Czech Technical University in Prague
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RESEARCH WORK

Equations of Eigenvalues for a Tridiagonal Operator

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Název práce:
Rovnice na vlastní čísla pro tridiagonální operátor

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Abstrakt: Definujeme speciální funkci $\mathfrak{F}$ a uvedeme některé její vlastnosti, zejména pak dokážeme platnost rekurentí formule, s jejíž pomocí rozšíříme definiční obor této speciální funkce. Shrneme vlastnosti takto rozšířené funkce a použijeme ji ke konstrukci speciální báze prostoru $\mathbb{C}^{2d+1}$. V této bázi má Jacobiho matice speciálního typu velice jednoduchý tvar, který umožňuje odvodit formuli pro charakteristický polynom této matice. Navíc nalezneme vzorec pro rezolventu uvažované Jacobiho matice.

Klíčová slova: Jacobiho matice, tridiagonální operátor, charakteristický polynom, spektrum, rezolventa

Title:
Equations of Eigenvalues for a Tridiagonal Operator

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Abstract: We define a special function $\mathfrak{F}$ and we present some of its properties, especially we prove a recurrent relation which allows to extend this special function. We summarize properties of this expanded function and with the aid of this function we construct a special basis of the space $\mathbb{C}^{2d+1}$. A Jacobi matrix of a special type written in that basis has a very simple form which allows us to derive a formula for a characteristic polynomial of this matrix. Furthermore we find a formula for a resolvent operator of the Jacobi matrix under consideration.

Key words: Jacobi matrix, tridiagonal operator, characteristic function, spectrum, resolvent

Contents

Introduction

This research work is a continuation to my Bachelor thesis [2] where I have started to investigate a spectrum of, so called, Jacobi matrices of a special type. The Jacobi matrix is a complex tridiagonal matrix. Fundamental definitions and a main result of my Bachelor thesis will be reminded in the first chapter, especially a special function $\mathfrak{F}$ will be defined there (Definition 1). Next, some of properties of the function $\mathfrak{F}$ and its relation to a characteristic reduced function of a Jacobi matrix of a special type will be stated.

In the second chapter I will introduce a formula for the function $\mathfrak{F}$ and I will also examine asymptotic properties of the function $\mathfrak{F}$ for a small argument. The asymptotic relations will allow me to evaluate the characteristic reduced function in some spacial cases which will be illustrated in the next chapter 3. At the end of the third chapter I will introduce an interesting expression for particular values of the characteristic reduced function which will be generalized in the next chapter.

In the fourth chapter I will extend the function $\mathfrak{F}$ with the aid of a recurrent relation from the first chapter. The expanded function is denoted $\mathfrak{J}$ and some algebraic identities concern with the function $\mathfrak{J}$ are summarized. They often are generalizations of the properties of the function $\mathfrak{F}$. At the end of the chapter I will derive an explicit formula for the function $\mathfrak{J}$ and I will shortly discuss a proposition dealing with particular values of the characteristic reduced function.

Main results will be obtained in the final chapter. With the aid of the function $\mathfrak{J}$ I will construct a special basis of a space $\mathbb{C}^{2d+1}$ in which the Jacobi matrix of a special type will have simple and appropriate form to find an explicit formula for the characteristic reduced function. Furthermore a formula for a resolvent will be presented.

Chapter 1

Some results from the Bachelor's thesis

In this chapter we will recall some definitions and general results which can be mostly found in my bachelor's thesis [2] and which we will need in following chapters.

1.1 Function $\mathfrak{F}$ and its properties

Definition 1. Define $\mathfrak{F} : D \to \mathbb{C}$,

$$\mathfrak{F}(x) := 1 + \sum_{m=1}^{\infty}(-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \dots x_{k_m} x_{k_m+1}$$

where

$$D = \left\{ x = \{x_k\}_{k=1}^{\infty} \middle| \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}$$

Remark 1. Obviously, if all but finitely many elements of x are zeroes then $\mathfrak{F}(x)$ reduces to a finite sum. For a finite number of variables we often will write $\mathfrak{F}(x_1, x_2, \dots, x_k)$ instead of $\mathfrak{F}(x)$ where $x = (x_1, x_2, \dots, x_k, 0, 0, 0, \dots)$.

Definition 2. The operator T_1 defined on the space of all sequences indexed by $\mathbb{N}$ such that

$$T_1(\{x_k\}_{k=1}^{\infty}) := \{x_{k+1}\}_{k=1}^{\infty}$$

in called the operator of *truncation from the left*. Next, set $T_n := (T_1)^n$, $n = 0, 1, 2, \dots$, hence

$$T_n(\{x_k\}_{k=1}^{\infty}) = \{x_{k+n}\}_{k=1}^{\infty}$$

In particular, T_0 is the identity.

Proposition 1. It holds

$$\mathfrak{F}(T_n x) - \mathfrak{F}(T_{n+1} x) + x_{n+1} x_{n+2} \mathfrak{F}(T_{n+2} x) = 0, \quad n = 0, 1, 2, \dots \tag{1.1}$$

Proof. The proof can be found in [2]. ◻

Remark 2. Especially, it holds

$$\mathfrak{F}(x_k, \dots, x_{k+n}) = \mathfrak{F}(x_{k+1}, \dots, x_{k+n}) - x_k x_{k+1} \mathfrak{F}(x_{k+2}, \dots, x_{k+n}) \tag{1.2}$$

for all $k, n \in \mathbb{N}$.

Proposition 2. It holds

$$\mathfrak{F}(x_1, x_2, \dots, x_n) = \mathfrak{F}(x_n, x_{n-1}, \dots, x_1) \tag{1.3}$$

for all $n \in \mathbb{N}$.

Proof. Let us denote $y_j := x_{n-j+1}$ for $j = 1, \dots, n$, then

$$\begin{aligned}
&\mathfrak{F}(x_n, x_{n-1}, \dots, x_1) = \mathfrak{F}(y_1, y_2, \dots, y_n) = \\
&= 1 + \sum_{m=1}^{\left[\frac{n+1}{2}\right]} (-1)^m \sum_{k_1=1}^{n-2(m-1)} \sum_{k_2=k_1+2}^{n-2(m-2)} \cdots \sum_{k_m=k_{m-1}+2}^{n} y_{k_1} y_{k_1+1} y_{k_2} y_{k_2+1} \dots y_{k_m} y_{k_m+1} = \\
&= 1 + \sum_{m=1}^{\left[\frac{n+1}{2}\right]} (-1)^m \sum_{l_1=2m-1}^{n} \sum_{l_2=2m-3}^{l_1-2} \cdots \sum_{l_m=1}^{l_{m-1}-2} x_{l_1} x_{l_1+1} x_{l_2} x_{l_2+1} \dots x_{l_m} x_{l_m+1}
\end{aligned}$$

where we have substituted $l_j = n - k_j + 1$, $j = 1, \dots, n$. To proceed further we just denote the summand index l_j as l_{m-j+1} for all $j = 1, \dots, n$ and change the order of the sums such that the sum with the summand index l_1 will be at the first place, the sum with the summand index l_2 will be at the second place, etc.

$$\begin{aligned}
&= 1 + \sum_{m=1}^{\left[\frac{n+1}{2}\right]} (-1)^m \sum_{l_m=2m-1}^{n} \sum_{l_{m-1}=2m-3}^{l_m-2} \cdots \sum_{l_1=1}^{l_2-2} x_{l_1} x_{l_1+1} x_{l_2} x_{l_2+1} \dots x_{l_m} x_{l_m+1} = \\
&= 1 + \sum_{m=1}^{\left[\frac{n+1}{2}\right]} (-1)^m \sum_{l_{m-1}=2m-3}^{n-2} \sum_{l_{m-2}=2m-5}^{l_{m-1}-2} \cdots \sum_{l_1=1}^{l_2-2} \sum_{l_m=l_{m-1}+2}^{n} x_{l_1} x_{l_1+1} \dots x_{l_m} x_{l_m+1} = \\
&= 1 + \sum_{m=1}^{\left[\frac{n+1}{2}\right]} (-1)^m \sum_{l_{m-2}=2m-5}^{n-4} \cdots \sum_{l_1=1}^{l_2-2} \sum_{l_{m-1}=l_{m-2}+2}^{n-2} \sum_{l_m=l_{m-1}+2}^{n} x_{l_1} x_{l_1+1} \dots x_{l_m} x_{l_m+1} = \\
&= \cdots = \\
&= 1 + \sum_{m=1}^{\left[\frac{n+1}{2}\right]} (-1)^m \sum_{l_1=1}^{n-2(m-1)} \sum_{l_2=l_1+2}^{n-2(m-2)} \cdots \sum_{l_{m-1}=l_{m-2}+2}^{n-2} \sum_{l_m=l_{m-1}+2}^{n} x_{l_1} x_{l_1+1} \dots x_{l_m} x_{l_m+1} = \\
&= \mathfrak{F}(x_1, x_2, \dots, x_n).
\end{aligned}$$

◻

Remark 3. By using the last symmetry property of the function $\mathfrak{F}$ and the recurrent relation (1.2) one can easily derive the following identity

$$\mathfrak{F}(x_k, \dots, x_{k+n}) = \mathfrak{F}(x_k, \dots, x_{k+n-1}) - x_{k+n}x_{k+n-1}\mathfrak{F}(x_k, \dots, x_{k+n-2}) \tag{1.4}$$

for $k, n \in \mathbb{N}$

1.2 Function $\mathfrak{F}$ and the characteristic function of a Jacobi matrix

Let us denote

$$K_S := \begin{pmatrix} -\lambda_d & w & & & & & & & \\ w & -\lambda_{d-1} & w & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & w & -\lambda_1 & w & & & & \\ & & & w & 0 & w & & & \\ & & & & w & \lambda_1 & w & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & w & \lambda_{d-1} & w \\ & & & & & & & w & \lambda_d \end{pmatrix}$$

and set

$$\chi_S(z) := \det(zI - K_S).$$

The main result of my Bachelor's thesis follows.

Proposition 3. It holds

$$\begin{aligned} (-1)^d \frac{1}{z}\chi_S(z) = &\left(\prod_{k=1}^{d}(\lambda_k^2 - z^2)\right) \mathfrak{F}\left(\frac{w}{\lambda_1 - z}, \dots, \frac{w}{\lambda_d - z}\right) \mathfrak{F}\left(\frac{w}{\lambda_1 + z}, \dots, \frac{w}{\lambda_d + z}\right) \\ &+ 2\sum_{j=1}^{d} w^{2j} \left(\prod_{k=j+1}^{d}(\lambda_k^2 - z^2)\right) \mathfrak{F}\left(\frac{w}{\lambda_{j+1} - z}, \dots, \frac{w}{\lambda_d - z}\right) \mathfrak{F}\left(\frac{w}{\lambda_{j+1} + z}, \dots, \frac{w}{\lambda_d + z}\right) \end{aligned} \tag{1.5}$$

Proof. One can find the proof in [2]. □

Chapter 2

More on function $\mathfrak{F}$

2.1 A formula for $\mathfrak{F}(\frac{w}{m}, \frac{w}{m+1}, \dots, \frac{w}{n})$

Proposition 4. It holds

$$\sum_{k=m}^{n} \frac{(n+1-k)(n+2-k)\dots(n+s-1-k)}{k(k+1)\dots(k+s)} = \frac{(n-m+1)(n-m+2)\dots(n-m+s)}{s(n+s)m(m+1)\dots(m+s-1)} \tag{2.1}$$

for all $m, n, s \in \mathbb{N}, m \leq n$.

Proof. Denote the LHS by $Y_s(m,n)$. One can verify the statement by induction in s. The case $s = 1$ gives the identity

$$\sum_{k=m}^{n} \frac{1}{k(k+1)} = \frac{n-m+1}{(n+1)m}$$

which is easy to verify (since $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$).
In the induction step $s-1 \to s$, with $s > 1$, observe that

$$Y_s(m,n) = -\frac{n+2s-1}{s} Y_{s-1}(m+1, n+1) + \frac{n+s-1}{s} Y_{s-1}(m,n).$$

Now apply the induction hypothesis.

$$\begin{aligned} Y_s(m,n) = &-\frac{n+2s-1}{s} \frac{(n-m+1)(n-m+2)\dots(n-m+s-1)}{(s-1)(n+s)(m+1)(m+2)\dots(m+s-1)} + \\ &+ \frac{n+s-1}{s} \frac{(n-m+1)(n-m+2)\dots(n-m+s-1)}{(s-1)(n+s-1)m(m+1)\dots(m+s-2)} = \\ &= \frac{(n-m+1)(n-m+2)\dots(n-m+s)}{s(n+s)m(m+1)(m+2)\dots(m+s-1)} = RHS \end{aligned}$$

□

Proposition 5. It holds

$$\sum_{k_1=m}^{n-2s+2}\sum_{k_2=k_1+2}^{n-2s+4}\cdots\sum_{k_s=k_{s-1}+2}^{n}\frac{1}{k_1(k_1+1)k_2(k_2+1)\dots k_s(k_s+1)}$$
$$=\frac{(n-m-2s+3)(n-m-2s+4)\dots(n-m-s+2)}{s!m(m+1)\dots(m+s-1)(n-s+2)(n-s+3)\dots(n+1)} \quad (2.2)$$

for all $m,n,s\in\mathbb{N}, m\le n-2s+2$.

Proof. This can be again proved by the induction in s. For $s=1$ we get the well known identity

$$\sum_{k=m}^{n}\frac{1}{k(k+1)}=\frac{n-m+1}{(n+1)m}.$$

Denote the LHS by X_s. To carry out the induction step from $s-1$ to s, with $s>1$, we apply the induction hypothesis(IH) to the summation in the indices $k_2\dots k_s$ and find that (when writing k instead of k_1 and k_{j-1} instead of k_j)

$$X_s=\sum_{k=m}^{n-2s+2}\frac{1}{k(k+1)}\sum_{k_1=k+2}^{n-2s+4}\cdots\sum_{k_{s-1}=k_{s-2}+2}^{n}\frac{1}{k_1(k_1+1)k_2(k_2+1)\dots k_{s-1}(k_{s-1}+1)}\overset{IH}{=}$$
$$\overset{IH}{=}\sum_{k-m}^{n-2s+2}\frac{(n-k-2s+3)(n-k-2s+4)\dots(n-k-s+1)}{(s-1)!(n-s+3)\dots(n+1)k(k+1)(k+2)\dots(k+s)}.$$

To conclude the proof it suffices to apply the preceding identity (2.1), with n being replaced by $n-2s+2$, to the RHS in (2.1). Then

$$X_s=\frac{(n-2s-m+3)(n-2s-m+4)\dots(n-s-m+2)}{(s-1)!(n-s+3)\dots(n+1)s(n-s+2)m(m+1)\dots(m+s-1)}=$$
$$=\frac{(n-2s-m+3)(n-2s-m+4)\dots(n-s-m+2)}{s!m(m+1)\dots(m+s-1)(n-s+2)\dots(n+1)}=RHS.$$

□

Corollary 1. It holds

$$\mathfrak{F}\left(\frac{w}{m},\frac{w}{m+1},\dots,\frac{w}{n}\right)=\frac{(m-1)!}{n!}\sum_{s=0}^{[\frac{n-m+1}{2}]}(-1)^s\frac{(n-s)!(n-m-s+1)!}{s!(m+s-1)!(n-m-2s+1)!}w^{2s}$$

for all $m,n\in\mathbb{N}, m\le n$.

Proof. Using the definition of the function $\mathfrak{F}$ we have

$$\mathfrak{F}\left(\frac{w}{m},\frac{w}{m+1},\dots,\frac{w}{n}\right)=$$
$$=1+\sum_{s=1}^{\infty}(-1)^s\sum_{k_1=m}^{n-2s+1}\sum_{k_2=k_1+2}^{n-2s+3}\cdots\sum_{k_s=k_{s-1}+2}^{n-1}\frac{w^{2s}}{k_1(k_1+1)k_2(k_2+1)\dots k_s(k_s+1)}.$$

Now observe that s is restricted by the inequality $2s \leq n - m + 1$ and use identity (2.2) with n replaced by $n - 1$.Then

$$\mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \dots, \frac{w}{n}\right) = \frac{(m-1)!}{n!} \sum_{s=0}^{[\frac{n-m+1}{2}]} (-1)^s \frac{(n-s)!(n-m-s+1)!}{s!(m+s-1)!(n-m-2s+1)!} w^{2s}.$$

□

2.2 A relation to Bessel function

In this section we will investigate an asymptotic behaviour of the function $\mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \dots, \frac{w}{n}\right)$ for w small. There appears a relation between the function $\mathfrak{F}$ and Bessel function of the first and second kind.

Remark 4. The Bessel function of the first kind $J_m(z)$ and the second kind $Y_n(z)$ appears in this and the following chapter. We will often use the expansions of $J_m(z)$ and $Y_n(z)$ (found in [1] 9.1.10, 9.1.11):

$$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{k=0}^{\infty} \frac{1}{k!(m+k)!} \left(-\frac{1}{4}z^2\right)^k \tag{2.3}$$

$$Y_n(z) = -\frac{1}{\pi}\left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{1}{4}z^2\right)^k + \frac{2}{\pi} \ln\left(\frac{z}{2}\right) J_n(z) + O(z^n) \tag{2.4}$$

for $m, n \in \mathbb{N}_0$. Also generalized hypergeometric functions will be used a lot. The F_{pq} function has the series expansion

$$F_{pq}(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k, \dots, (a_p)_k}{(b_1)_k, \dots, (b_q)_k} \frac{z^k}{k!} \tag{2.5}$$

where $(c)_k = c(c+1)\dots(c+k-1)$ and $(c)_0 = 1$.

2.2.1 Case $m = 1$

Proposition 6. It holds

$$\mathfrak{F}\left(w, \frac{w}{2}, \dots, \frac{w}{n}\right) = -\frac{\pi}{n!} w^{n+1} J_0(2w) Y_{n+1}(2w) + O(w^{2n+2} \ln w), \tag{2.6}$$

where $n \in \mathbb{N}$.

Proof. It holds

$$F_{23}\left(a, a + \frac{1}{2}; d, 2a, 2a - d + 1; z\right) = F_{01}\left(d; \frac{z}{4}\right) F_{01}\left(2a - d + 1; \frac{z}{4}\right). \tag{2.7}$$

This identity can be found in http://functions.wolfram.com/07.26.03.6005.01. And by using another identity ([1] 9.1.69)

$$F_{01}\left(\nu+1;-\frac{1}{4}z^2\right)=J_\nu(z)\Gamma(\nu+1)\left(\frac{z}{2}\right)^{-\nu}$$

we get

$$F_{23}\left(-\frac{\mu}{2},-\frac{\mu}{2}+\frac{1}{2};1,-\mu,-\mu;-4w^2\right)=J_0(2w)F_{01}(-\mu;-w^2)$$

where $a=-\frac{\mu}{2}$, $d=1$, $z=-4w^2$. By using the series expansion (2.5) and (2.3) one can write the last identity as

$$\sum_{k=0}^{\infty}\frac{(\frac{1}{2}-\frac{\mu}{2})_k(-\frac{\mu}{2})_k}{(1)_k(-\mu)_k(-\mu)_k}(-1)^k\frac{(2w)^{2k}}{k!}=\sum_{s=0}^{\infty}\sum_{l=0}^{s}\frac{(-1)^s}{(-\mu)_l l![(s-l)!]^2}w^{2s}\qquad(\mu\notin\mathbb{N}).$$

Next, take into consideration only terms with $w^{2j}, j=0,1,\dots,n$

$$\sum_{k=0}^{n}\frac{(\frac{1}{2}-\frac{\mu}{2})_k(-\frac{\mu}{2})_k}{(1)_k(-\mu)_k(-\mu)_k}(-1)^k\frac{(2w)^{2k}}{k!}=\sum_{s=0}^{n}\sum_{l=0}^{s}\frac{(-1)^s}{(-\mu)_l l![(s-l)!]^2}w^{2s}$$

and make a limit $\mu\to n$

$$\sum_{k=0}^{n}\frac{(\frac{1}{2}-\frac{n}{2})_k(-\frac{n}{2})_k}{(1)_k(-n)_k(-n)_k}(-1)^k\frac{(2w)^{2k}}{k!}=\sum_{s=0}^{n}\sum_{l=0}^{s}\frac{(-1)^s}{(-n)_l l![(s-l)!]^2}w^{2s}.$$

$$LHS=\sum_{k=0}^{n}\frac{(\frac{1}{2}-\frac{n}{2})_k(-\frac{n}{2})_k}{(1)_k(-n)_k(-n)_k}(-1)^k\frac{(2w)^{2k}}{k!}=\frac{1}{n!}\sum_{s=0}^{[\frac{n}{2}]}(-1)^s\left(\frac{(n-s)!}{s!}\right)^2\frac{w^{2s}}{(n-2s)!}=$$
$$=\mathfrak{F}\left(w,\frac{w}{2},\dots,\frac{w}{n}\right)$$

The last equality holds due to the Corollary 1 with $m=1$. Thus, we have

$$\mathfrak{F}\left(w,\frac{w}{2},\dots,\frac{w}{n}\right)=\sum_{s=0}^{n}\sum_{l=0}^{s}\frac{(-1)^s}{(-n)_l l![(s-l)!]^2}w^{2s}$$
$$=\frac{1}{n!}\sum_{s=0}^{n}\left(\sum_{l=0}^{s}\frac{(-1)^l(n-l)!}{l![(s-l)!]^2}\right)(-1)^s w^{2s}.$$

Next, consider the RHS in the statement of the proposition. Taking into account the series expansion of Bessel functions (2.3) and (2.4) one arrives at the following expression

$$-\frac{\pi}{n!}w^{n+1}J_0(2w)Y_{n+1}(2w)=\frac{1}{n!}\sum_{s=0}^{n}\sum_{l=0}^{s}\frac{(n-l)!}{l!}\frac{(-1)^{s-l}}{[(s-l)!]^2}w^{2s}+O(w^{2n+2}\ln w)$$

and the statement is verified. □

2.2.2 General case $m \in \mathbb{N}$

In this section we will generalize the procedure used in the previous special case with $m = 1$.

Proposition 7. It holds

$$\mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \dots, \frac{w}{n}\right) = -\pi\frac{(m-1)!}{n!}w^{n-m+2}J_{m-1}(2w)Y_{n+1}(2w) + O(w^{2n-2m+4}) \tag{2.8}$$

for all $m, n \in \mathbb{N}$, $2 \le m \le n$.

Proof. Investigate the following generalized hypergeometric function (observe that the index k is restricted ($k \le \left[\frac{n-m+1}{2}\right]$) due to the nominator of the expression)

$$\begin{aligned}
&F_{23}\left(\frac{m-1-n}{2}, \frac{m-n}{2}; m, m-1-n, -n; -4w^2\right)\\
&\quad= \sum_{k=0}^{\infty} \frac{\left(\frac{m-1-n}{2}\right)_k \left(\frac{m-n}{2}\right)_k}{(m)_k(m-1-n)_k(-n)_k}\frac{(-4w^2)^k}{k!}\\
&\quad= \sum_{k=0}^{\left[\frac{n-m+1}{2}\right]} \frac{(-1)^k(n-m+1)(n-m+1)\dots(n-m+2-2k)}{m\dots(m+k-1)(n-m+1)\dots(n-m+2-k)n\dots(n+1-k)}\frac{w^{2k}}{k!}\\
&\quad= \frac{(m-1)!}{n!}\sum_{k=0}^{\left[\frac{n-m+1}{2}\right]} (-1)^k\frac{(n-m+1-k)!(n-k)!}{(n-m+1-2k)!(m+k-1)!}\frac{w^{2k}}{k!}\\
&\quad= \mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \dots, \frac{w}{n}\right).
\end{aligned}$$

The last equality holds due to Corollary 1. By using identity (2.7) with $d = m$, $a = \frac{m-1-\mu}{2}$ and $z = -4w^2$ we get

$$F_{23}\left(\frac{m-1-\mu}{2}, \frac{m-\mu}{2}; m, m-1-\mu, -\mu; -4w^2\right) = F_{01}(m; -w^2)F_{01}(-\mu; -w^2).$$

The parameter μ is near n but μ is not an integer. Then, by using definitions of the functions F_{23} and F_{01} (2.5), restricting both sides of the last equation such that there remains only terms with w^{2j}, $j = 0, \dots, n-m+1$ and making a limit $\mu \to n$ (very similar procedure was made in the proof of Proposition 6) one easily arrives at the expression

$$\begin{aligned}
\mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \dots, \frac{w}{n}\right) &= \sum_{s=0}^{n-m+1}\sum_{l=0}^{s}\frac{1}{(m)_{s-l}(-n)_l}\frac{(-1)^s w^{2s}}{l!(s-l)!}\\
&= \frac{(m-1)!}{n!}\sum_{s=0}^{n-m+1}\sum_{l=0}^{s}(-1)^{s-l}\frac{(n-l)!}{l!(s-l)!(m+s-l-1)!}w^{2s}.
\end{aligned} \tag{2.9}$$

By using (2.3) and (2.4) again we find the asymptotic expansion of the Bessel functions on the RHS in the statement.

$$
\begin{aligned}
&-\pi w^{n-m+2} J_{m-1}(2w)Y_{n+1}(2w) = \\
&\quad = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m-1+k)!} w^{2k} \left(\sum_{l=0}^{n} \frac{(n-l)!}{l!} w^{2k} + O(w^{2n+2}\ln w) \right) = \\
&\quad = \sum_{l=0}^{n} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m-1+k)!} \frac{(n-l)!}{l!} w^{2(k+l)} + O(w^{2n+2}\ln w) = \\
&\quad = \sum_{l=0}^{n} \sum_{s=l}^{\infty} \frac{(-1)^{s-l}}{(s-l)!(m-1+s-l)!} \frac{(n-l)!}{l!} w^{2s} + O(w^{2n+2}\ln w) = \\
&\quad = \sum_{s=0}^{\infty} \sum_{l=0}^{s} \frac{(-1)^{s-l}(n-l)!}{l!(s-l)!(m-1+s-l)!} w^{2s} + O(w^{2n+2}\ln w) = \\
&\quad = \sum_{s=0}^{n-m+1} \sum_{l=0}^{s} (-1)^{s-l} \frac{(n-l)!}{l!(s-l)!(m-1+s-l)!} w^{2s} + O(w^{2n-2m+4}) \qquad (2.10)
\end{aligned}
$$

Finally, expression (2.9) together with the last expansion (2.10) prove the statement.

□

Remark 5. One can see from the proof of Proposition 7 that if we were more precise in expression (2.10) we would obtain a more precise asymptotic expansion for the function $\mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \ldots, \frac{w}{n}\right)$. If we do this we get

$$
\begin{aligned}
\frac{n!}{(m-1)!} \mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \ldots, \frac{w}{n}\right) &= -\pi w^{n-m+2} J_{m-1}(2w)Y_{n+1}(2w) \\
&\quad - \sum_{s=n-m+2}^{n} \sum_{l=0}^{s} (-1)^{s-l} \frac{(n-l)!}{l!(s-l)!(m-1+s-l)!} w^{2s} + O(w^{2n+2}\ln w)
\end{aligned}
\qquad (2.11)
$$

where $1 \leq m \leq n$, $m, n \in \mathbb{N}$.

Lemma 1. It holds

$$
\sum_{k=n}^{\infty} \binom{k}{n} x^{k-n} = \frac{1}{(1-x)^{n+1}} \qquad (2.12)
$$

for all $x \in \mathbb{R}, |x| < 1$ and all $n \in \mathbb{N}_0$.

Proof.

$$
\begin{aligned}
\sum_{k=n}^{\infty} \binom{k}{n} x^{k-n} &= \sum_{k=0}^{\infty} \binom{n+k}{n} x^k = \frac{1}{n!} \sum_{k=0}^{\infty} (n+k)(n+k-1)\ldots(k+1)x^k = \\
&= \frac{1}{n!} \frac{d^n}{dx^n} \left(\sum_{k=0}^{\infty} x^k \right) = \frac{1}{n!} \frac{d^n}{dx^n} \left(\frac{1}{1-x} \right) = \frac{1}{(1-x)^{n+1}}
\end{aligned}
$$

Since $|x| < 1$ the sum $\sum_{k=0}^{\infty}\binom{n+k}{n}x^k$ converges and the change of order of the derivation $\frac{d^n}{dx^n}$ and the sum $\sum_{k=0}^{\infty}x^k$ is correct. □

Lemma 2. It holds

$$\sum_{l=0}^{s}(-1)^l\binom{s+m}{l+m}\binom{n+l}{n} = \binom{m+s-n-1}{s} \tag{2.13}$$

for all $m, n, s \in \mathbb{N}_0$, $m > n$.

Proof. We will investigate this power serie

$$\sum_{s=0}^{\infty}\sum_{l=0}^{s}(-1)^l\binom{s+m}{l+m}\binom{n+l}{n}x^s$$

where $x \in \mathbb{R}$, $|x| < \frac{1}{2}$.

$$\sum_{s=0}^{\infty}\sum_{l=0}^{s}(-1)^l\binom{s+m}{l+m}\binom{n+l}{n}x^s = \sum_{l=0}^{\infty}(-1)^l\binom{n+l}{n}x^l\sum_{s=l}^{\infty}\binom{s+m}{l+m}x^{s-l}$$

Next, by applying Lemma 8 to the inner sum we get

$$\sum_{s=0}^{\infty}\sum_{l=0}^{s}(-1)^l\binom{s+m}{l+m}\binom{n+l}{n}x^s = \sum_{l=0}^{\infty}(-1)^l\binom{n+l}{n}\frac{x^l}{(1-x)^{m+l+1}}.$$

Then we can use Lemma 8 again and we arrive at the expression

$$\sum_{s=0}^{\infty}\sum_{l=0}^{s}(-1)^l\binom{s+m}{l+m}\binom{n+l}{n}x^s = \frac{1}{(1-x)^{m+1}}\frac{1}{(1+\frac{x}{1-x})^{n+1}} = \frac{1}{(1-x)^{m-n}}.$$

Note that if $|x| < \frac{1}{2}$ then $|\frac{x}{1-x}| < 1$. Thus

$$\sum_{l=0}^{s}(-1)^l\binom{s+m}{l+m}\binom{n+l}{n} = \frac{1}{s!}\frac{d^s}{dx^s}(1-x)^{n-m}\mid_{x=0}=$$

$$= \frac{(-1)^s}{s!}(n-m)(n-m-1)\dots(n-m-s+1) = \binom{m+s-n-1}{s}.$$

□

Now we can obtain a more precise asymptotic expansion for the function $\mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \dots, \frac{w}{n}\right)$.

Proposition 8. It holds

$$\frac{n!}{(m-1)!}\mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \dots, \frac{w}{n}\right) = -\pi w^{n-m+2}J_{m-1}(2w)Y_{n+1}(2w)-$$
$$-\sum_{s=0}^{m-2}\frac{s!(m+n-2s-2)!}{(n+m-s-1)!(m-s-2)!(n-s)!}w^{2n-2s} + O(w^{2n+2}\ln w) \tag{2.14}$$

for all $m, n \in \mathbb{N}, 1 \le m \le n$.

Proof. The case $m = 1$ have already been proved in Proposition 6. Let $1 < m \le n$, start with expression (2.11):

$$\frac{n!}{(m-1)!}\mathfrak{F}\left(\frac{w}{m},\frac{w}{m+1},\dots,\frac{w}{n}\right) = -\pi w^{n-m+2}J_{m-1}(2w)Y_{n+1}(2w)$$
$$-\sum_{s=n-m+2}^{n}\sum_{l=0}^{s}(-1)^{s-l}\frac{(n-l)!}{l!(s-l)!(m-1+s-l)!}w^{2s} + O(w^{2n+2}\ln w).$$

Next, apply Lemma 2 to the inner sum in the second term on the RHS.

$$\sum_{l=0}^{s}(-1)^{s-l}\frac{(n-l)!}{l!(s-l)!(m-1+s-l)!} = \sum_{l=0}^{s}(-1)^{l}\frac{(n-s+l)!}{l!(s-l)!(m-1+l)!} =$$
$$= \frac{(n-s)!}{(s+m-1)!}\sum_{l=0}^{s}(-1)^{l}\binom{s+m-1}{l+m-1}\binom{n-s+l}{n-s} =$$
$$= \frac{(n-s)!}{(s+m-1)!}\binom{m+2s-n-2}{s}$$

Then we have

$$\frac{n!}{(m-1)!}\mathfrak{F}\left(\frac{w}{m},\frac{w}{m+1},\dots,\frac{w}{n}\right) = -\pi w^{n-m+2}J_{m-1}(2w)Y_{n+1}(2w) -$$
$$-\sum_{s=n-m+2}^{n}\frac{(n-s)!}{(s+m-1)!}\binom{m+2s-n-2}{s}w^{2s} + O(w^{2n+2}\ln w) =$$
$$= -\pi w^{n-m+2}J_{m-1}(2w)Y_{n+1}(2w) -$$
$$-\sum_{s=0}^{m-2}\frac{s!}{(n+m-s-1)!}\binom{m+n-2s-2}{n-s}w^{2n-2s} + O(w^{2n+2}\ln w) =$$
$$= -\pi w^{n-m+2}J_{m-1}(2w)Y_{n+1}(2w) -$$
$$-\sum_{s=0}^{m-2}\frac{s!(m+n-2s-2)!}{(n+m-s-1)!(m-s-2)!(n-s)!}w^{2n-2s} + O(w^{2n+2}\ln w).$$

□

Chapter 3

The reduced characteristic function

Let us denote

$$K := \begin{pmatrix} -d & w & & & & & & & \\ w & -d+1 & w & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & w & -1 & w & & & & \\ & & & w & 0 & w & & & \\ & & & & w & 1 & w & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & w & d-1 & w \\ & & & & & & & w & d \end{pmatrix}$$

and set

$$\chi_{red}(z) := \frac{(-1)^d}{z} \det(zI - K).$$

$\chi_{red}(z)$ is a polynomial in w of order $2d$. Further $\chi_{red}(z)$ will be called the *characteristic reduced function*. By substituteing λ_j by j in (1.5) one gets

$$\chi_{red}(z) = \left(\prod_{k=1}^{d}(k^2 - z^2)\right) \mathfrak{F}\left(\frac{w}{1-z}, \ldots, \frac{w}{d-z}\right) \mathfrak{F}\left(\frac{w}{1+z}, \ldots, \frac{w}{d+z}\right) + \\ + 2\sum_{j=1}^{d} w^{2j} \left(\prod_{k=j+1}^{d}(k^2 - z^2)\right) \mathfrak{F}\left(\frac{w}{j+1-z}, \ldots, \frac{w}{d-z}\right) \mathfrak{F}\left(\frac{w}{j+1+z}, \ldots, \frac{w}{d+z}\right) \tag{3.1}$$

Remark 6. Note that the characteristic reduced function is an even polynomial in the variable z, i.e.,

$$\chi_{red}(-z) = \chi_{red}(z) \qquad \text{for all } z \in \mathbb{C}. \tag{3.2}$$

3.1 Particular values of $\chi_{red}(z)$

In this section we would like to obtain an expression for $\chi_{red}(n)$ for $n = 0, 1, \dots, d$ by simplifying formula (3.1). But we will not arrive at the general expression in this way as it will be seen further. We start from examining cases $n = 0$ and $n = 1$.

3.1.1 Case $n = 0$

The value $\chi_{red}(0)$ is to be treated separately. Formula (3.1) with $z = 0$ gives

$$\chi_{red}(0) = (d!)^2 \mathfrak{F}\left(w, \frac{w}{2}, \dots, \frac{w}{d}\right)^2 + 2\sum_{j=1}^{d} w^{2j} \left(\frac{d!}{j!}\right)^2 \mathfrak{F}\left(\frac{w}{j+1}, \frac{w}{j+2}, \dots, \frac{w}{d}\right)^2. \tag{3.3}$$

Next, we use the asymptotic expression for the function $\mathfrak{F}$ (2.6) and (2.8) obtained in the previous chapter and we arrive at the expression

$$\chi_{red}(0) = \pi^2 w^{2d+2} Y_{d+1}^2(2w) \left(J_0^2(2w) + 2\sum_{j=1}^{d} J_j^2(2w) \right) + O(w^{2d+2} \ln w).$$

To proceed further we use the identity

$$J_0^2(z) + 2\sum_{j=1}^{\infty} J_j^2(z) = 1$$

which can be found in [1] (9.1.76). Then it holds

$$J_0^2(z) + 2\sum_{j=1}^{d} J_j^2(z) = 1 + O(w^{2d+2})$$

and we have

$$\chi_{red}(0) = \pi^2 w^{2d+2} Y_{d+1}^2(2w) + O(w^{2d+2} \ln w).$$

To continue we take into consideration the asymptotic expansion of the Bessel function Y (2.4) and we get

$$\chi_{red}(0) = \left(\sum_{k=0}^{d} \frac{(d-k)!}{k!} w^{2k} \right)^2 + O(w^{2d+2} \ln w).$$

Since there is a polynomial in w of order $2d$ on the LHS the Landau symbol on the RHS will be omitted (only terms with $w^{2j}, 0 \le j \le d$ remain) and we can write

$$\chi_{red}(0) = \sum_{s=0}^{d} \sum_{k=0}^{s} \frac{(d-k)!(d-s+k)!}{k!(s-k)!} w^{2s}. \tag{3.4}$$

Lemma 3. It holds

$$\sum_{k=0}^{m} \binom{s+k}{s} \binom{p+m-k}{p} = \binom{s+p+m+1}{s+p+1}$$

for $m, p, s \in \mathbb{N}_0$.

Proof. Let us define function

$$f_s(z) = \sum_{k=0}^{\infty} \binom{s+k}{s} z^k$$

for all $z \in \mathbb{C}$, $|z| < 1$. Since

$$\binom{s+k}{k} = \frac{1}{k!}(s+k)\dots(s+1) = \frac{(-1)^k}{k!}(-s-1)\dots(-s-k) = (-1)^k \binom{-s-1}{k}$$

we have

$$f_s(z) = \sum_{k=0}^{\infty} \binom{s+k}{k} z^k = \sum_{k=0}^{\infty} \binom{-s-1}{k} (-z)^k = (1-z)^{-s-1}.$$

It follows that the identity

$$f_s(z) f_p(z) = f_{s+p+1}(z)$$

holds for all $z, |z| < 1$. Then by using the previous identity we get

$$\sum_{m=0}^{\infty} \binom{s+p+1+m}{s+p+1} z^m \equiv f_{s+p+1}(z) = f_s(z) f_p(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{s+k}{s} \binom{p+l}{p} z^{k+l} =$$
$$= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \binom{s+k}{s} \binom{p+m-k}{p} \right) z^m$$

which proves the statement. □

Now we can write the final expression for $\chi_{red}(0)$.

Corollary 2. It holds

$$\chi_{red}(0) = \sum_{s=0}^{d} \frac{[(d-s)!]^2 (2d-s+1)!}{s!(2d-2s+1)!} w^{2s}. \tag{3.5}$$

Proof. To verify the statement it is enough to compute the inner sum in expression (3.4).

$$\sum_{k=0}^{s} \frac{(d-k)!(d-s+k)!}{k!(s-k)!} = [(d-s)!]^2 \sum_{k=0}^{s} \frac{(d-k)!}{(d-s)!(s-k)!} \frac{(d-s+k)!}{(d-s)!k!} =$$
$$= [(d-s)!]^2 \sum_{k=0}^{s} \binom{d-k}{d-s} \binom{d-s+k}{d-s}$$

The last sum is a special case of the sum of the previous lemma. Thus, writing s instead of m, $(d-s)$ instead of s and $(d-s)$ instead of p the previous lemma follows that the identity

$$\sum_{k=0}^{s} \binom{d-k}{d-s} \binom{d-s+k}{d-s} = \binom{2d-s+1}{2d-2s+1} = \frac{(2d-s+1)!}{s!(2d-2s+1)!}$$

holds and the statement is proved. ☐

3.1.2 Case $n = 1$

Lemma 4. The identity

$$\frac{\chi_{red}(1)}{(d-1)!(d+1)!} = -\frac{w^{2s}}{(s-1)!s!} \mathfrak{F}\left(\frac{w}{s+1}, \dots, \frac{w}{d-1}\right) \mathfrak{F}\left(\frac{w}{s+1}, \dots, \frac{w}{d+1}\right) +$$
$$+ \frac{w^{2s}}{(s-1)!s!} \mathfrak{F}\left(\frac{w}{s}, \dots, \frac{w}{d-1}\right) \mathfrak{F}\left(\frac{w}{s+2}, \dots, \frac{w}{d+1}\right) +$$
$$+ 2\sum_{j=s+1}^{d} \frac{w^{2j}}{(j-1)!(j+1)!} \mathfrak{F}\left(\frac{w}{j}, \dots, \frac{w}{d-1}\right) \mathfrak{F}\left(\frac{w}{j+2}, \dots, \frac{w}{d+1}\right)$$

holds for all $s = 1, 2, \dots, d$.

Proof. The statement will be proved by induction in s. For s=1 we start with formula (3.1) for $\chi_{red}(z)$. Notice that the recurrent relation (1.2) implies

$$(1-z)\mathfrak{F}\left(\frac{w}{1-z}, \dots, \frac{w}{d-z}\right)\Bigg|_{z=1} = -\frac{w^2}{2-z}\mathfrak{F}\left(\frac{w}{3-z}, \dots, \frac{w}{d-z}\right)\Bigg|_{z=1} =$$
$$= -w^2 \mathfrak{F}\left(\frac{w}{2}, \dots, \frac{w}{d-1}\right)$$

and thus the formula for $\chi_{red}(z)$ (3.1) leads to the expression

$$\chi_{red}(1) = (d-1)!(d+1)!(-w^2)\mathfrak{F}\left(\frac{w}{2}, \dots, \frac{w}{d-1}\right) \mathfrak{F}\left(\frac{w}{2}, \dots, \frac{w}{d+1}\right) +$$
$$+ 2\sum_{j=1}^{d} \frac{(d-1)!}{(j-1)!} \frac{(d+1)!}{(j+1)!} \mathfrak{F}\left(\frac{w}{j}, \dots, \frac{w}{d-1}\right) \mathfrak{F}\left(\frac{w}{j+2}, \dots, \frac{w}{d-1}\right)$$

which satisfy the statement of the lemma for $s = 1$. To carry out the induction step $s \to s+1$, with $1 \leq s < d$, we start from the expression

$$
\begin{aligned}
&- \frac{w^{2s}}{(s-1)!s!}\mathfrak{F}\left(\frac{w}{s+1},\ldots,\frac{w}{d-1}\right)\mathfrak{F}\left(\frac{w}{s+1},\ldots,\frac{w}{d+1}\right)+\\
&+ \frac{w^{2s}}{(s-1)!s!}\mathfrak{F}\left(\frac{w}{s},\ldots,\frac{w}{d-1}\right)\mathfrak{F}\left(\frac{w}{s+2},\ldots,\frac{w}{d+1}\right)+\\
&+ 2\frac{w^{2(s+1)}}{s!(s+2)!}\mathfrak{F}\left(\frac{w}{s+1},\ldots,\frac{w}{d-1}\right)\mathfrak{F}\left(\frac{w}{s+3},\ldots,\frac{w}{d-1}\right).
\end{aligned}
$$

Applying the recurrent relation (1.2) to the term $\mathfrak{F}\left(\frac{w}{s+1},\ldots,\frac{w}{d+1}\right)$ in the first summand and, at the same time, to the term $\mathfrak{F}\left(\frac{w}{s},\ldots,\frac{w}{d-1}\right)$ in the second summand we arrive at the expression

$$
\begin{aligned}
&- \frac{w^{2s+2}}{(s+1)!s!}\mathfrak{F}\left(\frac{w}{s+2},\ldots,\frac{w}{d-1}\right)\mathfrak{F}\left(\frac{w}{s+2},\ldots,\frac{w}{d+1}\right)+\\
&+ \frac{w^{2s+2}}{(s-1)!(s+2)!}\left(1+\frac{2}{s}\right)\mathfrak{F}\left(\frac{w}{s+1},\ldots,\frac{w}{d-1}\right)\mathfrak{F}\left(\frac{w}{s+3},\ldots,\frac{w}{d+1}\right).
\end{aligned}
$$

The induction step follows.

$$
\begin{aligned}
\frac{\chi_{red}(1)}{(d-1)!(d+1)!} \overset{IH}{=} &- \frac{w^{2s}}{(s-1)!s!}\mathfrak{F}\left(\frac{w}{s+1},\ldots,\frac{w}{d-1}\right)\mathfrak{F}\left(\frac{w}{s+1},\ldots,\frac{w}{d+1}\right)+\\
&+ \frac{w^{2s}}{(s-1)!s!}\mathfrak{F}\left(\frac{w}{s},\ldots,\frac{w}{d-1}\right)\mathfrak{F}\left(\frac{w}{s+2},\ldots,\frac{w}{d+1}\right)+\\
&+ 2\frac{w^{2(s+1)}}{s!(s+2)!}\mathfrak{F}\left(\frac{w}{s+1},\ldots,\frac{w}{d-1}\right)\mathfrak{F}\left(\frac{w}{s+3},\ldots,\frac{w}{d-1}\right)+\\
&+ 2\sum_{j=s+2}^{d}\frac{w^{2j}}{(j-1)!(j+1)!}\mathfrak{F}\left(\frac{w}{j},\ldots,\frac{w}{d-1}\right)\mathfrak{F}\left(\frac{w}{j+2},\ldots,\frac{w}{d+1}\right)=\\
&- \frac{w^{2s+2}}{s!(s+1)!}\mathfrak{F}\left(\frac{w}{s+2},\ldots,\frac{w}{d-1}\right)\mathfrak{F}\left(\frac{w}{s+2},\ldots,\frac{w}{d+1}\right)+\\
&+ \frac{w^{2s+2}}{s!(s+1)!}\mathfrak{F}\left(\frac{w}{s+1},\ldots,\frac{w}{d-1}\right)\mathfrak{F}\left(\frac{w}{s+3},\ldots,\frac{w}{d+1}\right)+\\
&+ 2\sum_{j=s+2}^{d}\frac{w^{2j}}{(j-1)!(j+1)!}\mathfrak{F}\left(\frac{w}{j},\ldots,\frac{w}{d-1}\right)\mathfrak{F}\left(\frac{w}{j+2},\ldots,\frac{w}{d+1}\right)
\end{aligned}
$$

□

With the aid of the previous lemma we can obtain the following formula for $\chi_{red}(1)$.

Proposition 9.

$$\chi_{red}(1) = (d+1)w^{2d} \tag{3.6}$$

Proof. To prove this it is enough to use the previous lemma for $s = d$. Pay attention, one has to set

$$\mathfrak{F}\left(\frac{w}{d+1}, \ldots, \frac{w}{d-1}\right) = 0$$

while

$$\mathfrak{F}\left(\frac{w}{d}, \ldots, \frac{w}{d-1}\right) = \mathfrak{F}\left(\frac{w}{d+2}, \ldots, \frac{w}{d+1}\right) = 1$$

(for satisfying the recurrent relation). □

3.1.3 General case $n \in \mathbb{N}$

We will generalize the relation from Lemma 4.

Proposition 10. Let $n \in \{1, \ldots, d\}$ then the identity

$$\begin{aligned}
\frac{\chi_{red}(n)}{(d-n)!(d+n)!} &= \frac{(-1)^n w^{2s}}{n(s-1)!s!}\mathfrak{F}\left(\frac{w}{s+1}, \ldots, \frac{w}{d-n}\right)\mathfrak{F}\left(\frac{w}{s+1}, \ldots, \frac{w}{d+n}\right) + \\
&+ 2s\frac{w^{2s}}{n}\sum_{j=1}^{n-1}\frac{(-1)^{n+j}}{(s-j)!(s+j)!}\mathfrak{F}\left(\frac{w}{s+1-j}, \ldots, \frac{w}{d-n}\right)\mathfrak{F}\left(\frac{w}{s+1+j}, \ldots, \frac{w}{d+n}\right) + \\
&+ \frac{w^{2s}}{n(s-n)!(s+n-1)!}\mathfrak{F}\left(\frac{w}{s+1-n}, \ldots, \frac{w}{d-n}\right)\mathfrak{F}\left(\frac{w}{s+1+n}, \ldots, \frac{w}{d+n}\right) + \\
&+ 2\sum_{j=s+1}^{d}\frac{w^{2j}}{(j-n)!(j+n)!}\mathfrak{F}\left(\frac{w}{j+1-n}, \ldots, \frac{w}{d-n}\right)\mathfrak{F}\left(\frac{w}{j+1+n}, \ldots, \frac{w}{d+n}\right)
\end{aligned} \tag{3.7}$$

holds for $s \in \{n, n+1, \ldots, d-n+1\}$.

We will not verify the statement here because we will prove a more general statement with the same identity but with larger set of index s in section 4.4.

Remark 7. Although this proposition is a generalization of Lemma 4 we can't derive a formula for $\chi_{red}(n)$ as easy as in the previous special case with $n = 1$. The formula for $\chi_{red}(n)$ will be derive by another way later (in chapter 5).

Chapter 4

An extension of the function $\mathfrak{F}$

In this chapter we will extend the function $\mathfrak{F}$ with the aid of the recursive identity (1.2). The expanded function is denoted $\mathfrak{J}$. Then we will find a formula for the function $\mathfrak{J}$ and at the end of this chapter we will prove a generalization of Proposition 10.

4.1 Function $\mathfrak{G}$

Let us denote

$$\mathfrak{G}(m,n) := \mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \dots \frac{w}{n}\right)$$

for $m, n \in \mathbb{N}, m \leq n+1$. Notice that $\mathfrak{G}(n+1,n) = \mathfrak{F}(0) = 1$. Then the recurrent relations (1.2) and (1.4) have the form

$$\mathfrak{G}(m,n) = \mathfrak{G}(m+1,n) - \frac{w^2}{m(m+1)}\mathfrak{G}(m+2,n), \tag{4.1}$$

$$\mathfrak{G}(m,n) = \mathfrak{G}(m,n-1) - \frac{w^2}{n(n-1)}\mathfrak{G}(m,n-2) \tag{4.2}$$

where $m, n \in \mathbb{N}, m \leq n$.
Next, with the aid of the first recurrent relation, we define function $\mathfrak{G}(m,n)$ also for $m, n \in \mathbb{N}, m > n+1$. To satisfy identity (4.1) one must set

$$\mathfrak{G}(n+2,n) := 0$$

and

$$\mathfrak{G}(m,n) := -\frac{(m-1)!(m-2)!}{n!(n+1)!}w^{-2(m-n)+4}\mathfrak{G}(n+2,m-2)$$

for $m > n+2$.

Remark 8. Consequently, it is easy to see that the relation

$$\mathfrak{G}(m,n) = -\frac{(m-1)!(m-2)!}{n!(n+1)!} w^{-2(m-n)+4} \mathfrak{G}(n+2, m-2) \tag{4.3}$$

holds for all $m, n \in \mathbb{N}$, $m > 2$.

Remark 9. In this and the following sections we will usually use algebraic identities like (4.1) and (4.2) while adjusting a term. In that case we will note, what relation we will use, above the equal sign. For example, an equal sign $\stackrel{4.27}{=}$ means that we use a relation (4.27) to adjust the LHS of an equation.

Let us verify the validity of the recurrent relation (4.1) for $m \geq n+2$:

$$\begin{aligned}
RHS &\equiv \mathfrak{G}(m+1,n) - \frac{w^2}{m(m+1)}\mathfrak{G}(m+2,n) = \\
&= -\frac{m!(m-1)!}{n!(n+1)!}w^{-2(m-n)+2}\mathfrak{G}(n+2,m-1) + \frac{m!(m-1)!}{n!(n+1)!}w^{-2(m-n)+2}\mathfrak{G}(n+2,m) = \\
&= -\frac{m!(m-1)!}{n!(n+1)!}w^{-2(m-n)+2}\left[\mathfrak{G}(n+2,m-1) - \mathfrak{G}(n+2,m)\right] \stackrel{4.2}{=} \\
&\stackrel{4.2}{=} -\frac{(m-1)!(m-2)!}{n!(n+1)!}w^{-2(m-n)+4}\mathfrak{G}(n+2,m-2) \equiv LHS.
\end{aligned}$$

Thus the recurrent relation

$$\mathfrak{G}(m,n) = \mathfrak{G}(m+1,n) - \frac{w^2}{m(m+1)}\mathfrak{G}(m+2,n) \tag{4.4}$$

holds for all $m, n \in \mathbb{N}$.

Proposition 11. The recurrent relation

$$\mathfrak{G}(m,n) = \mathfrak{G}(m,n-1) - \frac{w^2}{n(n-1)}\mathfrak{G}(m,n-2) \tag{4.5}$$

holds for all $m, n \in \mathbb{N}, n > 2$.

Proof. The case $m \leq n$ is treated in (4.2). Let $m > n$ then

$$\begin{aligned}
RHS &\stackrel{4.3}{=} -\frac{(m-1)!(m-2)!}{(n-1)!n!}w^{-2(m-n)+2}\mathfrak{G}(n+1,m-2) + \\
&\quad + \frac{(m-1)!(m-2)!}{(n-1)!n!}w^{-2(m-n)+2}\mathfrak{G}(n,m-2) = \\
&= -\frac{(m-1)!(m-2)!}{(n-1)!n!}w^{-2(m-n)+2}\left[\mathfrak{G}(n+1,m-2) - \mathfrak{G}(n,m-2)\right] \stackrel{4.4}{=} \\
&\stackrel{4.4}{=} -\frac{(m-1)!(m-2)!}{n!(n+1)!}w^{-2(m-n)+4}\mathfrak{G}(n+2,m-2) \equiv LHS.
\end{aligned}$$

□

4.2 Function $\mathfrak{J}$

Let us denote

$$\mathfrak{J}(m,n) := \frac{n!}{(m-1)!} w^{m-n} \mathfrak{G}(m,n) \tag{4.6}$$

for all $m, n \in \mathbb{N}$.
Then the recurrent relation (4.4) has the form

$$\mathfrak{J}(m,n) = \frac{m}{w}\mathfrak{J}(m+1,n) - \mathfrak{J}(m+2,n) \tag{4.7}$$

where $m, n \in \mathbb{N}$. With the aid of this recurrent relation we can define the function $\mathfrak{J}(m,n)$ even for $m \in \mathbb{Z}$ in this way:

$$\mathfrak{J}(-k,n) := (-1)^{k+1}\mathfrak{J}(k+2,n)$$

for $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$.

Remark 10. It is easy to verify that the identity

$$\mathfrak{J}(-k,n) = (-1)^{k+1}\mathfrak{J}(k+2,n) \tag{4.8}$$

holds for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$.

Let us verify the validity of the recurrent relation (4.7) for $m \leq 0$:
let $m \in \mathbb{N}$ then

$$\begin{aligned} &RHS \equiv -\frac{m}{w}\mathfrak{J}(-m+1,n) - \mathfrak{J}(-m+2,n) \overset{4.8}{=} \\ &\overset{4.8}{=} (-1)^{m+1}(\frac{m}{w}\mathfrak{J}(m+1,n) - \mathfrak{J}(m,n)) \overset{4.7}{=} (-1)^{m+1}\mathfrak{J}(m+2,n) \overset{4.8}{=} \\ &\overset{4.8}{=} \mathfrak{J}(-m,n) \equiv LHS \end{aligned}$$

and for $m = 0$ we have

$$RHS \equiv -\mathfrak{J}(2,n) \overset{4.8}{=} \mathfrak{J}(0,n) \equiv LHS.$$

Thus the recurrent relation

$$\mathfrak{J}(m,n) = \frac{m}{w}\mathfrak{J}(m+1,n) - \mathfrak{J}(m+2,n) \tag{4.9}$$

holds for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$.
Identity (4.3) written in the language of the function $\mathfrak{J}$ is

$$\mathfrak{J}(m,n) = -\mathfrak{J}(n+2,m-2) \tag{4.10}$$

where $m, n \in \mathbb{N}, m > 2$.

Proposition 12. The recurrent relation

$$\mathfrak{J}(m,n) = \frac{n}{w}\mathfrak{J}(m,n-1) - \mathfrak{J}(m,n-2) \tag{4.11}$$

holds for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}, n > 2$.

Proof. The case $m, n \in \mathbb{N}, n > 2$ have already been treated in Proposition 11 where we use the new function $\mathfrak{J}$ according to the definition relation (4.6).
Let $m \geq 0$ and $n > 2$ then

$$RHS \equiv \frac{n}{w}\mathfrak{J}(-m,n-1) - \mathfrak{J}(-m,n-2) \stackrel{4.8}{=}$$
$$\stackrel{4.8}{=} (-1)^{m+1}\left(\frac{n}{w}\mathfrak{J}(m+2,n-1) - \mathfrak{J}(m+2,n-2)\right) \stackrel{4.5}{=} (-1)^{m+1}\mathfrak{J}(m+2,n) \stackrel{4.8}{=}$$
$$\stackrel{4.8}{=} \mathfrak{J}(-m,n) \equiv LHS.$$

□

Finally with the aid of the previous recurrent identity (4.11) we can extend the function $\mathfrak{J}(m,n)$ even for $n \leq 0$. Thus the recurrent relation

$$\mathfrak{J}(m,n) = \frac{n}{w}\mathfrak{J}(m,n-1) - \mathfrak{J}(m,n-2) \tag{4.12}$$

holds for all $m, n \in \mathbb{Z}$.

Proposition 13. The recurrent relation

$$\mathfrak{J}(m,n) = \frac{m}{w}\mathfrak{J}(m+1,n) - \mathfrak{J}(m+2,n) \tag{4.13}$$

holds for all $m, n \in \mathbb{Z}$.

Proof. It remains to prove the statement for $n \leq 0$ because all other cases have already been treated before (see (4.9)). It will be proved by induction in n. For $n = 0$ we have

$$RHS \equiv \frac{m}{w}\mathfrak{J}(m+1,0) - \mathfrak{J}(m+2,0) \stackrel{4.12}{=}$$
$$\stackrel{4.12}{=} \frac{m}{w}\left(\frac{2}{w}\mathfrak{J}(m+1,1) - \mathfrak{J}(m+1,2)\right) - \frac{2}{w}\mathfrak{J}(m+2,1) + \mathfrak{J}(m+2,2).$$

By applying the recurrent relation (4.9) to the third and the fourth term we arrive at the expression

$$RHS = \frac{2}{w}\mathfrak{J}(m,1) - \mathfrak{J}(m,2) \stackrel{4.12}{=} \mathfrak{J}(m,0) \equiv LHS.$$

The induction step $\mathbb{N}_0 \ni n \to n+1$:

$$RHS = \frac{m}{w}\mathfrak{J}(m+1,-n-1) - \mathfrak{J}(m+2,-n-1) \overset{4.12}{=}$$
$$\overset{4.12}{=} \frac{m}{w}\left(\frac{-n+1}{w}\mathfrak{J}(m+1,-n) - \mathfrak{J}(m+1,-n+1)\right) -$$
$$- \frac{-n+1}{w}\mathfrak{J}(m+2,-n) + \mathfrak{J}(m+2,-n+1)$$

To proceed further it is needed to apply the induction hypothesis to the third and the fourth term and we get

$$RHS = \frac{-n+1}{w}\mathfrak{J}(m,-n) - \mathfrak{J}(m,-n+1) \overset{4.12}{=} \mathfrak{J}(m,-n-1) = LHS.$$

□

Proposition 14. The identity

$$\mathfrak{J}(-k,n) := (-1)^{k+1}\mathfrak{J}(k+2,n) \tag{4.14}$$

holds for all $k, n \in \mathbb{Z}$.

Proof. It remains to verify the identity for a non-positive second argument (other cases have been treated in Remark 10). It can be proved by mathematical induction in n.

$$n = 0: \quad \mathfrak{J}(-k,0) \overset{4.12}{=} \frac{2}{w}\mathfrak{J}(-k,1) - \mathfrak{J}(-k,2) \overset{4.8}{=}$$
$$\overset{4.8}{=} (-1)^{k+1}\left(\frac{2}{w}\mathfrak{J}(k+2,1) - \mathfrak{J}(k+2,2)\right) \overset{4.12}{=} (-1)^{k+1}\mathfrak{J}(k+2,0)$$

$$\mathbb{N}_0 \ni n \to n+1: \quad \mathfrak{J}(-k,-n-1) \overset{4.12}{=} \frac{-n+1}{w}\mathfrak{J}(-k,-n) - \mathfrak{J}(-k,-n+1) \overset{IH}{=}$$
$$\overset{IH}{=} (-1)^{k+1}\left(\frac{-n+1}{w}\mathfrak{J}(k+2,-n) - \mathfrak{J}(k+2,-n+1)\right) \overset{4.12}{=}$$
$$\overset{4.12}{=} (-1)^{k+1}\mathfrak{J}(k+2,-n-1)$$

The sign $\overset{IH}{=}$ shows the place where we use the induction hypothesis. □

Proposition 15. The identity

$$\mathfrak{J}(m,n) = -\mathfrak{J}(n+2,m-2) \tag{4.15}$$

holds for all $m, n \in \mathbb{Z}$.

Proof. The identity holds for $m, n \in \mathbb{N}$, $m > 2$, see (4.10).
1)In the first step we will show that the identity

$$\mathfrak{J}(m,n) = -\mathfrak{J}(n+2, m-2) \tag{4.16}$$

holds for $n \in \mathbb{Z}$ and $m > 2$ by mathematical induction in n.

$$\begin{aligned} n = 0: \quad & \mathfrak{J}(m,0) \overset{4.12}{=} \frac{2}{w}\mathfrak{J}(m,1) - \mathfrak{J}(m,2) \overset{4.10}{=} \\ & \overset{4.10}{=} -\frac{2}{w}\mathfrak{J}(3,m-2) + \mathfrak{J}(4,m-2) \overset{4.13}{=} -\mathfrak{J}(2,m-2) \end{aligned}$$

$$\begin{aligned} \mathbb{N}_0 \ni n \to n+1: \quad & \mathfrak{J}(m,-n-1) \overset{4.12}{=} \frac{-n+1}{w}\mathfrak{J}(m,-n) - \mathfrak{J}(m,-n+1) \overset{IH}{=} \\ & \overset{IH}{=} -\frac{-n+1}{w}\mathfrak{J}(-n+2,m-2) + \mathfrak{J}(-n+3,m-2) \overset{4.13}{=} \\ & \overset{4.13}{=} -\mathfrak{J}(-n+1,m-2) \end{aligned}$$

2)Now we will show the validity of the identity for all $m, n \in \mathbb{Z}$ by using mathematical induction in m.

$$\begin{aligned} m = 2: \quad & RHS \equiv -\mathfrak{J}(n+2,0) \overset{4.12}{=} -\frac{2}{w}\mathfrak{J}(n+2,1) + \mathfrak{J}(n+2,2) \overset{4.16}{=} \\ & \overset{4.16}{=} \frac{2}{w}\mathfrak{J}(3,n) - \mathfrak{J}(4,n) \overset{4.13}{=} \mathfrak{J}(2,n) \equiv LHS \end{aligned}$$

$$\begin{aligned} 2 \geq m \to m-1: \quad & RHS \equiv -\mathfrak{J}(n+2,m-3) \overset{4.12}{=} \\ & \overset{4.12}{=} -\frac{m-1}{w}\mathfrak{J}(n+2,m-2) + \mathfrak{J}(n+2,m-1) \overset{IH}{=} \\ & \overset{IH}{=} \frac{m-1}{w}\mathfrak{J}(m,n) - \mathfrak{J}(m+1,n) \overset{4.13}{=} \mathfrak{J}(m-1,n) \equiv LHS \end{aligned}$$

□

Corollary 3. It holds

$$\mathfrak{J}(m,-k) = (-1)^{k+1}\mathfrak{J}(m,k-2) \tag{4.17}$$

for all $k, m \in \mathbb{Z}$.

Proof.

$$\mathfrak{J}(m,-k) \overset{4.15}{=} -\mathfrak{J}(-k+2,m-2) \overset{4.14}{=} (-1)^{k+1}\mathfrak{J}(k,m-2) \overset{4.15}{=} (-1)^{k+1}\mathfrak{J}(m,k-2)$$

□

Corollary 4. It holds

$$\mathfrak{J}(-m,-n) = (-1)^{n+m+1}\mathfrak{J}(n,m) \tag{4.18}$$

for all $m, n \in \mathbb{Z}$.

Proof.

$$\mathfrak{J}(-m,-n) \overset{4.17}{=} (-1)^{n+1}\mathfrak{J}(-m,n-2) \overset{4.14}{=} (-1)^{m+n}\mathfrak{J}(m+2,n-2) \overset{4.15}{=} (-1)^{n+m+1}\mathfrak{J}(n,m)$$

□

4.3 A formula for the function $\mathfrak{J}$

Proposition 16. It holds

$$\mathfrak{J}(n-k,n) = \sum_{s=0}^{\left[\frac{k+1}{2}\right]} (-1)^s \binom{n-s}{k-2s+1} \frac{(k-s+1)!}{s!} w^{2s-k} \tag{4.19}$$

for all $n, k \in \mathbb{Z}$, $k \geq -2$.

Proof. The proof is split into 3 parts.
1)Let $n \in \mathbb{N}$ and $-2 \leq k < n$. By using the formula for the function $\mathfrak{F}\left(\frac{w}{m}, \ldots, \frac{w}{n}\right) \equiv \mathfrak{G}(m,n)$ from Corollary 1 (note that the formula in the Corollary 1 holds also for $m = n+1$ and $m = n+2$) and the definition of the function $\mathfrak{J}$ (4.6) we easily arrive at the expression

$$\mathfrak{J}(n-k,n) = \frac{n!}{(n-k-1)!} w^{-k} \mathfrak{G}(n-k,n) = \sum_{s=0}^{\left[\frac{k+1}{2}\right]} (-1)^s \frac{(n-s)!(k-s+1)!}{s!(n-k+s-1)!(k-2s+1)!} w^{2s-k} =$$
$$= \sum_{s=0}^{\left[\frac{k+1}{2}\right]} (-1)^s \binom{n-s}{k-2s+1} \frac{(k-s+1)!}{s!} w^{2s-k}.$$

Thus we have verified the validity of formula (4.19) for $n \in \mathbb{N}, -2 \leq k < n$.
2)In the second step we will verify formula (4.19) for all $k \geq -2, n \in \mathbb{N}$. We will proceed this by mathematical induction in k. Bearing in mind step 1), only the induction step $n-1 \leq k \to k+1$ is to be treated ($n \in \mathbb{N}$ is fixed):

$$\mathfrak{J}(n-(k+1),n) \overset{4.13}{=} \frac{n-k-1}{w} \mathfrak{J}(n-k,n) - \mathfrak{J}(n-k+1,n) \overset{IH}{=}$$
$$\overset{IH}{=} (n-k-1) \sum_{s=0}^{\left[\frac{k+1}{2}\right]} (-1)^s \frac{(n-s)!(k-s+1)!}{s!(n-k+s-1)!(k-2s+1)!} w^{2s-k-1} -$$
$$- \sum_{s=0}^{\left[\frac{k}{2}\right]} (-1)^s \frac{(n-s)!(k-s)!}{s!(n-k+s)!(k-2s)!} w^{2s-k+1} =$$
$$= (n-k-1) \sum_{s=0}^{\left[\frac{k+1}{2}\right]} (-1)^s \frac{(n-s)!(k-s+1)!}{s!(n-k+s-1)!(k-2s+1)!} w^{2s-k-1} +$$
$$+ \sum_{s=1}^{\left[\frac{k}{2}\right]+1} (-1)^s \frac{(n-s+1)!(k-s+1)!}{(s-1)!(n-k+s-1)!(k-2s+2)!} w^{2s-k-1}.$$

To proceed further, one must realize that the upper bound of the first sum can be changed to $\left[\frac{k}{2}\right]+1$ because if k is odd then $\left[\frac{k+1}{2}\right] = \left[\frac{k}{2}\right]+1$ and if k is even then $\left[\frac{k+1}{2}\right] = \frac{k}{2}$ but the added term ($s = \frac{k}{2}+1$) is 0 due to the term $\frac{1}{(-1)!}$ which must be set 0. For a similar reason the lower bound of the second sum can be changed to 0.

Then we have an expression

$$\begin{aligned}
&\mathfrak{J}(n-(k+1),n) = \\
&= \sum_{s=0}^{\left[\frac{k}{2}\right]+1} (-1)^s \frac{(n-s+1)!(k-s+1)!}{s!(n-k+s-1)!(k-2s+2)!} \left[(n-k+1)(k-2s+2)+s(n-s+1)\right] w^{2s-k-1} \\
&= \sum_{s=0}^{\left[\frac{k+2}{2}\right]} (-1)^s \frac{(n-s+1)!(k-s+2)!}{s!(n-k+s-2)!(k-2s+2)!} w^{2s-k-1}
\end{aligned}$$

which was to be shown.
3)Finally we will verify the validity of the formula (4.19) for all $k \geq -2, n \in \mathbb{Z}$. Again we will proceed by mathematical induction in n:

$$\begin{aligned}
&n=0: \\
&\mathfrak{J}(-k,0) \overset{4.12}{=} \frac{2}{w}\mathfrak{J}(1-(k+1),1)-\mathfrak{J}(2-(k+2),2) = \\
&= 2\sum_{s=0}^{\left[\frac{k+2}{2}\right]} (-1)^s \binom{1-s}{k-2s+2} \frac{(k-s+2)!}{s!} w^{2s-k-2} - \sum_{s=0}^{\left[\frac{k+3}{2}\right]} (-1)^s \binom{2-s}{k-2s+3} \frac{(k-s+3)!}{s!} w^{2s-k-2}
\end{aligned}$$

where we have used the result of step 2). We can change the upper bound of the first sum to $\left[\frac{k+3}{2}\right]$ because of similar reasons as discussed in step 2). Next note that the first term of the first sum ($s=0$) together with the first term of the second sum ($s=0$) give 0. Thus we have

$$\begin{aligned}
&\mathfrak{J}(-k,0) = \\
&= 2\sum_{s=1}^{\left[\frac{k+3}{2}\right]} (-1)^{s+k} \frac{(k-s)!(k-s+2)!}{s!(k-2s+2)!(s-2)!} w^{2s-k-2} - \sum_{s=1}^{\left[\frac{k+3}{2}\right]} (-1)^{s+k+1} \frac{(k-s)!(k-s+3)!}{s!(k-2s+3)!(s-3)!} w^{2s-k-2} \\
&= \sum_{s=0}^{\left[\frac{k+1}{2}\right]} (-1)^{s+k+1} \frac{(k-s-1)!(k-s+1)!}{(s+1)!(k-2s+1)!(s-1)!} \left[2(k-2s+1)+(s-1)(k-s+2)\right] w^{2s-k} \\
&= \sum_{s=0}^{\left[\frac{k+1}{2}\right]} (-1)^{s+k+1} \frac{(k-s)!(k-s+1)!}{s!(k-2s+1)!(s-1)!} w^{2s-k} = \sum_{s=0}^{\left[\frac{k+1}{2}\right]} (-1)^s \binom{-s}{k-2s+1} \frac{(k-s+1)!}{s!} w^{2s-k}
\end{aligned}$$

$$\begin{aligned}
&\mathbb{N}_0 \ni n \to n+1: \\
&\mathfrak{J}(-n-1-k,-n-1) \overset{4.12}{=} \frac{-n+1}{w}\mathfrak{J}(-n-(k+1),-n)-\mathfrak{J}(-n+1-(k+2),-n+1) \overset{IH}{=} \\
&\overset{IH}{=} (-n+1)\sum_{s=0}^{\left[\frac{k+2}{2}\right]} (-1)^s \binom{-n-s}{k-2s+2} \frac{(k-s+2)!}{s!} w^{2s-k-2} - \\
&- \sum_{s=0}^{\left[\frac{k+3}{2}\right]} (-1)^s \binom{-n+1-s}{k-2s+3} \frac{(k-s+3)!}{s!} w^{2s-k-2}.
\end{aligned}$$

To proceed further we will make similar steps as we have done in the case $n=0$.We will change the upper bound of the first sum to $\left[\frac{k+3}{2}\right]$ because of similar reasons as discussed in step 2) and also the first terms of both sums are subtracting to 0. Then we arrive at an expression

$$\mathfrak{J}(-n-1-k,-n-1) = -(n-1)\sum_{s=1}^{\left[\frac{k+3}{2}\right]}(-1)^{k+s}\frac{(n+k-s+1)!(k-s+2)!}{s!(k-2s+2)!(n+s-1)!}w^{2s-k-2}+$$
$$+\sum_{s=1}^{\left[\frac{k+3}{2}\right]}(-1)^{k+s}\frac{(n+k-s+1)!(k-s+3)!}{s!(k-2s+3)!(n+s-2)!}w^{2s-k-2} =$$
$$=\sum_{s=0}^{\left[\frac{k+1}{2}\right]}(-1)^{k+s}\frac{(n+k-s)!(k-s+1)!}{(s+1)!(k-2s+1)!(n+s)!}\left[(n-1)(k-2s+1)-(k-s+2)(n+s)\right]w^{2s-k}$$
$$=\sum_{s=0}^{\left[\frac{k+1}{2}\right]}(-1)^{s}\binom{-n-1-s}{k-2s+1}\frac{(k-s+1)!}{s!}w^{2s-k}$$

which concludes the proof. □

Proposition 17. It holds

$$\mathfrak{J}(n+k+2,n) = \sum_{s=1}^{\left[\frac{k+1}{2}\right]}(-1)^{s}\binom{k+n-s+1}{k-2s+1}\frac{(k-s)!}{(s-1)!}w^{2s-k} \tag{4.20}$$

for all $n,k\in\mathbb{Z},\ k\geq 0$.

Proof.

$$\mathfrak{J}(n+k+2,n) \overset{4.14}{=} (-1)^{n+k+1}\mathfrak{J}(-n-k,n) \overset{4.17}{=} (-1)^{k}\mathfrak{J}(-n-k,-n-2) \overset{4.19}{=}$$
$$\overset{4.19}{=}\sum_{s=0}^{\left[\frac{k-1}{2}\right]}(-1)^{k+s}\binom{-n-2-s}{k-2s-1}\frac{(k-s-1)!}{s!}w^{2s-k+2} =$$
$$=\sum_{s=0}^{\left[\frac{k-1}{2}\right]}(-1)^{s+1}\binom{k+n-s}{k-2s-1}\frac{(k-s-1)!}{s!}w^{2s-k+2} = \sum_{s=1}^{\left[\frac{k+1}{2}\right]}(-1)^{s}\binom{k+n-s+1}{k-2s+1}\frac{(k-s)!}{(s-1)!}w^{2s-k}$$

□

Example 1. By using the previous results of the function $\mathfrak{J}$ we can introduce some special examples:

$$\mathfrak{J}(n+2,n) = \mathfrak{J}(n,-n) = 0,$$

$$\mathfrak{J}(n,n) = n, \qquad \mathfrak{J}(n+1,n) = w,$$
$$\mathfrak{J}(n+3,n) = -w, \qquad \mathfrak{J}(n+4,n) = -n-2,$$
$$\mathfrak{J}(n-1,n) = \frac{(n-1)n}{w} - w, \qquad \mathfrak{J}(n-2,n) = \frac{n(n-1)(n-2)}{w^2} - 2(n-1)$$

where $n\in\mathbb{Z}$.

4.4 Function $\mathfrak{J}$ and particular values of $\chi_{red}(z)$

In this section we will return to the expressions for a particular value of the characteristic reduced function. As we promised in section 3.1.3 we will prove a statement which will generalize relation (3.7), that means relation (3.7) will be a special case of the statement.
We start with expression (3.1)

$$\chi_{red}(z) = \left(\prod_{k=1}^{d}(k^2-z^2)\right)\mathfrak{G}(1-z,d-z)\mathfrak{G}(1+z,d+z)+$$
$$+2\sum_{j=1}^{d} w^{2j}\left(\prod_{k=j+1}^{d}(k^2-z^2)\right)\mathfrak{G}(j+1-z,d-z)\mathfrak{G}(j+1+z,d+z) \qquad (4.21)$$

where we denote

$$\mathfrak{G}(j+1\pm z,d\pm z) := \mathfrak{F}\left(\frac{w}{j+1\pm z},\dots,\frac{w}{d\pm z}\right).$$

Although we have defined the function $\mathfrak{G}$ only for integer arguments there isn't any problem.

Lemma 5. It holds

$$(n-z)\mathfrak{G}(n-j-z,d-z)\Big|_{z=n} = -\frac{w^{2j+2}}{(j+1)!j!}\mathfrak{G}(j+2,d-n)$$

for $j=0,\dots,n-1$.

Proof. We will proved the statement by finite mathematical induction in j. Case $j=0$:

$$(n-z)\mathfrak{G}(n-z,d-z)\Big|_{z=n} = (n-z)\mathfrak{G}(n+1-z,d-z)\Big|_{z=n} -$$
$$-\frac{w^2}{n+1-z}\mathfrak{G}(n+2-z,d-z)\Big|_{z=n} = -w^2\mathfrak{G}(2,d-n)$$

where the recurrent relation (1.2) was used. The case $j=1$ will be treated similarly:

$$(n-z)\mathfrak{G}(n-1-z,d-z)\Big|_{z=n} = (n-z)\mathfrak{G}(n-z,d-z)\Big|_{z=n} -$$
$$-\frac{w^2}{n-1-z}\mathfrak{G}(n+2-z,d-z)\Big|_{z=n} = -w^2\mathfrak{G}(2,d-n)+w^2\mathfrak{G}(1,d-n) =$$
$$= -\frac{w^4}{2}\mathfrak{G}(3,d-n).$$

Let $0 \leq j < n-1$. To proceed the induction step $j \to j+1$ the well known recurrent relation (1.2) is to be used again:

$$(n-z)\mathfrak{G}(n-(j+1)-z, d-z)\Big|_{z=n} = (n-z)\mathfrak{G}(n-j-z, d-z)\Big|_{z=n} -$$
$$-(n-z)\frac{w^2}{(n-j-1-z)(n-j-z)}\mathfrak{G}(n-j+1-z, d-z)\Big|_{z=n} \overset{IH}{=}$$
$$\overset{IH}{=} -\frac{w^{2j+2}}{(j+1)!j!}\mathfrak{G}(j+2, d-n) + \frac{w^{2j+2}}{(j+1)!j!}\mathfrak{G}(j+1, d-n) =$$
$$= -\frac{w^{2j+4}}{(j+2)!(j+1)!}\mathfrak{G}(j+3, d-n).$$

□

With the aid of the previous lemma we can evaluate the characteristic reduced function (4.21) in $n \in \{1, \dots, d\}$. Thus

$$\chi_{red}(n) = 2n\left(\prod_{k=1,k\neq n}^{d} k^2 - n^2\right)(n-z)\mathfrak{G}(1-z, d-z)\Big|_{z=n}\mathfrak{G}(1+n, d+n) +$$
$$+2\sum_{j=1}^{n-1} w^{2j} 2n\left(\prod_{k=j+1,k\neq n}^{d} k^2 - n^2\right)(n-z)\mathfrak{G}(j+1-z, d-z)\Big|_{z=n}\mathfrak{G}(j+1+n, d+n) +$$
$$+2\sum_{j=n}^{d} w^{2j}\left(\prod_{k=j+1}^{d} k^2 - n^2\right)\mathfrak{G}(j+1-n, d-n)\mathfrak{G}(j+1+n, d+n) =$$
$$= (-1)^n\frac{(d-n)!(d+n)!}{(n!)^2}w^{2n}\mathfrak{G}(n+1, d-n)\mathfrak{G}(1+n, d+n) +$$
$$+2\sum_{j=1}^{n-1}(-1)^{n-j}\frac{(d-n)!(d+n)!}{(n-j)!(n+j)!}w^{2n}\mathfrak{G}(n-j+1, d-n)\mathfrak{G}(n+j+1, d+n) +$$
$$+2\sum_{j=n}^{d}\frac{(d-n)!(d+n)!}{(j-n)!(j+n)!}w^{2j}\mathfrak{G}(j+1-n, d-n)\mathfrak{G}(j+1+n, d+n).$$

So we have derived an expression

$$\frac{\chi_{red}(n)}{(d-n)!(d+n)!} = \frac{(-1)^n}{(n!)^2}w^{2n}\mathfrak{G}(n+1, d-n)\mathfrak{G}(1+n, d+n) +$$
$$+2w^{2n}\sum_{j=1}^{n-1}\frac{(-1)^{n-j}}{(n-j)!(n+j)!}\mathfrak{G}(n-j+1, d-n)\mathfrak{G}(n+j+1, d+n) +$$
$$+2\sum_{j=n}^{d}\frac{w^{2j}}{(j-n)!(j+n)!}\mathfrak{G}(j+1-n, d-n)\mathfrak{G}(j+1+n, d+n). \qquad (4.22)$$

Remark 11. 1)Note that expression (4.22) coincides with the recurrent relation (3.7) with $s = n$.
2)By using the definition relation (4.6) of the fuction $\mathfrak{J}$ we can write expression (4.22) as

$$\frac{\chi_{red}(n)}{w^{2d-2}} = (-1)^n \mathfrak{J}(n+1, d-n)\mathfrak{J}(1+n, d+n)+ \\ + 2\sum_{j=1}^{n-1}(-1)^{n-j}\mathfrak{J}(n-j+1, d-n)\mathfrak{J}(n+j+1, d+n)+ \\ + 2\sum_{j=n}^{d}\mathfrak{J}(j+1-n, d-n)\mathfrak{J}(j+1+n, d+n). \tag{4.23}$$

Lemma 6. Let us denote

$$[k, l] := \mathfrak{J}(k, d-n)\mathfrak{J}(l, d+n)$$

and

$$\Omega(n) := 2(-1)^n \sum_{j=1}^{n-1}(-1)^j \{(s+1)[s+2-j, s+2+j] - s[s+1-j, s+1+j]\} + \\ + (-1)^n(s+1)[s+2, s+2] - (-1)^n s[s+1, s+1]$$

for some $n \in \mathbb{N}$ and $k, l, s \in \mathbb{Z}$. Then the identity

$$\Omega(n) = (s+n)[s+1-n, s+1+n] - (s-n+1)[s+2-n, s+2+n] \tag{4.24}$$

holds for all $n \in \mathbb{N}$ and $s \in \mathbb{Z}$.

Proof. The identity will be proved by mathematical induction in n. Considering the case $n = 1$ we have

$$\Omega(1) = -(s+1)[s+2, s+2] + s[s+1, s+1]$$

where is needed to apply the basic recurrent relation, namely relation (4.13) is applied to the first argument of the first bracket and to the second argument of the second bracket. Then we arrive at an expression

$$\Omega(1) = \\ -\frac{(s+1)s}{w}[s+1, s+2] + (s+1)[s, s+2] + \frac{s(s+1)}{w}[s+1, s+2] - s[s+1, s+3] = \\ = (s+1)[s, s+2] - s[s+1, s+3]$$

which coincides with the RHS of identity (4.24). Now the induction step $1 \leq n \to n+1$ is to be treated. Since

$$\Omega(n+1) = -\Omega(n) - 2(s+1)[s+2-n, s+2+n] + 2s[s+1-n, s+1+n]$$

we can apply the induction hypothesis and we get

$$\Omega(n+1) = (s-n)[s+1-n, s+1+n] - (n+s+1)[s+2-n, s+2+n].$$

By using the recurrent relation (4.13) on the second argument of the first bracket and on the first argument of the second bracket we obtain an expression

$$\begin{aligned}\Omega(n+1) = \frac{(s-n)(s+1+n)}{w}&[s+1-n, s+2+n] - (s-n)[s+1-n, s+3+n]- \\ - \frac{(n+s+1)(s-n)}{w}&[s+1-n, s+2+n] + (n+s+1)[s-n, s+2+n] = \\ = -(s-n)&[s+1-n, s+3+n] + (n+s+1)[s-n, s+2+n]\end{aligned}$$

which was to be verified. □

Proposition 18. Let $n \in \{1, \ldots, d\}$ and denote $[k, l] := \mathfrak{J}(k, d-n)\mathfrak{J}(l, d+n)$ as in the previous lemma. Then the relation

$$\begin{aligned}\frac{n\chi_{red}(n)}{w^{2d-2}} = (-1)^n s[s+1, s+1] + 2(-1)^n s \sum_{j=1}^{n-1} (-1)^j [s+1-j, s+1+j]+ \\ + (s+n)[s+1-n, s+1+n] + 2n \sum_{j=s+1}^{d} [j+1-n, j+1+n] \qquad (4.25)\end{aligned}$$

holds for all $s \in \mathbb{Z}$.

Proof. The statement will be verified by mathematical induction in s. Expression (4.23) follows that the statement holds for $s = n$. It is clear that the induction step $n \leq s \to s+1$ will be completed if the equation

$$\begin{aligned}&(-1)^n s[s+1, s+1] + 2(-1)^n s \sum_{j=1}^{n-1} (-1)^j [s+1-j, s+1+j]+ \\ &+ (s+n)[s+1-n, s+1+n] + 2n[s+2-n, s+2+n] = \\ &= (-1)^n (s+1)[s+2, s+2] + 2(-1)^n (s+1) \sum_{j=1}^{n-1} (-1)^j [s+2-j, s+2+j]+ \\ &+ (s+1+n)[s+2-n, s+2+n]\end{aligned}$$

holds. But it is true because this equation is identity (4.24) (only rearranged) from the previous lemma. The induction step $n \geq s \to s-1$ can be treated in a similar way with the aid of Lemma 6 (writing $s-1$ instead of s). □

Remark 12. 1)By plugging the definition relation of the function $\mathfrak{J}$ (4.6) into relation (3.7) one can easily arrive at expression (4.25) with only restricted index s.

Thus Proposition 10 is a special case of Proposition 18.
2)Especially for $s = d$ it holds

$$\frac{n\chi_{red}(n)}{w^{2d-2}} = (-1)^n d[d+1, d+1] + 2(-1)^n d \sum_{j=1}^{n-1} (-1)^j [d+1-j, d+1+j] + \\ + (d+n)[d+1-n, d+1+n]$$

and for $s = 0$ it holds

$$\frac{\chi_{red}(n)}{w^{2d-2}} = [1-n, 1+n] + 2\sum_{j=1}^{d} [j+1-n, j+1+n]. \tag{4.26}$$

Note that this relation for $\chi_{red}(n)$ holds also for $n = 0$ because the expression

$$w^{2d-2}\left(\mathfrak{I}(1,d)^2 + 2\sum_{j=1}^{d} \mathfrak{I}(j+1,d)^2\right)$$

coincides with identity (3.3).

Chapter 5

The characteristic function and the resolvent of a Jacobi matrix

We have prepared enough so we can derive the exact formula for $\chi_{red}(n)$ with $n = 1, \ldots, d$. Then by using the formula for $\chi_{red}(n)$ we will reconstruct the whole reduced characteristic function $\chi_{red}(z)$ for $z \in \mathbb{C}$. Furthermore we will derive a formula for the resolvent operator of a Jacobi matrix of a special type.

Recall that $K \in \mathbb{C}^{2d+1 \times 2d+1}$ is a complex Jacobi matrix of the form

$$K := \begin{pmatrix} -d & w & & & & & & & \\ w & -d+1 & w & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & w & -1 & w & & & & \\ & & & w & 0 & w & & & \\ & & & & w & 1 & w & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & w & d-1 & w \\ & & & & & & & w & d \end{pmatrix}.$$

Definition 3. Let $n, s \in \mathbb{Z}$. Then define a column vector $x_{s,n} \in \mathbb{C}^{2d+1}$ as

$$x_{s,n}^T := (\mathfrak{J}(d+s+1,n), \mathfrak{J}(d+s,n), \ldots, \mathfrak{J}(s+1,n), \ldots, \mathfrak{J}(-d+s+1,n)) \quad (5.1)$$

By putting $-j+s$ instead of m into the recurrent relation (4.13) we get an identity

$$w\mathfrak{J}(-j+s+2,n) + (j-s)\mathfrak{J}(-j+s+1,n) + w\mathfrak{J}(+j+s,n) = 0$$

for all $j, n, s \in \mathbb{Z}$. Thus we can write

$$wx_{n,s}^{j-1} + (j-s)x_{n,s}^{j} + wx_{n,s}^{j+1} = 0 \quad (5.2)$$

for all $n, s \in \mathbb{Z}$, $j = -d, \ldots, d$, but we have to set $x_{n,s}^{-d-1} := \mathfrak{J}(d+s+2,n)$ and $x_{n,s}^{d+1} := \mathfrak{J}(-d+s,n)$. Then it holds

$$(K-s)x_{n,s} = -w\mathfrak{J}(d+s+2,n)e_{-d} - w\mathfrak{J}(-d+s,n)e_{d} \quad (5.3)$$

where $n, s \in \mathbb{Z}$ and $(e_{-d}, \dots, e_d)$ is a standard basis in $\mathbb{C}^{2d+1}$ (that is $e_j^k = \delta_{jk}$) which is easy to verify if we consider identity (5.2).
Next, we set $n = d + s$. Since $\mathfrak{J}(d+s+2, d+s) = 0$ which can be seen, for example, from (4.15) we can eliminate the first term on the RHS in identity (5.3) and we arrive at an expression

$$Kv_s = sv_s - w\mathfrak{J}(-d+s, d+s)e_d \tag{5.4}$$

where we denote $v_s := x_{s,d+s}$ for all $s \in \mathbb{Z}$.

Remark 13. Since

$$\mathfrak{J}(-d, d) \overset{4.14}{=} (-1)^{d+1}\mathfrak{J}(d+2, d) = 0$$

the vector v_0 is the eigenvector of K for eigenvalue 0 which is easy to see from relation (5.4) if we set $s = 0$.

Lemma 7. Let $n \in \mathbb{N}$ and p is a polynomial of degree $s \le n - 1$. Then

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} p(j) = 0.$$

Proof. The statement will be verified by finite mathematical induction in the degree s of a polynomial. The case $s = 0$ is nothing but the binomial theorem. Let q is a polynomial of degree $s + 1 \le n - 1$, that is

$$q(j) = \sum_{l=0}^{s+1} a_l j^l, \qquad a_l \in \mathbb{C}, a_{s+1} \neq 0.$$

Since

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} q(j) = a_{s+1} \sum_{j=0}^{n} (-1)^j \binom{n}{j} j^{s+1} + \sum_{j=0}^{n} (-1)^j \binom{n}{j} \sum_{l=0}^{s} a_l j^l$$

it is clear that it suffices to show that

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} j^{s+1} = 0.$$

But

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} j^{s+1} = -n \sum_{j=1}^{n} (-1)^{j-1} \binom{n-1}{j-1} j^s = -n \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} (j+1)^s$$

and since $(j+1)^s$ is a polynomial in j of degree $s \le n-2$ we can apply the induction hypothesis and we get

$$\sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} (j+1)^s = 0$$

which concludes the proof. □

Proposition 19. Let $V \in \mathbb{C}^{2d+1\times 2d+1}$ where

$$V_{ks} := v_s^k \equiv \mathfrak{J}(-k+s+1, d+s), \qquad k, s \in (-d, -d+1, \dots, d).$$

Then

$$\det V = \frac{\prod_{k=1}^{2d} k!}{w^{(d-1)(2d+1)}}. \tag{5.5}$$

Proof. Let us define a matrix $W \in \mathbb{C}^{2d+1\times 2d+1}$ such that

$$W_{jk} := (-1)^{j+d}\binom{k+d}{j+d}.$$

Since $W_{jk} = 0$ for $k < j$ we can easily compute the determinant of W

$$\det W = \prod_{k=-d}^{d} W_{kk} = \prod_{k=-d}^{d} (-1)^{k+d}.$$

Let us investigate a matrix VW.

$$(VW)_{kl} = \sum_{s=-d}^{d} V_{ks}W_{sl} = \sum_{s=-d}^{d} (-1)^{d+s}\binom{d+l}{d+s}\mathfrak{J}(-k+s+1, d+s) =$$
$$= \sum_{j=0}^{2d}(-1)^j\binom{d+l}{j}\mathfrak{J}(j-(d+k-1), j)$$

Since $\binom{d+l}{j} = 0$ for $j > d+l$ we can change the upper bound of the sum to $d+l$. Next we apply formula (4.19) and get

$$(VW)_{kl} = \sum_{j=0}^{d+l}(-1)^j\binom{d+l}{j}\sum_{t=0}^{\left[\frac{d+k}{2}\right]}(-1)^t\binom{j-t}{d+k-2t}\frac{(d+k-t)!}{t!}w^{2t-d-k+1} =$$
$$= \sum_{t=0}^{\left[\frac{d+k}{2}\right]}(-1)^t\frac{(d+k-t)!}{t!}w^{2t-d-k+1}\sum_{j=0}^{d+l}(-1)^j\binom{d+l}{j}\binom{j-t}{d+k-2t}$$

Since $\binom{j-t}{d+k-2t}$ is a polynomial in j of degree $d+k-2t$ the previous Lemma 7 gives

$$\sum_{j=0}^{d+l}(-1)^j\binom{d+l}{j}\binom{j-t}{d+k-2t} = 0$$

if $k-l < 2t$.
Let $k < l$ then, with the aid of Lemma 7, we arrive at an equation

$$(VW)_{kl} = 0.$$

For $k = l$ the only nonzero term is the term with $t = 0$ and thus

$$(VW)_{kk} = (d+k)!w^{-d-k+1}\sum_{j=0}^{d+k}(-1)^j\binom{d+k}{j}\binom{j}{d+k} = (-1)^{d+k}(d+k)!w^{-d-k+1}.$$

The last equation holds since $\binom{j}{d+k} \neq 0$ only for $j = d+k$.
Finally we can derive the determinant of the matrix V

$$\det V = \frac{\det(VW)}{\det W} = \frac{\prod_{k=-d}^{d}(VW)_{kk}}{\prod_{k=-d}^{d}(W)_{kk}} = \frac{\prod_{k=-d}^{d}(-1)^{d+k}(d+k)!w^{-d-k+1}}{\prod_{k=-d}^{d}(-1)^{d+k}} =$$
$$= \frac{\prod_{k=1}^{2d}k!}{w^{(d-1)(2d+1)}}.$$

□

Corollary 5. A set of vectors $\vartheta := (v_{-d}, v_{-d+1}, \ldots, v_d)$ is a basis in $\mathbb{C}^{2d+1}$.

Proof. ϑ is basis in $\mathbb{C}^{2d+1} \Leftrightarrow$ matrix V is regular $\Leftrightarrow \det V \equiv \frac{\prod_{k=1}^{2d}k!}{w^{(d-1)(2d+1)}} \neq 0$.

□

Lemma 8. It holds

$$e_d = \sum_{s=-d}^{d}\frac{(-1)^{d+s}}{(d+s)!(d-s)!}w^{2d-1}v_s \tag{5.6}$$

where $e_d \in \mathbb{C}^{2d+1}, e_d^k = \delta_{dk}$.

Proof.

$$\sum_{s=-d}^{d}\frac{(-1)^{d+s}}{(d+s)!(d-s)!}w^{2d-1}\mathfrak{J}(-k+s+1, d+s) \overset{4.19}{=}$$

$$\overset{4.19}{=} \sum_{s=-d}^{d}\frac{(-1)^{d+s}}{(d+s)!(d-s)!}w^{2d-1}\sum_{l=0}^{\left[\frac{d+k}{2}\right]}(-1)^l\binom{d+s-l}{d+k-2l}\frac{(d+k-l)!}{l!}w^{2l-d-k+1} =$$
$$= \sum_{l=0}^{\left[\frac{d+k}{2}\right]}(-1)^l\frac{(d+k-l)!}{l!}w^{2l+d-k}\frac{1}{(2d)!}\sum_{s=0}^{2d}(-1)^s\binom{2d}{s}\binom{s-l}{d+k-2l}$$

According to Lemma 7 the inner sum

$$\sum_{s=0}^{2d}(-1)^s\binom{2d}{s}\binom{s-l}{d+k-2l} = 0$$

if $k < d + 2l$ and this inequality holds for all $k \in \{-d, \ldots, d-1\}$. If $k = d$ then the inner sum is not zero only if $l = 0$, thus

$$\sum_{s=-d}^{d}\frac{(-1)^{d+s}}{(d+s)!(d-s)!}w^{2d-1}\mathfrak{J}(-k+s+1, d+s) = \sum_{s=0}^{2d}(-1)^s\binom{2d}{s}\binom{s}{2d} = 1.$$

Since $\binom{s}{2d} = 0$ for all $s < 2d$ the last equality holds.
So for $k \in \{-d, \dots, d\}$ we have an equality

$$\sum_{s=-d}^{d} \frac{(-1)^{d+s}}{(d+s)!(d-s)!} w^{2d-1} \mathfrak{J}(-k+s+1, d+s) = \delta_{kd}$$

which proves the statement. □

Finally we can express the operator K in the basis ϑ (denoted $K^{\vartheta} \in \mathbb{C}^{2d+1 \times 2d+1}$). Starting with (5.4) and considering the statement of the previous Lemma 8 we obtain an expression

$$(K^{\vartheta})_{ts} = s\delta_{ts} - w\mathfrak{J}(-d+s, d+s)e_d(t) \tag{5.7}$$

where

$$e_d(t) = \frac{(-1)^{d+t}}{(d+t)!(d-t)!} w^{2d-1}$$

and $t, s \in \{-d, \dots, d\}$. Next let us denote $K_0 \in \mathbb{C}^{2d+1 \times 2d+1}$,

$$(K_0)_{ts} := s\delta_{ts}$$

and $a, e_d^{\vartheta} \in \mathbb{C}^{2d+1}$,

$$(e_d^{\vartheta})^T := (e_d(-d), e_d(-d+1), \dots, e_d(d)), \qquad a^T := (\alpha_{-d}, \alpha_{-d+1}, \dots, \alpha_d)$$

where $\alpha_s := -w\mathfrak{J}(-d+s, d+s) \overset{4.14}{=} (-1)^{d+s} w\mathfrak{J}(d-s+2, d+s)$. Then we can rewrite relation (5.7) to a simple expression

$$K^{\vartheta} = K_0 + e_d^{\vartheta} a^T. \tag{5.8}$$

Remark 14. 1)Note that $e_d^{\vartheta} a^T \in \mathbb{C}^{2d+1 \times 2d+1}$. It is not a scalar product.
2)The inverse operator $(K_0 - z)^{-1}$ exists for all $z \in \mathbb{C} \setminus \{-d. \dots, d\}$ and

$$((K_0 - z)^{-1})_{ts} = \frac{1}{s-z}\delta_{ts}.$$

3)It holds $\alpha_{-s} = -\alpha_s$ for all $s \in \{-d \dots, d\}$. Especially it follows that $\alpha_0 = 0$.
Verification:

$$\alpha_{-s} = -w\mathfrak{J}(-d-s, d-s) \overset{4.18}{=} w\mathfrak{J}(-d+s, d+s) = -\alpha_s.$$

5.1 The resolvent $(K^{\vartheta} - z)^{-1}$

In the following proposition we will find a formula for the resolvent operator $(K^{\vartheta} - z)^{-1}$.

Proposition 20. Let $z \in \mathbb{C} \setminus \{-d, \dots, d\}$ such that an inequality

$$1 + a^T (K_0 - z)^{-1} e_d^\vartheta \neq 0$$

holds. Then

$$(K^\vartheta - z)^{-1} = (K_0 - z)^{-1} - \frac{1}{1 + a^T (K_0 - z)^{-1} e_d^\vartheta} (K_0 - z)^{-1} e_d^\vartheta a^T (K_0 - z)^{-1}. \quad (5.9)$$

Proof. From formula (5.8) it follows that

$$K^\vartheta - z = (K_0 - z)(1 + (K_0 - z)^{-1} e_d^\vartheta a^T), \quad (5.10)$$

note that 1 stands for an identity operator. Then

$$\left[(K_0 - z)^{-1} - \frac{1}{1 + a^T (K_0 - z)^{-1} e_d^\vartheta} (K_0 - z)^{-1} e_d^\vartheta a^T (K_0 - z)^{-1} \right] (K^\vartheta - z) =$$
$$= \left[1 - \frac{1}{1 + a^T (K_0 - z)^{-1} e_d^\vartheta} (K_0 - z)^{-1} e_d^\vartheta a^T \right] (1 + (K_0 - z)^{-1} e_d^\vartheta a^T) =$$
$$= 1 + (K_0 - z)^{-1} e_d^\vartheta a^T - \frac{1}{1 + a^T (K_0 - z)^{-1} e_d^\vartheta} (K_0 - z)^{-1} e_d^\vartheta a^T -$$
$$- \frac{a^T (K_0 - z)^{-1} e_d^\vartheta}{1 + a^T (K_0 - z)^{-1} e_d^\vartheta} (K_0 - z)^{-1} e_d^\vartheta a^T = 1$$

which was to be verified. □

Remark 15. By multiplying the equality

$$1 + a^T (K_0 - z)^{-1} e_d^\vartheta = 0$$

by a term $\prod_{k=-d}^{d} (k - z)$ we obtain a polynomial equation in z.Thus the inequality

$$1 + a^T (K_0 - z)^{-1} e_d^\vartheta \neq 0$$

holds for all $z \in \mathbb{C}$ with exception of a finite number of z.

5.2 A formula for $\chi_{red}(z)$

Lemma 9. Let $a, b \in \mathbb{C}^n$ then

$$\det(1 + ba^T) = 1 + a^T b.$$

Proof. The definition of the determinant gives

$$\det(1 + ba^T) = \sum_{\pi \in S_n} \mathrm{sgn}\pi \prod_{k=1}^{n} (\delta_{k\pi(k)} + a_{\pi(k)} b_k) \quad (5.11)$$

where S_n is a set of permutations of the set $\{1,\ldots,n\}$. Since a determinant of a matrix with all entries equal to 1 is zero the identity

$$\sum_{\pi\in S_n}\mathrm{sgn}\pi=0$$

holds for all $n\geq 2$.Then a lot of the terms in sum (5.11) are zeros which can be seen from following relations

$$\sum_{\pi\in S_n}\mathrm{sgn}\pi\prod_{k=1}^{n}a_{\pi(k)}b_k=\prod_{k=1}^{n}a_kb_k\sum_{\pi\in S_n}\mathrm{sgn}\pi=0,$$

$$\sum_{\pi\in S_n}\mathrm{sgn}\pi\delta_{\pi(i)i}\prod_{k=1,k\neq i}^{n}a_{\pi(k)}b_k=\prod_{k=1,k\neq i}^{n}a_kb_k\sum_{\pi\in S_n,\pi(i)=i}\mathrm{sgn}\pi=$$
$$=\prod_{k=1,k\neq i}^{n}a_kb_k\sum_{\sigma\in S_{n-1}}\mathrm{sgn}\sigma=0,$$
$$\sum_{\pi\in S_n}\mathrm{sgn}\pi\delta_{\pi(i)i}\delta_{\pi(j)j}\prod_{k\neq i,k\neq j}a_{\pi(k)}b_k=\prod_{k\neq i,k\neq j}a_kb_k\sum_{\tau\in S_{n-2}}\mathrm{sgn}\tau=0,$$

etc., until we arrive at the following cases

$$\sum_{\pi\in S_n}\mathrm{sgn}\pi a_{\pi(i)}b_i\prod_{k\neq i}\delta_{\pi(k)k}=a_ib_i\sum_{\pi=\mathrm{id}}\mathrm{sgn}\pi=a_ib_i$$

and

$$\sum_{\pi\in S_n}\mathrm{sgn}\pi\prod_{k=1}^{n}\delta_{\pi(k)k}=1.$$

These relations together with the formula for the determinant (5.11) give

$$\det(1+ba^T)=1+\sum_{i=1}^{n}a_ib_i$$

which concludes the proof.

□

Now we can use relation (5.10) and the previous lemma to find a formula for $\chi_{red}(z)$:

$$\chi_{red}(z)=\frac{(-1)^{d+1}}{z}\det(K-z)=\frac{(-1)^{d+1}}{z}\det(K^{\vartheta}-z)=$$
$$=\frac{(-1)^{d+1}}{z}\det(K_0-z)\det(1+(K_0-z)^{-1}e_d^{\vartheta}a^T)=$$
$$=\frac{(-1)^{d+1}}{z}\prod_{k=-d}^{d}(k-z)(1+a^T(K_0-z)^{-1}e_d^{\vartheta})=\prod_{k=1}^{d}(k^2-z^2)(1+a^T(K_0-z)^{-1}e_d^{\vartheta})=$$
$$=\prod_{k=1}^{d}(k^2-z^2)\left(1+w^{2d-1}\sum_{s=-d}^{d}\frac{(-1)^{d+s}}{(d+s)!(d-s)!}\frac{\alpha_s}{s-z}\right).$$

Since $\alpha_{-s} = -\alpha_s$ (see Remark 14) we can further adjust the sum in the previous expression

$$\sum_{s=-d}^{d} \frac{(-1)^{d+s}}{(d+s)!(d-s)!} \frac{\alpha_s}{s-z} = \sum_{s=1}^{d} \frac{(-1)^{d+s}}{(d+s)!(d-s)!} \alpha_s \left(\frac{1}{s-z} - \frac{1}{-s-z} \right) =$$

$$= \sum_{s=1}^{d} \frac{2s}{s^2 - z^2} \frac{w\mathfrak{J}(d-s+2, d+s)}{(d+s)!(d-s)!}.$$

Finally we arrive at a formula

$$\chi_{red}(z) = \prod_{k=1}^{d} (k^2 - z^2) \left(1 + w^{2d} \sum_{s=1}^{d} \frac{2s}{s^2 - z^2} \frac{\mathfrak{J}(d-s+2, d+s)}{(d+s)!(d-s)!} \right). \tag{5.12}$$

Next for $z = n \in \{1, 2, \dots, d\}$ we obtain a formula for $\chi_{red}(n)$

$$\chi_{red}(n) = \prod_{k=1, k\neq n}^{d} (k^2 - n^2) \frac{2n}{(d-n)!(d+n)!} \mathfrak{J}(d+n-(2n-2), d+n) w^{2d} \overset{4.19}{=}$$

$$\overset{4.19}{=} \frac{(-1)^{n+1}}{n} w^{2d-2n+2} \sum_{k=0}^{n-1} (-1)^k \binom{d+n-k}{2n-2k-1} \frac{(2n-k-1)!}{k!} w^{2k} =$$

$$= \frac{1}{n} \sum_{l=0}^{n-1} (-1)^l \binom{n+l}{2l+1} \frac{(d+l+1)!}{(d-l)!} w^{2d-2l} \tag{5.13}$$

where we have done a substitution $l = n - k - 1$.
At the end we will introduce a more convenient expression for the reduced characteristic function then formula (5.12).

Proposition 21. It holds

$$\chi_{red}(z) = \sum_{s=0}^{d} \frac{(2d-s+1)!}{s!(2d-2s+1)!} w^{2s} \prod_{k=1}^{d-s} (k^2 - z^2) \tag{5.14}$$

for all $z \in \mathbb{C}$.

Proof. Since $\chi_{red}(z)$ is an even polynomial in z of the degree $2d$ (see (3.2)) it is enough to check that the values of the RHS for $z = 0, 1, \dots, d$ coincide with $\chi_{red}(0), \chi_{red}(1), \dots, \chi_{red}(d)$. The expression

$$\chi_{red}(0) = \sum_{s=0}^{d} \frac{(2d-s+1)!}{s!(2d-2s+1)!} w^{2s} [(d-s)!]^2$$

is exactly the formula (3.5). Let $n \in \{1, \dots, d\}$ then

$$\prod_{k=1}^{d-s} (k^2 - n^2) = \begin{cases} 0 & s \le d-n \\ \frac{(-1)^{d-s}}{n} \frac{(n+d-s)!}{(n-d+s-1)!} & d-n < s \le d \end{cases}$$

hence

$$RHS = \frac{1}{n}\sum_{s=d-n+1}^{d}\frac{(2d-s+1)!}{s!(2d-2s+1)!}\frac{(-1)^{d-s}(n+d-s)!}{(n-d+s-1)!}w^{2s} =$$
$$= \frac{1}{n}\sum_{l=0}^{n-1}(-1)^l\frac{(d+l+1)!}{(d-l)!(2l+1)!}\frac{(n+l)!}{(n-l-1)!}w^{2d-2l}$$

which coincides with the formula for $\chi_{red}(n)$ (5.13). $\square$

Remark 16. From formula (5.14) it is obvious that $\chi_{red}(z)$ has no roots for $|z| < 1$ and $\chi_{red}(\pm 1) = 0$ if and only if $w = 0$.

5.3 More on particular values of $\chi_{red}(z)$

In this section we will come back to examine the particular values of $\chi_{red}(z)$ ones more.

Proposition 22. It holds

$$\frac{\chi_{red}(n)}{w^{2d-2}} = v_{-n}^T(I-S)v_n$$

where $S \in \mathbb{C}^{2d+1}$ such that

$$S := \begin{pmatrix} -I_d & & \\ & 0 & \\ & & I_d \end{pmatrix}$$

and $n \in \{0, 1, \ldots, d\}$.

Proof. Since

$$I - S = \begin{pmatrix} 2I_d & & \\ & 1 & \\ & & 0_d \end{pmatrix}$$

the RHS of the statement

$$(\mathfrak{J}(d-n+1, d-n), \ldots, \mathfrak{J}(-d-n+1, d-n))(I-S)\begin{pmatrix} \mathfrak{J}(d+n+1, d+n) \\ \vdots \\ \mathfrak{J}(-d+n+1, d+n) \end{pmatrix} =$$

$$= 2\sum_{j=1}^{d}\mathfrak{J}(j-n+1, d-n)\mathfrak{J}(j+n+1, d+n) + \mathfrak{J}(-n+1, d-n)\mathfrak{J}(n+1, d+n)$$

is exactly the RHS of identity (4.26). $\square$

By using the symmetry of the matrix K ($K = K^T$) and expression (5.4) we get

$$sv_t^T v_s + \alpha_s v_t^d = v_t^T K v_s = (Kv_t)^T v_s = tv_t^T v_s + \alpha_t v_s^d$$

where $\alpha_s = -w\mathfrak{J}(-d+s, d+s)$ and $t, s \in \{-d, \dots, d\}$. It follows that it holds

$$v_t^T v_s = \frac{1}{t-s}(\alpha_s v_t^d - \alpha_t v_s^d). \tag{5.15}$$

Further, by similar way ($S = S^T$), we can write

$$v_t^T SK v_s = v_t^T S(sv_s + \alpha_s e_d) = sv_t S v_s + \alpha_s v_t^d$$

and

$$v_t^T KS v_s = SK v_t^T v_s = tv_t S v_s + \alpha_t v_s^d.$$

Finally by subtracting last two equations and by using the expression (5.15) we arrive at the relation

$$v_t^T [K, S] v_s = (t-s) v_t^T (S - I) v_s \tag{5.16}$$

where $[K, S] = KS - SK$.

Corollary 6. The identity

$$\chi_{red}(n) = \frac{w^{2d-2}}{2n} v_{-n}^T [K, S] v_n$$

holds for $n \in \{0, 1, \dots, d\}$.

Proof. The statement follows from Proposition 22 and relation (5.16) immediately. □

Proposition 23. It holds

$$\chi_{red}(n) = \frac{w^{2d-1}}{2n}\Big(\mathfrak{J}(-n+2, d-n)\mathfrak{J}(n+1, d+n) - \mathfrak{J}(-n+1, d-n)\mathfrak{J}(n+2, d+n) + \mathfrak{J}(-n+1, d-n)\mathfrak{J}(n, d+n) - \mathfrak{J}(-n, d-n)\mathfrak{J}(n+1, d+n)\Big) \tag{5.17}$$

for $n \in \{0, 1, \dots, d\}$.

Proof. It suffices to compute the commutator $[K, S]$ and use the previous Corollary. Since

$$KS = \begin{pmatrix} d & -w & & & & & & & \\ -w & d-1 & -w & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & -w & 1 & 0 & & & & \\ & & & -w & 0 & w & & & \\ & & & & 0 & 1 & w & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & w & d-1 & w \\ & & & & & & & w & d \end{pmatrix}$$

and

$$SK = \begin{pmatrix} d & -w & & & & & & & \\ -w & d-1 & -w & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & -w & 1 & -w & & & & \\ & & & 0 & 0 & 0 & & & \\ & & & & w & 1 & w & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & w & d-1 & w \\ & & & & & & & w & d \end{pmatrix}$$

thus the commutator has only four nonzero entries

$$[K,S] = \begin{pmatrix} 0 & 0 & & & & & & & \\ 0 & & & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & 0 & 0 & w & & & & \\ & & & -w & 0 & w & & & \\ & & & & -w & 0 & 0 & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & & & 0 \\ & & & & & & & 0 & 0 \end{pmatrix}.$$

Then

$$\chi_{red}(n) = \frac{w^{2d-2}}{2n} v_{-n}^T [K,S] v_n = \frac{w^{2d-1}}{2n} (v_{-n}^{-1} v_n^0 - v_{-n}^0 v_n^{-1} + v_{-n}^0 v_n^1 - v_{-n}^1 v_n^0)$$

which concludes the proof. □

Conclusion

The aim of this paper was to describe properties of a spectrum of a finite-dimensional operator with Jacobi matrix of a special type. With the aid of an expanded special function $\mathfrak{F}$ I was able to derive a formula for the characteristic polynomial of the Jacobi matrix of a special type and I have also found a formula for a resolvent of this matrix.

In my Bachelor thesis [2] I have shown what a problem arises if one tries to obtain a global description of the spectrum of an infinite-dimensional operator with the Jacobi matrix by using a regular perturbation theory. Results acquired in this paper could be used in my future work in which I'm going to try to derive a global description of the spectrum of an infinite-dimensional operator with the Jacobi matrix by dividing the infinite matrix to finite blocks. In this afford the gained knowledge of the spectrum of finite dimensional Jacobi matrices should be useful.

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