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DIPLOMA THESIS

Spectral Problem of Jacobi Matrices of a Certain Type

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Název práce: **Spektrální problém pro jistou třídu Jacobiho matic**

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Abstrakt: Na vektorovém prostoru komplexních posloupností definujeme funkci $\mathfrak{F}$ a shrneme několik jednoduchých algebraických vlastností této funkce. Některé speciální funkce lze vyjádřit pomocí $\mathfrak{F}$, například Besselovu funkci prvního druhu. Dále je vyšetřována jistá třída Jacobiho matic liché dimenze, jejichž diagonála zavisí lineárně na indexu a vedlejší diagonály tvoří konstantní parametr. Je odvozena formule pro charakteristický polynom, který je také vyjádřen pomocí funkce $\mathfrak{F}$ v jednoduchém a uceleném tvaru. Je zkonstruována speciální báze v níž studovaná Jacobiho matice přejde v součet diagonální matice a tak zvané „rank-one" matice. Uvedeme vektotovou funkci v komplexní rovině, jejíž hodnoty ve vlastních číslech Jacobiho matice jsou odpovídající vlatní vektory. Nakonec dokážeme, že za jistých předpokladů, je každá vlastní hodnota tridiagonálního operátoru limitou posloupnosti vlastních čísel matic vzniklých ořezáním tohoto operátoru a naopak.

Klíčová slova: Jacobiho matice, tridiagonální operátor, úloha na vlastní čísla, charakteristický polynom

Title: **Spectral Problem of Jacobi Matrices of a Certain Type**

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Abstract: A function $\mathfrak{F}$ with simple and nice algebraic properties is defined on a subset of the space of complex sequences. Some special functions are expressible in terms of $\mathfrak{F}$, first of all the Bessel functions of first kind. Further we focus on the particular class of Jacobi matrices of odd dimension whose parallels to the diagonal are constant and whose diagonal depends linearly on the index. A formula is derived for the characteristic function. Yet another formula is presented in which the characteristic function is expressed in terms of the function $\mathfrak{F}$ in a simple and compact manner. A special basis is constructed in which the Jacobi matrix becomes a sum of a diagonal matrix and a rank-one matrix operator. A vector-valued function on the complex plain is constructed having the property that its values on spectral points of the Jacobi matrix are equal to corresponding eigenvectors.Under certain assumptions, it is shown that every eigenvalue of tridiagonal operator is a limit point of a sequence of eigenvalues of a truncated finitedimensional operator and vice versa.

Key words: finite Jacobi matrix, tridiagonal operator, eigenvalue problem, characteristic function

Contents

Introduction

The results of the current thesis are related to the eigenvalue problem for finite-dimensional symmetric tridiagonal (Jacobi) matrices. Notably, the eigenvalue problem for finite Jacobi matrices is solvable explicitly in terms of generalized hypergeometric series [4]. Here we focus on a very particular class of Jacobi matrices which makes it possible to derive some expressions in a comparatively simple and compact form. We do not aim at all, however, at a complete solution of the eigenvalue problem. We restrict ourselves to derivation of several explicit formulas, first of all that for the characteristic function, as explained in more detail below.

First, we introduce a function, called $\mathfrak{F}$, defined on a subset of the space of complex sequences. In the remainder of the paper it is intensively used in various formulas. The function $\mathfrak{F}$ has remarkably simple and nice algebraic properties. Among others, function $\mathfrak{F}$ satisfies a tree term recurrent relation which plays an essential role in the remainder of the paper. Let us also note that some special functions are expressible in terms of $\mathfrak{F}$. First of all this concerns the Bessel functions of first kind. We examine the relationship between $\mathfrak{F}$ and the Bessel functions and provide some supplementary details on it. Bessel functions (of first and second kind) also arise while studying an asymptotic behaviour of $\mathfrak{F}$.

Next, we introduce a Jacobi matrix of a special type, more precisely, a Jacobi matrix with a sequence (antisymmetric with respect to its center) on the diagonal with constant neighboring parallels. As an application of the usage of $\mathfrak{F}$ we present a comparatively simple formula for the characteristic function of the Jacobi matrix and also an explicit formula for the eigenvector corresponding to zero eigenvalue. These results have been derived in former work [3].

From the third chapter, we focus on the rather particular class of Jacobi matrices of odd dimension whose parallels to the diagonal are constant and whose diagonal depends linearly on the index. With the aid of the function $\mathfrak{F}$ we define a symbol, denoted $\mathfrak{J}$, depending on two integers and we study its algebraic properties. In addition, we find a general formula for the symbol $\mathfrak{J}$.

One of the main goal is obtained in the fourth chapter. It is a formula for characteristic function of the Jacobi matrix under investigation. With the aid of the symbol $\mathfrak{J}$ we construct a special basis in which the Jacobi matrix becomes a sum of a diagonal matrix and a rank-one matrix operator. This form is rather suitable for various computations. Particularly, one can compute a determinant which leads us to find the formula for the characteristic function. Moreover, we are able to express the resolvent operator of the studied matrix, again due to the convenient form of the matrix written in the constructed basis. Finally, we provide several information dealing with a distribution of the spectrum following directly from the formula for characteristic function.

The fifth chapter brings us to a slight generalization of the symbol $\mathfrak{J}$, we add a new continuous dependent variable and define a function also called $\mathfrak{J}$. We present one more identity for characteristic function by using the new function $\mathfrak{J}$. Further, a vector-valued function on the complex plain is constructed having the property that its values on spectral points of the Jacobi matrix are equal to corresponding eigenvectors. At the end of the chapter we modify the presented results to obtain the characteristic function and the vector-valued function (with the same property mentioned above) for the Jacobi matrix of even dimension.

In the last chapter we collect information dealing with a relationship between the spectrum of the infinitedimensional Jacobi matrix and the spectrum of its truncations which is described mostly in [7] and [8]. More precisely, we show that under certain assumptions, every eigenvalue of tridiagonal operator is a limit point of a sequence of eigenvalues of a truncated finitedimensional operator and vice versa.

Chapter 1

Function $\mathfrak{F}$

Almost all important results obtained in this paper are based on an establishing of a special function, simply called function $\mathfrak{F}$. This function satisfies a very important recurrent relation which shows to be essential.

1.1 Function $\mathfrak{F}$ and its Basic Properties

Definition 1. Define $\mathfrak{F} : D \to \mathbb{C}$,

$$\mathfrak{F}(x) := 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \dots x_{k_m} x_{k_m+1}$$

where

$$D = \left\{ x = \{x_k\}_{k=1}^{\infty} \,\middle|\, \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}.$$

Remark 2.
1) Note that the summation indices satisfy $k_j \geq 2j - 1$.
2) Note that the definition domain D is not a linear space. One has, however, $\ell^2(\mathbb{N}) \subset D$ (use Schwarz inequality).
3) Obviously, if all but finitely many elements of x are zeroes then $\mathfrak{F}(x)$ reduces to a finite sum. For a finite number of variables we often will write $\mathfrak{F}(x_1, x_2, \dots, x_k)$ instead of $\mathfrak{F}(x)$ where $x = (x_1, x_2, \dots, x_k, 0, 0, 0, \dots)$.

Example 3. Let us introduce some simple examples,

$$\begin{aligned}
&\mathfrak{F}(\emptyset) = 1,\\
&\mathfrak{F}(x_1) = 1,\\
&\mathfrak{F}(x_1, x_2) = 1 - x_1x_2,\\
&\mathfrak{F}(x_1, x_2, x_3) = 1 - x_1x_2 - x_2x_3,\\
&\mathfrak{F}(x_1, x_2, x_3, x_4) = 1 - x_1x_2 - x_2x_3 - x_3x_4 + x_1x_2x_3x_4,\\
&\mathfrak{F}(x_1, x_2, x_3, x_4, x_5) = 1 - x_1x_2 - x_2x_3 - x_3x_4 + x_1x_2x_3x_4 - x_4x_5 + x_1x_2x_4x_5\\
&\qquad + x_2x_3x_4x_5,\\
&\mathfrak{F}(x_1, x_2, x_3, x_4, x_5, x_6) = 1 - x_1x_2 - x_2x_3 - x_3x_4 + x_1x_2x_3x_4 - x_4x_5 + x_1x_2x_4x_5\\
&\qquad + x_2x_3x_4x_5 - x_5x_6 + x_1x_2x_5x_6 + x_2x_3x_5x_6 + x_3x_4x_5x_6 - x_1x_2x_3x_4x_5x_6.
\end{aligned}$$

Definition 4. The operator T_1 defined on the space of all sequences indexed by $\mathbb{N}$ such that

$$T_1(\{x_k\}_{k=1}^{\infty}) := \{x_{k+1}\}_{k=1}^{\infty}$$

is called the operator of *truncation from the left.* Next, set $T_n := (T_1)^n$, $n = 0, 1, 2, \dots$, hence

$$T_n(\{x_k\}_{k=1}^{\infty}) = \{x_{k+n}\}_{k=1}^{\infty}$$

In particular, T_0 is the identity.

Proposition 5. It holds

$$\mathfrak{F}(T_nx) - \mathfrak{F}(T_{n+1}x) + x_{n+1}x_{n+2}\mathfrak{F}(T_{n+2}x) = 0, \tag{1.1}$$

for all $n \in \mathbb{N}_0$ and $x \in D$.

Proof. To verify this identity, note that after the substitution $x' = T_nx$ one can restrict oneself to the particular case $n = 0$. Consider that

$$\mathfrak{F}(T_nx) = 1 + \sum_{m=1}^{\infty}(-1)^m \sum_{k_1=n+1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1}x_{k_1+1}x_{k_2}x_{k_2+1}\dots x_{k_m}x_{k_m+1}$$

and so

$$\begin{aligned}
\mathfrak{F}(x) - \mathfrak{F}(T_1x) &= -x_1x_2 + \sum_{m=2}^{\infty}(-1)^m \sum_{k_2=3}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_1x_2x_{k_2}x_{k_2+1}\dots x_{k_m}x_{k_m+1}\\
&= -x_1x_2\mathfrak{F}(T_2x).
\end{aligned}$$

□

Remark 6. Especially, it holds

$$\mathfrak{F}(x_1, \dots, x_n) = \mathfrak{F}(x_2, \dots, x_n) - x_1x_2\mathfrak{F}(x_3, \dots, x_n) \tag{1.2}$$

for $n \in \mathbb{N} \setminus \{1\}$.

Remark 7. Since $\mathfrak{F}(\emptyset) = \mathfrak{F}(x_1) = 1$, relation (1.2) determines recursively and unambiguously $\mathfrak{F}(x_1, \dots, x_n)$ for any finite number of variables $n \in \mathbb{N}_0$.

Proposition 8. It holds

$$\mathfrak{F}(x_1, x_2, \dots, x_n) = \mathfrak{F}(x_n, x_{n-1}, \dots, x_1) \tag{1.3}$$

for all $n \in \mathbb{N}_0$.

Proof. Let us denote $y_j := x_{n-j+1}$ for $j = 1, \dots, n$, then

$$\begin{aligned}
&\mathfrak{F}(x_n, x_{n-1}, \dots, x_1) = \mathfrak{F}(y_1, y_2, \dots, y_n) \\
&= 1 + \sum_{m=1}^{\left[\frac{n+1}{2}\right]} (-1)^m \sum_{k_1=1}^{n-2(m-1)} \sum_{k_2=k_1+2}^{n-2(m-2)} \cdots \sum_{k_m=k_{m-1}+2}^{n} y_{k_1} y_{k_1+1} y_{k_2} y_{k_2+1} \dots y_{k_m} y_{k_m+1} \\
&= 1 + \sum_{m=1}^{\left[\frac{n+1}{2}\right]} (-1)^m \sum_{l_1=2m-1}^{n} \sum_{l_2=2m-3}^{l_1-2} \cdots \sum_{l_m=1}^{l_{m-1}-2} x_{l_1} x_{l_1+1} x_{l_2} x_{l_2+1} \dots x_{l_m} x_{l_m+1}
\end{aligned}$$

where we have substituted $l_j = n - k_j + 1$, $j = 1, \dots, n$. To proceed further we just denote the summand index l_j as l_{m-j+1} for all $j = 1, \dots, n$ and change the order of the sums such that the sum with the summand index l_1 will be at the first place, the sum with the summand index l_2 will be at the second place, etc.

$$\begin{aligned}
&= 1 + \sum_{m=1}^{\left[\frac{n+1}{2}\right]} (-1)^m \sum_{l_m=2m-1}^{n} \sum_{l_{m-1}=2m-3}^{l_m-2} \cdots \sum_{l_1=1}^{l_2-2} x_{l_1} x_{l_1+1} x_{l_2} x_{l_2+1} \dots x_{l_m} x_{l_m+1} \\
&= 1 + \sum_{m=1}^{\left[\frac{n+1}{2}\right]} (-1)^m \sum_{l_{m-1}=2m-3}^{n-2} \sum_{l_{m-2}=2m-5}^{l_{m-1}-2} \cdots \sum_{l_1=1}^{l_2-2} \sum_{l_m=l_{m-1}+2}^{n} x_{l_1} x_{l_1+1} \dots x_{l_m} x_{l_m+1} \\
&= 1 + \sum_{m=1}^{\left[\frac{n+1}{2}\right]} (-1)^m \sum_{l_{m-2}=2m-5}^{n-4} \cdots \sum_{l_1=1}^{l_2-2} \sum_{l_{m-1}=l_{m-2}+2}^{n-2} \sum_{l_m=l_{m-1}+2}^{n} x_{l_1} x_{l_1+1} \dots x_{l_m} x_{l_m+1} \\
&= \cdots = \\
&= 1 + \sum_{m=1}^{\left[\frac{n+1}{2}\right]} (-1)^m \sum_{l_1=1}^{n-2(m-1)} \sum_{l_2=l_1+2}^{n-2(m-2)} \cdots \sum_{l_{m-1}=l_{m-2}+2}^{n-2} \sum_{l_m=l_{m-1}+2}^{n} x_{l_1} x_{l_1+1} \dots x_{l_m} x_{l_m+1} \\
&= \mathfrak{F}(x_1, x_2, \dots, x_n).
\end{aligned}$$

□

Remark 9. By using the stated symmetry property of the function $\mathfrak{F}$ and recurrent relation (1.2) one can easily derive the following identity

$$\mathfrak{F}(x_1, \dots, x_n) = \mathfrak{F}(x_1, \dots, x_{n-1}) - x_n x_{n-1} \mathfrak{F}(x_1, \dots, x_{n-2}) \tag{1.4}$$

which holds for $n \in \mathbb{N} \setminus \{1\}$.

A generalization of Proposition 5 can be found.

Proposition 10. For every $x \in D$ and $k \in \mathbb{N}$ one has

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \dots, x_k)\,\mathfrak{F}(T_k x) - \mathfrak{F}(x_1, \dots, x_{k-1}) x_k x_{k+1}\,\mathfrak{F}(T_{k+1} x). \tag{1.5}$$

Proof. Let us proceed by induction in k. For $k = 1$, equality (1.5) coincides with (1.1). Suppose (1.5) is true for $k \in \mathbb{N}$. Applying Proposition 5 to the sequence $T_k x$ and using (1.4) one finds that the RHS of (1.5) equals

$$\begin{aligned}
&\mathfrak{F}(x_1, \dots, x_k)\,\mathfrak{F}(T_{k+1} x) - \mathfrak{F}(x_1, \dots, x_k) x_{k+1} x_{k+2}\,\mathfrak{F}(T_{k+2} x) \\
&- \mathfrak{F}(x_1, \dots, x_{k-1}) x_k x_{k+1}\,\mathfrak{F}(T_{k+1} x) \\
&= \mathfrak{F}(x_1, \dots, x_k, x_{k+1})\,\mathfrak{F}(T_{k+1} x) - \mathfrak{F}(x_1, \dots, x_k) x_{k+1} x_{k+2} \mathfrak{F}(T_{k+2} x).
\end{aligned} \tag{1.6}$$

This concludes the verification. □

1.2 Examples

Example 11. The Bessel functions of the first kind can be expressed in terms of function $\mathfrak{F}$. More precisely, for $\nu \notin -\mathbb{N}$, one has

$$J_\nu(2w) = \frac{w^\nu}{\Gamma(\nu+1)}\,\mathfrak{F}\left(\left\{\frac{w}{\nu+k}\right\}_{k=1}^{\infty}\right). \tag{1.7}$$

To prove (1.7) one can proceed by (finite) induction in $j = 0, 1, \dots, m-1$, to show that

$$\begin{aligned}
&\sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} \\
&\quad \times \frac{1}{(\nu+k_1)(\nu+k_1+1)(\nu+k_2)(\nu+k_2+1)\dots(\nu+k_m)(\nu+k_m+1)} \\
&= \frac{1}{j!} \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_{m-j}=k_{m-j-1}+2}^{\infty} \\
&\quad \times \frac{1}{(\nu+k_1)(\nu+k_1+1)(\nu+k_2)(\nu+k_2+1)\dots(\nu+k_{m-j})(\nu+k_{m-j}+1)} \\
&\quad \times \frac{1}{(\nu+k_{m-j}+2)(\nu+k_{m-j}+3)\dots(\nu+k_{m-j}+j+1)}.
\end{aligned} \tag{1.8}$$

Case $j = 0$ is immediate. To proceed the induction step $j \to j+1$ one have to count the inner sum in the RHS of (1.8) which is easy due to the decomposition

$$\begin{aligned}
&\frac{1}{(\nu+k_{m-j})(\nu+k_{m-j}+1)(\nu+k_{m-j}+2)\dots(\nu+k_{m-j}+j+1)} \\
&= \frac{1}{j+1}\Bigg(\frac{1}{(\nu+k_{m-j})(\nu+k_{m-j}+1)(\nu+k_{m-j}+2)\dots(\nu+k_{m-j}+j)} \\
&\quad - \frac{1}{(\nu+k_{m-j}+1)(\nu+k_{m-j}+2)(\nu+k_{m-j}+3)\dots(\nu+k_{m-j}+j+1)}\Bigg).
\end{aligned}$$

In particular, for $j = m-1$, the RHS of (1.8) equals

$$\frac{1}{(m-1)!}\sum_{k_1=1}^{\infty}\frac{1}{(\nu+k_1)(\nu+k_1+1)(\nu+k_1+2)\dots(\nu+k_1+m)}$$
$$=\frac{1}{m!}\sum_{k_1=1}^{\infty}\frac{1}{(\nu+k_1)\dots(\nu+k_1+m-1)}-\frac{1}{(\nu+k_1+1)\dots(\nu+k_1+m)}$$
$$=\frac{1}{m!(\nu+1)(\nu+2)\dots(\nu+m)}=\frac{\Gamma(\nu+1)}{m!\Gamma(\nu+m+1)} \tag{1.9}$$

and so

$$\frac{w^{\nu}}{\Gamma(\nu+1)}\mathfrak{F}\left(\left\{\frac{w}{\nu+k}\right\}_{k=1}^{\infty}\right)=\sum_{m=0}^{\infty}(-1)^m\frac{w^{2m+\nu}}{m!\Gamma(\nu+m+1)}=J_{\nu}(2w),$$

as claimed. The last equality is an expansion of the Bessel function of the first kind, see for example [1], chap. 9. Furthermore, the recurrence relation (1.1) transforms to the well known identity

$$zJ_{\nu}(z)-2(\nu+1)J_{\nu+1}(z)+zJ_{\nu+2}(z)=0$$

which can be found also in [1], chap. 9. A formula for $\mathfrak{F}$ applied on the truncated sequence $\{w/(m+k)\}_{k=0}^{n}$ can be found. Its derivation will be described later (see (1.17) or (5.30)).

Example 12. One can also find the value of $\mathfrak{F}$ on the geometric sequence $\{t^{k-1}w\}_{k=1}^{n}$ for $t, w \in \mathbb{C}$, $|t|<1$. First, one can prove the identity

$$\sum_{k_1=n}^{\infty}\sum_{k_2=k_1+2}^{\infty}\cdots\sum_{k_m=k_{m-1}+2}^{\infty}t^{2k_1-1}t^{2k_2-1}\dots t^{2k_m-1}=\frac{t^{m(2m-3)}t^{2mn}}{(1-t^2)(1-t^4)\dots(1-t^{2m})} \tag{1.10}$$

by mathematical induction in m. Case $m=1$ is easy to verify. Let us proceed the induction step $m \to m+1$,

$$\sum_{k_1=n}^{\infty}\sum_{k_2=k_1+2}^{\infty}\cdots\sum_{k_{m+1}=k_m+2}^{\infty}t^{2k_1-1}t^{2k_2-1}\dots t^{2k_{m+1}-1}$$
$$\overset{IH}{=}\sum_{k_1=n}^{\infty}t^{2k_1-1}\frac{t^{m(2m-3)}t^{2m(k_1+2)}}{(1-t^2)(1-t^4)\dots(1-t^{2m})}=\frac{t^{(m+1)(2m-1)}t^{2(m+1)n}}{(1-t^2)(1-t^4)\dots(1-t^{2m+2})}.$$

Next, by using identity (1.10) and the definition of $\mathfrak{F}$ one has the equality

$$\mathfrak{F}\left(\{t^{k-1}w\}_{k=1}^{\infty}\right)=1+\sum_{m=1}^{\infty}(-1)^m\frac{t^{m(2m-1)}w^{2m}}{(1-t^2)(1-t^4)\dots(1-t^{2m})} \tag{1.11}$$

This function can be identified with a basic hypergeometric series (also called q-hypergeometric series) defined by

$$ {}_r\phi_s(a;b;q,z) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k \dots (a_r;q)_k}{(b_1;q)_k \dots (b_s;q)_k} \left((-1)^k q^{\frac{1}{2}k(k-1)}\right)^{1+s-r} \frac{z^k}{(q;q)_k} $$

where $r, s \in \mathbb{N}_0$ and

$$ (\alpha;q)_k = \prod_{j=0}^{k-1} \left(1 - \alpha q^j\right), \ k = 0, 1, 2, \dots, $$

see [5]. In fact, the RHS in (1.11) equals ${}_0\phi_1(;0;t^2,-tw^2)$ where

$$ {}_0\phi_1(;0;q,z) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)}}{(q;q)_k} z^k = \sum_{k=0}^{\infty} \frac{q^{k(k-1)}}{(1-q)(1-q^2)\dots(1-q^k)} z^k, $$

with $q, z \in \mathbb{C}$, $|q| < 1$, and the recursive rule (1.1) takes the form

$$ {}_0\phi_1(;0;q,z) = {}_0\phi_1(;0;q,qz) + z\, {}_0\phi_1(;0;q,q^2 z). \tag{1.12} $$

Similarly as in the previous example one can try to find a formula for $\mathfrak{F}$ applied on the truncated sequence $\{wt^k\}_{k=m}^n$ for $w, t \in \mathbb{C}$ and $m \le n$. The formula has the form

$$ \mathfrak{F}(wt^m, wt^{m+1}, \dots, wt^n) = 1 + \sum_{s=1}^{\left[\frac{n-m+1}{2}\right]} \frac{\prod_{k=s-1}^{2s-2}(t^{2m+2k} - t^{2n})}{\prod_{j=1}^{s}(t^{2j}-1)} \frac{w^{2s}}{t^{s(s-2)}}. \tag{1.13} $$

Since the cases when $m = n+1$ and $m = n$ give 1 it is sufficient to show the RHS in (1.13) satisfies the recurrence relation

$$ \mathfrak{F}(wt^m, wt^{m+1} \dots, wt^n) = \mathfrak{F}(wt^{m+1}, \dots, wt^n) - w^2 t^{2m+1} \mathfrak{F}(wt^{m+2}, \dots, wt^n) \tag{1.14} $$

(see Remark 7). Thus

$$ \begin{aligned} &1 + \sum_{s=1}^{\left[\frac{n-m}{2}\right]} \frac{\prod_{k=s-1}^{2s-2}(t^{2m+2k+2} - t^{2n})}{\prod_{j=1}^{s}(t^{2j}-1)} \frac{w^{2s}}{t^{s(s-2)}} \\ &- t^{2m+1} \left(w^2 + \sum_{s=1}^{\left[\frac{n-m-1}{2}\right]} \frac{\prod_{k=s-1}^{2s-2}(t^{2m+2k+4} - t^{2n})}{\prod_{j=1}^{s}(t^{2j}-1)} \frac{w^{2s+2}}{t^{s(s-2)}} \right) \\ &= 1 + \sum_{s=1}^{\left[\frac{n-m}{2}\right]} \frac{\prod_{k=s}^{2s-1}(t^{2m+2k} - t^{2n})}{\prod_{j=1}^{s}(t^{2j}-1)} \frac{w^{2s}}{t^{s(s-2)}} - \sum_{s=1}^{\left[\frac{n-m+1}{2}\right]} \frac{\prod_{k=s}^{2s-2}(t^{2m+2k} - t^{2n})}{\prod_{j=1}^{s-1}(t^{2j}-1)} \frac{w^{2s}}{t^{(s-1)(s-3)}}. \end{aligned} $$

To proceed further, one have to notice that if $n-m+1$ is an odd number then $\left[\frac{n-m+1}{2}\right] = \left[\frac{n-m}{2}\right]$ and if $n-m+1$ is an even number then $\left[\frac{n-m+1}{2}\right] = \left[\frac{n-m}{2}\right]+1$ but if the index of summation s of the first sum goes up to $\left[\frac{n-m}{2}\right]+1$ the last term of the sum would be zero. Thus we can change the upper bound of the first sum to $\left[\frac{n-m+1}{2}\right]$ and we arrive at the expression

$$\begin{aligned}
&1+\sum_{s=1}^{\left[\frac{n-m+1}{2}\right]} \frac{w^{2s}\prod_{k=s}^{2s-2}(t^{2m+2k}-t^{2n})}{t^{s(s-2)}\prod_{j=1}^{s}(t^{2j}-1)}\left[(t^{2m+4s-2}-t^{2n})-t^{2m+2s-2}(t^{2s}-1)\right] \\
&=1+\sum_{s=1}^{\left[\frac{n-m+1}{2}\right]} \frac{\prod_{k=s-1}^{2s-2}(t^{2m+2k}-t^{2n})}{\prod_{j=1}^{s}(t^{2j}-1)}\frac{w^{2s}}{t^{s(s-2)}}
\end{aligned}$$

which concludes the proof.

1.3 A Formula for $\mathfrak{F}(\frac{w}{m}, \frac{w}{m+1}, \ldots, \frac{w}{n})$

In later chapters it will be useful to know an explicit formula for $\mathfrak{F}(\frac{w}{m}, \frac{w}{m+1}, \ldots, \frac{w}{n})$ where w is a real parameter and $m, n \in \mathbb{N}$, $m \le n$. The formula will be derived throughout this section.

Lemma 13. It holds

$$\begin{aligned}
&\sum_{k=m}^{n} \frac{(n+1-k)(n+2-k)\ldots(n+s-1-k)}{k(k+1)\ldots(k+s)} \\
&\qquad = \frac{(n-m+1)(n-m+2)\ldots(n-m+s)}{s(n+s)m(m+1)\ldots(m+s-1)} \qquad (1.15)
\end{aligned}$$

for all $m, n, s \in \mathbb{N}, m \le n$.

Proof. Denote the LHS by $Y_s(m,n)$. One can verify the statement by induction in s. The case $s=1$ gives the identity

$$\sum_{k=m}^{n} \frac{1}{k(k+1)} = \frac{n-m+1}{(n+1)m}$$

which is easy to verify (by using decomposition $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$).
In the induction step $s-1 \to s$, with $s>1$, observe that

$$Y_s(m,n) = -\frac{n+2s-1}{s}Y_{s-1}(m+1,n+1) + \frac{n+s-1}{s}Y_{s-1}(m,n).$$

Now, by applying the induction hypothesis we have

$$\begin{aligned}
Y_s(m,n) &= -\frac{n+2s-1}{s}\frac{(n-m+1)(n-m+2)\dots(n-m+s-1)}{(s-1)(n+s)(m+1)(m+2)\dots(m+s-1)} \\
&+ \frac{n+s-1}{s}\frac{(n-m+1)(n-m+2)\dots(n-m+s-1)}{(s-1)(n+s-1)m(m+1)\dots(m+s-2)} \\
&= \frac{(n-m+1)(n-m+2)\dots(n-m+s)}{s(n+s)m(m+1)(m+2)\dots(m+s-1)} = RHS.
\end{aligned}$$

□

Proposition 14. It holds

$$\sum_{k_1=m}^{n-2s+2}\sum_{k_2=k_1+2}^{n-2s+4}\cdots\sum_{k_s=k_{s-1}+2}^{n}\frac{1}{k_1(k_1+1)k_2(k_2+1)\dots k_s(k_s+1)}$$
$$= \frac{(n-m-2s+3)(n-m-2s+4)\dots(n-m-s+2)}{s!m(m+1)\dots(m+s-1)(n-s+2)(n-s+3)\dots(n+1)} \tag{1.16}$$

for all $m, n, s \in \mathbb{N}, m \leq n-2s+2$.

Proof. This can be again proved by the induction in s. For $s=1$ we get the well known identity

$$\sum_{k=m}^{n}\frac{1}{k(k+1)} = \frac{n-m+1}{(n+1)m}.$$

Denote the LHS of (1.16) by X_s. To carry out the induction step from $s-1$ to s, with $s>1$, we apply the induction hypothesis to the summation in the indices $k_2 \dots k_s$ and find that (when writing k instead of k_1 and k_{j-1} instead of k_j)

$$\begin{aligned}
X_s &= \sum_{k=m}^{n-2s+2}\frac{1}{k(k+1)}\sum_{k_1=k+2}^{n-2s+4}\cdots\sum_{k_{s-1}=k_{s-2}+2}^{n}\frac{1}{k_1(k_1+1)k_2(k_2+1)\dots k_{s-1}(k_{s-1}+1)} \\
&\overset{IH}{=} \sum_{k=m}^{n-2s+2}\frac{(n-k-2s+3)(n-k-2s+4)\dots(n-k-s+1)}{(s-1)!(n-s+3)\dots(n+1)k(k+1)(k+2)\dots(k+s)}.
\end{aligned}$$

To conclude the proof it suffices to apply identity (1.15), with n being replaced by $n-2s+2$, to the RHS in (1.15). Then we get

$$\begin{aligned}
X_s &= \frac{(n-2s-m+3)(n-2s-m+4)\dots(n-s-m+2)}{(s-1)!(n-s+3)\dots(n+1)s(n-s+2)m(m+1)\dots(m+s-1)} \\
&= \frac{(n-2s-m+3)(n-2s-m+4)\dots(n-s-m+2)}{s!m(m+1)\dots(m+s-1)(n-s+2)\dots(n+1)} = RHS.
\end{aligned}$$

□

Corollary 15. It holds

$$\mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \dots, \frac{w}{n}\right) = \frac{(m-1)!}{n!} \sum_{s=0}^{[\frac{n-m+1}{2}]} (-1)^s \frac{(n-s)!(n-m-s+1)!}{s!(m+s-1)!(n-m-2s+1)!} w^{2s} \tag{1.17}$$

for all $m, n \in \mathbb{N}, m \leq n$.

Proof. Using the definition of the function $\mathfrak{F}$ we have

$$\mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \dots, \frac{w}{n}\right) =$$
$$= 1 + \sum_{s=1}^{\infty} (-1)^s \sum_{k_1=m}^{n-2s+1} \sum_{k_2=k_1+2}^{n-2s+3} \cdots \sum_{k_s=k_{s-1}+2}^{n-1} \frac{w^{2s}}{k_1(k_1+1)k_2(k_2+1)\dots k_s(k_s+1)}.$$

Now observe that s is restricted by the inequality $2s \leq n-m+1$ and use identity (1.16) with n replaced by $n-1$.Then

$$\mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \dots, \frac{w}{n}\right) = \frac{(m-1)!}{n!} \sum_{s=0}^{[\frac{n-m+1}{2}]} (-1)^s \frac{(n-s)!(n-m-s+1)!}{s!(m+s-1)!(n-m-2s+1)!} w^{2s}.$$

□

1.4 Asymptotic Properties of $\mathfrak{F}(\frac{w}{m}, \frac{w}{m+1}, \dots, \frac{w}{n})$

In this section we will investigate an asymptotic behaviour of the function $\mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \dots, \frac{w}{n}\right)$ for w small. There appears a relation between the function $\mathfrak{F}$ and Bessel function of the first and second kind.

Remark 16. The Bessel function of the first kind $J_m(z)$ and the second kind $Y_n(z)$ arises a lot in this section. Their series expansions are (found in [1] 9.1.10, 9.1.11):

$$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{k=0}^{\infty} \frac{1}{k!(m+k)!} \left(-\frac{1}{4}z^2\right)^k, \tag{1.18}$$

$$Y_n(z) = -\frac{1}{\pi}\left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{1}{4}z^2\right)^k + \frac{2}{\pi} \ln\left(\frac{z}{2}\right) J_n(z) + O(z^n) \tag{1.19}$$

where $m, n \in \mathbb{N}_0$. Also generalized hypergeometric functions will be used. The F_{pq} function has the series expansion

$$F_{pq}(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k, \dots, (a_p)_k}{(b_1)_k, \dots, (b_q)_k} \frac{z^k}{k!} \tag{1.20}$$

where $(c)_k = c(c+1)\dots(c+k-1)$ is the Pochhammer symbol (found in [1] 15.1.1).

1.4.1 Case $m = 1$

Proposition 17. It holds

$$\mathfrak{F}\left(w, \frac{w}{2}, \ldots, \frac{w}{n}\right) = -\frac{\pi}{n!} w^{n+1} J_0(2w) Y_{n+1}(2w) + O(w^{2n+2} \ln w), \tag{1.21}$$

where $n \in \mathbb{N}$.

Proof. It holds

$$F_{23}\left(a, a+\frac{1}{2}; d, 2a, 2a-d+1; z\right) = F_{01}\left(d; \frac{z}{4}\right) F_{01}\left(2a-d+1; \frac{z}{4}\right). \tag{1.22}$$

This identity can be found in http://functions.wolfram.com/07.26.03.6005.01 (or it can be obtained by combining identity (2.2.2.12) from [2] with the identity stated below). Next, by using the identity ([1] 9.1.69)

$$F_{01}\left(\nu+1; -\frac{1}{4}z^2\right) = J_\nu(z)\Gamma(\nu+1)\left(\frac{z}{2}\right)^{-\nu}$$

we get

$$F_{23}\left(-\frac{\mu}{2}, -\frac{\mu}{2}+\frac{1}{2}; 1, -\mu, -\mu; -4w^2\right) = J_0(2w) F_{01}(-\mu; -w^2)$$

where $a = -\frac{\mu}{2}$, $d = 1$, $z = -4w^2$. By using the series expansion (1.20) and (1.18) one can write the last identity as

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2}-\frac{\mu}{2})_k(-\frac{\mu}{2})_k}{(1)_k(-\mu)_k(-\mu)_k}(-1)^k\frac{(2w)^{2k}}{k!} = \sum_{s=0}^{\infty}\sum_{l=0}^{s}\frac{(-1)^s}{(-\mu)_l l![(s-l)!]^2} w^{2s} \qquad (\mu \notin \mathbb{N}).$$

Next, take into consideration only terms with $w^{2j}, j = 0, 1, \ldots, n$

$$\sum_{k=0}^{n} \frac{(\frac{1}{2}-\frac{\mu}{2})_k(-\frac{\mu}{2})_k}{(1)_k(-\mu)_k(-\mu)_k}(-1)^k\frac{(2w)^{2k}}{k!} = \sum_{s=0}^{n}\sum_{l=0}^{s}\frac{(-1)^s}{(-\mu)_l l![(s-l)!]^2} w^{2s}$$

and make a limit $\mu \to n$

$$\sum_{k=0}^{n} \frac{(\frac{1}{2}-\frac{n}{2})_k(-\frac{n}{2})_k}{(1)_k(-n)_k(-n)_k}(-1)^k\frac{(2w)^{2k}}{k!} = \sum_{s=0}^{n}\sum_{l=0}^{s}\frac{(-1)^s}{(-n)_l l![(s-l)!]^2} w^{2s}.$$

The LHS of this expression can be further adjusted, so

$$\begin{aligned} LHS \equiv \sum_{k=0}^{n} \frac{(\frac{1}{2}-\frac{n}{2})_k(-\frac{n}{2})_k}{(1)_k(-n)_k(-n)_k}(-1)^k\frac{(2w)^{2k}}{k!} &= \frac{1}{n!}\sum_{s=0}^{[\frac{n}{2}]}(-1)^s\left(\frac{(n-s)!}{s!}\right)^2\frac{w^{2s}}{(n-2s)!} \\ &= \mathfrak{F}\left(w, \frac{w}{2}, \ldots, \frac{w}{n}\right). \end{aligned}$$

The last equality holds due to Corollary 15 with $m = 1$. Thus, we have

$$\mathfrak{F}\left(w, \frac{w}{2}, \dots, \frac{w}{n}\right) = \sum_{s=0}^{n}\sum_{l=0}^{s} \frac{(-1)^s}{(-n)_l l![(s-l)!]^2} w^{2s}$$
$$= \frac{1}{n!}\sum_{s=0}^{n}\left(\sum_{l=0}^{s} \frac{(-1)^l (n-l)!}{l![(s-l)!]^2}\right)(-1)^s w^{2s}.$$

Next, consider the RHS in the statement of the proposition. Taking into account the series expansion of Bessel functions (1.18) and (1.19) one arrives at the following expression

$$-\frac{\pi}{n!} w^{n+1} J_0(2w) Y_{n+1}(2w) = \frac{1}{n!}\sum_{s=0}^{n}\sum_{l=0}^{s} \frac{(n-l)!}{l!}\frac{(-1)^{s-l}}{[(s-l)!]^2} w^{2s} + O(w^{2n+2}\ln w)$$

and the statement is verified. □

1.4.2 General Case $m \in \mathbb{N}$

In this section we will generalize the procedure used in the previous special case with $m = 1$.

Proposition 18. It holds

$$\mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \dots, \frac{w}{n}\right) = -\pi \frac{(m-1)!}{n!} w^{n-m+2} J_{m-1}(2w) Y_{n+1}(2w) + O(w^{2n-2m+4}) \tag{1.23}$$

for all $m, n \in \mathbb{N}$, $2 \le m \le n$.

Proof. Investigate the following generalized hypergeometric function (observe that the index k is restricted ($k \le \left[\frac{n-m+1}{2}\right]$) due to the nominator of the expression)

$$F_{23}\left(\frac{m-1-n}{2}, \frac{m-n}{2}; m, m-1-n, -n; -4w^2\right)$$
$$= \sum_{k=0}^{\infty} \frac{\left(\frac{m-1-n}{2}\right)_k \left(\frac{m-n}{2}\right)_k}{(m)_k (m-1-n)_k (-n)_k} \frac{(-4w^2)^k}{k!}$$
$$= \sum_{k=0}^{\left[\frac{n-m+1}{2}\right]} \frac{(-1)^k (n-m+1)(n-m)\dots(n-m+2-2k)}{m\dots(m+k-1)(n-m+1)\dots(n-m+2-k)n\dots(n+1-k)} \frac{w^{2k}}{k!}$$
$$= \frac{(m-1)!}{n!} \sum_{k=0}^{\left[\frac{n-m+1}{2}\right]} (-1)^k \frac{(n-m+1-k)!(n-k)!}{(n-m+1-2k)!(m+k-1)!} \frac{w^{2k}}{k!}$$
$$= \mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \dots, \frac{w}{n}\right).$$

The last equality holds due to Corollary 15. By using identity (1.22) with $d = m$, $a = \frac{m-1-\mu}{2}$ and $z = -4w^2$ we get

$$F_{23}\left(\frac{m-1-\mu}{2}, \frac{m-\mu}{2}; m, m-1-\mu, -\mu; -4w^2\right) = F_{01}(m; -w^2)F_{01}(-\mu; -w^2).$$

The parameter μ is near n but μ is not an integer. Then, by using expansions of F_{23} and F_{01} (see 1.20), restricting both sides of the last equation such that there remains only terms with w^{2j}, $j = 0, \ldots, n-m+1$ and making a limit $\mu \to n$ (very similar procedure was made in the proof of Proposition 17) one easily arrives at the expression

$$\begin{aligned}\mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \ldots, \frac{w}{n}\right) &= \sum_{s=0}^{n-m+1}\sum_{l=0}^{s}\frac{1}{(m)_{s-l}(-n)_l}\frac{(-1)^s}{l!(s-l)!}w^{2s} \\ &= \frac{(m-1)!}{n!}\sum_{s=0}^{n-m+1}\sum_{l=0}^{s}(-1)^{s-l}\frac{(n-l)!}{l!(s-l)!(m+s-l-1)!}w^{2s}.\end{aligned} \tag{1.24}$$

By using (1.18) and (1.19) again, we find the asymptotic expansion of the Bessel functions on the RHS in the statement

$$\begin{aligned}&-\pi w^{n-m+2}J_{m-1}(2w)Y_{n+1}(2w) \\ &\quad= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(m-1+k)!}w^{2k}\left(\sum_{l=0}^{n}\frac{(n-l)!}{l!}w^{2k} + O(w^{2n+2}\ln w)\right) \\ &\quad= \sum_{l=0}^{n}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(m-1+k)!}\frac{(n-l)!}{l!}w^{2(k+l)} + O(w^{2n+2}\ln w) \\ &\quad= \sum_{l=0}^{n}\sum_{s=l}^{\infty}\frac{(-1)^{s-l}}{(s-l)!(m-1+s-l)!}\frac{(n-l)!}{l!}w^{2s} + O(w^{2n+2}\ln w) \\ &\quad= \sum_{s=0}^{\infty}\sum_{l=0}^{s}\frac{(-1)^{s-l}(n-l)!}{l!(s-l)!(m-1+s-l)!}w^{2s} + O(w^{2n+2}\ln w) \\ &\quad= \sum_{s=0}^{n-m+1}\sum_{l=0}^{s}(-1)^{s-l}\frac{(n-l)!}{l!(s-l)!(m-1+s-l)!}w^{2s} + O(w^{2n-2m+4}).\end{aligned} \tag{1.25}$$

Finally, expression (1.24) together with the last expansion (1.25) prove the statement. □

Remark 19. One can see from the proof of Proposition 18 that if we were more precise in expression (1.25) we would obtain a more precise asymptotic expansion for the function $\mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \ldots, \frac{w}{n}\right)$. If we do this we get

$$\frac{n!}{(m-1)!}\mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \ldots, \frac{w}{n}\right) = -\pi w^{n-m+2} J_{m-1}(2w) Y_{n+1}(2w)$$
$$- \sum_{s=n-m+2}^{n} \sum_{l=0}^{s} (-1)^{s-l} \frac{(n-l)!}{l!(s-l)!(m-1+s-l)!} w^{2s} + O(w^{2n+2} \ln w) \tag{1.26}$$

where $1 \le m \le n$, $m, n \in \mathbb{N}$.

Lemma 20. It holds

$$\sum_{k=n}^{\infty} \binom{k}{n} x^{k-n} = \frac{1}{(1-x)^{n+1}} \tag{1.27}$$

for all $x \in \mathbb{R}, |x| < 1$ and all $n \in \mathbb{N}_0$.

Proof. To prove the statement we will proceed by a straightforward computation

$$\sum_{k=n}^{\infty} \binom{k}{n} x^{k-n} = \sum_{k=0}^{\infty} \binom{n+k}{n} x^k = \frac{1}{n!} \sum_{k=0}^{\infty} (n+k)(n+k-1)\ldots(k+1)x^k$$
$$= \frac{1}{n!} \frac{d^n}{dx^n} \left(\sum_{k=0}^{\infty} x^k \right) = \frac{1}{n!} \frac{d^n}{dx^n} \left(\frac{1}{1-x} \right) = \frac{1}{(1-x)^{n+1}}.$$

Since $|x| < 1$ the sum $\sum_{k=0}^{\infty} \binom{n+k}{n} x^k$ converges and the change of order of the derivation $\frac{d^n}{dx^n}$ and the sum $\sum_{k=0}^{\infty} x^k$ is correct. □

Lemma 21. It holds

$$\sum_{l=0}^{s} (-1)^l \binom{s+m}{l+m} \binom{n+l}{n} = \binom{m+s-n-1}{s} \tag{1.28}$$

for all $m, n, s \in \mathbb{N}_0$, $m > n$.

Proof. We will investigate the power serie

$$\sum_{s=0}^{\infty} \sum_{l=0}^{s} (-1)^l \binom{s+m}{l+m} \binom{n+l}{n} x^s$$

where $x \in \mathbb{R}$, $|x| < \frac{1}{2}$. Since

$$\sum_{s=0}^{\infty} \sum_{l=0}^{s} (-1)^l \binom{s+m}{l+m} \binom{n+l}{n} x^s = \sum_{l=0}^{\infty} (-1)^l \binom{n+l}{n} x^l \sum_{s=l}^{\infty} \binom{s+m}{l+m} x^{s-l},$$

we can apply Lemma 20 to the inner sum and we get

$$\sum_{s=0}^{\infty} \sum_{l=0}^{s} (-1)^l \binom{s+m}{l+m} \binom{n+l}{n} x^s = \sum_{l=0}^{\infty} (-1)^l \binom{n+l}{n} \frac{x^l}{(1-x)^{m+l+1}}.$$

Then we can use Lemma 20 again and we arrive at the expression

$$\sum_{s=0}^{\infty}\sum_{l=0}^{s}(-1)^l\binom{s+m}{l+m}\binom{n+l}{n}x^s=\frac{1}{(1-x)^{m+1}}\frac{1}{(1+\frac{x}{1-x})^{n+1}}=\frac{1}{(1-x)^{m-n}}.$$

Note that if $|x|<\frac{1}{2}$ then $|\frac{x}{1-x}|<1$. Thus

$$\sum_{l=0}^{s}(-1)^l\binom{s+m}{l+m}\binom{n+l}{n}=\frac{1}{s!}\frac{d^s}{dx^s}(1-x)^{n-m}\ |_{x=0}$$
$$=\frac{(-1)^s}{s!}(n-m)(n-m-1)\dots(n-m-s+1)=\binom{m+s-n-1}{s}.$$

□

Now we can obtain a more precise asymptotic expansion for the function $\mathfrak{F}\left(\frac{w}{m},\frac{w}{m+1},\dots,\frac{w}{n}\right)$.

Proposition 22. It holds

$$\frac{n!}{(m-1)!}\mathfrak{F}\left(\frac{w}{m},\frac{w}{m+1},\dots,\frac{w}{n}\right)=-\pi w^{n-m+2}J_{m-1}(2w)Y_{n+1}(2w)$$
$$-\sum_{s=0}^{m-2}\frac{s!(m+n-2s-2)!}{(n+m-s-1)!(m-s-2)!(n-s)!}w^{2n-2s}+O(w^{2n+2}\ln w) \tag{1.29}$$

for all $m,n\in\mathbb{N},1\le m\le n$.

Proof. By applying Lemma 21 to the inner sum in the second term on the RHS in equation (1.26) we get

$$\sum_{l=0}^{s}(-1)^{s-l}\frac{(n-l)!}{l!(s-l)!(m-1+s-l)!}=\sum_{l=0}^{s}(-1)^l\frac{(n-s+l)!}{l!(s-l)!(m-1+l)!}$$
$$=\frac{(n-s)!}{(s+m-1)!}\sum_{l=0}^{s}(-1)^l\binom{s+m-1}{l+m-1}\binom{n-s+l}{n-s}=\frac{(n-s)!}{(s+m-1)!}\binom{m+2s-n-2}{s}.$$

Thus we have

$$\frac{n!}{(m-1)!}\mathfrak{F}\left(\frac{w}{m},\frac{w}{m+1},\dots,\frac{w}{n}\right)=-\pi w^{n-m+2}J_{m-1}(2w)Y_{n+1}(2w)$$
$$-\sum_{s=n-m+2}^{n}\frac{(n-s)!}{(s+m-1)!}\binom{m+2s-n-2}{s}w^{2s}+O(w^{2n+2}\ln w)$$
$$=-\pi w^{n-m+2}J_{m-1}(2w)Y_{n+1}(2w)$$
$$-\sum_{s=0}^{m-2}\frac{s!(m+n-2s-2)!}{(n+m-s-1)!(m-s-2)!(n-s)!}w^{2n-2s}+O(w^{2n+2}\ln w).$$

□

Chapter 2

The Jacobi Matrix and Particular Values of χ_{red}

In this chapter we will introduce a Jacobi matrix which spectral properties we are interesting in.

2.1 A Jacobi Matrix of a Special Type

In general a Jacobi matrix is a tridiagonal complex matrix but we will restrict ourself to Jacobi matrices of a form

$$S := \begin{pmatrix} -\lambda_d & w & 0 & & & & & & \\ w & -\lambda_{d-1} & w & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & w & -\lambda_1 & w & & & & \\ & & & w & 0 & w & & & \\ & & & & w & \lambda_1 & w & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & w & \lambda_{d-1} & w \\ & & & & & & & w & \lambda_d \end{pmatrix} \tag{2.1}$$

where $\lambda_k > 0$ for all $k = 1, \dots d$, $w \in \mathbb{R} \setminus \{0\}$ and $d \in \mathbb{N}$. Next let us denote a characteristic function of the matrix S as

$$\chi_S(z) := \det(S - zI). \tag{2.2}$$

Since S is a Hermitian matrix the spectrum of S is real. Let us summarize other spectral properties of the matrix S (worked out in detail in [3] chap.4):

1. The spectrum of the matrix S is simple (since if we know an element of an eigenvector of S we can compute others recursively).

2. The characteristic function $\chi_S(z)$ is an odd polynomial of degree $2d+1$. It follows that 0 is always an eigenvalue of S. Vector $a = (\alpha_{-d}, \alpha_{-d+1}, \ldots, \alpha_d)$ where $\alpha_{-k} = (-1)^k \alpha_k$ and

$$\alpha_k = (-1)^k \frac{w^k}{\prod_{j=1}^k \lambda_j} \mathfrak{F}(wT_k\kappa) \quad k = 0, 1, \ldots, d \tag{2.3}$$

with

$$\kappa = (\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_d}, 0, 0, 0, \ldots)$$

is an eigenvector with eigenvalue 0 of the matrix S.

3. A relation between the function $\mathfrak{F}$ and the characteristic function χ_S

$$\begin{aligned}(-1)^{d+1}\frac{1}{z}\chi_S(z) &= \left(\prod_{k=1}^{d}(\lambda_k^2 - z^2)\right)\mathfrak{F}\left(\frac{w}{\lambda_1 - z}, \ldots, \frac{w}{\lambda_d - z}\right)\mathfrak{F}\left(\frac{w}{\lambda_1 + z}, \ldots, \frac{w}{\lambda_d + z}\right)\\ &+ 2\sum_{j=1}^{d} w^{2j}\left(\prod_{k=j+1}^{d}(\lambda_k^2 - z^2)\right)\mathfrak{F}\left(\frac{w}{\lambda_{j+1} - z}, \ldots, \frac{w}{\lambda_d - z}\right)\mathfrak{F}\left(\frac{w}{\lambda_{j+1} + z}, \ldots, \frac{w}{\lambda_d + z}\right)\end{aligned} \tag{2.4}$$

holds.

Let us consider a special case of the matrix S with $\lambda_k = k$ and let us denote it as K, i.e.,

$$K := \begin{pmatrix} -d & w & & & & & & & \\ w & -d+1 & w & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & w & -1 & w & & & & \\ & & & w & 0 & w & & & \\ & & & & w & 1 & w & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & w & d-1 & w \\ & & & & & & & w & d \end{pmatrix} \tag{2.5}$$

and

$$\chi_K(z) := \det(K - zI). \tag{2.6}$$

Let us set

$$\chi_{red}(z) := \frac{(-1)^{d+1}}{z}\chi_K(z). \tag{2.7}$$

The function $\chi_{red}(z)$ will be called the *characteristic reduced function.*

Remark 23. Note that the characteristic reduced function is an even polynomial in the variable z, i.e.,

$$\chi_{red}(-z) = \chi_{red}(z) \qquad \text{for all } z \in \mathbb{C}. \tag{2.8}$$

2.2 Particular Values of the Characteristic Reduced Function

One of main goals of this work is to find an explicit formula for characteristic function of K. The formula is derived in chapter 4 but it would not be found without knowledge of particular values $\chi_{red}(n)$ where $n \in \{0, 1, \ldots, d\}$.
By substituteing λ_j by j into (2.4) we get an expression

$$\chi_{red}(z) = \left(\prod_{k=1}^{d}(k^2 - z^2)\right) \mathfrak{F}\left(\frac{w}{1-z}, \ldots, \frac{w}{d-z}\right) \mathfrak{F}\left(\frac{w}{1+z}, \ldots, \frac{w}{d+z}\right)$$
$$+ 2\sum_{j=1}^{d} w^{2j} \left(\prod_{k=j+1}^{d}(k^2 - z^2)\right) \mathfrak{F}\left(\frac{w}{j+1-z}, \ldots, \frac{w}{d-z}\right) \mathfrak{F}\left(\frac{w}{j+1+z}, \ldots, \frac{w}{d+z}\right). \tag{2.9}$$

2.2.1 Case $n = 0$

First, the case with $n = 0$ is to be treated with the aid of expression (2.9) and asymptotic expansions of the function $\mathfrak{F}$ derived in section 1.4. Formula (2.9) with $z = 0$ gives

$$\chi_{red}(0) = (d!)^2 \mathfrak{F}\left(w, \frac{w}{2}, \ldots, \frac{w}{d}\right)^2 + 2\sum_{j=1}^{d} w^{2j} \left(\frac{d!}{j!}\right)^2 \mathfrak{F}\left(\frac{w}{j+1}, \frac{w}{j+2}, \ldots, \frac{w}{d}\right)^2. \tag{2.10}$$

Next, we use the asymptotic expression for the function $\mathfrak{F}$ (1.21) and (1.23) and we arrive at the expression

$$\chi_{red}(0) = \pi^2 w^{2d+2} Y_{d+1}^2(2w) \left(J_0^2(2w) + 2\sum_{j=1}^{d} J_j^2(2w)\right) + O(w^{2d+2}\ln w).$$

To proceed further we use the identity

$$J_0^2(z) + 2\sum_{j=1}^{\infty} J_j^2(z) = 1$$

which can be found in [1] (9.1.76). Then it holds

$$J_0^2(2w) + 2\sum_{j=1}^{d} J_j^{2w}(z) = 1 + O(w^{2d+2})$$

and we have

$$\chi_{red}(0) = \pi^2 w^{2d+2} Y_{d+1}^2(2w) + O(w^{2d+2}\ln w).$$

To continue we take into consideration the asymptotic expansion of the Bessel function Y (1.19). It leads us to an expression

$$\chi_{red}(0) = \left(\sum_{k=0}^{d} \frac{(d-k)!}{k!} w^{2k} \right)^2 + O(w^{2d+2} \ln w).$$

Since there is a polynomial in w of order $2d$ on the LHS the Landau symbol on the RHS will be omitted (only terms with $w^{2j}, 0 \leq j \leq d$ remain) and we can write

$$\chi_{red}(0) = \sum_{s=0}^{d} \sum_{k=0}^{s} \frac{(d-k)!(d-s+k)!}{k!(s-k)!} w^{2s}. \tag{2.11}$$

Lemma 24. It holds

$$\sum_{k=0}^{m} \binom{s+k}{s} \binom{p+m-k}{p} = \binom{s+p+m+1}{s+p+1}$$

for $m, p, s \in \mathbb{N}_0$.

Proof. Let us define a function

$$f_s(z) = \sum_{k=0}^{\infty} \binom{s+k}{s} z^k$$

for all $z \in \mathbb{C}$, $|z| < 1$. Since

$$\binom{s+k}{k} = \frac{1}{k!}(s+k)\dots(s+1) = \frac{(-1)^k}{k!}(-s-1)\dots(-s-k) = (-1)^k \binom{-s-1}{k}$$

we have

$$f_s(z) = \sum_{k=0}^{\infty} \binom{s+k}{k} z^k = \sum_{k=0}^{\infty} \binom{-s-1}{k} (-z)^k = (1-z)^{-s-1}.$$

It follows that the identity

$$f_s(z) f_p(z) = f_{s+p+1}(z)$$

holds for all $z, |z| < 1$. Then by using the previous identity we get

$$\sum_{m=0}^{\infty} \binom{s+p+1+m}{s+p+1} z^m \equiv f_{s+p+1}(z) = f_s(z) f_p(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{s+k}{s} \binom{p+l}{p} z^{k+l}$$
$$= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \binom{s+k}{s} \binom{p+m-k}{p} \right) z^m$$

which proves the statement. □

Now we can write the final expression for $\chi_{red}(0)$.

Corollary 25. It holds

$$\chi_{red}(0) = \sum_{s=0}^{d} \frac{[(d-s)!]^2(2d-s+1)!}{s!(2d-2s+1)!} w^{2s}. \tag{2.12}$$

Proof. To verify the statement it suffices to compute the inner sum in expression (2.11). Thus

$$\sum_{k=0}^{s} \frac{(d-k)!(d-s+k)!}{k!(s-k)!} = [(d-s)!]^2 \sum_{k=0}^{s} \frac{(d-k)!}{(d-s)!(s-k)!} \frac{(d-s+k)!}{(d-s)!k!}$$
$$= [(d-s)!]^2 \sum_{k=0}^{s} \binom{d-k}{d-s}\binom{d-s+k}{d-s}.$$

The last sum can be simplify with the aid of Lemma 24. Writing s instead of m, $(d-s)$ instead of s and $(d-s)$ instead of p, Lemma 24 follows that the identity

$$\sum_{k=0}^{s} \binom{d-k}{d-s}\binom{d-s+k}{d-s} = \binom{2d-s+1}{2d-2s+1} = \frac{(2d-s+1)!}{s!(2d-2s+1)!}$$

holds and the statement is proved. □

2.2.2 Case $n \in \{1, 2, \ldots, d\}$

Proposition 26. If $n \in \{1, 2, \ldots, d\}$ then an identity for particular value of characteristic reduced function

$$\chi_{red}(n) = \frac{1}{n} \sum_{l=0}^{n-1} (-1)^l \binom{n+l}{2l+1} \frac{(d+l+1)!}{(d-l)!} w^{2d-2l} \tag{2.13}$$

holds.

The verification of the identity for a general $n \in \{1, 2, \ldots, d\}$ is not as straightforward as in the special case when $n = 0$. The identity will be proved by using another method in chapter 4 (the method will be purely algebraic and we will not need to use the asymptotic properties of $\mathfrak{F}$).

Chapter 3

New Symbols Derived from the Function $\mathfrak{F}$

With the aid of the function $\mathfrak{F}$ we will define symbols $\mathfrak{G}$ and $\mathfrak{J}$ depending on two integers. To satisfy a tree term recurrent rule, symbol $\mathfrak{J}$ will be extended to the whole plain $\mathbb{Z} \times \mathbb{Z}$. Further, several algebraic properties of $\mathfrak{J}$ will be discussed and some formulas will be found. The usefulness of the symbol $\mathfrak{J}$ will follow in the next chapter while deriving a formula for characteristic reduced function.

3.1 Symbol $\mathfrak{G}$

Let us denote

$$\mathfrak{G}(m,n) := \mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \dots \frac{w}{n}\right)$$

for $m, n \in \mathbb{N}, m \le n+1$. Notice that $\mathfrak{G}(n+1,n) = \mathfrak{F}(0) = 1$. Then the recurrent relations (1.2) and (1.4) have the form

$$\mathfrak{G}(m,n) = \mathfrak{G}(m+1,n) - \frac{w^2}{m(m+1)}\mathfrak{G}(m+2,n) \tag{3.1}$$

where $m, n \in \mathbb{N}, m < n$ and

$$\mathfrak{G}(m,n) = \mathfrak{G}(m,n-1) - \frac{w^2}{n(n-1)}\mathfrak{G}(m,n-2) \tag{3.2}$$

where $m, n \in \mathbb{N}, m < n, n > 2$.
Next, with the aid of the first recurrent relation, we define function $\mathfrak{G}(m,n)$ also for $m, n \in \mathbb{N}, m > n+1$. To satisfy identity (3.1) one must set

$$\mathfrak{G}(n+2,n) := 0$$

and

$$\mathfrak{G}(m,n) := -\frac{(m-1)!(m-2)!}{n!(n+1)!}w^{-2(m-n)+4}\mathfrak{G}(n+2,m-2)$$

for $m > n+2$.

Remark 27. Consequently, it is easy to see that the relation

$$\mathfrak{G}(m,n) = -\frac{(m-1)!(m-2)!}{n!(n+1)!} w^{-2(m-n)+4} \mathfrak{G}(n+2, m-2) \tag{3.3}$$

holds for all $m, n \in \mathbb{N}$, $m > 2$.

Remark 28. In this and the following sections we will usually use algebraic identities like (3.1) and (3.2) while adjusting a term. In that case we will note, what relation we will use, above the equal sign. For example, an equal sign $\stackrel{4.27}{=}$ means that we use a relation (4.27) to adjust the LHS of an equation.

Let us verify the validity of the recurrent relation (3.1) for $m > n+2$:

$$\begin{aligned}
RHS &\equiv \mathfrak{G}(m+1,n) - \frac{w^2}{m(m+1)} \mathfrak{G}(m+2,n) \\
&\stackrel{3.3}{=} -\frac{m!(m-1)!}{n!(n+1)!} w^{-2(m-n)+2} \left[\mathfrak{G}(n+2,m-1) - \mathfrak{G}(n+2,m)\right] \\
&\stackrel{3.2}{=} -\frac{(m-1)!(m-2)!}{n!(n+1)!} w^{-2(m-n)+4} \mathfrak{G}(n+2,m-2) \stackrel{3.3}{=} \mathfrak{G}(m,n) \equiv LHS.
\end{aligned}$$

Thus the recurrent relation

$$\mathfrak{G}(m,n) = \mathfrak{G}(m+1,n) - \frac{w^2}{m(m+1)} \mathfrak{G}(m+2,n) \tag{3.4}$$

holds for all $m, n \in \mathbb{N}$. Similar situation occurs to the second relation.

Proposition 29. The recurrent relation

$$\mathfrak{G}(m,n) = \mathfrak{G}(m,n-1) - \frac{w^2}{n(n-1)} \mathfrak{G}(m,n-2) \tag{3.5}$$

holds for all $m, n \in \mathbb{N}, n > 2$.

Proof. The case $m < n$ is treated in (3.2). Let $m \geq n$ then

$$\begin{aligned}
RHS &\stackrel{3.3}{=} -\frac{(m-1)!(m-2)!}{(n-1)!n!} w^{-2(m-n)+2} \left[\mathfrak{G}(n+1,m-2) - \mathfrak{G}(n,m-2)\right] \\
&\stackrel{3.4}{=} -\frac{(m-1)!(m-2)!}{n!(n+1)!} w^{-2(m-n)+4} \mathfrak{G}(n+2,m-2) \stackrel{3.3}{=} LHS.
\end{aligned}$$

□

3.2 Symbol $\mathfrak{J}$

Definition 30. Let us denote

$$\mathfrak{J}(m,n) := \frac{(n-1)!}{m!} w^{m-n+2} \mathfrak{G}(m+1, n-1) \tag{3.6}$$

for all $m, n \in \mathbb{N}_0, n \geq 2$.

Remark 31. Then the recurrent relations (3.4) and (3.5) have the form

$$\mathfrak{J}(m-1,n) = \frac{m}{w}\mathfrak{J}(m,n) - \mathfrak{J}(m+1,n) \tag{3.7}$$

where $m \geq 1, n \geq 2$ and

$$\mathfrak{J}(m,n+1) = \frac{n}{w}\mathfrak{J}(m,n) - \mathfrak{J}(m,n-1) \tag{3.8}$$

where $m \geq 0, n \geq 3$.

Remark 32. Identity (3.3) can be now rewrite as

$$\mathfrak{J}(m,n) = -\mathfrak{J}(n,m) \tag{3.9}$$

where $m \geq 2$, $n \geq 2$.

Recurrent relation (3.7) allows us to define the symbol $\mathfrak{J}(m,n)$ even for $m \in \mathbb{Z}$. Let us define

$$\mathfrak{J}(-k,n) := (-1)^k \mathfrak{J}(k,n)$$

for $k \in \mathbb{N}$ and $n \geq 2$.

Remark 33. It is easy to see that the identity

$$\mathfrak{J}(-k,n) = (-1)^k \mathfrak{J}(k,n) \tag{3.10}$$

holds for all $k \in \mathbb{Z}$ and $n \geq 2$.

Proposition 34. The recurrent relation

$$\mathfrak{J}(m-1,n) = \frac{m}{w}\mathfrak{J}(m,n) - \mathfrak{J}(m+1,n) \tag{3.11}$$

holds for all $m \in \mathbb{Z}$ and $n \geq 2$.

Proof. Bearing in mind validity of relation (3.7) it suffices to verify the statement for $m \leq 0$. Case $m = 0$ is nothing but definition relation (3.10) with $k = 1$. Let $m \in \mathbb{N}$ then

$$\begin{aligned} RHS &\equiv -\frac{m}{w}\mathfrak{J}(-m,n) - \mathfrak{J}(-m+1,n) \overset{3.10}{=} (-1)^{m+1}(\frac{m}{w}\mathfrak{J}(m,n) - \mathfrak{J}(m-1,n)) \\ &\overset{3.7}{=} (-1)^{m+1}\mathfrak{J}(m+1,n) \overset{3.10}{=} \mathfrak{J}(-m-1,n) \equiv LHS. \end{aligned}$$

□

Proposition 35. The recurrent relation

$$\mathfrak{J}(m,n+1)=\frac{n}{w}\mathfrak{J}(m,n)-\mathfrak{J}(m,n-1) \tag{3.12}$$

holds for all $m\in\mathbb{Z}$ and $n>2$.

Proof. Considering validity of relation (3.8) it suffices to prove the statement for $m<0$. Let $m>0$ then

$$\begin{aligned} RHS &\equiv \frac{n}{w}\mathfrak{J}(-m,n)-\mathfrak{J}(-m,n-1) \overset{3.10}{=} (-1)^m\left(\frac{n}{w}\mathfrak{J}(m,n)-\mathfrak{J}(m,n-1)\right) \\ &\overset{3.8}{=} (-1)^m\mathfrak{J}(m,n+1) \overset{3.10}{=} \mathfrak{J}(-m,n+1)\equiv LHS. \end{aligned}$$

□

Finally, with the aid of the previous recurrent identity (3.12) we can extend the symbol $\mathfrak{J}(m,n)$ even for $n<2$. We can do that by putting

$$\mathfrak{J}(m,-k):=(-1)^k\mathfrak{J}(m,k) \tag{3.13}$$

for $m\in\mathbb{Z}$ and $k\geq 2$. Unfortunately relation (3.13) do not define the symbol $\mathfrak{J}(m,k)$ for $k=-1,0,1$. To keep the validity of the recurrent relation (which we prefer) we have to set

$$\mathfrak{J}(m,1)=-\mathfrak{J}(m,-1):=\frac{2}{w}\mathfrak{J}(m,2)-\mathfrak{J}(m,3), \tag{3.14}$$

$$\mathfrak{J}(m,0):=\frac{1}{w}\mathfrak{J}(m,1)-\mathfrak{J}(m,2). \tag{3.15}$$

Thus the symbol $\mathfrak{J}(m,n)$ is now define for all $(m,n)\in\mathbb{Z}\times\mathbb{Z}$.

Remark 36. The identity

$$\mathfrak{J}(m,-k):=(-1)^k\mathfrak{J}(m,k) \tag{3.16}$$

holds for all $m,k\in\mathbb{Z}$.

Proposition 37. The recurrent relations

$$\mathfrak{J}(m-1,n)=\frac{m}{w}\mathfrak{J}(m,n)-\mathfrak{J}(m+1,n) \tag{3.17}$$

and

$$\mathfrak{J}(m,n+1)=\frac{n}{w}\mathfrak{J}(m,n)-\mathfrak{J}(m,n-1) \tag{3.18}$$

holds for all $m,n\in\mathbb{Z}$.

Proof. 1) It suffices to show the validity of the first relation for $m \in \mathbb{Z}$ and $n < 2$. The other cases have already been discussed before (see (3.11)). Let $n \geq 2$ then

$$RHS \equiv \frac{m}{w}\mathfrak{J}(m,-n) - \mathfrak{J}(m+1,-n) \stackrel{3.16}{=} (-1)^n \Big(\frac{m}{w}\mathfrak{J}(m,n) - \mathfrak{J}(m+1,n)\Big)$$
$$\stackrel{3.11}{=} (-1)^n \mathfrak{J}(m-1,n) \stackrel{3.16}{=} \mathfrak{J}(m-1,-n) \equiv LHS.$$

2) Similarly, to prove the validity of the second recurrent relation we can restrict ourself to cases when $m \in \mathbb{Z}$ and $n < 0$ (due to (3.12) and definition relations (3.14) and (3.15)). If $n = -1$ then

$$RHS \equiv -\frac{1}{w}\mathfrak{J}(m,-1) - \mathfrak{J}(m,-2) \stackrel{3.16}{=} \frac{1}{w}\mathfrak{J}(m,1) - \mathfrak{J}(m,2) \stackrel{3.15}{=} \mathfrak{J}(m,0) \equiv LHS.$$

If $n = -2$ then

$$RHS \equiv -\frac{2}{w}\mathfrak{J}(m,-2) - \mathfrak{J}(m,-3) \stackrel{3.16}{=} -\frac{2}{w}\mathfrak{J}(m,2) + \mathfrak{J}(m,3) \stackrel{3.14}{=} \mathfrak{J}(m,-1) \equiv LHS.$$

Finally, if $n \geq 3$ then relation (3.12) can be used and we can write

$$RHS \equiv -\frac{n}{w}\mathfrak{J}(m,-n) - \mathfrak{J}(m,-n-1) \stackrel{3.16}{=} (-1)^{n+1} \Big(\frac{n}{w}\mathfrak{J}(m,n) + \mathfrak{J}(m,n+1)\Big)$$
$$\stackrel{3.12}{=} (-1)^{n+1}\mathfrak{J}(m,n-1) \stackrel{3.16}{=} \mathfrak{J}(m,-n+1) \equiv LHS.$$

□

Proposition 38. The identity

$$\mathfrak{J}(-m,n) = (-1)^m \mathfrak{J}(m,n) \tag{3.19}$$

holds for all $m, n \in \mathbb{Z}$.

Proof. Considering Remark 33 it suffices to prove the identity for $m \in \mathbb{Z}$ and $n < 2$. Let $m \in \mathbb{Z}$ and $n > -2$. We can write

$$\mathfrak{J}(-m,-1) \stackrel{3.14}{=} -\mathfrak{J}(-m,1) \stackrel{3.18}{=} -\frac{2}{w}\mathfrak{J}(-m,2) + \mathfrak{J}(-m,3) \stackrel{3.10}{=}$$
$$\stackrel{3.10}{=} (-1)^{m+1}\left(\frac{2}{w}\mathfrak{J}(m,2) - \mathfrak{J}(m,3)\right) \stackrel{3.18}{=} (-1)^{m+1}\mathfrak{J}(m,1) \stackrel{3.14}{=} (-1)^m \mathfrak{J}(m,-1)$$

which proves the cases when $n = \pm 1$. The case when $n = 0$ is to be treated similarly

$$\mathfrak{J}(-m,0) \stackrel{3.15}{=} \frac{1}{w}\mathfrak{J}(-m,1) + \mathfrak{J}(-m,2) = (-1)^m \left(\frac{1}{w}\mathfrak{J}(m,1) - \mathfrak{J}(m,2)\right)$$
$$\stackrel{3.15}{=} (-1)^m \mathfrak{J}(m,0).$$

Finally, let $n \geq 2$ then

$$\mathfrak{J}(-m,-n) \stackrel{3.16}{=} (-1)^n \mathfrak{J}(-m,n) \stackrel{3.10}{=} (-1)^{m+n}\mathfrak{J}(m,n) \stackrel{3.16}{=} (-1)^m \mathfrak{J}(m,-n).$$

□

Proposition 39. The identity

$$\mathfrak{J}(m,n) = -\mathfrak{J}(n,m) \tag{3.20}$$

holds for all $m, n \in \mathbb{Z}$.

Proof. The case $m, n \geq 2$ have already been discussed in Remark 32. Let $m, n \in \mathbb{Z}$ and $m, n > -2$. First we must check those cases when $-1 \leq -m, -n \leq 1$. Since identities (3.16) and (3.19) holds we can restrict ourself to cases when $(m, n) \in \{(0,0), (1,0), (0,1), (1,1)\}$. By using the definition of the symbol $\mathfrak{J}$ we get

$$\mathfrak{J}(0,1) = w, \quad \mathfrak{J}(1,0) = -w$$

and

$$\mathfrak{J}(0,0) = \mathfrak{J}(1,1) = 0.$$

Thus the proven identity holds for $-1 \leq m, n \leq 1$. Let be $m, n \geq 2$ then

$$\mathfrak{J}(-m,-n) \overset{3.16,3.19}{=} (-1)^{m+n}\mathfrak{J}(m,n) \overset{3.9}{=} (-1)^{m+n+1}\mathfrak{J}(n,m) \overset{3.16,3.19}{=} -\mathfrak{J}(-n,-m).$$

□

3.3 A Formula for the Symbol $\mathfrak{J}$

Proposition 40. It holds

$$\mathfrak{J}(n-k,n) = \sum_{s=1}^{\left[\frac{k+1}{2}\right]} (-1)^{s+1} \binom{n-s}{k-2s+1} \frac{(k-s)!}{(s-1)!} w^{2s-k} \tag{3.21}$$

for all $n \in \mathbb{Z}$, $k \in \mathbb{N}_0$ and

$$\mathfrak{J}(n+k,n) = \sum_{s=1}^{\left[\frac{k+1}{2}\right]} (-1)^{s} \binom{k+n-s}{k-2s+1} \frac{(k-s)!}{(s-1)!} w^{2s-k} \tag{3.22}$$

for all $n \in \mathbb{Z}$, $k \in \mathbb{N}_0$.

Proof. 1)At the beginning let us proof the first identity. The proof is split into 3 parts.
i)Let $n \geq 2$ and $0 \leq k \leq n$. By using the formula for the function $\mathfrak{F}\left(\frac{w}{m}, \frac{w}{m+1}, \ldots, \frac{w}{n}\right) \equiv \mathfrak{G}(m,n)$ from Corollary 15 (note that the formula in the Corollary 15 holds also for $m = n+1$ and $m = n+2$) and the definition of the function $\mathfrak{J}$ (5.6) we easily arrive

at an expression

$$\begin{aligned}\mathfrak{I}(n-k,n) &= \frac{(n-1)!}{(n-k)!}w^{-k+2}\mathfrak{G}(n-k+1,n-1)\\ &= \sum_{s=0}^{\left[\frac{k-1}{2}\right]}(-1)^s\frac{(n-1-s)!(k-s-1)!}{s!(n-k+s)!(k-2s-1)!}w^{2s-k+2}\\ &= \sum_{s=1}^{\left[\frac{k+1}{2}\right]}(-1)^{s+1}\frac{(n-s)!(k-s)!}{(s-1)!(n-k+s-1)!(k-2s+1)!}w^{2s-k}\\ &= \sum_{s=1}^{\left[\frac{k+1}{2}\right]}(-1)^{s+1}\binom{n-s}{k-2s+1}\frac{(k-s)!}{(s-1)!}w^{2s-k}.\end{aligned}$$

Thus we have verified the validity of formula (3.21) for $n \geq 2$ and $0 \leq k \leq n$.
ii)In the second step we will verify formula (3.21) for all $n \geq 2, k \geq 0$. We will proceed this by mathematical induction in k. Bearing in mind step (i), only the induction step $n \leq k \to k+1$ is to be treated ($n \in \mathbb{N}$ is fixed)

$$\begin{aligned}&\mathfrak{I}(n-(k+1),n) \overset{3.16}{=} \frac{n-k}{w}\mathfrak{I}(n-k,n) - \mathfrak{I}(n-k+1,n)\\ &\overset{IH}{=} (n-k)\sum_{s=1}^{\left[\frac{k+1}{2}\right]}(-1)^{s+1}\frac{(n-s)!(k-s)!}{(s-1)!(n-k+s-1)!(k-2s+1)!}w^{2s-k-1}\\ &-\sum_{s=1}^{\left[\frac{k}{2}\right]}(-1)^{s+1}\frac{(n-s)!(k-1-s)!}{(s-1)!(n-k+s)!(k-2s)!}w^{2s-k+1}\\ &= (n-k)\sum_{s=1}^{\left[\frac{k+1}{2}\right]}(-1)^{s+1}\frac{(n-s)!(k-s)!}{(s-1)!(n-k+s-1)!(k-2s+1)!}w^{2s-k-1}\\ &+\sum_{s=2}^{\left[\frac{k}{2}\right]+1}(-1)^{s+1}\frac{(n-s+1)!(k-s)!}{(s-2)!(n-k+s-1)!(k-2s+2)!}w^{2s-k-1}.\end{aligned}$$

To proceed further, one must realize that the upper bound of the first sum can be changed to $\left[\frac{k}{2}\right]+1$ because if k is odd then $\left[\frac{k+1}{2}\right] = \left[\frac{k}{2}\right]+1$ and if k is even then $\left[\frac{k+1}{2}\right] = \frac{k}{2}$ but the added term ($s = \frac{k}{2}+1$) is 0 due to the term $\frac{1}{(-1)!}$ which must be set to 0. For a similar reason the lower bound of the second sum can be changed to

1. Then we have an expression

$$\mathfrak{J}(n-(k+1),n) = (n-k)\sum_{s=1}^{\left[\frac{k}{2}\right]+1}(-1)^{s+1}\frac{(n-s)!(k-s)!}{(s-1)!(n-k+s-1)!(k-2s+1)!}w^{2s-k-1}$$
$$+\sum_{s=1}^{\left[\frac{k}{2}\right]+1}(-1)^{s+1}\frac{(n-s+1)!(k-s)!}{(s-2)!(n-k+s-1)!(k-2s+2)!}w^{2s-k-1}$$
$$=\sum_{s=1}^{\left[\frac{k+2}{2}\right]}(-1)^{s}\frac{(n-s)!(k-s+1)!}{(s-1)!(n-k+s-2)!(k-2s+2)!}w^{2s-k-1}$$

which was to be shown.
iii)Finally we will verify the validity of formula (3.21) for all $k \geq 0, n \in \mathbb{Z}$. Again we will proceed by mathematical induction in n.Considering step (ii), only the induction step $2 \geq n \to n-1$ is to be treated

$$\mathfrak{J}(n-1-k,n-1) \overset{3.17}{=} \frac{n}{w}\mathfrak{J}(n-(k+1),n) - \mathfrak{J}(n+1-(k+2),n+1)$$
$$\overset{IH}{=} \frac{n}{w}\sum_{s=1}^{\left[\frac{k+2}{2}\right]}(-1)^{s+1}\frac{(n-s)!(k-s+1)!}{(s-1)!(n-k+s-2)!(k-2s+2)!}w^{2s-k-1}$$
$$-\sum_{s=1}^{\left[\frac{k+3}{2}\right]}(-1)^{s+1}\frac{(n+1-s)!(k-s+2)!}{(s-1)!(n-k+s-2)!(k-2s+3)!}w^{2s-k-2}. \tag{3.23}$$

We can change the upper bound of the first sum to $\left[\frac{k+3}{2}\right]$ because of similar reasons as discussed in step (ii). Next note that the first term of the first sum ($s=1$) together with the first term of the second sum ($s=1$) gives 0. Thus we have

$$\mathfrak{J}(n-1-k,n-1) = n\sum_{s=2}^{\left[\frac{k+3}{2}\right]}(-1)^{s+1}\frac{(n-s)!(k-s+1)!}{(s-1)!(n-k+s-2)!(k-2s+2)!}w^{2s-k-2}$$
$$-\sum_{s=2}^{\left[\frac{k+3}{2}\right]}(-1)^{s+1}\frac{(n+1-s)!(k-s+2)!}{(s-1)!(n-k+s-2)!(k-2s+3)!}w^{2s-k-2}$$
$$=-n\sum_{s=1}^{\left[\frac{k+1}{2}\right]}(-1)^{s+1}\frac{(n-s-1)!(k-s)!}{s!(n-k+s-1)!(k-2s)!}w^{2s-k}$$
$$+\sum_{s=1}^{\left[\frac{k+1}{2}\right]}(-1)^{s+1}\frac{(n-s)!(k-s+1)!}{s!(n-k+s-1)!(k-2s+1)!}w^{2s-k}$$
$$=\sum_{s=1}^{\left[\frac{k+1}{2}\right]}(-1)^{s+1}\frac{(n-s-1)!(k-s)!}{(s-1)!(n-k+s-2)!(k-2s+1)!}w^{2s-k}.$$

Step (iii) concludes the proof of identity (3.21).
2)Since a binomial coefficient satisfies a relation

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$$

we can easily obtain the second identity from the first one

$$\mathfrak{J}(n+k,n) \overset{3.16,3.19}{=} (-1)^k \mathfrak{J}(-n-k,-n) = \sum_{s=1}^{\left[\frac{k+1}{2}\right]} (-1)^{k+s+1} \binom{-n-s}{k-2s+1} \frac{(k-s)!}{(s-1)!} w^{2s-k}$$
$$= \sum_{s=1}^{\left[\frac{k+1}{2}\right]} (-1)^s \binom{k+n-s}{k-2s+1} \frac{(k-s)!}{(s-1)!} w^{2s-k}.$$

□

Example 41. By using the previous results of the symbol $\mathfrak{J}$ we can introduce some special examples:

$$\mathfrak{J}(n,n) = \mathfrak{J}(-n,n) = 0,$$

$$\begin{aligned} \mathfrak{J}(n+1,n) &= -w, \\ \mathfrak{J}(n+2,n) &= -n-1, \\ \mathfrak{J}(n+3,n) &= -\frac{(n+1)(n+2)}{w} + w, \\ \mathfrak{J}(n+4,n) &= -\frac{(n+1)(n+2)(n+3)}{w^2} + 2(n+2), \end{aligned}$$

$$\begin{aligned} \mathfrak{J}(n-1,n) &= w, \\ \mathfrak{J}(n-2,n) &= n-1, \\ \mathfrak{J}(n-3,n) &= \frac{(n-1)(n-2)}{w} - w, \\ \mathfrak{J}(n-4,n) &= \frac{(n-1)(n-2)(n-3)}{w^2} - 2(n-2), \end{aligned}$$

where $n \in \mathbb{Z}$.

Chapter 4

The Characteristic Function and the Resolvent of the Jacobi Matrix

In this chapter we will introduce a method, how to use the symbol $\mathfrak{J}$ and its main properties, to find the exact formula for $\chi_{red}(n)$ with $n = 1, 2 \ldots, d$. The formula have already been presented in Proposition 26, however, without a proof. Next, by using the expression for particular values of χ_{red} we will be able to reconstruct the whole reduced characteristic function $\chi_{red}(z)$, $z \in \mathbb{C}$. Furthermore, we will derive an equation for the resolvent operator of a Jacobi matrix under investigation. At the end, some observations dealing with a localization of the spectrum will be summarized.

Recall that $K \in \mathbb{C}^{2d+1 \times 2d+1}$ is a complex Jacobi matrix of the form

$$K = \begin{pmatrix} -d & w & & & & & & & \\ w & -d+1 & w & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & w & -1 & w & & & & \\ & & & w & 0 & w & & & \\ & & & & w & 1 & w & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & w & d-1 & w \\ & & & & & & & w & d \end{pmatrix}. \tag{4.1}$$

For $n, s \in \mathbb{Z}$, let us define a column vector $x_{s,n} \in \mathbb{C}^{2d+1}$ as

$$x_{s,n}^T := (\mathfrak{J}(d+s,n), \mathfrak{J}(d+s-1,n), \ldots, \mathfrak{J}(s,n), \ldots, \mathfrak{J}(-d+s,n)) \tag{4.2}$$

By putting $-j+s$ instead of m into the recurrent relation (3.17) we get an identity

$$w\mathfrak{J}(-j+s-1,n) + (j-s)\mathfrak{J}(-j+s,n) + w\mathfrak{J}(-j+s+1,n) = 0$$

which holds for all $j, n, s \in \mathbb{Z}$. Thus, we can write

$$wx_{s,n}^{j-1} + (j-s)x_{s,n}^{j} + wx_{s,n}^{j+1} = 0 \tag{4.3}$$

for all $n, s \in \mathbb{Z}$, $j = -d, \dots, d$, but we have to set $x_{s,n}^{-d-1} := \mathfrak{J}(d+s+1, n)$ and $x_{s,n}^{d+1} := \mathfrak{J}(-d+s-1, n)$. Then it is easy to see the equation

$$(K - s)x_{s,n} = -w\mathfrak{J}(d+s+1, n)e_{-d} - w\mathfrak{J}(-d+s-1, n)e_d \tag{4.4}$$

holds for $n, s \in \mathbb{Z}$ and where $(e_{-d}, \dots, e_d)$ is a standard canonical basis in $\mathbb{C}^{2d+1}$ (that is $e_j^k = \delta_{jk}$). Next, we set $n = d+s+1$. Since $\mathfrak{J}(d+s+1, d+s+1) = 0$ (see Example 41) we can eliminate the first term on the RHS in identity (4.4) and we arrive at an expression

$$Kv_s = sv_s - w\mathfrak{J}(-d+s-1, d+s+1)e_d \tag{4.5}$$

where we denote $v_s := x_{s,d+s+1}$ for all $s \in \mathbb{Z}$.

Remark 42. Since

$$\mathfrak{J}(-d-1, d+1) = 0$$

the vector v_0 is the eigenvector of K for eigenvalue 0 which is easy to see from relation (4.5) if we set $s = 0$.

Remark 43. One should mention that a similar procedure could be done by starting from the second recurrent relation (3.18). An identity

$$w\mathfrak{J}(m, -j+s-1) + (j-s)\mathfrak{J}(m, -j+s) + w\mathfrak{J}(m, -j+s+1) = 0$$

holds for all $j, m, s \in \mathbb{Z}$ and similarly as in (4.3) we could write

$$w\widetilde{x}_{s,m}^{j-1} + (j-s)\widetilde{x}_{s,m}^{j} + w\widetilde{x}_{s,m}^{j+1} = 0$$

where

$$\widetilde{x}_{s,m}^{T} := (\mathfrak{J}(m, d+s), \mathfrak{J}(m, d+s-1), \dots, \mathfrak{J}(m, -d+s)).$$

However, property (3.20) gives a relation

$$x_{s,n} = -\widetilde{x}_{s,n}$$

and we would arrive at equation (4.4) again.

Lemma 44. Let $n \in \mathbb{N}$ and p is a polynomial of degree $s \leq n-1$. Then

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} p(j) = 0.$$

Proof. Since every polynomial of a degree s has a form $p(z) = \sum_{j=0}^{s} \alpha_j z^j$ where $\alpha_j \in \mathbb{C}$, $\alpha_s \neq 0$ it is clear that it suffices to prove the identity

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} j^s = 0$$

for all $0 \leq s < n$. The statement will be verified by mathematical induction in n. Verification of the statement in the case with $n = 1$ is immediate. Let $n \in \mathbb{N}$ is fixed and the equation

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} j^r = 0$$

holds for all $0 \leq r < n$ as the induction hypothesis. For $s = 0$ the statement follows easily from the binomial theorem (with an arbitrary $n \in \mathbb{N}$). Let $0 < s \leq n$ then

$$\sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} j^s = (n+1) \sum_{j=1}^{n+1} (-1)^j \binom{n}{j-1} j^{s-1}$$
$$= -(n+1) \sum_{j=0}^{n} (-1)^j \binom{n}{j} (j+1)^{s-1} = -(n+1) \sum_{k=0}^{s-1} \binom{s-1}{k} \sum_{j=0}^{n} (-1)^j \binom{n}{j} j^k = 0$$

because, according to an induction hypothesis, the inner sums are all 0 and the induction step is concluded. □

Proposition 45. Let $V \in \mathbb{C}^{2d+1 \times 2d+1}$ where

$$V_{ks} := v_s^k \equiv \mathfrak{J}(-k+s, d+s+1), \qquad k, s \in (-d, -d+1, \ldots, d).$$

Then

$$\det V = \frac{\prod_{k=1}^{2d} k!}{w^{(d-1)(2d+1)}}. \tag{4.6}$$

Proof. Let us define a matrix $W \in \mathbb{C}^{2d+1 \times 2d+1}$ such that

$$W_{jk} := (-1)^{j+d} \binom{k+d}{j+d}.$$

Since $W_{jk} = 0$ for $k < j$ we can easily compute the determinant of W

$$\det W = \prod_{k=-d}^{d} W_{kk} = \prod_{k=-d}^{d} (-1)^{k+d}.$$

Investigating matrix VW we get

$$(VW)_{kl} = \sum_{s=-d}^{d} V_{ks} W_{sl} = \sum_{s=-d}^{d} (-1)^{d+s} \binom{d+l}{d+s} \mathfrak{J}(-k+s, d+s+1)$$
$$= \sum_{j=0}^{2d} (-1)^j \binom{d+l}{j} \mathfrak{J}(j+1-(d+k+1), j+1).$$

Since $\binom{d+l}{j} = 0$ for $j > d+l$ we can change the upper bound of the sum to $d+l$. Next, we apply formula (3.21) and get

$$
\begin{aligned}
&(VW)_{kl} = \\
&= \sum_{j=0}^{d+l} (-1)^j \binom{d+l}{j} \sum_{t=1}^{\left[\frac{d+k+1}{2}\right]} (-1)^{t+1} \binom{j+1-t}{d+k-2t+2} \frac{(d+k+1-t)!}{(t-1)!} w^{2t-d-k-1} \\
&= \sum_{t=1}^{\left[\frac{d+k+1}{2}\right]} (-1)^{t+1} \frac{(d+k+1-t)!}{(t-1)!} w^{2t-d-k-1} \sum_{j=0}^{d+l} (-1)^j \binom{d+l}{j} \binom{j+1-t}{d+k-2t+2}.
\end{aligned}
$$

Since $\binom{j+1-t}{d+k-2t+2}$ is a polynomial in j of degree $d+k-2t+2$ Lemma 44 gives

$$
\sum_{j=0}^{d+l} (-1)^j \binom{d+l}{j} \binom{j+1-t}{d+k-2t+2} = 0
$$

if $k - l < 2(t-1)$.
Let $k < l$ then, with the aid of Lemma 44, we arrive at an equation

$$
(VW)_{kl} = 0.
$$

For $k = l$ the only nonzero term is the term with $t = 1$ and thus

$$
(VW)_{kk} = (d+k)! w^{-d-k+1} \sum_{j=0}^{d+k} (-1)^j \binom{d+k}{j} \binom{j}{d+k} = (-1)^{d+k} (d+k)! w^{-d-k+1}.
$$

The last equation holds because $\binom{j}{d+k} \neq 0$ only for $j = d+k$.
Finally, we can derive the determinant of the matrix V

$$
\begin{aligned}
\det V &= \frac{\det(VW)}{\det W} = \frac{\prod_{k=-d}^{d} (VW)_{kk}}{\prod_{k=-d}^{d} (W)_{kk}} = \frac{\prod_{k=-d}^{d} (-1)^{d+k} (d+k)! w^{-d-k+1}}{\prod_{k=-d}^{d} (-1)^{d+k}} \\
&= \frac{\prod_{k=1}^{2d} k!}{w^{(d-1)(2d+1)}}.
\end{aligned}
$$

□

Corollary 46. A set of vectors $\vartheta := (v_{-d}, v_{-d+1}, \dots, v_d)$ is a basis in $\mathbb{C}^{2d+1}$.

Proof. ϑ is basis in $\mathbb{C}^{2d+1}$ ⇔ matrix V is regular ⇔ $\det V \equiv \frac{\prod_{k=1}^{2d} k!}{w^{(d-1)(2d+1)}} \neq 0$.

□

Next we would like to write matrix K in basis ϑ. To do that we must first express vector e_d in basis ϑ which is stated in the following lemma.

Lemma 47. It holds

$$e_d = \sum_{s=-d}^{d} \frac{(-1)^{d+s}}{(d+s)!(d-s)!} w^{2d-1} v_s \tag{4.7}$$

where $e_d \in \mathbb{C}^{2d+1}, e_d^k = \delta_{dk}$.

Proof. By adjusting the RHS of the statement we obtain the expression

$$\sum_{s=-d}^{d} \frac{(-1)^{d+s}}{(d+s)!(d-s)!} w^{2d-1} \mathfrak{J}(-k+s, d+s+1)$$
$$\overset{3.21}{=} \sum_{s=-d}^{d} \sum_{l=1}^{\left[\frac{d+k}{2}\right]+1} (-1)^{l+d+s+1} \binom{d+s+1-l}{d+k+2-2l} \frac{(d+k+1-l)!}{(d+s)!(d-s)!(l-1)!} w^{2l+d-k-2}$$
$$= \sum_{l=0}^{\left[\frac{d+k}{2}\right]} (-1)^l \frac{(d+k-l)!}{l!} w^{2l+d-k} \frac{1}{(2d)!} \sum_{s=0}^{2d} (-1)^s \binom{2d}{s} \binom{s-l}{d+k-2l}.$$

According to Lemma 44, the inner sum

$$\sum_{s=0}^{2d} (-1)^s \binom{2d}{s} \binom{s-l}{d+k-2l} = 0$$

if $k < d+2l$ and this inequality holds for all $k \in \{-d, \ldots, d-1\}$. If $k = d$ then the inner sum is not zero only if $l = 0$, thus

$$\sum_{s=-d}^{d} \frac{(-1)^{d+s}}{(d+s)!(d-s)!} w^{2d-1} \mathfrak{J}(-d+s, d+s+1) = \sum_{s=0}^{2d} (-1)^s \binom{2d}{s} \binom{s}{2d} = 1.$$

Since $\binom{s}{2d} = 0$ for all $s < 2d$ the last equality holds.
So for $k \in \{-d, \ldots, d\}$ we have verified validity of the equality

$$\sum_{s=-d}^{d} \frac{(-1)^{d+s}}{(d+s)!(d-s)!} w^{2d-1} \mathfrak{J}(-k+s, d+s+1) = \delta_{kd}$$

which proves the statement. □

Finally, we can express operator K in basis ϑ (denoted $K^\vartheta \in \mathbb{C}^{2d+1\times 2d+1}$). Starting with (4.5) and considering the statement of the previous lemma we obtain an expression

$$(K^\vartheta)_{ts} = s\delta_{ts} - w\mathfrak{J}(-d+s-1, d+s+1)e_d(t) \tag{4.8}$$

where

$$e_d(t) = \frac{(-1)^{d+t}}{(d+t)!(d-t)!} w^{2d-1}$$

and $t, s \in \{-d, \dots, d\}$. Next, let us denote diagonal matrix $K_0 \in \mathbb{C}^{2d+1 \times 2d+1}$ where

$$(K_0)_{ts} := s\delta_{ts}$$

and $a, e_d^\vartheta \in \mathbb{C}^{2d+1}$,

$$(e_d^\vartheta)^T := (e_d(-d), e_d(-d+1), \dots, e_d(d)), \qquad a^T := (\alpha_{-d}, \alpha_{-d+1}, \dots, \alpha_d)$$

with entries $\alpha_s := -w\mathfrak{J}(-d+s-1, d+s+1) \overset{3.19}{=} (-1)^{d+s} w\mathfrak{J}(d-s+1, d+s+1)$. Then we can rewrite relation (4.8) to a simple expression

$$K^\vartheta = K_0 + e_d^\vartheta a^T. \tag{4.9}$$

Remark 48.
1) Note that $e_d^\vartheta a^T \in \mathbb{C}^{2d+1 \times 2d+1}$ (it is not a scalar product).
2) The inverse operator $(K_0 - z)^{-1}$ exists for all $z \in \mathbb{C} \setminus \{-d. \dots, d\}$ and

$$((K_0 - z)^{-1})_{ts} = \frac{1}{s-z} \delta_{ts}.$$

3) It holds $\alpha_{-s} = -\alpha_s$ for all $s \in \{-d \dots, d\}$. Especially, it follows that $\alpha_0 = 0$. The equation can be verified by a straightforward computation

$$\begin{aligned}\alpha_{-s} &= -w\mathfrak{J}(-d-s-1, d-s+1) \overset{3.19}{=} (-1)^{d+s} w\mathfrak{J}(d+s+1, d-s+1) \\ &\overset{3.20}{=} (-1)^{d+s+1} w\mathfrak{J}(d-s+1, d+s+1) = -\alpha_s.\end{aligned}$$

4.1 The Resolvent $(K^\vartheta - z)^{-1}$

The simple form of the matrix K expressed in the basis ϑ allows us to find a formula for the resolvent operator $(K^\vartheta - z)^{-1}$.

Proposition 49. Let $z \in \mathbb{C} \setminus \{-d, \dots, d\}$ such that an inequality

$$1 + a^T (K_0 - z)^{-1} e_d^\vartheta \neq 0$$

holds. Then

$$(K^\vartheta - z)^{-1} = (K_0 - z)^{-1} - \frac{1}{1 + a^T (K_0 - z)^{-1} e_d^\vartheta} (K_0 - z)^{-1} e_d^\vartheta a^T (K_0 - z)^{-1}. \tag{4.10}$$

Proof. From formula (4.9) it follows that

$$K^\vartheta - z = (K_0 - z)(1 + (K_0 - z)^{-1} e_d^\vartheta a^T), \tag{4.11}$$

note that 1 stands for an identity operator. Then

$$\left[(K_0-z)^{-1}-\frac{1}{1+a^T(K_0-z)^{-1}e_d^\vartheta}(K_0-z)^{-1}e_d^\vartheta a^T(K_0-z)^{-1}\right](K^\vartheta-z)$$
$$=\left[1-\frac{1}{1+a^T(K_0-z)^{-1}e_d^\vartheta}(K_0-z)^{-1}e_d^\vartheta a^T\right](1+(K_0-z)^{-1}e_d^\vartheta a^T)$$
$$=1+(K_0-z)^{-1}e_d^\vartheta a^T-\frac{1}{1+a^T(K_0-z)^{-1}e_d^\vartheta}(K_0-z)^{-1}e_d^\vartheta a^T$$
$$-\frac{a^T(K_0-z)^{-1}e_d^\vartheta}{1+a^T(K_0-z)^{-1}e_d^\vartheta}(K_0-z)^{-1}e_d^\vartheta a^T=1$$

which was to be verified. □

Remark 50. By multiplying the equality

$$1+a^T(K_0-z)^{-1}e_d^\vartheta=0$$

by a term $\prod_{k=-d}^{d}(k-z)$ we obtain a polynomial equation in z.Thus the inequality

$$1+a^T(K_0-z)^{-1}e_d^\vartheta\neq 0$$

holds for all $z\in\mathbb{C}$ with exception of a finite number of z (in the following chapter will be seen that z which solves the equation is a point of the spectrum of K).

4.2 A Formula for $\chi_{red}(z)$

Lemma 51. Let $a,b\in\mathbb{C}^n$ then

$$\det(1+ba^T)=1+a^Tb.$$

Proof. Since a determinant of a matrix is a linear function of its columns $\det(1+ba^T)$ is equal to

$$\left|\begin{pmatrix} a_1b_1 & a_1b_2 & \dots & a_1b_n \\ a_2b_1 & a_2b_2+1 & \dots & a_nb_n \\ \vdots & \vdots & & \vdots \\ a_nb_1 & a_nb_2 & \dots & a_nb_n+1 \end{pmatrix}\right| + \left|\begin{pmatrix} 1 & a_1b_2 & \dots & a_1b_n \\ 0 & a_2b_2+1 & \dots & a_nb_n \\ \vdots & \vdots & & \vdots \\ 0 & a_nb_2 & \dots & a_nb_n+1 \end{pmatrix}\right|.$$

The first determinant can be decomposed similarly, however, with exception of the term

$$\begin{pmatrix} a_1b_1 & 0 & \dots & 0 \\ a_2b_1 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ a_nb_1 & 0 & \dots & 1 \end{pmatrix},$$

all other matrices are singular, hence

$$\det(1+ba^T) = a_1b_1 + \left| \begin{pmatrix} a_2b_2+1 & a_2b_3 & \dots & a_2b_n \\ a_3b_2 & a_3b_3+1 & \dots & a_nb_n \\ \vdots & \vdots & & \vdots \\ a_nb_1 & a_nb_2 & \dots & a_nb_n+1 \end{pmatrix} \right|.$$

By repeating this procedure one verifies the statement. □

Now we can use relation (4.11) and the previous lemma to find a formula for $\chi_{red}(z)$, thus,

$$\begin{aligned} \chi_{red}(z) &= \frac{(-1)^{d+1}}{z}\det(K-z) = \frac{(-1)^{d+1}}{z}\det(K^{\vartheta}-z) \\ &= \frac{(-1)^{d+1}}{z}\det(K_0-z)\det(1+(K_0-z)^{-1}e_d^{\vartheta}a^T) \\ &= \frac{(-1)^{d+1}}{z}\prod_{k=-d}^{d}(k-z)(1+a^T(K_0-z)^{-1}e_d^{\vartheta}) = \prod_{k=1}^{d}(k^2-z^2)(1+a^T(K_0-z)^{-1}e_d^{\vartheta}) \\ &= \prod_{k=1}^{d}(k^2-z^2)\left(1+w^{2d-1}\sum_{s=-d}^{d}\frac{(-1)^{d+s}}{(d+s)!(d-s)!}\frac{\alpha_s}{s-z}\right). \end{aligned}$$

Since $\alpha_{-s} = -\alpha_s$ (see Remark 48) we can further adjust the sum in the previous expression

$$\begin{aligned} \sum_{s=-d}^{d}\frac{(-1)^{d+s}}{(d+s)!(d-s)!}\frac{\alpha_s}{s-z} &= \sum_{s=1}^{d}\frac{(-1)^{d+s}}{(d+s)!(d-s)!}\alpha_s\left(\frac{1}{s-z}-\frac{1}{-s-z}\right) \\ &= \sum_{s=1}^{d}\frac{2s}{s^2-z^2}\frac{w\mathfrak{I}(d-s+1,d+s+1)}{(d+s)!(d-s)!}. \end{aligned}$$

Finally, we arrive at the equation

$$\chi_{red}(z) = \prod_{k=1}^{d}(k^2-z^2)\left(1+w^{2d}\sum_{s=1}^{d}\frac{2s}{s^2-z^2}\frac{\mathfrak{I}(d-s+1,d+s+1)}{(d+s)!(d-s)!}\right). \tag{4.12}$$

Next, for $z = n \in \{1,2,\dots,d\}$ we can easily find a value for $\chi_{red}(n)$

$$\begin{aligned} \chi_{red}(n) &= \prod_{k=1,k\neq n}^{d}(k^2-n^2)\frac{2n}{(d-n)!(d+n)!}\mathfrak{I}(d+n+1-2n),d+n+1)w^{2d} \\ &\overset{3.21}{=} \frac{(-1)^{n+1}}{n}w^{2d-2n}\sum_{k=1}^{n}(-1)^{k+1}\binom{d+n+1-k}{2n-2k+1}\frac{(2n-k)!}{(k-1)!}w^{2k} \\ &= \frac{1}{n}\sum_{l=0}^{n-1}(-1)^l\binom{n+l}{2l+1}\frac{(d+l+1)!}{(d-l)!}w^{2d-2l} \end{aligned} \tag{4.13}$$

where the substitution $l = n - k$ have been used. This procedure gives a proof for Proposition 26 stated before.
At the end we will introduce a more convenient expression for the reduced characteristic function then formula (4.12).

Proposition 52. It holds

$$\chi_{red}(z) = \sum_{s=0}^{d} \frac{(2d-s+1)!}{s!(2d-2s+1)!} w^{2s} \prod_{k=1}^{d-s} (k^2 - z^2) \tag{4.14}$$

for all $z \in \mathbb{C}$.

Proof. Since $\chi_{red}(z)$ is an even polynomial in z of the degree $2d$ (see (2.8)) it is enough to check that the values of the RHS for $z = 0, 1, \ldots, d$ coincide with $\chi_{red}(0), \chi_{red}(1), \ldots, \chi_{red}(d)$. The expression

$$\chi_{red}(0) = \sum_{s=0}^{d} \frac{(2d-s+1)!}{s!(2d-2s+1)!} w^{2s} [(d-s)!]^2$$

is exactly the formula (2.12). Let $n \in \{1, \ldots, d\}$ then

$$\prod_{k=1}^{d-s} (k^2 - n^2) = \begin{cases} 0 & s \le d-n \\ \frac{(-1)^{d-s}}{n} \frac{(n+d-s)!}{(n-d+s-1)!} & d-n < s \le d \end{cases}$$

hence

$$\begin{aligned} RHS &= \frac{1}{n} \sum_{s=d-n+1}^{d} \frac{(2d-s+1)!}{s!(2d-2s+1)!} \frac{(-1)^{d-s}(n+d-s)!}{(n-d+s-1)!} w^{2s} \\ &= \frac{1}{n} \sum_{l=0}^{n-1} (-1)^l \frac{(d+l+1)!}{(d-l)!(2l+1)!} \frac{(n+l)!}{(n-l-1)!} w^{2d-2l} \end{aligned}$$

which coincides with the formula for $\chi_{red}(n)$ (4.13). □

Remark 53. From formula (4.14) it is obvious that $\chi_{red}(z)$ has no roots for $|z| < 1$ and $\chi_{red}(\pm 1) = 0$ if and only if $w = 0$.

4.3 Notes on Localization of the Spectrum of K

We will introduce some features of distribution of eigenvalues of the matrix K which follows from the formula for the characteristic reduced function (4.14).

Lemma 54. Let $0 < |w| \le 1$, $d, n \in \mathbb{N}$ and $n \le d + 1$. Then $\chi_{red}(n) > 0$ if n is an odd number and $\chi_{red}(n) < 0$ if n is an even number.

Proof. 1)Let n is an even integer, $n \leq d+1$. Formula (4.13) can be rewritten as

$$
\begin{aligned}
n\chi_{red}(n) &= \sum_{l=0}^{\frac{n}{2}-1} \left[\binom{n+2l}{4l+1} \frac{(d+2l+1)!}{(d-2l)!} w^{2d-4l} - \binom{n+2l+1}{4l+3} \frac{(d+2l+2)!}{(d-2l-1)!} w^{2d-4l-2} \right] \\
&= \sum_{l=0}^{\frac{n}{2}-1} \frac{(d+2l+1)!}{(d-2l)!} \frac{(n+2l)!}{(n-2l-1)!(4l+3)!} \alpha_w(l) w^{2d-4l-2} \qquad (4.15)
\end{aligned}
$$

where we have denoted

$$\alpha_w(l) := (4l+3)(4l+2)w^2 - (n^2 - (2l+1)^2)(d+2l+2)(d-2l).$$

Let $l \in \{1, 2, \ldots, \frac{n}{2} - 1\}$ (note that the range of l is nonempty only if $n \geq 4$). Then

$$
\begin{aligned}
&\alpha_w(l) - \alpha_w(l-1) = (32l+4)w^2 + 8l(d^2 + 2d + n^2 - 1 - 8l^2) \\
&> d^2 + 2d + n^2 - 1 - 8(\frac{n}{2} - 1)^2 = d^2 + 2d - 9 + 8n - n^2 \qquad (4.16)
\end{aligned}
$$

and since we consider n even and $1 \leq n \leq d+1$ we get the estimate

$$\alpha_w(l) - \alpha_w(l-1) > d^2 + 2d - 9 + 16 - (d+1)^2 = 6 > 0.$$

As a consequence it holds

$$\alpha_w(l) \leq \alpha_w(\frac{n}{2} - 1)$$

for all $l \in \{0, 1, \ldots, \frac{n}{2} - 1\}$. Next, still considering $1 \leq n \leq d+1$ and $0 < |w| \leq 1$, we can estimate

$$
\begin{aligned}
&\alpha_w(\frac{n}{2} - 1) = (2n-1)(2n-3)w^2 - (2n-1)(d+n)(d-n+2) \\
&\leq (2n-1)[(2n-3) - (d^2 + 2d + 2n - n^2)] \leq -2(2n-1) < 0.
\end{aligned}
$$

It follows that $\alpha_w(l) < 0$ for all $l \in \{0, 1, \ldots, \frac{n}{2} - 1\}$ which, together with expression (4.15) proves that $\chi_{red}(n) < 0$ for n an even integer less or equal to $d+1$.
2)Let n is an odd integer, $n \leq d+1$. This case will be treated similarly as the case with even n. Formula (4.13) can be rewritten as

$$
\begin{aligned}
n\chi_{red}(n) &= \sum_{l=1}^{\frac{n-1}{2}} \left[\binom{n+2l}{4l+1} \frac{(d+2l+1)!}{(d-2l)!} w^{2d-4l} - \binom{n+2l-1}{4l-1} \frac{(d+2l)!}{(d-2l+1)!} w^{2d-4l+2} \right] \\
&+ n(d+1)w^{2d} \\
&= \sum_{l=1}^{\frac{n-1}{2}} \frac{(d+2l)!}{(d-2l+1)!} \frac{(n+2l-1)!}{(n-2l)!(4l+1)!} \beta_w(l) w^{2d-4l} + n(d+1)w^{2d} \qquad (4.17)
\end{aligned}
$$

where

$$\beta_w(l) := (n^2 - (2l)^2)(d+2l+1)(d-2l-1) - 4l(4l+1)w^2.$$

Let $l \in \{2, 3, \ldots, \frac{n-1}{2}\}$ (note that the range of l is nonempty only if $n \geq 5$). Then

$$\begin{aligned} &\beta_w(l) - \beta_w(l-1) = (-32l + 12)w^2 - 4(2l-1)(d^2 + 2d + n^2 - 3 + 8l - 8l^2) \\ &< -(d^2 + 2d - 3 + n^2 - 2(n-1)(n-3)) = -(d^2 + 2d - 9 + 8n - n^2) \leq -14 < 0 \end{aligned}$$

and

$$\begin{aligned} &\beta_w(\frac{n-1}{2}) = (2n-1)(d+n)(d-n+2) - (2n-2)(2n-1)w^2 \\ &> (2n-1)[(d^2 + 2d + 2n - n^2) - (2n-2)] \geq d^2 + 2d + 2 - n^2 \geq 1 > 0. \end{aligned}$$

As a consequence we can claim that $\beta_w(l) > 0$ for all $l \in \{1, 2, \ldots, \frac{n-1}{2}\}$ and therefore every term in (4.17) is positive which concludes the proof. □

Proposition 55. Let $0 < |w| \leq 1$, $d \in \mathbb{N}$. Then

$$\sigma(K) = \{0, \pm\lambda_1, \pm\lambda_2, \ldots, \pm\lambda_d\} \tag{4.18}$$

where $\lambda_k \in (k, k+1)$ for all $k \in \{1, 2, \ldots, d\}$. Next, an inequality

$$\lambda_{k+1} - \lambda_k > 1 \tag{4.19}$$

holds for all $k \in \{1, 2, \ldots, d\}$.

Proof. Since the characteristic function is an odd function zero is always an eigenvalue of K and the spectrum of K is symmetric around zero (see section 2.1). Let $k \in \{1, 2, \ldots, d\}$. Lemma 54 implies that

$$\operatorname{sgn}(\chi_{red}(k)) = -\operatorname{sgn}(\chi_{red}(k+1))$$

and therefore there is at least one eigenvalue in interval $(k, k+1)$. Since the last statement holds for all $k \in \{1, 2, \ldots, d\}$ there is exactly one eigenvalue in each interval $(k, k+1)$ (matrix K has exactly d positive eigenvalues).
The second part of the statement is a little bit more difficult to verify. Let $\lambda_k \in \sigma(K) \cap (k, k+1)$, $k \in \{1, 2, \ldots, d\}$. We will verify that $\chi_{red}(z)$ is always positive or always negative for every $z \in (\lambda_k, \lambda_k + 1)$. If it is true then the reduced characteristic polynomial has no root in $(\lambda_k, \lambda_k + 1)$ and inequality (4.19) will hold. The already checked properties of the spectrum of K follows that it suffices to show an equality

$$\operatorname{sgn}(\chi_{red}(k+1)) = \operatorname{sgn}(\chi_{red}(\lambda_k + 1)).$$

Since an identity

$$\prod_{l=1}^{d-s}(l^2 - (\lambda_k + 1)^2) = \frac{-\lambda_k(d - s + 1 + \lambda_k)}{(1 + \lambda_k)(d - s - \lambda_k)} \prod_{l=1}^{d-s}(l^2 - \lambda_k^2)$$

holds for $s \in \{0, 1, \dots, d\}$ we can write

$$\chi_{red}(\lambda_k + 1) = \sum_{s=0}^{d} \frac{(2d-s+1)!}{s!(2d-2s+1)!} w^{2s} \prod_{k=1}^{d-s} (k^2 - (\lambda_k + 1)^2)$$
$$= -\frac{\lambda_k}{1+\lambda_k} \sum_{s=0}^{d} \frac{(2d-s+1)!}{s!(2d-2s+1)!} w^{2s} \frac{d-s+1+\lambda_k}{d-s-\lambda_k} \prod_{l=1}^{d-s} (l^2 - \lambda_k^2). \tag{4.20}$$

Note that

$$\mathrm{sgn}(\prod_{l=1}^{d-s}(l^2 - \lambda_k^2)) = (-1)^k$$

for $s \in \{0, 1, \dots, d-k-1\}$. To proceed further we must treat two cases separately. 1) Let k is an odd integer. Since we investigate a distance between λ_k and λ_{k+1} we take $d \geq k+1$ (otherwise there is no λ_{k+1} in the spectrum). First we estimate an expression

$$\frac{d-s+1+\lambda_k}{d-s-\lambda_k} = 1 + \frac{2\lambda_k + 1}{d-s-\lambda_k} \leq 1 + \frac{2\lambda_k+1}{k+1-\lambda_k}$$

which holds for $s \in \{0, 1, \dots, d-k-1\}$. Next by splitting sum $\sum_{s=0}^{d}$ in expression (4.20) into to two sums $\sum_{s=0}^{d-k-1} + \sum_{s=d-k}^{d}$ and by considering the last inequality we obtain an estimation

$$\chi_{red}(\lambda_k+1) \geq -\frac{\lambda_k}{1+\lambda_k} \sum_{s=d-k}^{d} \frac{(2d-s+1)!}{s!(2d-2s+1)!} w^{2s} \frac{d-s+1+\lambda_k}{d-s-\lambda_k} \prod_{l=1}^{d-s}(l^2 - \lambda_k^2)$$
$$-\frac{\lambda_k}{1+\lambda_k}\left(1 + \frac{2\lambda_k+1}{k+1-\lambda_k}\right) \sum_{s=0}^{d-k-1} \frac{(2d-s+1)!}{s!(2d-2s+1)!} w^{2s} \prod_{l=1}^{d-s}(l^2 - \lambda_k^2)$$

which can be further adjusted

$$\chi_{red}(\lambda_k+1) \geq -\frac{\lambda_k}{1+\lambda_k} \sum_{s=d-k}^{d} \frac{(2d-s+1)!}{s!(2d-2s+1)!} w^{2s} \frac{d-s+1+\lambda_k}{d-s-\lambda_k} \prod_{l=1}^{d-s}(l^2 - \lambda_k^2)$$
$$-\frac{\lambda_k}{1+\lambda_k}\left(1 + \frac{2\lambda_k+1}{d-\lambda_k}\right) \left[-\sum_{s=d-k}^{d} \frac{(2d-s+1)!}{s!(2d-2s+1)!} w^{2s} \prod_{l=1}^{d-s}(l^2 - \lambda_k^2) + \chi_{red}(\lambda_k)\right].$$

λ_k is a nonzero eigenvalue of K hence $\chi_{red}(\lambda_k) = 0$. We can continue

$$\chi_{red}(\lambda_k+1)$$
$$\leq \frac{\lambda_k}{1+\lambda_k} \sum_{s=d-k}^{d} \frac{(2d-s+1)!}{s!(2d-2s+1)!} w^{2s} \left[\frac{2\lambda_k+1}{k+1-\lambda_k} - \frac{2\lambda_k+1}{d-s-\lambda_k}\right] \prod_{l=1}^{d-s}(l^2 - \lambda_k^2)$$
$$\frac{\lambda_k(2\lambda_k+1)}{(1+\lambda_k)(k+1-\lambda_k)} \sum_{s=d-k}^{d} \frac{(2d-s+1)!}{s!(2d-2s+1)!} w^{2s} \frac{d-s-k-1}{d-s-\lambda_k} \prod_{l=1}^{d-s}(l^2 - \lambda_k^2) \tag{4.21}$$

Since we are interested only in the sign of the previous expression we omit the positive constant which stands in front of the sum. Next, by doing substitution $j = d - s$ we arrive at an expression

$$\sum_{j=0}^{k} \binom{d+j+1}{2j+1} w^{2d-2j} \frac{j-k-1}{j-\lambda_k} \prod_{l=1}^{j} (l^2 - \lambda_k^2) \tag{4.22}$$

which can be rewritten as

$$\begin{aligned}
&\sum_{j=0}^{\frac{k-1}{2}} \Bigg[\binom{d+2j+1}{4j+1} w^{2d-4j} \frac{2j-k-1}{2j-\lambda_k} \prod_{l=1}^{2j} (l^2 - \lambda_k^2) \\
&\quad + \binom{d+2j+2}{4j+3} w^{2d-4j-2} \frac{2j-k}{2j+1-\lambda_k} \prod_{l=1}^{2j+1} (l^2 - \lambda_k^2) \Bigg] \\
&\quad = \sum_{j=0}^{\frac{k-1}{2}} \frac{(d+2j+1)!}{(4j+3)!(d-2j)!} w^{2d-4j-2} \Omega_w(j) \prod_{l=1}^{2j} (l^2 - \lambda_k^2) \qquad (4.23)
\end{aligned}$$

where we have denoted

$$\Omega_w(j) := (4j+3)(4j+2) \frac{2j-k-1}{2j-\lambda_k} w^2 + (d+2j+2)(d-2j) \frac{2j-k}{2j+1-\lambda_k} ((2j+1)^2 - \lambda_k^2).$$

Finally it suffices to show that $\Omega_w(j) < 0$ for all $j \in \{0, 1, \dots, \frac{k-1}{2}\}$ because in that case every term in (4.23) will be negative and also $\chi_{red}(\lambda_k + 1)$ will be negative. Let us check the negativity of $\Omega_w(j)$, by doing several estimations we can write

$$\begin{aligned}
\Omega_w(j) &= (4j+3)(4j+2) \frac{k+1-2j}{\lambda_k - 2j} w^2 - (d+2j+2)(d-2j)(k-2j)(2j+1+\lambda_k) \\
&< (4j+3)(4j+2)(k+1-2j) - (d+2j+2)(d-2j)(k-2j)(2j+1+k) \\
&\leq (4j+3)(4j+2) + (k-2j)[(4j+3)(4j+2) - (d+2j+2)(d-2j)(k+1)].
\end{aligned} \tag{4.24}$$

Since

$$\begin{aligned}
&(4j+3)(4j+2) - (d+2j+2)(d-2j)(k+1) < (k+1)(4k - (d+2)d) \\
&\quad \leq (k+1)(-d^2 + 2d - 4) < -(k+1)(d-1)^2 \leq 0,
\end{aligned}$$

the term in square brackets in (4.24) is negative and we can continue in estimations

$$\begin{aligned}
\Omega_w(j) &\leq (k+1)(8k - d(d+2)) \leq (k+1)(-d^2 + 6d - 8) = -(k+1)(d+2)(d+4) \\
&< 0
\end{aligned}$$

where we used that $0 \leq j \leq \frac{k-1}{2}$, $k+1 \leq d$ and $\min\{(d+2j+2)(d-2j) | j = 0, 1, \dots\} = (d+2)d$.

2) Let k is an even integer. This case is to be treated very similarly as case 1). By following the same procedure as in case 1) we obtain an estimation (it differs only in opposite inequalities)

$$\begin{aligned}&\chi_{red}(\lambda_k+1) \geq \\ &\geq \frac{\lambda_k(2\lambda_k+1)}{(1+\lambda_k)(k+1-\lambda_k)} \sum_{s=d-k}^{d} \frac{(2d-s+1)!}{s!(2d-2s+1)!} w^{2s} \frac{d-s-k-1}{d-s-\lambda_k} \prod_{l=1}^{d-s} (l^2-\lambda_k^2).\end{aligned}$$

Since we are interested only in the sign of the previous expression we omit the positive constant which stands in front of the sum. Next, by doing substitution $j = d - s$ we arrive at an expression

$$\sum_{j=1}^{k} \binom{d+j+1}{2j+1} w^{2d-2j} \frac{j-k-1}{j-\lambda_k} \prod_{l=1}^{j} (l^2-\lambda_k^2) + \frac{d(d+1)}{\lambda_k} w^{2d}. \tag{4.25}$$

The second term is obviously positive and the sum can be rewritten as

$$\begin{aligned}&\sum_{j=1}^{\frac{k}{2}} \Bigg[\binom{d+2j}{4j-1} w^{2d-4j+2} \frac{2j-k-2}{2j-1-\lambda_k} \prod_{l=1}^{2j-1} (l^2-\lambda_k^2) \\ &\quad + \binom{d+2j+1}{4j+1} w^{2d-4j} \frac{2j-k-1}{2j-\lambda_k} \prod_{l=1}^{2j} (l^2-\lambda_k^2) \Bigg] \\ &= \sum_{j=1}^{\frac{k}{2}} \frac{(d+2j)!}{(4j+1)!(d-2j+1)!} w^{2d-4j} \Psi_w(j) \prod_{l=1}^{2j-1} (l^2-\lambda_k^2) \qquad (4.26)\end{aligned}$$

where we have denoted

$$\Psi_w(j) := 4j(4j+1)\frac{2j-k-2}{2j-1-\lambda_k} w^2 + (d+2j+1)(d-2j+1)\frac{2j-k-1}{2j-\lambda_k}((2j)^2-\lambda_k^2).$$

Now it suffices to show that $\Psi_w(j) < 0$ for all $j \in \{1, 2, \ldots, \frac{k}{2}\}$ because in that case every term in (4.26) will be positive and therefore $\chi_{red}(\lambda_k + 1)$ will be positive. Let us check the negativity of $\Psi_w(j)$, by doing several estimations we get

$$\begin{aligned}\Psi_w(j) &= 4j(4j+1)\frac{k+2-2j}{\lambda_k-2j+1} w^2 - (d+2j+1)(d-2j+1)(k+1-2j)(2j+\lambda_k) \\ &< 4j(4j+1)(k+2-2j) - (d+2j+1)(d-2j+1)(k+1-2j)(2j+k) \\ &\leq 4j(4j+1) + (k+1-2j)[4j(4j+1) - (d+2j+1)(d-2j+1)(k+2)]. \qquad (4.27)\end{aligned}$$

Since

$$\begin{aligned}&4j(4j+1) - (d+2j+1)(d-2j+1)(k+2) < (k+2)(4k-(d+1)^2) \\ &\leq (k+2)(-d^2+2d-5) \leq -(k+2)(d-1)^2 \leq 0,\end{aligned}$$

the term in square brackets in (4.27) is negative and we can continue in estimations

$$\begin{aligned}\Psi_w(j) &\leq 4k(2k+1) - (d+1)^2(k+2) \leq (k+2)(8k-(d+1)^2) \\ &\leq (k+2)(-d+6d-9) \leq -(k+2)(d+2)(d+4) < 0\end{aligned} \tag{4.28}$$

where we used that $1 \leq j \leq \frac{k}{2}$, $k+1 \leq d$. □

Remark 56. Property (4.19) implies that $\lambda_k - k$ is increasing with k.

Chapter 5

Function $\mathfrak{J}$ and Eigenvectors of Matrix K

5.1 Functions $\mathfrak{G}$ and $\mathfrak{J}$

In chapter 3 symbols $\mathfrak{G}$ and $\mathfrak{J}$ was defined. These symbols depends on two integers. Now, we are going to add a third continuous dependent variable. By this way functions $\mathfrak{G}$ and $\mathfrak{J}$ will be defined. Lettering will remain the same but the functions will have three dependent variables while the symbols will have only two, for example $\mathfrak{G}(m, n, z)$ is the new defined function while $\mathfrak{G}(m, n)$ is the old defined symbol.

Definition 57. Let us denote

$$\mathfrak{G}(m, n, z) := \mathfrak{F}\left(\frac{w}{m+z}, \frac{w}{m+1+z}, \dots, \frac{w}{n+z}\right) \tag{5.1}$$

where $m, n \in \mathbb{Z}$, $m \leq n$ and $z \in \mathbb{C} \setminus \mathbb{Z}$. To satisfy the recurrent rule (1.2) we put

$$\mathfrak{G}(n+1, n, z) := 1$$

and

$$\mathfrak{G}(n+2, n, z) := 0$$

for $n \in \mathbb{Z}$, $z \in \mathbb{C}$. Finally, let us define

$$\mathfrak{G}(m, n, z) := -\frac{\Gamma(m+z)\Gamma(m-1+z)}{\Gamma(n+2+z)\Gamma(n+1+z)} w^{-2(m-n)+4} \mathfrak{G}(n+2, m-2, z) \tag{5.2}$$

for $m, n \in \mathbb{Z}$, $m > n+2$ and $z \in \mathbb{C} \setminus \mathbb{Z}$.

Consequently, we have defined function $\mathfrak{G}(m, n, z)$ for all $m, n \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus \mathbb{Z}$.

Remark 58. It is easy to verify that the relation (5.2) holds for all $m, n \in \mathbb{Z}$.

Remark 59. Note that
$$\mathfrak{G}(m,n,0)=\mathfrak{G}(m,n)$$
for $m,n\in\mathbb{N}$.

Remark 60. Recurrent relations (1.2) and (1.4) implies that a similar property holds for function $\mathfrak{G}$,

$$\mathfrak{G}(m,n,z)=\mathfrak{G}(m+1,n,z)-\frac{w^2}{(m+z)(m+1+z)}\mathfrak{G}(m+2,n,z), \tag{5.3}$$
$$\mathfrak{G}(m,n,z)=\mathfrak{G}(m,n-1,z)-\frac{w^2}{(n+z)(n-1+z)}\mathfrak{G}(m,n-2,z) \tag{5.4}$$

where $m,n\in\mathbb{Z}$, $m\le n$ and $z\in\mathbb{C}\setminus\mathbb{Z}$.

Proposition 61. Recurrent relations (5.3) and (5.4) holds for all $m,n\in\mathbb{Z}$ and $z\in\mathbb{C}\setminus\mathbb{Z}$.

Proof. Considering the last remark it suffices to prove the relations for $m>n$. Let $m=n+1$ then
$$LHS=\mathfrak{G}(n+1,n,z)=1$$
and

$$\begin{aligned}RHS&=\mathfrak{G}(n+2,n,z)-\frac{w^2}{(n+1+z)(n+2+z)}\mathfrak{G}(n+3,n,z)\\&=\frac{w^2}{(n+1+z)(n+2+z)}\frac{\Gamma(n+3+z)\Gamma(n+2+z)}{\Gamma(n+1+z)\Gamma(n+2+z)}w^{-2}\mathfrak{G}(n+2,n+1,z)=1.\end{aligned}$$

Let $m\ge n+2$ then

$$\begin{aligned}RHS&=\mathfrak{G}(m+1,n,z)-\frac{w^2}{(m+z)(m+1+z)}\mathfrak{G}(m+2,n,z)\\&=-\frac{\Gamma(m+1+z)\Gamma(m+z)}{\Gamma(n+1+z)\Gamma(n+2+z)}w^{-2(m-n)+2}(\mathfrak{G}(n+2,m-1,z)-\mathfrak{G}(n+2,m,z)).\end{aligned}$$

Since $m\ge n+2$ recurrent relation (5.4) can be applied to the last expression

$$\mathfrak{G}(n+2,m-1,z)-\mathfrak{G}(n+2,m,z)=\frac{w^2}{(m+z)(m-1+z)}\mathfrak{G}(n+2,m-2,z)$$

and then

$$\begin{aligned}RHS&=-\frac{\Gamma(m+z)\Gamma(m-1+z)}{\Gamma(n+1+z)\Gamma(n+2+z)}w^{-2(m-n)+4}\mathfrak{G}(n+2,m-2,z)\\&=\mathfrak{G}(m,n,z)=LHS.\end{aligned}$$

Thus, recurrent relation (5.3) holds for all $m, n \in \mathbb{Z}$.
The second recurrent relation is to be treated. Let $m > n$ then

$$\begin{aligned} RHS &= \mathfrak{G}(m, n-1, z) - \frac{w^2}{(n+z)(n-1+z)} \mathfrak{G}(m, n-2, z) \\ &= -\frac{\Gamma(m+z)\Gamma(m-1+z)}{\Gamma(n+z)\Gamma(n+1+z)} w^{-2(m-n)+2} (\mathfrak{G}(n+1, m-2, z) - \mathfrak{G}(n, m-2, z)). \end{aligned}$$

According to the first recurrent relation (which was already proved) the identity

$$\mathfrak{G}(n+1, m-2, z) - \mathfrak{G}(n, m-2, z) = \frac{w^2}{(n+z)(n+1+z)} \mathfrak{G}(n+2, m-2, z)$$

holds and finally

$$\begin{aligned} RHS &= -\frac{\Gamma(m+z)\Gamma(m-1+z)}{\Gamma(n+1+z)\Gamma(n+2+z)} w^{-2(m-n)+4} \mathfrak{G}(n+2, m-2, z) = \mathfrak{G}(m, n, z) \\ &= LHS. \end{aligned}$$

□

Lemma 62. It holds

$$(m+z)\mathfrak{G}(m-j, n, z)\Big|_{z=-m} = -\frac{w^{2j+2}}{(j+1)!j!} \mathfrak{G}(j+2, n-m) \tag{5.5}$$

for $j \in \mathbb{N}_0$ and $m, n \in \mathbb{Z}$, $m < n$.

Proof. We will proved the statement by mathematical induction in j. Case $j = 0$ is to be treated with the aid of recurrent relation (5.3)

$$\begin{aligned} &(m+z)\mathfrak{G}(m, n, z)\Big|_{z=-m} = (m+z)\mathfrak{G}(m+1, n, z) - \frac{w^2}{m+1+z} \mathfrak{G}(m+2, n, z)\Big|_{z=-m} \\ &= -w^2 \mathfrak{G}(2, n-m). \end{aligned}$$

The case $j = 1$ will be proved similarly

$$\begin{aligned} &(m+z)\mathfrak{G}(m-1, n, z)\Big|_{z=-m} = (m+z)\mathfrak{G}(m, n, z) - \frac{w^2}{m-1+z} \mathfrak{G}(m+1, n, z)\Big|_{z=-m} \\ &= -w^2 \mathfrak{G}(2, n-m) + w^2 \mathfrak{G}(1, n-m) = -\frac{w^4}{2} \mathfrak{G}(3, n-m). \end{aligned}$$

Let $0 \leq j$. To proceed the induction step $j \to j+1$ the well known recurrent relation (5.3) is to be used again

$$\begin{aligned}
&(m+z)\mathfrak{G}(m-(j+1),n,z)\Big|_{z=-m} \\
&= (m+z)\left(\mathfrak{G}(m-j,n,z) - \frac{w^2}{(m-j-1+z)(m-j+z)}\mathfrak{G}(m-j+1,n,z)\right)\Bigg|_{z=-m} \\
&\overset{IH}{=} -\frac{w^{2j+2}}{(j+1)!j!}\mathfrak{G}(j+2,n-m) + \frac{w^{2j+2}}{(j+1)!j!}\mathfrak{G}(j+1,n-m) \\
&= -\frac{w^{2j+4}}{(j+2)!(j+1)!}\mathfrak{G}(j+3,n-m).
\end{aligned}$$

□

Definition 63. Let us define

$$\mathfrak{J}(m,n,z) := \frac{\Gamma(n+z)}{\Gamma(m+1+z)}w^{m-n+2}\mathfrak{G}(m+1,n-1,z) \tag{5.6}$$

where $m,n \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus \mathbb{Z}$.

Remark 64. For $m = n-1$ and $m = n$ function $\mathfrak{J}$ can be defined for all $z \in \mathbb{C}$ as

$$\mathfrak{J}(n-1,n,z) := w$$

and

$$\mathfrak{J}(n,n,z) := 0.$$

Remark 65. The relation (5.2) rephrased with the aid of the function $\mathfrak{J}$ has the form

$$\mathfrak{J}(m,n,z) = -\mathfrak{J}(n,m,z) \tag{5.7}$$

where $m,n \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus \mathbb{Z}$.

Proposition 66. Let $z_0 \in \mathbb{Z}$ then

$$\lim_{z \to z_0} \mathfrak{J}(m,n,z) = \mathfrak{J}(m+z_0, n+z_0) \tag{5.8}$$

for all $m,n \in \mathbb{Z}$.

Proof. (i) Let $n \geq m+2$. Since for $z \notin \mathbb{Z}$

$$\begin{aligned}
\mathfrak{J}(m,n,z) &= \frac{\Gamma(n+z)}{\Gamma(m+1+z)}w^{m-n+2}\mathfrak{G}(m+1,n-1,z) \\
&= \prod_{k=m+1}^{n-1}(k+z)w^{m-n+2}\mathfrak{F}\left(\frac{w}{m+1+z}, \frac{w}{m+2+z}, \dots, \frac{w}{n-1+z}\right)
\end{aligned} \tag{5.9}$$

there is no problem to do a limit $z \to z_0$ and check the validity of the proposition if $z_0 \notin \{-n+1, -n+2, \ldots, -m-1\}$. Let $z_0 \in \{-n+1, -n+2, \ldots, -m-1\}$ then with the aid of lemma 62 we can do a limit $z \to z_0$ in expression (5.9) (where the function $\mathfrak{F}$ is replaced by relevant function $\mathfrak{G}$) and we obtain a result

$$\begin{aligned}
&\lim_{z\to z_0} \mathfrak{J}(m,n,z) = \\
&= -\prod_{k=m+1, k\neq -z_0}^{n-1} (k+z_0) \frac{w^{-m-n-2z_0+2}}{(-m-1-z_0)!(-m-z_0)!} \mathfrak{G}(-m-z_0+1, n-1+z_0) \\
&= (-1)^{m+z_0} \frac{(n-1+z_0)!}{(-m-z_0)!} w^{(-m-z_0)-(n+z_0)+2} \mathfrak{G}(-m-z_0+1, n-1+z_0) \\
&= (-1)^{m+z_0} \mathfrak{J}(-m-z_0, n+z_0) = \mathfrak{J}(m+z_0, n+z_0).
\end{aligned}$$

The last equation holds due to identity (3.19).
(ii) Let $m = n-1$ then $\mathfrak{J}(n-1, n, z) = w$ for all $z \in \mathbb{C}$ and $\mathfrak{J}(n-1+z_0, n+z_0) = w$. Similarly, if $m = n$ then $\mathfrak{J}(n, n, z) = 0$ for all $z \in \mathbb{C}$ and $\mathfrak{J}(n+z_0, n+z_0) = 0$.
(iii) Finally, if $m > n+2$ then it suffices to use the identity (5.7) and the already proved cases

$$\lim_{z\to z_0} \mathfrak{J}(m,n,z) = -\lim_{z\to z_0} \mathfrak{J}(n,m,z) = -\mathfrak{J}(n+z_0, m+z_0) = \mathfrak{J}(m+z_0, n+z_0).$$

□

Remark 67. The function $\mathfrak{J}(m,n,z)$ can be defined also for $z \in \mathbb{Z}$ as

$$\mathfrak{J}(m,n,z) := \mathfrak{J}(m+z, n+z).$$

Then the function $\mathfrak{J}(m,n,z)$ is defined for all $z \in \mathbb{C}$ and it is continuous in z.

Proposition 68. Recurrent relations

$$\mathfrak{J}(m-1,n,z) = \frac{m+z}{w}\mathfrak{J}(m,n,z) - \mathfrak{J}(m+1,n,z), \tag{5.10}$$

$$\mathfrak{J}(m,n+1,z) = \frac{n+z}{w}\mathfrak{J}(m,n,z) - \mathfrak{J}(m,n-1,z) \tag{5.11}$$

holds for all $m, n \in \mathbb{Z}$ and all $z \in \mathbb{C}$.

Proof. For $z \in \mathbb{C} \setminus \mathbb{Z}$ it suffices to take into account recurrent relations (5.3) and (5.4) and use the definition of function $\mathfrak{J}$. Next, for $z \in \mathbb{Z}$ it suffices to use the continuity of function $\mathfrak{J}$. □

5.2 Characteristic Function and Eigenvectors of the Jacobi Matrix

In section 4.2 we have derived a formula for the characteristic function of K. Now, we will find another useful identity for the characteristic function with the aid of function $\mathfrak{J}$. Next, we will describe formulas for eigenvectors of K.

Proposition 69. Function

$$w^{2d}\mathfrak{J}(-d-1, d+1, z) \tag{5.12}$$

is the characteristic function of matrix K. Next if $\lambda \in \sigma(K)$ then vector

$$v_\lambda^T := (\mathfrak{J}(d, d+1, \lambda), \mathfrak{J}(d-1, d+1, \lambda), \ldots, \mathfrak{J}(-d, d+1, \lambda)) \tag{5.13}$$

is an eigenvector of K respective to eigenvalue λ.

Proof. By using recurrent relation (5.10) we easily arrive at the identity

$$(K-z)v_z = -w\mathfrak{J}(d+1, d+1, z)e_{-d} - w\mathfrak{J}(-d-1, d+1, z)e_d$$

where $e_{\pm d}$ are vectors of standard canonical basis in $\mathbb{C}^{2d+1}$. Since $\mathfrak{J}(d+1, d+1, z) \equiv 0$ the first term vanishes and we obtain a formula

$$(K-z)v_z = -w\mathfrak{J}(-d-1, d+1, z)e_d \tag{5.14}$$

which both statements follow from. First it is important to notice that $v_z \neq 0$ for all $z \in \mathbb{C}$ because

$$v_z^1 = \mathfrak{J}(d, d+1, z) = w \neq 0.$$

Next, it is necessary to find out that function

$$w^{2d}\mathfrak{J}(-d-1, d+1, z) = \prod_{k=-d}^{d}(k+z)\mathfrak{F}\left(\frac{w}{-d+z}, \frac{w}{-d+1+z}, \ldots, \frac{w}{d+z}\right)$$

is a polynomial in z of degree $2d+1$ and the coefficient respective to the term z^{2d+1} is 1. That can be seen either from Definition 1 of function $\mathfrak{F}$ or from a fact that function $\mathfrak{J}(-d-1, d+1, z)$ can be reconstruct by using recurrent relation (5.10) starting from $\mathfrak{J}(d, d+1, z) = w$ and $\mathfrak{J}(d+1, d+1, z) = 0$. Next, all roots of function $\mathfrak{J}(-d-1, d+1, z)$ are eigenvalues of the matrix K which arises from identity (5.14). Finally, it suffices to show that $\mathfrak{J}(-d-1, d+1, z)$ has no multiple roots. In fact, suppose $\mathfrak{J}(-d-1, d+1, \lambda) = \mathfrak{J}'(-d-1, d+1, \lambda) = 0$ for some $\lambda \in \mathbb{R}$ (here $\mathfrak{J}'(-d-1, d+1, \lambda)$ is the derivative of $\mathfrak{J}(-d-1, d+1, z)$ in λ). From (5.14) one deduces that $(K-\lambda)v_\lambda = 0$, $(K-\lambda)v_\lambda' = v_\lambda$ (v_z' stands for the derivative of v_z). Hence

$$v_\lambda \in \operatorname{Ker}(K-\lambda) \cap \operatorname{Ran}(K-\lambda).$$

Since $K-\lambda$ is a hermitian matrix spaces $\operatorname{Ker}(K-\lambda)$ and $\operatorname{Ran}(K-\lambda)$ are orthogonal and so $v_\lambda = 0$. This contradicts the fact, however, that $v_\lambda^1 = w \neq 0$. One concludes that the set of roots of $\mathfrak{J}(-d-1, d+1, z)$ coincides with $\sigma(K)$. Necessarily, $w^{2d}\mathfrak{J}(-d-1, d+1, z)$ is equal to the characteristic function of $K(w)$. The second statement of the proposition also follows directly from identity (5.14). $\square$

Remark 70. Notice that polynomial $w^{2d}\mathfrak{J}(-d-1,d+1,z)$ differs from $\chi_K(z)$ in a sign. Thus identity

$$w^{2d}\mathfrak{J}(-d-1,d+1,z) = -\chi_K(z) = (-1)^d z\sum_{s=0}^{d}\frac{(2d-s+1)!}{s!(2d-2s+1)!}w^{2s}\prod_{k=1}^{d-s}(k^2-z^2) \tag{5.15}$$

holds for all $z\in\mathbb{C}$ which follows directly from the previous proposition and formula for characteristic reduced function (4.14).

Remark 71. The formula for particular values of characteristic reduced function (4.13) can be derived easily now. By using Lemma 62 we get

$$\begin{aligned}\chi_K(n) &= \prod_{k=-d,k\neq -n}^{d}(k+n)(z-n)\mathfrak{G}(-d,d,z)\Bigg|_{z=n}\\ &= (-1)^{d+n+1}\frac{(d+n)!}{(d+1-n)!}w^{2d-2n+2}\mathfrak{G}(d-n+2,d+n)\\ &= (-1)^{d+n+1}w^{2d}\mathfrak{J}(d-n+1,d+n+1).\end{aligned} \tag{5.16}$$

Next, by substituting for $n := d+n+1$ and $k := 2n$ in identity (3.21) we obtain an expression

$$\begin{aligned}\chi_K(n) &= (-1)^{d+n+1}w^{2d}\sum_{s=1}^{n}(-1)^{s+1}\binom{d+n+1-s}{2n-2s+1}\frac{(2n-s)!}{(s-1)!}w^{2s-2n}\\ &= (-1)^d\sum_{k=0}^{n-1}(-1)^k\binom{d+k+1}{2k+1}\frac{(n+k)!}{(n-k-1)!}w^{2d-2k}\\ &= (-1)^d\sum_{k=0}^{n-1}(-1)^k\binom{n+k}{2k+1}\frac{(d+k+1)!}{(d-k)!}w^{2d-2k}\end{aligned} \tag{5.17}$$

which holds for an arbitrary $n\in\mathbb{N}$.

In the rest of this chapter we will deal with finding formulas for particular components of vector

$$v_z^T \equiv (\mathfrak{J}(d,d+1,z),\mathfrak{J}(d-1,d+1,z),\ldots,\mathfrak{J}(-d,d+1,z)) \tag{5.18}$$

where $z\in\mathbb{C}$. Let us denote

$$\xi_k(z) := w^{k-1}\mathfrak{J}(d-k,d+1,z) \tag{5.19}$$

where $k\in\{0,1,\ldots,2d\}$ and $z\in\mathbb{C}$. Next, we will introduce some properties of function $\xi_k(z)$.

Remark 72. Note that $\xi_k(z)$ is a polynomial in z of degree k and the coefficient respective to the term z^k is 1. It can be seen from the fact that $\mathfrak{J}(d-k,d+1,z)$ can be reconstruct by using recurrent relation (5.10) starting from $\mathfrak{J}(d,d+1,z)=w$ and $\mathfrak{J}(d+1,d+1,z)=0$.

Example 73. Let us introduce examples of $\xi_k(z)$ for $k=0,1,2,3$:

$$\begin{aligned}
&\xi_0(z)=1,\\
&\xi_1(z)=d+z,\\
&\xi_2(z)=(d+z)(d-1+z)-w^2,\\
&\xi_3(z)=(d+z)(d-1+z)(d-2+z)-2w^2(d-1+z).
\end{aligned}$$

Remark 74. Let $n\in\mathbb{Z}$. By using identity (3.21) we can express $\xi_k(n)$ as

$$\begin{aligned}
\xi_k(n)&=w^{k-1}\mathfrak{J}(d-k,d+1,n)=w^{k-1}\mathfrak{J}(d+n-k,d+n+1)\\
&=\sum_{s=0}^{\left[\frac{k}{2}\right]}(-1)^s\binom{d+n-s}{k-2s}\frac{(k-s)!}{s!}w^{2s}. \qquad (5.20)
\end{aligned}$$

Lemma 75. The identity

$$\xi_k\left(-d+\left[\frac{k-1}{2}\right]+l\right)=(-1)^k\xi_k\left(-d+\left[\frac{k}{2}\right]-l\right) \qquad (5.21)$$

holds for $k\in\{0,1,\dots,2d\}$ and $l\in\mathbb{Z}$.

Proof. To prove the statement we can proceed straightforward

$$\begin{aligned}
LHS&\equiv\xi_k\left(-d+\left[\frac{k-1}{2}\right]+l\right)=w^{k-1}\mathfrak{J}\left(\left[\frac{k-1}{2}\right]-k+l,\left[\frac{k+1}{2}\right]+l\right)\\
&=w^{k-1}\mathfrak{J}\left(-\left[\frac{k}{2}\right]-1+l,k-\left[\frac{k}{2}\right]+l\right)\\
&=(-1)^kw^{k-1}\mathfrak{J}\left(-k+\left[\frac{k}{2}\right]-l,\left[\frac{k}{2}\right]+1-l\right)=(-1)^k\xi_k\left(-d+\left[\frac{k}{2}\right]-l\right)\equiv RHS
\end{aligned}$$

where identities (3.16), (3.19) and (3.20) have been used. □

Remark 76. By putting $l=0$ and considering k an odd integer in the previous lemma we find out that $-d+\frac{k-1}{2}$ is a root of polynomial ξ_k.

Next we will reconstruct function $\xi_k(.)$ with the aid of knowledge of particular values $\xi_k(\alpha_j)$ where $\alpha_j\in\mathbb{R}$, $\alpha_i\neq\alpha_j$ for $i\neq j$ and $i,j=0,1,\dots,k$ (recall that $\xi_k(.)$ is a polynomial of degree k). Polynomial $\xi_k(.)$ is uniquely determined through identity

$$\xi_k(z)=\sum_{j=0}^{k}\xi_k(\alpha_j)\frac{\prod_{i=0,i\neq j}^{k}(z-\alpha_i)}{\prod_{i=0,i\neq j}^{k}(\alpha_j-\alpha_i)}. \qquad (5.22)$$

Until we will do that, we will need a technical identity which is stated in the following lemma.

Lemma 77. Let $r \in \mathbb{N}$ then it holds

$$\sum_{j=r+1-s}^{r+1} (-1)^{j+1} \frac{2j(r+j-s)!}{(s+j-r-1)!(r+j+1)!(r-j+1)!} \prod_{i=r+1-s, i\neq j}^{r+1} (z^2 - i^2) = (-1)^r \tag{5.23}$$

for all $s \in \{0, 1, \ldots, r\}$ and $z \in \mathbb{C}$.

Proof. To prove the statement we will proceed indirectly. First since

$$\xi_{2d+1}(z) = w^{2d} \mathfrak{J}(-d-1, d+1, z)$$

from Proposition 69 it follows that $\xi_{2d+1}(.)$ is the characteristic function of matrix K. Although we have already know the exact formula for $\xi_{2d+1}(z)$ we will try to reconstruct polynomial $\xi_{2d+1}(z)$ by using expression (5.22) where we will put $\alpha_j := j$ and $j \in \{-d-1, -d, \ldots, -1, 1, \ldots, d, d+1\}$. Thus

$$\xi_{2d+1}(z) = \sum_{j=-d-1, j\neq 0}^{d+1} \xi_{2d+1}(j) \frac{\prod_{i=-1, i\neq d+j, i\neq d}^{2d+1} (z+d-i)}{\prod_{i=-1, i\neq d+j, i\neq d}^{2d+1} (d+j-i)},$$

since $\xi_{2d+1}(-j) = -\xi_{2d+1}(j)$ the sum can be modified further as

$$\begin{aligned} \xi_{2d+1}(z) &= \sum_{j=1}^{d+1} \xi_{2d+1}(j) \frac{j(-1)^{d+j+1}}{(d+1+j)!(d+1-j)!} \left[\frac{1}{z+j} + \frac{1}{z-j} \right] \prod_{i=1}^{d+1} (z^2 - i^2) \\ &= z(-1)^d \sum_{j=1}^{d+1} \xi_{2d+1}(j) \frac{2j(-1)^{j+1}}{(d+1+j)!(d+1-j)!} \prod_{i=1, i\neq j}^{d+1} (z^2 - i^2). \end{aligned}$$

Next, by applying identity (5.20) to $\xi_{2d+1}(j)$ and by switching sums we arrive at the expression

$$\xi_{2d+1}(z) = z(-1)^d \sum_{s=0}^{d} (-1)^s \frac{(2d-s+1)!}{(2d-2s+1)!s!} \omega_s(z) w^{2s} \prod_{i=1}^{d-s} (z^2 - i^2) \tag{5.24}$$

where we have denoted

$$\omega_s(z) := \sum_{j=d+1-s}^{d+1} \frac{(d+j-s)!}{(s+j-d-1)!} \frac{2j(-1)^{j+1}}{(d+1+j)!(d+1-j)!} \prod_{i=d+1-s, i\neq j}^{d+1} (z^2 - i^2).$$

Finally, it suffices to compare equation (5.24) with formula (5.15) to find out that

$$\omega_s(z) = (-1)^d$$

and this holds for an arbitrary $d \in \mathbb{N}$. □

Proposition 78. It holds

$$\xi_k(z) = \sum_{s=0}^{\left[\frac{k}{2}\right]} (-1)^s \frac{(k-s)!}{(k-2s)!s!} w^{2s} \prod_{j=s}^{k-1-s} (d+z-j) \tag{5.25}$$

where $k \in \{0, 1, \dots, 2d\}$, $d \in \mathbb{N}$ and $z \in \mathbb{C}$.

Proof. The proof is split into two parts.
1) Let k be an even integer. We will try to find a formula for $\xi_{k+1}(z)$ by using expression (5.22) with

$$\alpha_j := -d + \frac{k}{2} + j$$

and

$$j \in \left\{ -\frac{k}{2} - 1, -\frac{k}{2}, \dots, -1, 1, \dots, \frac{k}{2}, \frac{k}{2} + 1 \right\}.$$

Thus

$$\begin{aligned}\xi_{k+1}(z) &= \sum_{j=-\frac{k}{2}-1, j\neq 0}^{\frac{k}{2}+1} \xi_{k+1}\left(-d + \frac{k}{2} + j\right) \frac{\prod_{i=-1, i\neq \frac{k}{2}+j, i\neq \frac{k}{2}}^{k+1} (z+d-i)}{\prod_{i=-1, i\neq \frac{k}{2}+j, i\neq \frac{k}{2}}^{k+1} (\frac{k}{2}+j-i)} \\ &= \sum_{j=-\frac{k}{2}-1, j\neq 0}^{\frac{k}{2}+1} \xi_{k+1}\left(-d + \frac{k}{2} + j\right) \frac{j(-1)^{\frac{k}{2}+j+1}}{(\frac{k}{2}+1+j)!(\frac{k}{2}+1-j)!} \prod_{i=-1, i\neq \frac{k}{2}+j, i\neq \frac{k}{2}}^{k+1} (z+d-i).\end{aligned} \tag{5.26}$$

Next, identity (5.21) specified to our case follows that

$$\xi_{k+1}\left(-d + \frac{k}{2} + j\right) = -\xi_{k+1}\left(-d + \frac{k}{2} - j\right)$$

which allows us to modify the sum in (5.26) further as

$$\begin{aligned}\xi_{k+1}(z) = &\sum_{j=1}^{\frac{k}{2}+1} \xi_{k+1}\left(-d + \frac{k}{2} + j\right) \frac{j(-1)^{\frac{k}{2}+j+1}}{(\frac{k}{2}+1+j)!(\frac{k}{2}+1-j)!} \\ &\times \left[\frac{1}{z+d-\frac{k}{2}+j} + \frac{1}{z+d-\frac{k}{2}-j} \right] \prod_{i=-1, i\neq \frac{k}{2}}^{k+1} (z+d-i) \\ = &\sum_{j=1}^{\frac{k}{2}+1} \xi_{k+1}\left(-d + \frac{k}{2} + j\right) \frac{j(-1)^{\frac{k}{2}+j+1}}{(\frac{k}{2}+1+j)!(\frac{k}{2}+1-j)!} \\ &\times (2d+2z-k) \prod_{i=1, i\neq j}^{\frac{k}{2}+1} \left(\left(z+d-\frac{k}{2} \right)^2 - i^2 \right).\end{aligned}$$

Next, by applying identity (5.20) to $\xi_{k+1}\left(-d+\frac{k}{2}+j\right)$ and by switching sums we arrive at expression

$$(-1)^{\frac{k}{2}}\left(d+z-\frac{k}{2}\right)\sum_{s=0}^{\frac{k}{2}}\frac{(-1)^s(k-s+1)!}{(k-2s+1)!s!}\rho_s(z)w^{2s}\prod_{i=1}^{\frac{k}{2}-s}\left(\left(z+d-\frac{k}{2}\right)^2-i^2\right)$$

where we have denoted

$$\rho_s(z):=$$
$$\sum_{j=\frac{k}{2}+1-s}^{\frac{k}{2}+1}\frac{(\frac{k}{2}+j-s)!}{(s+j-\frac{k}{2}-1)!}\frac{2j(-1)^{j+1}}{(\frac{k}{2}+1+j)!(\frac{k}{2}+1-j)!}\prod_{i=\frac{k}{2}+1-s,i\neq j}^{\frac{k}{2}+1}\left(\left(z+d-\frac{k}{2}\right)^2-i^2\right)$$

Finally, by applying Lemma 77 we find out that

$$\rho_s(z)=(-1)^{\frac{k}{2}}$$

and thus

$$\xi_{k+1}(z)=(-1)^{\frac{k}{2}}\left(d+z-\frac{k}{2}\right)\sum_{s=0}^{\frac{k}{2}}\frac{(k-s+1)!}{(k-2s+1)!s!}w^{2s}\prod_{i=1}^{\frac{k}{2}-s}\left(i^2-\left(z+d-\frac{k}{2}\right)^2\right). \tag{5.27}$$

At the end, since

$$\prod_{i=1}^{\frac{k}{2}-s}\left(i^2-\left(z+d-\frac{k}{2}\right)^2\right)=(-1)^{\frac{k}{2}+s}\prod_{i=-\frac{k}{2}+s,i\neq 0}^{\frac{k}{2}-s}\left(z+d+i-\frac{k}{2}\right)$$
$$=(-1)^{\frac{k}{2}+s}\prod_{j=s,j\neq\frac{k}{2}}^{k-s}(z+d-j)$$

formula (5.27) can be rewritten as

$$\xi_{k+1}(z)=\sum_{s=0}^{\frac{k}{2}}(-1)^s\frac{(k-s+1)!}{(k-2s+1)!s!}w^{2s}\prod_{j=s}^{k-s}(d+z-j). \tag{5.28}$$

2)Suppose that k is still an even integer. By applying recurrent relation (5.10) to function $\xi_k(z)$ we get a similar relation

$$\xi_{k+1}(z)-(d-k+z)\xi_k(z)+w^2\xi_{k-1}(z)=0,$$

from which we will compute $\xi_k(z)$ with using formula (5.28). Thus

$$\begin{aligned}(d-k+z)\xi_k(z) &= \xi_{k+1}(z) + w^2\xi_{k-1}(z)\\ &= \sum_{s=0}^{\frac{k}{2}}(-1)^s\frac{(k-s+1)!}{(k-2s+1)!s!}w^{2s}\prod_{j=s}^{k-s}(d+z-j)\\ &+ \sum_{s=0}^{\frac{k}{2}-1}(-1)^s\frac{(k-s-1)!}{(k-2s-1)!s!}w^{2s+2}\prod_{j=s}^{k-2-s}(d+z-j).\end{aligned}$$

Next, we can change the lower bound of the second sum to -1 because the added term is zero due to term $\frac{1}{(-1)!}$ which has to be set to zero. Then we shift index s by 1 and we get expression

$$\begin{aligned}(d-k+z)\xi_k(z) &= \sum_{s=0}^{\frac{k}{2}}(-1)^s\frac{(k-s+1)!}{(k-2s+1)!s!}w^{2s}\prod_{j=s}^{k-s}(d+z-j)\\ &- \sum_{s=0}^{\frac{k}{2}}(-1)^s\frac{(k-s)!}{(k-2s+1)!(s-1)!}w^{2s}\prod_{j=s-1}^{k-1-s}(d+z-j)\\ &= \sum_{s=0}^{\frac{k}{2}}(-1)^s\frac{(k-s)!}{(k-2s+1)!s!}w^{2s}\tau_s(z)\prod_{j=s}^{k-1-s}(d+z-j)\end{aligned}$$

where we denote

$$\tau_s(z) := (k-s+1)(d+z-k+s) - s(d+z-s+1).$$

Since

$$\tau_s(z) = (d+z-k)(k-2s+1)$$

we arrive at formula

$$\xi_k(z) = \sum_{s=0}^{\frac{k}{2}}(-1)^s\frac{(k-s)!}{(k-2s)!s!}w^{2s}\prod_{j=s}^{k-1-s}(d+z-j). \tag{5.29}$$

Finally, formulas (5.28) and (5.29) can be written down for an arbitrary $k \in \{0,1,\ldots,2d\}$ at once as

$$\xi_k(z) = \sum_{s=0}^{[\frac{k}{2}]}(-1)^s\frac{(k-s)!}{(k-2s)!s!}w^{2s}\prod_{j=s}^{k-1-s}(d+z-j).$$

□

Remark 79. Although we restricted ourself only for $k \in \{0,1,\ldots,2d\}$, it is clear that the formula (5.25) holds for any $k \in \mathbb{N}_0$.

Remark 80. Formula (5.25) admits a generalization of equation (1.17). By putting $k := d, \nu := z$ in (5.25) and using the definition of the function $\mathfrak{J}$ one easily arrive at the identity

$$\mathfrak{F}\left(\frac{w}{\nu+1}, \frac{w}{\nu+2}, \dots, \frac{w}{\nu+d}\right) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+d+1)} \sum_{s=0}^{[d/2]} (-1)^s \frac{(d-s)!}{s!(d-2s)!} w^{2s} \prod_{j=s}^{d-1-s} (\nu+d-j) \tag{5.30}$$

which holds for $d \in \mathbb{N}_0$ and $\nu \in \mathbb{C} \setminus \{-n, -n+1, \dots, -1\}$.

Corollary 81. If $z \in \sigma(K)$ then $v_z \in \mathbb{C}^{2d+1}$ is the respective eigenvector with components given by formula

$$v_z^{k-d} = \sum_{s=0}^{\left[\frac{k}{2}\right]} (-1)^s \frac{(k-s)!}{(k-2s)!s!} w^{2s-k+1} \prod_{j=s}^{k-1-s} (d+z-j) \tag{5.31}$$

where $k \in \{0, 1, \dots, 2d\}$.

Proof. The statement follows directly from Proposition 69 and Proposition 78. $\square$

5.3 Modified Results for the Jacobi Matrix of Even Dimension

In previous chapters the characteristic function (4.14) and the vector valued function (5.31) which values in points of spectrum correspond to respective eigenvectors was found. Both results were derived for the special Jacobi matrix (4.1) which dimension is odd $(2d+1)$. Now, a slight modification of the already known formulas will allow us to introduce similar results for a special Jacobi matrix of an even dimension.

Let us denote L a Jacobi matrix of dimension $2d$ (for some $d \in \mathbb{N}$) such that

$$L = \begin{pmatrix} -d+\frac{1}{2} & w & & & & & & \\ w & -d+\frac{3}{2} & w & & & & & \\ & \ddots & \ddots & \ddots & & & & \\ & & w & -\frac{1}{2} & w & & & \\ & & & w & \frac{1}{2} & w & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & w & d-\frac{3}{2} & w \\ & & & & & & w & d-\frac{1}{2} \end{pmatrix}. \tag{5.32}$$

Next, define matrix $\widetilde{L} := L + \frac{1}{2}I$ which spectrum is shifted (by a half) with respect to the spectrum of L. Taking into account the recurrence rule (5.10) and equation $\mathfrak{J}(d, d, z) = 0$ one can easily find out that

$$(\widetilde{L} - zI)u_z = -w\mathfrak{J}(-d-1, d, z)e_d \tag{5.33}$$

where

$$u_z^T = (\mathfrak{J}(d-1,d,z), \mathfrak{J}(d-2,d,z), \ldots \mathfrak{J}(-d,d,z))$$

and

$$e_d^j = \delta_{jd}, \quad j = -d+1, -d+2, \ldots, d.$$

For similar reasons as discussed in the proof of Proposition 69, the function $w^{2d-1}\mathfrak{J}(-d-1,d,z)$ is the characteristic polynomial of $\widetilde{L}$ and u_λ is an eigenvector of $\widetilde{L}$ respective to eigenvalue λ. Let $\chi_{\widetilde{L}}$ be the characteristic function of $\widetilde{L}$. Rephrasing formula (5.25) (writing d instead of $d+1$ and $k-1$ instead of k) we have, for $k \in \{1, 2, \ldots, 2d\}$, equation

$$u_z^{k-d} = \mathfrak{J}(d-k,d,z) = \sum_{s=1}^{\left[\frac{k+1}{2}\right]} (-1)^{s+1} \frac{(k-s)!}{(k+1-2s)!(s-1)!} w^{2s-k} \prod_{j=s}^{k-s} (d+z-j). \quad (5.34)$$

Let us summarize the results in the following proposition.

Proposition 82. The formula for characteristic function of L

$$\chi_L(z) = (-1)^d \sum_{s=0}^{d} \frac{(2d-s)!}{(2d-2s)!s!} w^{2s} \prod_{j=1}^{d-s} \left[\left(j - \frac{1}{2}\right)^2 - z^2 \right] \quad (5.35)$$

holds. Furthermore, if $z \in \sigma(L)$ then $u_{z+\frac{1}{2}} \in \mathbb{C}^{2d}$ is the respective eigenvector with components given by formula

$$u_{z+\frac{1}{2}}^{k-d} = \sum_{s=1}^{\left[\frac{k+1}{2}\right]} (-1)^{s+1} \frac{(k-s)!}{(k+1-2s)!(s-1)!} w^{2s-k} \prod_{j=s}^{k-s} \left(d+z+\frac{1}{2}-j\right) \quad (5.36)$$

where $k \in \{1, 2, \ldots, 2d\}$.

Proof. The characteristic function of L is the characteristic function of $\widetilde{L}$ with shifted argument, more precisely

$$\chi_L(z) = \chi_{\widetilde{L}}\left(z + \frac{1}{2}\right).$$

Bearing in mind the above discussion and by using identity (5.34) we get

$$\begin{aligned}
\chi_L(z) &= w^{2d-1}\mathfrak{J}\left(-d-1, d, z+\frac{1}{2}\right) \\
&= \sum_{s=0}^{d} (-1)^s \frac{(2d-s)!}{(2d-2s)!s!} w^{2s} \prod_{j=s}^{2d-1-s} \left(d - \frac{1}{2} + z - j\right) \\
&= (-1)^d \sum_{s=0}^{d} \frac{(2d-s)!}{(2d-2s)!s!} w^{2s} \prod_{j=1}^{d-s} \left[\left(j-\frac{1}{2}\right)^2 - z^2\right],
\end{aligned}$$

thus the first part of the proposition is proved. Consequently, by (5.33), if $z \in \sigma(L)$ then $u_{z+\frac{1}{2}}$ is an eigenvector of L. □

Remark 83. From formula (5.35) it is obvious that $\chi_L(z)$ is an even function and it has no roots for $|z| < \frac{1}{2}$. $\chi_L(\pm\frac{1}{2}) = 0$ if and only if $w = 0$.

Chapter 6

A Set of Limit Points of a Tridiagonal Operator

In an effort to collect information about a relation between spectrum of infinite Jacobi matrices and respective truncations we add this chapter. It will be shown that, under certain assumptions, every eigenvalue of tridiagonal operator is a limit point of a sequence of eigenvalues of a truncated finite-dimensional operator and vice versa. The most of the following results are taken over from [6], [7] and [8] where some other related information can be found (we will focus only on the equality of the spectrum and the set of limit points for (possibly) unbounded tridiagonal operators).

Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. We consider a tridiagonal operator T which corresponds to a positive sequence $\{a_n\}_{n\in\mathbb{N}}$ and a real sequence $\{b_n\}_{n\in\mathbb{N}}$ as

$$Te_n := a_n e_{n+1} + a_{n-1}e_{n-1} + b_n e_n, \quad n \in \mathbb{N}, \quad (a_0 := 0). \tag{6.1}$$

This operator can be expressed as

$$T = VA + AV^* + B$$

where A, B are diagonal operators, $Ae_n := a_n e_n$, $Be_n := b_n e_n$, V is the unitary shift operator, $Ve_n := e_{n+1}$ and V^* its adjoint ($V^*e_n = e_{n-1}$, $V^*e_1 = 0$). If the sequence $\{a_n\}$ is bounded then T is self-adjoint with the definition domain of the operator B, i.e.

$$\mathrm{Dom}(T) = \mathrm{Dom}(B) = \{x \in \mathcal{H} \mid Bx \in \mathcal{H}\}.$$

With the aid of sequences $\{a_n\}$ and $\{b_n\}$ we define polynomials P_n, $n \in \mathbb{N}_0$ by recurrence relation

$$\begin{gathered} a_n P_{n+1}(x) + a_{n-1}P_{n-1}(x) + b_n P_n(x) = xP_n(x), \quad n \in \mathbb{N} \\ P_0(x) = 0, \quad P_1(x) = 1. \end{gathered} \tag{6.2}$$

Note that P_n is a polynomial of degree $n-1$ (since $a_n \neq 0$). More information about such polynomials can be found for example in [9] and [10].

6.1 Inclusion $\sigma \subseteq \Lambda$

Let $N \in \mathbb{N}$ and let us define truncated finite-dimensional operators

$$T_N := P_N T P_N$$

where P_N is the orthogonal projection operator on the subspace $\mathcal{H}_N$ spanned by $\{e_1, \ldots, e_N\}$. Suppose that T is self-adjoint. First, we will show the operator T can be strongly approximated by a sequence of operators $\{T_N\}$, i.e.

$$\lim_{N\to\infty} \|Tf - T_N f\| = 0 \tag{6.3}$$

for every $f \in \mathrm{Dom}(T)$. We prove this as follows. Let $f \in \mathrm{Dom}(T)$ then we have

$$\begin{aligned}
\|Tf - T_N f\|^2 &= \sum_{n=1}^{\infty} |(Tf - P_N T P_N f, e_n)|^2 = \sum_{n=1}^{\infty} |(f, (T - P_N T P_N) e_n)|^2 \\
&= \sum_{n=N+1}^{\infty} |(f, Te_n)|^2 + \sum_{n=1}^{N} |(f, (I - P_N) T e_n)|^2 = \sum_{n=N+1}^{\infty} |(Tf, e_n)|^2 + a_N^2 |(f, e_{N+1})|^2 \\
&= \sum_{n=N+1}^{\infty} |(Tf, e_n)|^2 + |(AV^* f, e_N)|^2 \qquad (6.4)
\end{aligned}$$

where the definition relation (6.1) and self-adjoint of T was used. The first member on the RHS of (6.4) tends to zero as $N \to \infty$ because $\|Tf\|^2 = \sum_{n=1}^{\infty} |(Tf, e_n)|^2 < \infty$ and the second because $AV^* f \in \mathcal{H}$ (since $\mathrm{Dom}(AV^*) \supseteq \mathrm{Dom}(T)$) and the sequence $\{e_N\}$ converges weakly to zero.
Second, we will show that the eigenvalues of T_N are zeros of the polynomial $P_{N+1}(x)$ defined by (6.2). Let us denote

$$\xi_N^T(x) := (P_1(x), P_2(x), \ldots, P_N(x)).$$

Since $P_1(x) = 1$ the vector-valued function $\xi_N(x) \neq 0$ for all $x \in \mathbb{R}$. From the relation

$$\begin{pmatrix}
b_1 & a_1 & & & & \\
a_1 & b_2 & a_2 & & & \\
 & \ddots & \ddots & \ddots & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & a_{N-2} & b_{N-1} & a_{N-1} \\
 & & & & a_{N-1} & b_N
\end{pmatrix} \xi_N(x) = x\xi_N(x) - a_N P_{N+1}(x) e_N$$

it follows that all roots of the polynomial P_{N+1} are eigenvalues of T_N. On the other hand, if $\mu \in \sigma(T_N)$ then $\exists x \neq 0$ such that $T_N x = \mu x$ and $x^1 \neq 0$ (otherwise $x = 0$). Without loss of generality we can assume $x^1 = 1$ then $x^k = P_k(\mu)$ for $k = 1, 2, \ldots, N$ and since

$$a_{N-1} P_{N-1}(\mu) + b_N P_N(\mu) = \mu P_N(\mu)$$

we get

$$a_N P_{N+1}(\mu) = 0.$$

The last equality together with the positivity of the sequence $\{a_n\}$ implies that μ is root of P_{N+1}.
Denote by $\Lambda(T)$ the set of all points which are limit points of eigenvalues of T_N when N tends to infinity.

Theorem 84. If T is self-adjoint then every point in the spectrum of T is a limit point of the set of all zeros of all polynomials P_N, i.e. $\sigma(T) \subseteq \Lambda(T)$.

Proof. Let $\lambda \in \sigma(T)$ and let λ is not a limit point of the zeros of the polynomials P_{n+1}, $n \in \mathbb{N}$ or equivalently $\lambda \notin \Lambda(T)$. Then there exists $d > 0$ and subsequence of $\{T_n\}$ which we denote by $\{T_N\}$ such that $|\lambda - \rho| \geq d$ for every eigenvalue ρ belonging to any of the operators T_N, $N = 1, 2, \ldots$.
Let $T_N x_k = \lambda_k x_k$, $k = 1, 2, \ldots, N$ and $(x_i, x_j) = \delta_{ij}$. Then for every $f \in \mathrm{Dom}(T)$ we have

$$(\lambda P_N - T_N)f = (\lambda P_N - P_N T P_N)f = \sum_{k=1}^{N} (\lambda - \lambda_k)(f, x_k)x_k$$

and

$$\|(\lambda P_N - P_N T P_N)f\|^2 = \sum_{k=1}^{N} |\lambda - \lambda_k|^2 |(f, x_k)|^2 \geq d^2 \|P_N f\|^2$$

which implies that

$$\|P_N(\lambda - T)P_N f\| \geq d\|P_N f\|$$

for all $f \in \mathrm{Dom}(T)$. Since $P_N T P_N$ converges strongly to T and P_N converges strongly to I the last inequality for $N \to \infty$ leads to an inequality

$$\|(\lambda - T)f\| \geq d\|f\|$$

which holds for every $f \in \mathrm{Dom}(T)$. This means that $\lambda \notin \sigma(T)$ (because T is self-adjoint), contrary to the assumption. □

6.2 Inclusion $\sigma \supseteq \Lambda$

In the last section we have shown that if T is self-adjoint then $\sigma(T) \subseteq \Lambda(T)$ but the equality $\sigma(T) = \Lambda(T)$ does not hold in general. For the invalidity of the equality see [6]. Next we will show some sufficient conditions for the validity of the equality $\sigma(T) = \Lambda(T)$.

Theorem 85. Assume that the sequence $\{a_n\}$ is positive and $\lim_{n\to\infty} a_n = 0$. Then (for any sequence $\{b_n\}$) we have $\sigma(T) = \Lambda(T)$.

Proof. Since $\{a_n\}$ is bounded T is self-adjoint ($\text{Dom}(T) = \text{Dom}(B)$) and from Theorem 84 it follows the validity of the inclusion $\sigma(T) \subseteq \Lambda(T)$. Thus it suffices to show the validity of the reverse inclusion $\sigma(T) \supseteq \Lambda(T)$.
Let $\lambda \in \Lambda(T)$ then there exists a subsequence of the sequence $\{T_N\}$ (which we denote also with $\{T_N\}$) such that

$$T_N x_N = \lambda_N x_N, \quad \|x_N\| = 1, \quad x_N \in \mathcal{H}_N \tag{6.5}$$

and

$$\lim_{N\to\infty} \lambda_N = \lambda. \tag{6.6}$$

From (6.5) we have

$$(T_N x_N, x_N) = (P_N T P_N x_N, x_N) = \lambda_N$$

and since $P_N x_N = x_N$ we obtain

$$(T x_N, x_N) = \lambda_N. \tag{6.7}$$

Next

$$\begin{aligned} \lambda_N^2 &= (T_N^2 x_N, x_N) = (P_N T x_N, T x_N) = ((I - Q_N) T x_N, T x_N) = \\ &= (T^2 x_N, x_N) - (Q_N T x_N, T x_N) \end{aligned} \tag{6.8}$$

where Q_N is the orthogonal projection on the subspace spanned by $\{e_{N+1}, e_{N+2}, \dots\}$.
Since $Te_k = a_k e_{k+1} + a_{k-1} e_{k-1} + b_k e_k$ and $(x_N, e_k) = 0$ for $k \geq N+1$ we can adjust the second term in (6.8)

$$Q_N T x_N = \sum_{N+1}^{\infty} (T x_N, e_k) e_k = \sum_{N+1}^{\infty} (x_N, T e_k) e_k = a_N (x_N, e_N) e_{N+1},$$

thus we find

$$(Q_N T x_N, T x_N) = a_N (x_N, e_N)(T e_{N+1}, x_N) = a_N^2 |(x_N, e_N)|^2. \tag{6.9}$$

By taking (6.8) together with (6.9) we get

$$(T^2 x_N, x_N) = \lambda_N^2 + a_N^2 |(x_N, e_N)|^2. \tag{6.10}$$

Due to Schwarz inequality it is $|(x_N, e_N)| \leq 1$ and the relation (6.10) gives

$$(T^2 x_N, x_N) \leq \lambda_N^2 + a_N^2.$$

Finally, due to (6.6) and the assumption $\lim_{n\to\infty} a_n = 0$ we arrive at an inequality

$$\limsup_{N\to\infty} (T^2 x_N, x_N) \leq \lambda^2. \tag{6.11}$$

Now we have

$$\begin{aligned}\|(T-\lambda)x_N\|^2 &= ((T-\lambda)x_N,(T-\lambda)x_N) = ((T-\lambda)^2x_N,x_N)\\ &= ((T^2+\lambda^2-2\lambda T)x_N,x_N) = (T^2x_N,x_N)+\lambda^2-2\lambda(Tx_N,x_N)\\ &= (T^2x_N,x_N)+\lambda^2-2\lambda\lambda_N.\end{aligned}$$

The above relation due to (6.6) and (6.11) gives

$$\limsup_{N\to\infty}\|(T-\lambda)x_N\|^2 \le 0.$$

So

$$\lim_{N\to\infty}\|(T-\lambda)x_N\| = 0$$

which means that $\lambda\in\sigma(T)$. This proves that $\Lambda(T)\subseteq\sigma(T)$. □

Theorem 86. Suppose that $T = VA + AV^* + B$ is self-adjoint and the sequences $\{a_n\}$ and $\{b_n\}$ satisfy

$$\lim_{n\to\infty}\frac{a_na_{n-1}}{b_n} = 0, \quad \lim_{n\to\infty} b_n = \infty. \tag{6.12}$$

Then $\Lambda(T) = \sigma(T)$.

Proof. Due to self-adjoint of T it suffices to show the validity of the inclusion $\Lambda(T)\subseteq\sigma(T)$. From (6.5) we obtain

$$\lambda_N(x_N,e_N) = (T_Nx_N,e_N) = (x_N,T_Ne_N) = (x_N,a_{N-1}e_{N-1}+b_Ne_N)$$

because $x_N\in\mathcal{H}_N$, thus $(x_N,e_{N+1}) = 0$. The last equation can be rewritten as

$$(\lambda_N-b_N)(x_N,e_N) = a_{N-1}(x_N,e_{N-1}).$$

Since $\lambda_N\to\lambda<\infty$ and $b_N\to\infty$, $n\to\infty$ for sufficiently large N we have $\lambda_N\neq b_N$. The relation (6.10) takes the form

$$(T^2x_N,x_N) = \lambda_N^2+\frac{a_N^2a_{N-1}^2}{(\lambda_N-b_N)^2}|(x_N,e_{N-1})|^2 \le \lambda_N^2+\frac{a_N^2a_{N-1}^2}{(\lambda_N-b_N)^2}. \tag{6.13}$$

Finally, from (6.13), due to (6.6) and (6.12) it follows that

$$\limsup_{N\to\infty}(T^2x_N,x_N)\le\lambda^2.$$

Now the proof follows in a similar way to Theorem 85. □

Corollary 87. Let the sequence $\{a_n\}$ is bounded and the sequence $\{b_n\}$ is divergent. Then $\Lambda(T) = \sigma(T)$.

Proof. Since $\{a_n\}$ is bounded the operator $T = VA + AV^* + B$ is self-adjoint and the condition (6.12) is satisfied. $\square$

Remark 88. A lot of similar (but more complicated) conditions to (6.12) can be derived. For example the condition (6.12) can be replaced by

$$\lim_{n\to\infty} \frac{a_n a_{n-1}(a_{n-1} + a_{n-2})}{b_n b_{n-1}} = 0, \quad \lim_{n\to\infty} b_n = \infty$$

or

$$\lim_{n\to\infty} \frac{a_n a_{n-1}}{b_n b_{n-1}} \left[\frac{a_{n-1}^2}{b_n} + \frac{a_{n-2}(a_{n-2} + a_{n-3})}{b_{n-2}} \right] = 0, \quad \lim_{n\to\infty} b_n = \infty$$

and the equation $\Lambda(T) = \sigma(T)$ holds too (see [7] for details).

Conclusion

In this work we studied the function $\mathfrak{F}$, its algebraic and asymptotic properties and its relationship to the eigenvalue problem for finitedimensional symmetric tridiagonal (Jacobi) matrices. With the aid of the symbol $\mathfrak{J}$ defined with using the function $\mathfrak{F}$ we construct a special basis in which the studied matrix becomes a sum of a diagonal matrix and a rank-one matrix operator. This form is suitable for various computation which allows us to derive the formula for characteristic function and the comparatively simple expression for resolvent operator of the Jacobi matrix whose parallels to the diagonal are constant and whose diagonal depends linearly on the index. Further, some information about a distribution of the spectrum of such matrix are summarized.

By slight generalization of the symbol $\mathfrak{J}$ we define a function also called $\mathfrak{J}$. This function arises in components of eigenvectors of the Jacobi matrix and also it forms one more formula for the characteristic function. Next, we present vector-valued function on the complex plain having the property that its values on spectral points of the Jacobi matrix are equal to corresponding eigenvectors. At the end it is shown that, under certain assumptions, every eigenvalue of tridiagonal operator is a limit point of a sequence of eigenvalues of a truncated finitedimensional operator and vice versa.

To continue in this work one could try to examine some of the following suggestions. The function $\mathfrak{F}$ deserves further investigation, for example, one could try to find values of the function $\mathfrak{F}$ applied on some other sequences then those which are discussed in section 1.2. The main results related to the eigenvalue problem (e.g., the formula for the characteristic function, for eigenvectors, etc.) could be possibly derived for a larger family of Jacobi matrices. Last but not least, one could use the results obtained for finitedimensional Jacobi matrices to explore the spectrum of infinitedimensional Jacobi matrices of a similar form.

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