matrix with all units on the neighboring parallels to the diagonal,

$$\tilde{J} = \begin{pmatrix} \tilde{\lambda}_1 & 1 & & & & \\ 1 & \tilde{\lambda}_2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & \tilde{\lambda}_{n-1} & 1 \\ & & & & 1 & \tilde{\lambda}_n \end{pmatrix}.$$

Hence $\det(J) = \left(\prod_{k=1}^{n} \gamma_k^2\right) \det(\tilde{J})$, and one can employ formula (27) (in the case of odd dimension) or (26) to evaluate $\det(\tilde{J})$. In more detail, one can put

$$\gamma_{2k-1} = \prod_{j=1}^{k-1} \frac{w_{2j}}{w_{2j-1}}\,,\ \gamma_{2k} = w_1 \prod_{j=1}^{k-1} \frac{w_{2j+1}}{w_{2j}}\,,\ k = 1, 2, 3, \ldots. \tag{29}$$

Alternatively, the sequence $\{\gamma_k\}_{k=1}^{n}$ is defined recursively by $\gamma_1 = 1$, $\gamma_{k+1} = w_k/\gamma_k$. Furthermore, $\tilde{\lambda}_k = \lambda_k/\gamma_k^2$. With this choice, (28) is clearly true.

The characteristic function of a general finite symmetric Jacobi matrix can be also expressed with the aid of $\mathfrak{F}$.

Proposition 29. *Let J be the Jacobi matrix defined in Remark 28. Then it holds*

$$\det(J - zI) = \left(\prod_{k=1}^{d} (\lambda_k - \gamma_k^2 z)\right) \mathfrak{F}\!\left(\frac{\gamma_1^2}{\lambda_1 - \gamma_1^2 z}, \frac{\gamma_2^2}{\lambda_2 - \gamma_2^2 z}, \ldots, \frac{\gamma_d^2}{\lambda_d - \gamma_d^2 z}\right) \tag{30}$$

where $z \in \mathbb{C}$ and the sequence $\{\gamma_k\}_{k=1}^{d}$ is defined in (29). In the case when $w_k = w \in \mathbb{C}$, for $k = 1, 2, \ldots, d$, one has

$$\det(J - zI) = \left(\prod_{k=1}^{d} (\lambda_k - z)\right) \mathfrak{F}\!\left(\frac{w}{\lambda_1 - z}, \frac{w}{\lambda_2 - z}, \ldots, \frac{w}{\lambda_d - z}\right). \tag{31}$$

Proof. Equality (31) follows from (26). Next, in view of Remark 28, one finds out $\det(J - zI)$ is equal to

$$\begin{aligned}
&= \left(\prod_{k=1}^{d} \gamma_k^2\right) \det(\tilde{J} - zI) = \left(\prod_{k=1}^{d} \gamma_k^2(\tilde{\lambda}_k - z)\right) \mathfrak{F}\!\left(\frac{1}{\tilde{\lambda}_1 - z}, \frac{1}{\tilde{\lambda}_2 - z}, \ldots, \frac{1}{\tilde{\lambda}_d - z}\right) \\
&= \left(\prod_{k=1}^{d} (\lambda_k - \gamma_k^2 z)\right) \mathfrak{F}\!\left(\frac{\gamma_1^2}{\lambda_1 - \gamma_1^2 z}, \frac{\gamma_2^2}{\lambda_2 - \gamma_2^2 z}, \ldots, \frac{\gamma_d^2}{\lambda_d - \gamma_d^2 z}\right).
\end{aligned}$$

□

Next we aim to derive another formula for the characteristic function of a Jacobi matrix with an antisymmetric diagonal. Suppose $\lambda = (\lambda_{-d}, \lambda_{-d+1}, \lambda_{-d+2}, \ldots, \lambda_d) \in \mathbb{C}^{2d+1}$ and $\lambda_{-k} = -\lambda_k$ for $-d \le k \le d$; in particular, $\lambda_0 = 0$. We consider the Jacobi matrix $K = \operatorname{diag}(\lambda) + wE_+ + wE_-$. Let us denote, temporarily, by S the diagonal matrix with alternating signs on the diagonal, $S = \operatorname{diag}(1, -1, 1, \ldots, 1)$, and by Q the permutation matrix with the entries $Q_{j,k} = \delta_{j+k,0}$ for $-d \le j, k \le d$. The commutation relations

$$SQKQS = -K,\ S^2 = Q^2 = I,$$

imply

$$\det(K - zI) = \det\big(SQ(K - zI)QS\big) = -\det(K + zI).$$

Hence the characteristic function of K is an odd polynomial in the variable z. This can be also seen from the explicit formula (32) derived below.

Proposition 30. *Suppose $d \in \mathbb{N}$, $w \in \mathbb{C}$, $\lambda \in \mathbb{C}^{2d+1}$ and $\lambda_{-k} = -\lambda_k$ for $k = -d, -d+1, -d+2, \ldots, d$. Then*

$$\frac{(-1)^{d+1}}{z}\,\det\big(\operatorname{diag}(\lambda) + wE_+ + wE_- - zI\big) \tag{32}$$
$$= \left(\prod_{k=1}^{d}(\lambda_k^2 - z^2)\right)\mathfrak{F}\!\left(\frac{w}{\lambda_1 - z}, \ldots, \frac{w}{\lambda_d - z}\right)\mathfrak{F}\!\left(\frac{w}{\lambda_1 + z}, \ldots, \frac{w}{\lambda_d + z}\right)$$
$$+ 2\sum_{j=1}^{d} w^{2j}\left(\prod_{k=j+1}^{d}(\lambda_k^2 - z^2)\right)\mathfrak{F}\!\left(\frac{w}{\lambda_{j+1} - z}, \ldots, \frac{w}{\lambda_d - z}\right)\mathfrak{F}\!\left(\frac{w}{\lambda_{j+1} + z}, \ldots, \frac{w}{\lambda_d + z}\right).$$

Proof. This is a particular case of (27) where one has to set $y_k = \lambda_k - z$ for $k > 0$, $y_0 = -z$, $y_k = -(\lambda_{-k} + z)$ for $k < 0$. To complete the proof it suffices to verify that

$$\frac{w^2}{z\,(\lambda_1 - z)}\,\mathfrak{F}\!\left(\frac{w}{\lambda_2 - z}, \ldots, \frac{w}{\lambda_d - z}\right)\mathfrak{F}\!\left(\frac{w}{\lambda_1 + z}, \frac{w}{\lambda_2 + z}, \ldots, \frac{w}{\lambda_d + z}\right)$$
$$-\frac{w^2}{z\,(\lambda_1 + z)}\,\mathfrak{F}\!\left(\frac{w}{\lambda_1 - z}, \frac{w}{\lambda_2 - z}, \ldots, \frac{w}{\lambda_d - z}\right)\mathfrak{F}\!\left(\frac{w}{\lambda_2 + z}, \ldots, \frac{w}{\lambda_d + z}\right)$$
$$= 2\sum_{j=1}^{d} w^{2j}\left(\prod_{k=1}^{j}\frac{1}{\lambda_k^2 - z^2}\right)\mathfrak{F}\!\left(\frac{w}{\lambda_{j+1} - z}, \ldots, \frac{w}{\lambda_d - z}\right)\mathfrak{F}\!\left(\frac{w}{\lambda_{j+1} + z}, \ldots, \frac{w}{\lambda_d + z}\right).$$

To this end, one can apply (7), with $n = d$, $u_k = w/(\lambda_k - z)$, $v_k = w/(\lambda_k + z)$. Note that $u_j - v_j = 2zu_jv_j/w$. □

Zero always belongs to spectrum of the Jacobi matrix K for the characteristic function is odd. Moreover, as is well known and as it simply follows from the analysis of the eigenvalue equation, if $w \neq 0$ then to every eigenvalue of K there belongs exactly one linearly independent eigenvector.

Proposition 31. *Suppose $w \in \mathbb{C}$, $\lambda \in \mathbb{C}^{2d+1}$, $\lambda_{-k} = -\lambda_k$ for $-d \leq k \leq d$, and $\prod_{k=1}^{d}\lambda_k \neq 0$. Then the vector $v \in \mathbb{C}^{2d+1}$, $v^T = (\theta_{-d}, \theta_{-d+1}, \theta_{-d+2}, \ldots, \theta_d)$, with the entries*

$$\theta_k = (-1)^k w^k\left(\prod_{j=k+1}^{d}\lambda_j\right)\mathfrak{F}\!\left(\frac{w}{\lambda_{k+1}}, \frac{w}{\lambda_{k+2}}, \ldots, \frac{w}{\lambda_d}\right)\ \ \text{for } k = 0, 1, 2, \ldots, d, \tag{33}$$

$\theta_{-k} = (-1)^k\,\theta_k$ for $-d \leq k \leq d$, belongs to the kernel of the Jacobi matrix $\operatorname{diag}(\lambda) + wE_+ + wE_-$. *In particular, $\theta_0 = \lambda_1\lambda_2\ldots\lambda_d\,\mathfrak{F}(w/\lambda_1, w/\lambda_2, \ldots, w/\lambda_d)$, $\theta_d = (-1)^d w^d$, and so $v \neq 0$.*

Remark. Clearly, formulas (33) can be extended to the case $\prod_{k=1}^{d}\lambda_k = 0$ as well provided one makes the obvious cancellations.

Proof. One has to show that

$$w\theta_{k-1} + \lambda_k\theta_k + w\theta_{k+1} = 0,\ \ k = -d+1, -d+2, \ldots, d-1,$$

and $\lambda_{-d}\theta_{-d} + w\theta_{-d+1} = 0$, $w\theta_{d-1} + \lambda_d\theta_d = 0$. Owing to the symmetries $\lambda_{-k} = -\lambda_k$, $\theta_{-k} = (-1)^k\theta_k$, it suffices to verify the equalities only for indices $0 \le k \le d$. This can be readily carried out using the explicit formulas (33) and the rule (2). $\square$

5 Jacobi matrices with a linear diagonal

Finally we focus on finite-dimensional Jacobi matrices of odd dimension whose diagonal depends linearly on the index and whose parallels to the diagonal are constant. Without loss of generality one can assume that the diagonal equals $(-d, -d+1, -d+2, \ldots, d)$, $d \in \mathbb{Z}_+$. For $w \in \mathbb{C}$ put

$$K_0 = \operatorname{diag}(-d, -d+1, -d+2, \ldots, d),\ K(w) = K_0 + wE_+ + wE_-.$$

Concerning the characteristic function $\chi(z) = \det(K(w) - z)$, we know that this is an odd function. Put

$$\chi_{\mathrm{red}}(z) = \frac{(-1)^{d+1}}{z}\,\det(K(w) - z).$$

Hence $\chi_{\mathrm{red}}(z)$ is an even polynomial of degree $2d$. Further, denote by $\{e_{-d}, e_{-d+1}, e_{-d+2}, \ldots, e_d\}$ the standard basis in $\mathbb{C}^{2d+1}$.

Suppose $w \neq 0$. Let us consider a family of column vectors $x_{s,n} \in \mathbb{C}^{2d+1}$ depending on the parameters $s, n \in \mathbb{Z}$ and defined by

$$x_{s,n}^T = \big(\mathfrak{J}(s+d, n), \mathfrak{J}(s+d-1, n), \mathfrak{J}(s+d-2, n), \ldots, \mathfrak{J}(s-d, n)\big).$$

From the fact that the matrix $\mathfrak{J}$ obeys (17), (18) one derives that

$$\forall s, n \in \mathbb{Z},\ K(w)x_{s,n} = s\,x_{s,n} - w\,\mathfrak{J}(s+d+1, n)e_{-d} - w\,\mathfrak{J}(s-d-1, n)e_d.$$

Put

$$v_s = x_{s,s+d+1},\ s \in \mathbb{Z}.$$

Recalling that $\mathfrak{J}(m, m) = \mathfrak{J}(-m, m) = 0$ one has

$$K(w)v_s = s\,v_s - w\,\mathfrak{J}(s-d-1, s+d+1)e_d. \tag{34}$$

Remark 32. Putting $s = 0$ one gets $K(w)v_0 = 0$, and so v_0 spans the kernel of $K(w)$.

Lemma 33. *For every $\ell = -d, -d+1, -d+2, \ldots, d$, one has*

$$w^{d+\ell} \sum_{s=-d}^{\ell} \frac{(-1)^{\ell+s}}{(d+s)!\,(\ell-s)!}\, v_s \in e_\ell + \operatorname{span}\{e_{\ell+1}, e_{\ell+2}, \ldots, e_d\}.$$

In particular,

$$e_d = w^{2d} \sum_{s=-d}^{d} \frac{(-1)^{d+s}}{(d+s)!\,(d-s)!}\, v_s. \tag{35}$$

Consequently, $\mathcal{V} = \{v_{-d}, v_{-d+1}, v_{-d+2}, \ldots, v_d\}$ is a basis in $\mathbb{C}^{2d+1}$.

Proof. One has to show that

$$w^{d+\ell}\sum_{s=-d}^{\ell}\frac{(-1)^{\ell+s}}{(d+s)!\,(\ell-s)!}\,\mathfrak{J}(s-k,s+d+1)=\delta_{\ell,k}\ \text{ for }\ -d\leq k\leq\ell.$$

Note that for any $a\in\mathbb{C}$ and $n\in\mathbb{Z}_+$,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{a+k}{r}=0,\ r=0,1,2,\ldots,n-1,\ \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{a+k}{n}=(-1)^n$$

(see Appendix). Using these equalities and (23) one can readily show, more generally, that

$$\sum_{s=-d}^{\ell}\frac{(-1)^{\ell+s}}{(d+s)!\,(\ell-s)!}\,\mathfrak{J}(m+s,n+s)=0\ \text{ for } m,n\in\mathbb{Z},m\leq n\leq m+d+\ell,$$

and

$$\sum_{s=-d}^{\ell}\frac{(-1)^{\ell+s}}{(d+s)!\,(\ell-s)!}\,\mathfrak{J}(m+s,m+d+\ell+s+1)=w^{-d-\ell}.$$

This proves the lemma. □

Denote by $\tilde{K}(w)$ the matrix of $K(w)$ in the basis $\mathcal{V}$ introduced in Lemma 33. Let $a,b\in\mathbb{C}^{2d+1}$ be the column vectors defined by $a^T=(\alpha_{-d},\alpha_{-d+1},\alpha_{-d+2},\ldots\alpha_d)$, $b^T=(\beta_{-d},\beta_{-d+1},\beta_{-d+2},\ldots\beta_d)$,

$$\alpha_s=\mathfrak{J}(s-d-1,s+d+1),\ \beta_s=\frac{(-1)^{d+s}\,w^{2d+1}}{(d+s)!\,(d-s)!}\,,\ s=-d,-d+1,-d+2,\ldots,d. \tag{36}$$

Note that

$$\alpha_{-s}=-\alpha_s,\ \beta_{-s}=\beta_s. \tag{37}$$

The former equality follows from (21) and (18). From (34) and (35) one deduces that

$$\tilde{K}(w)=K_0-ba^T. \tag{38}$$

Note, however, that the components of the vectors a and b depend on w, too, though not indicated in the notation.

According to (38), $\tilde{K}(w)$ differs from the diagonal matrix K_0 by a rank-one correction. This form is suitable for various computations. Particularly, one can express the resolvent of $\tilde{K}(w)$ explicitly,

$$(\tilde{K}(w)-z)^{-1}=(K_0-z)^{-1}+\frac{1}{1+a^T(K_0-z)^{-1}b}\,(K_0-z)^{-1}ba^T(K_0-z)^{-1}. \tag{39}$$

The equality holds for any $z\in\mathbb{C}$ such that $z\notin\operatorname{spec}\{K_0\}=\{-d,-d+1,-d+2,\ldots,d\}$ and $1+a^T(K_0-z)^{-1}b\neq0$. Clearly, this set of excluded values of z is finite. By multiplying the LHS of (39) $(\tilde{K}(w) - z)$ one readily verifies that the result is the identity matrix.

Let us proceed to derivation of a formula for the characteristic function of $K(w)$. Proposition 30 is applicable to $K(w)$ and so

$$\begin{aligned}\chi_{\mathrm{red}}(z) \;&=\; \left(\prod_{k=1}^{d}(k^2-z^2)\right)\mathfrak{F}\!\left(\frac{w}{1-z},\ldots,\frac{w}{d-z}\right)\mathfrak{F}\!\left(\frac{w}{1+z},\ldots,\frac{w}{d+z}\right) \qquad (40)\\ &+2\sum_{j=1}^{d}w^{2j}\left(\prod_{k=j+1}^{d}(k^2-z^2)\right)\mathfrak{F}\!\left(\frac{w}{j+1-z},\ldots,\frac{w}{d-z}\right)\mathfrak{F}\!\left(\frac{w}{j+1+z},\ldots,\frac{w}{d+z}\right).\end{aligned}$$

Below we derive a more convenient formula for $\chi_{\mathrm{red}}(z)$.

Lemma 34. *One has*

$$\chi_{\mathrm{red}}(0) = \sum_{s=0}^{d}\frac{\big((d-s)!\big)^2\,(2d-s+1)!}{s!\,(2d-2s+1)!}\,w^{2s} \qquad (41)$$

and

$$\chi_{\mathrm{red}}(n) = \frac{1}{n}\sum_{k=0}^{n-1}(-1)^k(2k+1)!\binom{n+k}{2k+1}\binom{d+k+1}{2k+1}w^{2d-2k} \qquad (42)$$

for $n = 1, 2, \ldots, d$.

Proof. Let us first verify the formula for $\chi_{\mathrm{red}}(0)$. From (40) it follows that

$$\chi_{\mathrm{red}}(0) = (d!)^2\,\mathfrak{F}\!\left(w,\frac{w}{2},\ldots,\frac{w}{d}\right)^2 + 2\sum_{j=1}^{d}w^{2j}\left(\frac{d!}{j!}\right)^2\mathfrak{F}\!\left(\frac{w}{j+1},\frac{w}{j+2},\ldots,\frac{w}{d}\right)^2.$$

By Proposition 15,

$$\chi_{\mathrm{red}}(0) = \pi^2 w^{2d+2}\,Y_{d+1}(2w)^2\left(J_0(2w)^2 + 2\sum_{j=1}^{d}J_j(2w)^2\right) + O\big(w^{2d+2}\log(w)\big)\,.$$

Further we need some basic facts concerning Bessel functions; see, for instance, [1, Chp. 9]. Recall that

$$J_0(z)^2 + 2\sum_{j=1}^{\infty}J_j(z)^2 = 1.$$

Hence

$$\begin{aligned}\chi_{\mathrm{red}}(0) \;&=\; \pi^2 w^{2d+2}\,Y_{d+1}(2w)^2 + O\big(w^{2d+2}\log(w)\big)\\ &=\; \left(\sum_{k=0}^{d}\frac{(d-k)!}{k!}\,w^{2k}\right)^2 + O\big(w^{2d+2}\log(w)\big)\,.\end{aligned}$$

Note that $\chi_{\rm red}(0)$ is a polynomial in the variable w of degree $2d$, and so

$$\chi_{\rm red}(0)=\sum_{s=0}^{d}\sum_{k=0}^{s}\frac{(d-k)!\,(d-s+k)!}{k!\,(s-k)!}\,w^{2s}.$$

Using the identity (proved in Appendix)

$$\begin{aligned}\sum_{k=0}^{s}\frac{(d-k)!\,(d-s+k)!}{k!\,(s-k)!} &= \big((d-s)!\big)^2\sum_{k=0}^{s}\binom{d-k}{d-s}\binom{d-s+k}{d-s}\\ &= \frac{\big((d-s)!\big)^2\,(2d-s+1)!}{s!\,(2d-2s+1)!}\end{aligned}$$

one arrives at (41).

To show (42) one can make use of (38). One has

$$\chi_{\rm red}(z)=\frac{(-1)^{d+1}}{z}\,\det(\tilde{K}(w)-z)=\frac{(-1)^{d+1}}{z}\,\det(K_0-z)\det\!\big(I-(K_0-z)^{-1}ba^T\big)\,.$$

Note that $\det(I+ba^T)=1+a^Tb$ (see Appendix). Hence, in view of (37),

$$\chi_{\rm red}(z)=\prod_{k=1}^{d}(k^2-z^2)\left(1-\sum_{s=-d}^{d}\frac{\beta_s\alpha_s}{s-z}\right)=\prod_{k=1}^{d}(k^2-z^2)\left(1-2\sum_{s=1}^{d}\frac{s\beta_s\alpha_s}{s^2-z^2}\right).$$

Using (36) one gets

$$\chi_{\rm red}(n)=-2n\beta_n\alpha_n\prod_{\substack{k=1\\k\neq n}}^{d}(k^2-z^2)=\frac{(-1)^d}{n}\,w^{2d+1}\,\mathfrak{J}(n-d-1,n+d+1).$$

Formula (42) then follows from (23). ☐

Proposition 35. *For every $d\in\mathbb{Z}_+$ one has*

$$\chi_{\rm red}(z)=\sum_{s=0}^{d}\frac{(2d-s+1)!}{s!\,(2d-2s+1)!}\,w^{2s}\prod_{k=1}^{d-s}(k^2-z^2). \qquad (43)$$

Proof. Since $\chi_{\rm red}(z)$ is an even polynomial in z of degree $2d$ it is enough to check that the RHS of (43) coincides, for $z=0,1,2,\ldots,d$, with $\chi_{\rm red}(0)$, $\chi_{\rm red}(1)$, $\chi_{\rm red}(2)$, …, $\chi_{\rm red}(d)$. With the knowledge of values (41) and (42), this is a matter of straightforward computation. ☐

Remark 36. Using (43) it is not difficult to check that formula (42) is valid for any $n\in\mathbb{N}$, including $n>d$ (the summation index k runs from 1 to $\min\{n-1,d\}$).

Remark 37. If $w\in\mathbb{R}$, $w\neq 0$, then the spectrum of the Jacobi matrix $K(w)$ is real and simple, and formula (43) implies that the interval $[-1,1]$ contains no other eigenvalue except of 0.

Eigenvectors of $K(w)$ can be expressed in terms of the function $\mathfrak{F}$, too. Suppose $w \neq 0$. Let us introduce the vector-valued function $x(z) \in \mathbb{C}^{2d+1}$ depending on $z \in \mathbb{C}$, $x(z)^T = (\xi_{-d}(z), \xi_{-d+1}(z), \xi_{-d+2}(z), \ldots, \xi_d(z))$,

$$\xi_k(z) = w^{-d-k} \, \frac{\Gamma(z+d+1)}{\Gamma(z-k+1)} \, \mathfrak{F}\!\left(\frac{w}{z-k+1}, \frac{w}{z-k+2}, \ldots, \frac{w}{z+d}\right), \quad -d \leq k \leq d.$$

With the aid of (2) one derives the equality

$$\big(K(w)-z\big)x(z) = -w^{-2d} \, \frac{\Gamma(z+d+1)}{\Gamma(z-d)} \, \mathfrak{F}\!\left(\frac{w}{z-d}, \frac{w}{z-d+1}, \ldots, \frac{w}{z+d}\right) e_d. \qquad (44)$$

Remark 38. According to (11),

$$\xi_k(z) = w^{-d-k} \sum_{s=0}^{[(d+k)/2]} (-1)^s \, \frac{(d+k-s)!}{s!\,(d+k-2s)!} \, w^{2s} \prod_{j=s}^{d+k-s-1} (z+d-j).$$

Hence $\xi_k(z)$ is a polynomial in z of degree $d+k$. In particular, $\xi_{-d}(z) = 1$, and so $x(z) \neq 0$.

Proposition 39. *If $w \in \mathbb{C}$, $w \neq 0$, then for every eigenvalue $\lambda \in \operatorname{spec}(K(w))$, $x(\lambda)$ is an eigenvector corresponding to λ.*

Proof. According to (31), equation (44) can be rewrite as

$$\big(K(w)-z\big)x(z) = w^{-2d}\chi(z)e_d.$$

Consequently, if $\chi(\lambda) = 0$ then $x(\lambda)$ is an eigenvector of $K(w)$. □

Remark 40. Since

$$\chi(z) = -z \left(\prod_{k=1}^{d}(z^2-k^2)\right) \mathfrak{F}\!\left(\frac{w}{z-d}, \frac{w}{z-d+1}, \frac{w}{z-d+2}, \ldots, \frac{w}{z+d}\right) \qquad (45)$$

one can rederive equality (42). For $1 \leq n \leq d$, a straightforward computation gives

$$\chi(n) = (-1)^{d+n} w^{2d+1} \, \mathfrak{J}(d-n+1, d+n+1).$$

Equality (42) then follows from (23).

The formula for the characteristic function and for eigenvalues can be also derived for the Jacobi matrix with linear diagonal of even dimension. For $w \in \mathbb{C}$ and $d \in \mathbb{Z}_+$ put

$$L_0 = \operatorname{diag}\left(-d+\frac{1}{2}, -d+\frac{3}{2}, -d+\frac{5}{2}, \ldots, d-\frac{1}{2}\right), \;\; L(w) = L_0 + wE_+ + wE_-$$

and denote $\psi(z) = \det(L(w) - z)$ the characteristic function of $L(w)$. Suppose $w \neq 0$. Let us introduce the vector-valued function $y(z) \in \mathbb{C}^{2d}$ depending on $z \in \mathbb{C}$, $y(z)^T = (\eta_{-d+1}(z), \eta_{-d+2}(z), \eta_{-d+3}(z), \ldots, \eta_d(z))$,

$$\eta_k(z) = w^{-d-k+1}\,\frac{\Gamma(z+d)}{\Gamma(z-k+1)}\,\mathfrak{F}\biggl(\frac{w}{z-k+1}, \frac{w}{z-k+2}, \ldots, \frac{w}{z+d-1}\biggr)$$

where $-d+1 \leq k \leq d$. With the aid of (2) one derives the equality

$$\bigl(L(w) + \frac{1}{2} - z\bigr)y(z) = -w^{-2d+1}\,\frac{\Gamma(z+d)}{\Gamma(z-d)}\,\mathfrak{F}\biggl(\frac{w}{z-d}, \frac{w}{z-d+1}, \ldots, \frac{w}{z+d-1}\biggr)\,e_d. \quad (46)$$

Remark 41. According to (11),

$$\eta_k(z) = w^{-d-k} \sum_{s=1}^{[(d+k+1)/2]} (-1)^{s+1}\,\frac{(d+k-s)!}{(s-1)!\,(d+k+1-2s)!}\,w^{2s-1} \prod_{j=s}^{d+k-s} (z+d-j).$$

Hence $\eta_k(z)$ is a polynomial in z of degree $d+k-1$. In particular, $\eta_{-d+1}(z) = 1$, and so $y(z) \neq 0$.

Proposition 42. *One has*

$$\psi(z) = (-1)^d \sum_{s=0}^{d} \frac{(2d-s)!}{(2d-2s)!s!} w^{2s} \prod_{j=1}^{d-s} \left[\left(j - \frac{1}{2}\right)^2 - z^2\right]. \quad (47)$$

If $w \in \mathbb{C}$, $w \neq 0$, then for every eigenvalue $\lambda \in \operatorname{spec}(L(w))$, $y(\lambda + \frac{1}{2})$ is an eigenvector corresponding to λ.

Proof. According to (31), the function

$$\prod_{k=-d}^{d-1} (z+k)\mathfrak{F}\biggl(\frac{w}{z-d}, \frac{w}{z-d+1}, \frac{w}{z-d+2}, \ldots, \frac{w}{z+d-1}\biggr)$$

is the characteristic function of $L(w) + \frac{1}{2}$ and the second part of the statement follows from (46). To conclude the proof, it suffices to use (11) together with the fact that $\psi(z)$ is equal to the characteristic function of $L(w) + \frac{1}{2}$ in $z + \frac{1}{2}$, to derive (47). □

Remark 43. If $w \in \mathbb{R}$, $w \neq 0$, then the spectrum of the Jacobi matrix $L(w)$ is real and simple, and formula (47) implies that the characteristic function of $L(w)$ is even and the interval $[-\frac{1}{2}, \frac{1}{2}]$ contains no eigenvalue.

6 The spectrum of the infinite Jacobi matrix with a linear diagonal

Let w be a positive constant and J an infinite Jacobi matrix of the form

$$J = \begin{pmatrix} 1 & w & & & \\ w & 2 & w & & \\ & w & 3 & w & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{48}$$

In this section the spectrum of J is shortly discussed. Since J is a self-adjoint operator on $\ell^2(\mathbb{N})$ the spectrum of J is real. J can be written as $D + W + W^*$ where $W = wT$ (W^* its adjoint), T is the right-shift operator and D is the respective diagonal operator. Since the essential spectrum of D is empty D has a compact resolvent. Then $W + W^*$ is D-compact self-adjoint perturbation (i.e. $(W + W^*)(D - i)^{-1}$ is compact) and by the Weyl theorem, J has a discrete spectrum.

The main goal of this section is to show that the spectrum of J coincides with zeros of a Bessel function of the first kind as function of its order. More precisely, it will be shown that

$$\operatorname{spec}(J) = \{z \in \mathbb{R} : J_{-z}(2w) = 0\}.$$

Lemma 44. *Let $-\infty < a < b < +\infty$. Then there exist constants $L > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \in \mathbb{N}, n \geq n_0$ and $w > 0$, the inequality*

$$\sup_{z \in [a,b]} \left| \frac{1}{\Gamma(1-z)} \mathfrak{F}\left(\frac{w}{1-z}, \frac{w}{2-z}, \ldots, \frac{w}{n-z} \right) \right| \leq L \exp(2w^2) \tag{49}$$

holds.

Proof. According to (11), one has

$$\begin{aligned} &\left| \frac{1}{\Gamma(1-z)} \mathfrak{F}\left(\frac{w}{1-z}, \frac{w}{2-z}, \ldots, \frac{w}{n-z} \right) \right| \\ &\leq \frac{1}{|\Gamma(n+1-z)|} \sum_{s=0}^{[\frac{n}{2}]} \binom{n-s}{s} w^{2s} \left| \frac{\Gamma(n-s+1-z)}{\Gamma(s+1-z)} \right|. \end{aligned}$$

for any $n \in \mathbb{N}$ and $z \in [a,b]$. An inequality

$$\frac{1}{|\Gamma(z)|} \leq 2N!$$

holds for all $z \in \mathbb{R}, z \geq -N$ and $N \in \mathbb{Z}_+$, which can be proved by mathematical induction in N. Thus, there exists a constant $L > 0$ such that

$$\begin{aligned} &\left| \frac{1}{\Gamma(1-z)} \mathfrak{F}\left(\frac{w}{1-z}, \frac{w}{2-z}, \ldots, \frac{w}{n-z} \right) \right| \leq L \sum_{s=0}^{[\frac{n}{2}]} \binom{n-s}{s} w^{2s} \left| \frac{\Gamma(n-s+1-z)}{\Gamma(n+1-z)} \right| \\ &= L \sum_{s=0}^{[\frac{n}{2}]} \frac{w^{2s}}{s!} \prod_{j=1}^{s} \left| \frac{n-s-j+1}{n-z-j+1} \right|. \end{aligned} \tag{50}$$

Next, consider that, for any $s = 0, 1, \ldots, \left[\frac{n}{2}\right]$, one has

$$\prod_{j=1}^{s} \left| \frac{n-s-j+1}{n-z-j+1} \right| = \prod_{j=1}^{s} \left| 1 + \frac{z-s}{n-z-j+1} \right| \leq \left(1 + \frac{\max\{\frac{n}{2} - z, |z|\}}{\frac{n}{2} - z + 1} \right)^{s} < 2^{s},$$

for n sufficiently large. Hence, there exists n_0 such that, for all $n \in \mathbb{N}, n \geq n_0$ and all $z \in [a, b]$, it holds

$$\prod_{j=1}^{s} \left| \frac{n-s-j+1}{n-z-j+1} \right| \leq 2^{s}.$$

Consequently, the expression of the RHS in (50) is less or equal to $L \exp(2w^2)$. $\square$

Proposition 45. *The sequence of functions*

$$\left\{ \frac{1}{\Gamma(1-z)} \mathfrak{F}\left(\frac{w}{1-z}, \frac{w}{2-z}, \ldots, \frac{w}{n-z} \right) \right\}_{n=1}^{\infty}$$

is locally uniformly convergent and its limit function is $w^z J_{-z}(2w)$. That is, for arbitrary $a, b \in \mathbb{R}, a < b$, it holds

$$\lim_{n \to +\infty} \sup_{z \in [a,b]} \left| \frac{1}{\Gamma(1-z)} \mathfrak{F}\left(\frac{w}{1-z}, \frac{w}{2-z}, \ldots, \frac{w}{n-z} \right) - w^z J_{-z}(2w) \right| = 0.$$

Proof. By (10), one finds out

$$\lim_{n \to +\infty} \frac{1}{\Gamma(1-z)} \mathfrak{F}\left(\frac{w}{1-z}, \frac{w}{2-z}, \ldots, \frac{w}{n-z} \right) = w^z J_{-z}(2w), \tag{51}$$

pointwise in $z \in \mathbb{R}$. Next, with the aid of Lemma 44 and recurrence rule (4), one verifies the Cauchy's criterion for the local uniform convergence is fulfilled. Let $n, p \in \mathbb{N}, n > \max\{n_0, b\}$ (see Lemma 44) then

$$\begin{aligned}
&\left| \frac{1}{\Gamma(1-z)} \left[\mathfrak{F}\left(\frac{w}{1-z}, \frac{w}{2-z}, \ldots, \frac{w}{n+p-z} \right) - \mathfrak{F}\left(\frac{w}{1-z}, \frac{w}{2-z}, \ldots, \frac{w}{n-z} \right) \right] \right| \\
&\leq \frac{1}{|\Gamma(1-z)|} \sum_{k=0}^{p-1} \left| \mathfrak{F}\left(\frac{w}{1-z}, \ldots, \frac{w}{n+k+1-z} \right) - \mathfrak{F}\left(\frac{w}{1-z}, \ldots, \frac{w}{n+k-z} \right) \right| \\
&= \frac{1}{|\Gamma(1-z)|} \sum_{k=0}^{p-1} \frac{w^2}{|n+k+1-z||n+k-z|} \left| \mathfrak{F}\left(\frac{w}{1-z}, \frac{w}{2-z}, \ldots, \frac{w}{n+k-1-z} \right) \right| \\
&\leq L w^2 \exp(2w^2) \sum_{k=n}^{\infty} \frac{1}{|k+1-z||k-z|}.
\end{aligned}$$

The last term tends to zero with $n \to \infty$ uniformly in $z \in [a, b]$. $\square$

Remark 46. According to (31), equation (51) gives

$$\lim_{n\to+\infty}\frac{1}{\Gamma(n+1-z)}\left|\begin{pmatrix}1-z & w & & \\ w & 2-z & w & \\ & \ddots & \ddots & \ddots \\ & & w & n-z\end{pmatrix}\right| = w^z J_{-z}(2w), \tag{52}$$

and the convergence is locally uniform in z.

Remark 47. In [8], it has been shown that, under certain assumptions, every eigenvalue of a tridiagonal operator is a limit point of a sequence of eigenvalues of a truncated finite-dimensional operator and vice versa. Infinite matrix J satisfies these assumptions (the diagonal sequence is divergent and the neighboring parallel sequence is bounded (even constant here), see Corollary 2.3. in [8] for details). Thus, $\lambda\in\operatorname{spec}(J)$ if and only if

$$(\exists\{k_n\}_{n=1}^{\infty}\subset\mathbb{N}, k_{n+1}>k_n)(\exists\{\lambda_n\}_{n=1}^{\infty}\subset\mathbb{R}, \lambda_n\in\operatorname{spec}(J_{k_n}), \lim_{n\to\infty}\lambda_n=\lambda) \tag{53}$$

where J_N is the truncation of J, this means

$$J_N=\begin{pmatrix}1 & w & & & \\ w & 2 & w & & \\ & \ddots & \ddots & \ddots & \\ & & w & N-1 & w \\ & & & w & N\end{pmatrix}.$$

Proposition 48. *Let $w>0$ and J is the infinite Jacobi matrix defined in (48) then it holds*

$$\operatorname{spec}(J)=\{z\in\mathbb{R}: J_{-z}(2w)=0\}.$$

Proof. Let us, temporarily, denote the function in the limit in (52) by $D_n(z)$. Since $\Gamma(z)^{-1}$ is continuous in $\mathbb{R}$ (even holomorphic in the whole $\mathbb{C}$) $D_n(z)$ is continuous in $\mathbb{R}$ for all $n\in\mathbb{N}$. Also the limit function $w^z J_{-z}(2w)$ is continuous in all $z\in\mathbb{R}$ which is a well known fact from the theory of Bessel functions.

First, let $\lambda\in\operatorname{spec}(J)$ then, according to Remark 47, there exists a sequence $\{\lambda_n\}$ such that $\lambda_n\in\operatorname{spec}(J_{k_n})$ for all n and $\lim_{n\to\infty}\lambda_n=\lambda$. Since $D_n(z)$ converges to $w^z J_{-z}(2w)$ locally uniformly one easily checks that

$$\lim_{n\to\infty} D_{k_n}(\lambda_n)=w^\lambda J_{-\lambda}(2w).$$

However, $D_{k_n}(\lambda_n)=0$ for all n, hence, $J_{-\lambda}(2w)=0$.

Second, let $\lambda\in\mathbb{R}$ be such that $J_{-\lambda}(2w)=0$. Suppose that there exist $a,b\in\mathbb{R}$, $a<\lambda<b$, such that

$$\operatorname{sign}(J_{-x}(2w))=-\operatorname{sign}(J_{-y}(2w))$$

for all $x\in(a,\lambda)$ and $y\in(\lambda,b)$. This is possible if λ is not a multiple zero of $J_{-z}(2w)$. Then, due to the local uniform convergence of $D_n(z)$ and the continuity of $D_n(z), J_{-z}(2w)$ in z, one can claim that there exists a sequence $\{l_n\}_{n=1}^{\infty} \subset \mathbb{N}$ and $\lambda_n \in \mathbb{R}$ such that, for all $n \in \mathbb{N}$, $|\lambda_n - \lambda| < \frac{1}{n}$ and $D_{l_n}(\lambda_n) = 0$. Thus, one has, for all $n \in \mathbb{N}$ (n large enough), $\lambda_n \in \operatorname{spec}(J_{l_n})$ and $\lambda_n \to \lambda$ with $n \to \infty$. Finally, Remark 47 implies that $\lambda \in \operatorname{spec}(J)$.

To conclude the proof, one has to verify that there are no multiple zeros of $J_{-z}(2w)$ (in z). This means to show that there is no $\nu_0 \in \mathbb{R}$ such that

$$J_{\nu_0}(2w) = \frac{\partial}{\partial \nu}\Big|_{\nu=\nu_0} J_\nu(2w) = 0. \tag{54}$$

This follows from the identity

$$J_\nu(2w)\frac{\partial Y_\nu(2w)}{\partial \nu} - Y_\nu(2w)\frac{\partial J_\nu(2w)}{\partial \nu} = -\frac{4}{\pi}\int_0^\infty K_0(4w\sinh t)e^{-2\nu t}dt \tag{55}$$

where Y_ν is the Bessel function of the second kind and K_0 is the modified Bessel function of the second kind of order zero (see [9], p.444). For K_ν, one has the equation

$$K_\nu(z) = \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})}\left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-z\cosh t}\sinh^{2\nu} t dt \tag{56}$$

(see [9], p.172). Thus, since the RHS of (55) is obviously negative for all $\nu \in \mathbb{R}$ and $w > 0$ the equality (54) would lead to a contradiction. □

7 Appendix

Some supplementary computations, mostly proving algebraic identities used in the paper, are made in this section.

A 1. *Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{n\times m}$, $A \in \mathbb{C}^{m\times m}$ and A is regular. Then*

$$\det\begin{pmatrix} A & C \\ B & D\end{pmatrix} = \det(A)\det(D - BA^{-1}C).$$

Proof. The statement follows directly from the equality

$$\begin{pmatrix} A & C \\ B & D\end{pmatrix} = \begin{pmatrix} I & 0 \\ BA^{-1} & I\end{pmatrix}\begin{pmatrix} A & C \\ 0 & D - BA^{-1}C\end{pmatrix}.$$

□

A 2. *For any $n \in \mathbb{Z}_+$, $a \in \mathbb{C}$ and $r = 0, 1, 2, \dots, n-1$, the identities*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{a+k}{r} = 0, \qquad \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{a+k}{n} = (-1)^n$$

hold.

Proof. To verify the first identity it suffices to prove the equality

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^r=0$$

for all $0\le r<n$. The statement will be verified by mathematical induction in n. The case $n=1$ is immediate. Let $n\in\mathbb{N}$ is fixed and the equation

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^r=0 \tag{57}$$

holds for all $0\le r<n$ as the induction hypothesis. For $r=0$, (57) follows easily from the binomial theorem (with an arbitrary $n\in\mathbb{N}$). Let $0<r\le n$ then

$$\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}j^r=(n+1)\sum_{j=1}^{n+1}(-1)^j\binom{n}{j-1}j^{r-1}$$

$$=-(n+1)\sum_{j=0}^{n}(-1)^j\binom{n}{j}(j+1)^{r-1}=-(n+1)\sum_{k=0}^{r-1}\binom{r-1}{k}\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^k=0$$

because, according to an induction hypothesis, the inner sums are all 0 and the induction step is concluded. Next, consider

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{a+k}{n}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[\binom{k}{n}+p_a(k)\right]$$

where $p_a(k)$ is a polynomial in k of degree less then n. According to the first part of the proof,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}p_a(k)=0$$

and, obviously

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k}{n}=(-1)^n,$$

hence the second identity is proved. □

A 3. *Let $n\in\mathbb{N}$ and $a,b\in\mathbb{R}^n$ then*

$$\det(I+ba^T)=1+a^Tb.$$

Proof. Since a determinant of a matrix is a linear function of its columns $\det(I+ba^T)$ is equal to

$$\left|\begin{pmatrix}a_1b_1 & a_1b_2 & \dots & a_1b_n\\ a_2b_1 & a_2b_2+1 & \dots & a_nb_n\\ \vdots & \vdots & & \vdots\\ a_nb_1 & a_nb_2 & \dots & a_nb_n+1\end{pmatrix}\right|+\left|\begin{pmatrix}1 & a_1b_2 & \dots & a_1b_n\\ 0 & a_2b_2+1 & \dots & a_nb_n\\ \vdots & \vdots & & \vdots\\ 0 & a_nb_2 & \dots & a_nb_n+1\end{pmatrix}\right|.$$

The first determinant can be decomposed similarly, however, with exception of the term

$$\begin{pmatrix} a_1b_1 & 0 & \dots & 0 \\ a_2b_1 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ a_nb_1 & 0 & \dots & 1 \end{pmatrix},$$

all other matrices are singular, hence

$$\det(I + ba^T) = a_1b_1 + \left| \begin{pmatrix} a_2b_2 + 1 & a_2b_3 & \dots & a_2b_n \\ a_3b_2 & a_3b_3 + 1 & \dots & a_nb_n \\ \vdots & \vdots & & \vdots \\ a_nb_1 & a_nb_2 & \dots & a_nb_n + 1 \end{pmatrix} \right|.$$

By repeating this procedure one verifies the statement. □

A 4. *It holds*

$$\sum_{k=0}^{m} \binom{s+k}{s} \binom{p+m-k}{p} = \binom{s+p+m+1}{s+p+1}$$

for $m, p, s \in \mathbb{Z}_+$.

Proof. Let us define a function

$$f_s(z) = \sum_{k=0}^{\infty} \binom{s+k}{s} z^k$$

for all $z \in \mathbb{C}$, $|z| < 1$. Since

$$\binom{s+k}{k} = (-1)^k \binom{-s-1}{k}$$

we have

$$f_s(z) = \sum_{k=0}^{\infty} \binom{s+k}{k} z^k = \sum_{k=0}^{\infty} \binom{-s-1}{k} (-z)^k = (1-z)^{-s-1}.$$

It follows the identity

$$f_s(z) f_p(z) = f_{s+p+1}(z)$$

holds for all $z, |z| < 1$ and so

$$\sum_{m=0}^{\infty} \binom{s+p+1+m}{s+p+1} z^m = f_{s+p+1}(z) = f_s(z) f_p(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{s+k}{s} \binom{p+l}{p} z^{k+l}$$
$$= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \binom{s+k}{s} \binom{p+m-k}{p} \right) z^m$$

which proves the statement. □

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