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BACHELOR´S THESIS

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Spektrální analýza Jacobiho matic speciálního typu

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Abstrakt: Uvedeme některé výsledky z funkcionální analýzy neomezených operátorů a seznámíme se základy poruchové teorie pro lineární operátory na Hilbertových prostorech. Použijeme získané znalosti na úlohu určení spektra operátoru s Jacobiho maticí. Shrneme výsledky a seznámíme s problémem, který vyvstane při použití teorie na naši úlohu. V závěru popíšeme některé vlastnosti konečno-rozměrných Jacobiho matic.

Klíčová slova: Jacobiho matice, perturbace, spektrum

Title:
Spectral Analysis of Special Type of Jacobi's matrices

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Abstract: We present some results from functional analysis of unbounded operators and introduce the perturbation theory for linear operators in Hilbert spaces. We use the gained knowledge at the task of describing of the spectrum of the operator with the Jacobi matrix. We summarize the results and present the problem, which appears at using the theory at the task of finding the spectrum of the operator with the Jacobi matrix. Finally we describe some features of finite-dimensional Jacobi matrices.

Key words: Jacobi matrix, perturbation, spectrum

Contents

Introduction

The Jacobi matrix is a complex tridiagonal matrix (Definition 7). The finding of the spectrum of the Jacobi matrix is the problem of much physical interest. The spectrum of (finite) Jacobi matrices appears in many applications: from orthogonal polynomials and nearest-neighbours interaction models to solvable models of quantum mechanics. This paper summarize the basic knowledge about the spectrum of an operator with the Jacobi matrix.

In the first chapter we will introduce some definitions and theorems of functional analysis as such: an essential spectrum, a relative compactness, the Weyl theorem and the min-max principle, which we will use later in the main chapter 3.

In the second chapter we will present the perturbation theory for linear operators and we will compute the coefficients of the Rayleigh-Schrödinger series of perturbed eigenvalue, which we will use later again in the specific problem of the Jacobi matrix in chapter 3.

In the third chapter we will introduce the operators with the Jacobi matrix. With the help of the theory written in chapter 1 and 2 we summarize some features of the spectrum of the operator with the Jacobi matrix. Next we will show that in some cases (operators with shrinking spectral gaps) the perturbation theory does not allow the global description of the spectrum.

In the final chapter we will find the kernel of the operator with the Jacobi matrix of a special type (a linear and symmetric case). Next we will state some helpful statements considering the finite-dimensional Jacobi matrices as a concept for the future afford to describe the spectrum of the Jacobi matrix globally.

Chapter 1

Operators in Hilbert space

In this chapter we introduce some basic definitions and fundamental theorems which we will use in the main chapter 3. We deal with usually unbounded operators in a Hilbert space. For more detail functional analysis, we refer reader to ,e.g., [3].

1.1 The essential spectrum and the min-max principle

Definition 1. A set $\sigma_{ess}(A) := \{\lambda \in \mathbb{C} \big| \exists \{x_n\}_{n=1}^{\infty} \subset D(A), \|x_n\| = 1$ which has no convergent subsequence and $\lim_{n\to\infty}(A-\lambda)x_n = 0\}$ is called *essential spectrum* of an operator A.

In this text we are interested in operators which are usually self-adjoint. So lets introduce some properties of $\sigma_{ess}(A)$ where A is self-adjoint.

Theorem 1. Let A be self-adjoint and $\lambda \in \mathbb{R}$ (recall that $\sigma(A) \subset \mathbb{R}$) then following statements are equivalent:
(i) $\lambda \in \sigma_{ess}(A)$;
(ii) $\exists \{x_n\}_{n=1}^{\infty} \subset D(A)$, $\|x_n\| = 1$, $x_n \to 0$ weakly and $\lim_{n\to\infty} \|(A-\lambda)x_n\| = 0$;
(iii) λ is an accumulation point of the set $\sigma(A)$ or λ is an eigenvalue with infinite multiplicity.

The proof of this theorem can be found in [3] chap. 10.

Remark 1. (a) A set $\sigma_d(A) := \sigma(A) \backslash \sigma_{ess}(A)$ is called *discrete spectrum* and the statement (iii) from the previous theorem follows that the discrete spectrum is composed of isolated points of $\sigma(A)$ with finite multiplicity. If $\sigma_{ess}(A) = \emptyset$, i.e. $\sigma(A) = \sigma_d(A)$ one says that the self-adjoint operator A has *purely discrete spectrum.*
(b) It holds: $\sigma_{ess}(A) = \emptyset \Leftrightarrow (\forall \mu \in \rho(A))(R_\mu(A) \equiv (A-\mu)^{-1}$ is compact)
(proved again in [3] chap.10) Therefore operators with clearly discrete spectrum are

often called *operators with compact resolvent.*
(c) $\sigma_{ess}(A)$ is a closed set.

Next we will deal with a stability of $\sigma_{ess}(A)$ for A self-adjoint.

Definition 2. Let A be a self-adjoint operator. Operator T (possibly unbounded) is A*-compact* (or *relative compact* with respect to A) if $D(T) \supset D(A)$ and an operator $T(A-i)^{-1}$ is compact.

Theorem 2. (Weyl) Let A be a self-adjoint operator and S be symmetric and A-compact operator. Then

$$\sigma_{ess}(A) = \sigma_{ess}(A+S)$$

.

A reader can find the proof of this theorem in [3] §10.4.

Next very useful theorem gives us information about $\sigma_d(A)$ and $\sigma_{ess}(A)$ (for A self-adjoint) from knowledge of the expectation values $(\psi, A\psi)$.

Theorem 3. (min-max principle) Let A be a self-adjoint operator that is bounded from below, i.e., $A \geq cI$ for some c. Define

$$\mu_n(A) := \sup_{\varphi_1,\ldots,\varphi_{n-1}} U_A(\varphi_1,\ldots,\varphi_{n-1})$$

where

$$U_A(\varphi_1,\ldots,\varphi_m) = \inf\{(\psi, A\psi)|\psi \in D(A); \|\psi\| = 1; \psi \in [\varphi_1,\ldots,\varphi_m]^\perp\}$$

Then, for each fixed n, *either*:
(a) there are n eigenvalues $\lambda_1,\ldots,\lambda_n$ (counting degenerate eigenvalues a number of times equal to their multiplicity) such that $\lambda_k < \inf\{\lambda|\lambda \in \sigma_{ess}\}$ $\forall k \in \hat{n}$ and $\mu_n = \lambda_n$ (counting multiplicity);
or
(b) $\mu_n = \inf\{\lambda|\lambda \in \sigma_{ess}\}$ and in that case $\mu_n = \mu_{n+1} = \mu_{n+2} = \ldots$ and there are at most $n-1$ eigenvalues (counting multiplicity) below μ_n.

The proof can be found in [4] vol. IV chap. XIII.

Chapter 2

Introduction to the perturbation theory

In this chapter we shall examine the following situation: An operator H_0 has an eigenvalue E_0 (we usually assume that E_0 is in the discrete spectrum). Suppose that H_0 is perturbed a little; that is, consider $H_0 + \beta V$ where V is some other operator and $|\beta|$ is small (sometimes we will consider generalized case where $H(\beta)$ will be an operator-valued function and $H(\beta_0) = H_0$ for some β_0). Then we will study what eigenvalues of $H_0 + \beta V$ lie near E_0, how they are related to V and what their properties are as functions of β.

2.1 Regular perturbation theory

Theorem 4. Suppose that A is closed operator and let λ be an isolated point of $\sigma(A)$. Explicitly, suppose that there is $\epsilon > 0$ such that $\{\mu \big| |\mu - \lambda| < \epsilon\} \cap \sigma(A) = \{\lambda\}$. Then,
(a) For any r with $0 < r < \epsilon$,

$$P_\lambda = -\frac{1}{2i\pi} \oint_{|\mu-\lambda|=r} (A-\mu)^{-1} d\mu \tag{2.1}$$

exists and is independent of r.
(b) $P_\lambda^2 = P_\lambda$. Thus P_λ is a projection.
(c) if μ is an isolated point of $\sigma(A)$ and $\mu \neq \lambda$, then $P_\mu P_\lambda = 0$. Thus $P_\mu P_\lambda = \delta_{\mu\lambda} P_\lambda$.
(d) If $G_\lambda = \operatorname{Ran} P_\lambda$ and $F_\lambda = \ker P_\lambda$, then G_λ and F_λ are complementary closed subspaces; that is, $G_\lambda + F_\lambda = \mathcal{H}$ and $G_\lambda \bigcap F_\lambda = \{0\}$. Moreover, $G_\lambda \subset D(A)$, $AG_\lambda \subset G_\lambda$.
(e) If $B \equiv A \restriction F_\lambda$, then $\lambda \notin \sigma(B)$.

Proof. (a) We know that resolvent $(A-\mu)^{-1}$ is an analytic function on $\mathbb{C} \backslash \sigma(A) \equiv \rho(A)$. Thus the integral exists as a Banach-space-valued Riemann integral. That it

is independent of r is a consequence of the Cauchy integral theorem.
(b) Let $r < R < \epsilon$.

$$P_\lambda^2 = \frac{1}{(2i\pi)^2} \oint_{|\mu-\lambda|=r} \oint_{|\nu-\lambda|=R} (A-\mu)^{-1}(A-\nu)^{-1} d\nu d\mu$$

Using the first resolvent equation:

$$\begin{aligned} P_\lambda^2 &= \frac{1}{(2i\pi)^2} \oint_{|\mu-\lambda|=r} \oint_{|\nu-\lambda|=R} (\nu-\mu)^{-1}[(A-\nu)^{-1} - (A-\mu)^{-1}] d\nu d\mu \\ &= \frac{1}{(2i\pi)^2} \oint_{|\nu-\lambda|=R} d\nu (A-\nu)^{-1} \oint_{|\mu-\lambda|=r} d\mu (\nu-\mu)^{-1} \\ &- \frac{1}{(2i\pi)^2} \oint_{|\mu-\lambda|=r} d\mu (A-\mu)^{-1} \oint_{|\nu-\lambda|=R} d\nu (\nu-\mu)^{-1} \\ &= \frac{1}{(2i\pi)^2} \left[0 - (2i\pi) \oint_{|\mu-\lambda|=r} (A-\mu)^{-1} d\mu \right] = P_\lambda \end{aligned}$$

(c) Similar computation (as in (b)) shows that:

$$\begin{aligned} P_\mu P_\lambda = &\frac{1}{(2i\pi)^2} \oint_{|\nu-\lambda|=r'} d\nu (A-\nu)^{-1} \oint_{|\zeta-\mu|=r} d\zeta (\nu-\zeta)^{-1} \\ &- \frac{1}{(2i\pi)^2} \oint_{|\zeta-\mu|=r} d\zeta (A-\zeta)^{-1} \oint_{|\nu-\lambda|=r'} d\nu (\nu-\zeta)^{-1} = 0 \end{aligned}$$

because r and r' are chosen such that $\{\zeta \in \mathbb{C} \big| |\zeta-\mu| \le r\} \cap \{\nu \in \mathbb{C} \big| |\nu-\lambda| \le r'\} = \emptyset$.
(d) That $G_\lambda = \ker(1-P_\lambda)$ and $F_\lambda = \ker P_\lambda$ are closed complementary subspaces is clear.
Let $\psi \in \mathcal{H}$. $P_\lambda \psi = -\frac{1}{2i\pi} \oint_{|\mu-\lambda|=r} (A-\mu)^{-1}\psi d\mu$. Define $\psi(\mu) := (A-\mu)^{-1}\psi$ and $K := \{\mu \in \mathbb{C} \big| |\mu-\lambda| = r\}$. We shall show that $-\frac{1}{2i\pi} \oint_K \psi(\mu) d\mu \in D(A)$. Since P_λ is given by a Riemann integral, $P_\lambda \psi = \lim_{n\to\infty} \chi_n$ where $\chi_n = -\frac{1}{2i\pi} \sum_{i=1}^{k_n} \psi(\mu_i)\Delta_i$ and μ_i and Δ_i are chosen so that $\chi_n \to -\frac{1}{2i\pi} \oint_K \psi(\mu) d\mu$ for $n \to \infty$. Consider that $\forall n \in \mathbb{N}$, $\chi_n \in D(A)$ and

$$A\chi_n = -\frac{1}{2i\pi} \sum_{i=1}^{k_n} A\psi(\mu_i)\Delta_i = -\frac{1}{2i\pi} \sum_{i=1}^{k_n} (\psi + \mu_i (A-\mu_i)^{-1}\psi)\Delta_i$$

Since $\sum_{i=1}^{k_n} \Delta_i = 0$ (K is a circle), $A\chi_n = -\frac{1}{2i\pi} \sum_{i=1}^{k_n} \mu_i (A-\mu_i)^{-1}\psi\Delta_i \to -\frac{1}{2i\pi} \oint_K \mu\psi(\mu) d\mu$ for $n \to \infty$. Finally, since A is closed, $-\frac{1}{2i\pi} \oint_K \psi(\mu) d\mu \in D(A)$. Thus $G_\lambda \subset D(A)$.
Let $\phi = P_\lambda \phi \in G_\lambda$. Then $A\phi = AP_\lambda \phi = P_\lambda(A\phi)$, thus $A\phi \in G_\lambda$.
(e) Let

$$R_\lambda := -\frac{1}{2i\pi} \oint_{|\mu-\lambda|=r} (\lambda-\mu)^{-1}(A-\mu)^{-1} d\mu$$

Then

$$\begin{aligned} R_\lambda(A-\lambda) &= -\frac{1}{2i\pi}\oint_{|\mu-\lambda|=r}(\lambda-\mu)^{-1}(A-\mu)^{-1}(A-\mu+\mu-\lambda)d\mu \\ &= -\frac{1}{2i\pi}\oint_{|\mu-\lambda|=r}(\lambda-\mu)^{-1}d\mu + \frac{1}{2i\pi}\oint_{|\mu-\lambda|=r}(A-\mu)^{-1}d\mu \\ &= 1 - P_\lambda \end{aligned}$$

And by restriction on F_λ, one finds that $R_\lambda(B-\lambda) = I \restriction F_\lambda$. Thus $\lambda \notin \sigma(B)$. □

Definition 3. A point $\lambda \in \sigma(A)$ is called *discrete* if λ is isolated and P_λ (given by (2.1)) is finite dimensional; if P_λ is one dimensional, we say λ is a *nondegenerate* eigenvalue.

Remark 2. (a) Suppose that $A\psi = \nu\psi$. Then

$$P_\lambda\psi = -\frac{1}{2i\pi}\oint_{|\mu-\lambda|=r}(\nu-\mu)^{-1}\psi d\mu = \begin{cases} \psi & if\ \nu = \lambda \\ 0 & if\ \nu \neq \lambda \end{cases}$$

It follows that the only eigenvalue of $A \restriction \operatorname{Ran} P_\lambda$ is λ.
(b) The only eigenvalue of $(A-\lambda)P_\lambda$ is zero. Thus the spectral diameter of $(A-\lambda)P_\lambda$ is zero. Such operators are called *quasi-nilpotents*.
(c) If λ is a nondegenerate eigenvalue, then $(\forall\psi \in \operatorname{Ran} P_\lambda)(A\psi = \lambda\psi)$.
It should be obvious if we consider theorem 4 and the previous remark:
$\psi \in \operatorname{Ran} P_\lambda \Rightarrow (A-\lambda)\psi \in \operatorname{Ran} P_\lambda$. Since $\dim P_\lambda = 1$, there is a constant c such that $c\psi = (A-\lambda)\psi = (A-\lambda)P_\lambda\psi \Rightarrow c = 0 \Rightarrow A\psi = \lambda\psi$.

To complete our discussion of discrete spectrum, we prove a converse to previous Theorem.

Theorem 5. Let A be an operator with $\{\mu \big| |\mu-\lambda| = r\} \subset \rho(A)$, $(\rho(A) = \mathbb{C}\setminus\sigma(A))$. Then $P = (-2i\pi)^{-1}\oint_{|\mu-\lambda|=r}(A-\mu)^{-1}d\mu$ is a projection. If P has dimension $n < \infty$, then A has at most n points of its spectrum in $\{\mu \big| |\mu-\lambda| < r\}$ and each is discrete. If $n = 1$, there is exactly one spectral point in $\{\mu \big| |\mu-\lambda| < r\}$ and it is nondegenerate.

Proof. The proof of the Theorem 1(b) carries through without change to prove that P is a projection and according to (c) we know that $G = \operatorname{Ran} P$ and $F = \ker P$ are closed complementary invariant subspaces. Let $A_1 = A \restriction G$ and $A_2 = A \restriction F$. As in the proof of Theorem 1(d), $\nu \notin \sigma(A_2)$ if $|\nu - \lambda| < r$. If $\dim G = n < \infty$, A_1 has eigenvalues $\nu_1 \ldots, \nu_k$ $(k \leq n)$, so a set $\sigma(A) \cap \{\nu \big| |\nu-\lambda| < r\}$ has at most n elements. To see that each spectral point in the circle is discrete, we note that if P_ν is the spectral projection of Theorem 1 and if ν is in the circle $\{\nu \big| |\nu-\lambda| < r\}$, then $P_\nu P = PP_\nu = P_\nu$ because:

$$P_\nu P = \frac{1}{(2i\pi)^2}\oint_{|\mu-\lambda|=r}\oint_{|\zeta-\nu|=r'}(A-\mu)^{-1}(A-\zeta)^{-1}d\zeta d\mu$$

(r' is chosen such that $\{\zeta \big| |\zeta-\nu| \leq r'\} \subseteq \{\mu \big| |\mu-\lambda| < r\}$)

$$
\begin{aligned}
&= \frac{1}{(2i\pi)^2} \oint_{|\mu-\lambda|=r} \oint_{|\zeta-\nu|=r'} (\zeta-\mu)^{-1}[(A-\zeta)^{-1} - (A-\mu)^{-1}] d\zeta d\mu \\
&= \frac{1}{(2i\pi)^2} \Big[\oint_{|\zeta-\nu|=r'} d\zeta (A-\zeta)^{-1} \oint_{|\mu-\lambda|=r} d\mu (\zeta-\mu)^{-1} \\
&- \oint_{|\mu-\lambda|=r} d\mu (A-\mu)^{-1} \oint_{|\zeta-\nu|=r'} d\zeta (\zeta-\mu)^{-1} \Big] \\
&= -\frac{1}{(2i\pi)} \oint_{|\zeta-\nu|=r'} (A-\zeta)^{-1} d\zeta = P_\nu
\end{aligned}
$$

$P_\nu P = P P_\nu = P_\nu$, thus $\operatorname{Ran} P_\nu \subset \operatorname{Ran} P$, then $\dim P_\nu \leq \dim P < \infty$.
The last statement is clear because if $n = 1$, we already know that A has at most 1 point of its spectrum in $\{\mu \big| |\mu-\lambda| < r\}$. If there were no spectral point, then $P = 0$ and that is a contradiction with $n = 1$. □

Definition 4. A (possibly unbounded) operator-valued function $T(\beta)$ on complex domain R is called an *analytic family* or an *analytic family in the sense of Kato* if and only if:
(i) $(\forall \beta \in R)$ ($T(\beta)$ is closed and $\rho(T(\beta)) \neq \emptyset$)
(ii) $(\forall \beta_0 \in R)(\exists \lambda_0 \in \rho(T(\beta_0)))(\exists \epsilon > 0)(\forall \beta \in R, |\beta-\beta_0| < \epsilon)(\lambda_0 \in \rho(T(\beta)))$ and ($(T(\beta)-\lambda_0)^{-1}$ is an analytic operator-valued function of β)

The number λ_0 in the above definition does not play a special role:

Theorem 6. Let $T(\beta)$ be an analytic family on a domain R. Then

$$\Gamma = \{(\beta,\lambda) \in \mathbb{C} \times \mathbb{C} \big| \beta \in R, \lambda \in \rho(T(\beta))\}$$

is open and the function $(T(\beta)-\lambda)^{-1}$ defined on Γ is an analytic function of two variables.

The proof of this theorem can be found in [4] and we will not give it here.

Lemma 1. Let P and Q are projections (not necessarily orthogonal) and $\dim P \neq \dim Q$, then $\| P-Q \| \geq 1$. In particular, if $P(x)$ is a continuous projection-valued function of x on a connected set $R \subset \mathbb{C}$, then $\dim P(x)$ is a constant.

Proof. Without loss of generality suppose $\dim P < \dim Q$. Let $F = \ker P$ and let $E = \operatorname{Ran} Q$. Then $\dim(F^\perp) = \dim P < \dim E$. As a result, $F \cap E \neq \{0\}$ (see e.g. [3] lemma 5.4.7). Let $\psi \neq 0$, $\psi \in F \cap E$. Then $P\psi = 0$, $Q\psi = \psi$, so $\| (P-Q)\psi \| = \| \psi \|$. This implies that $\| (P-Q) \| \geq 1$. The final statement we will prove by contradiction. Let $\dim P(x) \neq const$ on R. $P(x)$ is continuous on R, that is

$$(\forall x_0 \in R)(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in R, |x - x_0| < \delta)(\|P(x) - P(x_0)\| < \epsilon)$$

Put $\epsilon = 1$. Since $\dim P(x) \neq const$ on connected set R ,$\exists x_1, x_2 \in R$ such that $|x_1 - x_2| < \delta$ and $\dim P(x_1) \neq \dim P(x_2)$. Then, by the previous statement $\|P(x_1) - P(x_2)\| \geq 1$ and this is a contradiction with continuity of $P(x)$. □

Now we can finally proof the most important theorem of this chapter:

Theorem 7. (Kato-Rellich) Let $T(\beta)$ be an analytic family in the sence of Kato. Let E_0 be a nondegenerate discrete eigenvalue of $T(\beta_0)$. Then for β near β_0, there is exactly one point $E(\beta) \in \sigma(T(\beta))$ near E_0 and this point is isolated and nondegenerate. $E(\beta)$ is an analytic function of β for β near β_0. Furthermore, there is an analytic eigenvector $\Omega(\beta)$ (respective to $E(\beta)$) for β near β_0. If $T(\beta)$ is self-adjoint for β real, then $\Omega_(\beta)$ can be chosen to be normalized for β real.

Proof. Pick $\epsilon > 0$ so that $\sigma(T(\beta_0)) \cap \{E \big| |E - E_0| \leq \epsilon\} = \{E_0\}$. The circle $\{E \big| |E - E_0| = \epsilon\}$ is compact and the set Γ of the Theorem 6 is open $\Rightarrow$

$$(\exists \delta > 0)(\forall \beta, |\beta - \beta_0| \leq \delta)(\forall E, |E - E_0| = \epsilon)(E \notin \sigma(T(\beta)))$$

Then

$$P(\beta) := -\frac{1}{2i\pi} \oint_{|E-E_0|=\epsilon} (T(\beta) - E)^{-1} dE$$

exists and is analytic for $\beta \in N := \{\beta \big| |\beta - \beta_0| \leq \delta\}$. E_0 is a nondegenerate eigenvalue of $T(\beta_0) \Rightarrow \dim P(\beta_0) = 1$. Then the last lemma implies that $\forall \beta \in N \dim P(\beta) = 1$. Thus, by Theorem 5, there is exactly one eigenvalue $E(\beta)$ of $T(\beta)$ with $|E(\beta) - E_0| < \epsilon$ when $\beta \in N$ and this eigenvalue is nondegenerate. Put $\Omega(\beta) := P(\beta)\Omega_0$, where Ω_0 is the unperturbated eigenvector. Then $\Omega(\beta)$ is an analytic eigenvector to $E(\beta)$ of $T(\beta)$ (see Remark 2(c)). Thus, $T(\beta)P(\beta)\Omega_0 = E(\beta)P(\beta)\Omega_0$. The analyticity of $E(\beta)$ for β near β_0 follows from the formula:

$$(E(\beta) - E_0 - \epsilon)^{-1} = \frac{(\Omega_0, (T(\beta) - E_0 - \epsilon)^{-1} P(\beta)\Omega_0)}{(\Omega_0, P(\beta)\Omega_0)}$$

Since $P(\beta)$ is analytic for $\beta \in N$ and $(\Omega_0, P(\beta_0)\Omega_0) = \|\Omega_0\|^2 \neq 0$, for β near β_0 the denominator $(\Omega_0, P(\beta)\Omega_0)$ is nonzero.
We obtain an analytic eigenvector by choosing $\Omega(\beta) := P(\beta)\Omega_0$ or

$$\Omega(\beta) := (\Omega_0, P(\beta)\Omega_0)^{-\frac{1}{2}} P(\beta)\Omega_0$$

in the real case.

$$\begin{aligned} \|\Omega(\beta)\|^2 &= (\Omega(\beta), \Omega(\beta)) = (\Omega_0, P(\beta)\Omega_0)^{-1}(P(\beta)\Omega_0, P(\beta)\Omega_0) \\ &= (\Omega_0, P(\beta)\Omega_0)^{-1}(\Omega_0, P(\beta)^2\Omega_0) = (\Omega_0, P(\beta)\Omega_0)^{-1}(\Omega_0, P(\beta)\Omega_0) = 1 \end{aligned}$$

If we consider that $T = T^* \Rightarrow P = P^*$. □

This would not be very useful if we did not have convenient criteria for $T(\beta)$ to be analytic. Fortunately there are some simple ones and we shall discuss one of them in detail.

Definition 5. Let R be a connected domain in $\mathbb{C}$ and let $T(\beta)$ be a closed operator with nonempty resolvent set for each $\beta \in R$. We say that $T(\beta)$ is an *analytic family of type (A)* if and only if
(i) The operator domain of $T(\beta)$ is some set D independent of β.
(ii) $\forall \psi \in D$, $T(\beta)\psi$ is a vector-valued analytic function of β.

Now we leave the general case of the problem and consider only the linear case $T(\beta) = H_0 + \beta V$. We first prove a lemma, which gives a convenient criterion for a family to be type (A).

Lemma 2. Let H_0 be a closed operator with nonempty resolvent set. Define $H_0+\beta V$ on $D(H_0) \cap D(V)$. Then $H_0 + \beta V$ is analytic family of type (A) near $\beta = 0$ if and only if:
(a) $D(H_0) \subset D(V)$
(b) For some $a, b \geq 0$ and $\forall \psi \in D(H_0)$,

$$\|V\psi\| \leq a\|H_0\psi\| + b\|\psi\|$$

(we say that V is H_0-bounded)

Proof. Suppose first that $H_0 + \beta V$ is an analytic family of type (A).
Then $D(H_0) = D(H_0 + \beta V) = D(H_0) \cap D(V)$ so (a) holds. $\mathcal{D} := (D(H_0), |||.|||)$, where $|||\psi||| := \|H_0\psi\| + \|\psi\|$ is a norm and since H_0 is closed, $\mathcal{D}$ is a Banach space. Fix $\beta > 0$ small so that β and $-\beta$ are both in the domain of analyticity. The operator $H_0 + \beta V : \mathcal{D} \to \mathcal{H}$ is everywhere defined and it is easy to verify that this operator is closed (it has a closed graph in $\mathcal{D} \times \mathcal{H}$ since the graph is closed in $\mathcal{H} \times \mathcal{H}$ with a weaker topology). Thus by the closed graph theorem,

$$\|(H_0 + \beta V)\psi\| \leq a_1|||\psi|||$$

and

$$\|(H_0 - \beta V)\psi\| \leq a_2|||\psi|||$$

where a_1 and a_2 are some positive constants.
Thus,

$$\|V\psi\| \leq \frac{1}{2\beta}[\|(H_0 + \beta V)\psi\| + \|(H_0 - \beta V)\psi\|] \leq \frac{a_1 + a_2}{2\beta}|||\psi|||$$

so that condition (b) holds.
Conversely, let (a) and (b) hold. Then, for $\psi \in D(H_0)$,

$$\|H_0\psi\| \leq \|(H_0 + \beta V)\psi\| + |\beta|\|V\psi\|$$

$$\leq \|(H_0+\beta V)\psi\| + |\beta|a\|H_0\psi\| + |\beta|b\|\psi\|$$

Thus, if $|\beta| < a^{-1}$, we have

$$\|H_0\psi\| \leq (1-|\beta|a)^{-1}\|(H_0+\beta V)\psi\| + (1-|\beta|a)^{-1}|\beta|b\|\psi\|$$

Let $\psi_n \in D(H_0)$, $\psi_n \to \psi$ in $\mathcal{H}$ and $(H_0+\beta V)\psi_n$ is Cauchy, then $H_0\psi_n$ is Cauchy by the above inequality. H_0 is closed on $D(H_0) \Rightarrow \psi \in D(H_0)$ and $H_0\psi_n \to H_0\psi$ in $\mathcal{H}$. If we consider:

$$\begin{aligned}\|(H_0+\beta V)(\psi_n-\psi)\| &\leq \|H_0(\psi_n-\psi)\| + |\beta|\|V(\psi_n-\psi)\| \\ &\leq (1+|\beta|a)\|H_0(\psi_n-\psi)\| + |\beta|b\|\psi_n-\psi\|\end{aligned}$$

we can see that $(H_0+\beta V)\psi_n \to (H_0+\beta V)\psi$ in $\mathcal{H}$ and therefore, $H_0+\beta V$ is closed on $D(H_0)$. That $(H_0+\beta V)\psi$ is analytic for $\psi \in D(H_0)$ is obvious. □

Theorem 8. Let $H_0+\beta V$ be an analytic family of type (A) in a region R and let $0 \in R$. Then $H_0+\beta V$ is an analytic family in the sense of Kato. In particular, if E_0 is an isolated nondegenerate eigenvalue of H_0, then there is a unique point $E(\beta) \in \sigma(H_0+\beta V)$ near E_0 when $|\beta|$ is small which is an isolated nondegenerate eigenvalue. Moreover, $E(\beta)$ is analytic near $\beta = 0$.

Proof. Since the analyticity is a local property, we first prove analyticity in the sense Kato near $\beta = 0$. Choose $\lambda \notin \sigma(H_0)$. Then $(H_0-\lambda)^{-1}$ and $H_0(H_0-\lambda)^{-1} = 1+\lambda(H_0-\lambda)^{-1}$ are bounded. Thus for any $\varphi \in \mathcal{H}$,

$$\begin{aligned}\|V(H_0-\lambda)^{-1}\varphi\| &\leq a\|H_0(H_0-\lambda)^{-1}\varphi\| + b\|(H_0-\lambda)^{-1}\varphi\| \\ &\leq (a\|H_0(H_0-\lambda)^{-1}\| + b\|(H_0-\lambda)^{-1}\|)\|\varphi\|\end{aligned}$$

Thus $V(H_0-\lambda)^{-1}$ is bounded; so for β small, operator $[1+\beta V(H_0-\lambda)^{-1}]^{-1}$ exists and is analytic in β (being given by a geometric series). Direct computation gives:

$$\begin{aligned}(H_0-\lambda)^{-1}[1+\beta V(H_0-\lambda)^{-1}]^{-1} &= [H_0-\lambda+\beta V(H_0-\lambda)^{-1}(H_0-\lambda)]^{-1} \\ &= (H_0+\beta V-\lambda)^{-1}\end{aligned}$$

So for β small, $\lambda \notin \sigma(H_0+\beta V)$ and $(H_0+\beta V-\lambda)^{-1}$ is analytic in β. This proves that $H_0+\beta V$ is an analytic family in the sense of Kato near $\beta = 0$. By writing $H_0+\beta V = (H_0+\beta_0 V)+(\beta-\beta_0)V$, we can similarly prove analyticity at $\beta = \beta_0$. Next statements follow directly from Kato-Rellich theorem. □

Finally, we show that one can obtain explicit lower bounds on the radius of convergence of the Taylor series:

Theorem 9. Suppose that $\|V\varphi\| \leq a\|H_0\varphi\| + b\|\varphi\|$, $\forall\varphi \in D(H_0) \subset D(V)$. Let H_0 be self-adjoint with an unperturbed isolated, nondegenerate eigenvalue E_0, and let $\epsilon = \frac{1}{2}\operatorname{dist}(E_0, \sigma(H_0) \setminus \{E_0\})$. Define

$$r(a,b,E_0,\epsilon) = [a+\epsilon^{-1}[b+a(|E_0|+\epsilon)]]^{-1}$$

Then the eigenvalue $E(\beta)$ of $H_0+\beta V$ near E_0 is analytic in the circle of radius $r(a,b,E_0,\epsilon)$.

Proof. Here i refer reader to the great Kato's book [1] (p.88-89, 379-381), where is shown that the eigenvalues $E(\beta)$ are analytic for such β that:

$$|\beta| < r_0 \equiv \min_{\zeta\in\Gamma}(a\|H_0R(\zeta,H_0)\| + b\|R(\zeta,H_0)\|)^{-1}$$

where, in our case, we can take $\Gamma = \{\zeta \in \mathbb{C}\big||E_0 - \zeta| = \epsilon\}$ and $R(\zeta, H_0) = (H_0 - \zeta)^{-1}$. If we consider that $\|R(\zeta,H_0)\| = (\text{dist}(\zeta,\sigma(H_0)))^{-1} = \epsilon^{-1}$, we have:

$$\begin{aligned}
r_0 &= \min_{\zeta\in\Gamma}(a\|H_0R(\zeta,H_0)\| + b\|R(\zeta,H_0)\|)^{-1}\\
&\geq \min_{\zeta\in\Gamma}(a(1+|\zeta|\|R(\zeta,H_0)\|) + b\|R(\zeta,H_0)\|)^{-1}\\
&= [a(1+\epsilon^{-1}(|E_0|+\epsilon)) + \epsilon^{-1}b]^{-1} = [a+\epsilon^{-1}[b+a(|E_0|+\epsilon)]]^{-1} = r(a,b,E_0,\epsilon)
\end{aligned}$$

□

2.2 Perturbation series

Consider the special case $H(\beta) = H_0 + \beta V$. Suppose that H_0 is self-adjoint and E_0 is a nondegenerate eigenvalue of H_0. From Kato-Rellich theorem we know that, for β small, $H_0 + \beta V$ has a unique eigenvalue $E(\beta)$ near E_0 and that $E(\beta)$ is analytic near $\beta = 0$. The coefficients of its Taylor series are called *Rayleigh-Schrödinger coefficients* and the Taylor series are called the *Rayleigh-Schrödinger series*. For sufficiently small ϵ (and β) $E(\beta)$ is the only eigenvalue of $H_0 + \beta V$ in the circle $\{E \in \mathbb{C}\big||E - E_0| < \epsilon\}$. As we already know,

$$P(\beta) = -\frac{1}{2i\pi}\oint_{|E-E_0|=\epsilon}(H_0+\beta V - E)^{-1}dE$$

is the projection onto the eigenvector with eigenvalue $E(\beta)$. Since $(H_0+\beta V-E)^{-1}$ is analytic in β near $\beta = 0$, $P(\beta)$ is analytic in β near $\beta = 0$. In particular, if Ω_0 is the unperturbated eigenvector, $P(\beta)\Omega_0 \neq 0$ for β small since $P(\beta)\Omega_0 \to \Omega_0 \neq 0$ as $\beta \to 0$. Since $P(\beta)\Omega_0$ is an unnormalized eigenvector for $H(\beta)$,

$$E(\beta) = \frac{(\Omega_0, H(\beta)P(\beta)\Omega_0)}{(\Omega_0, P(\beta)\Omega_0)} = E_0 + \beta\frac{(\Omega_0, VP(\beta)\Omega_0)}{(\Omega_0, P(\beta)\Omega_0)}$$

To find the Taylor series for $E(\beta)$, we need only find the Taylor series for $P(\beta)$. To do this, we need only find a Taylor series for $(H_0+\beta V-E)^{-1}$ and integrate it. But the Taylor series for $(H_0+\beta V-E)^{-1}$ is just a geometric series:

$$(H_0+\beta V-E)^{-1} = \sum_{n=0}^{\infty}(-1)^n\beta^n(H_0-E)^{-1}[V(H_0-E)^{-1}]^n$$

Thus, the Rayleigh-Schrödinger series for $E(\beta)$ is given by

$$E(\beta) = E_0 + \beta \frac{\sum_{n=0}^{\infty} a_n \beta^n}{\sum_{n=0}^{\infty} b_n \beta^n}$$

where

$$\begin{aligned} a_n &= \frac{(-1)^{n+1}}{2i\pi} \oint_{|E-E_0|=\epsilon} (\Omega_0, [V(H_0 - E)^{-1}]^{n+1}\Omega_0) dE \\ b_n &= \frac{(-1)^{n+1}}{2i\pi} \oint_{|E-E_0|=\epsilon} (\Omega_0, (H_0 - E)^{-1}[V(H_0 - E)^{-1}]^n \Omega_0) dE \end{aligned} \tag{2.2}$$

Let us compute $E(\beta)$ up to order β^4. Assume that there is an orthonormal basis of eigenvectors, $\Omega_0, \Omega_1, \dots$, with $H_0\Omega_i = E_i\Omega_i$ (it holds if for example H_0 has compact resolvent). Denote $V_{ij} := (\Omega_i, V\Omega_j)$. Then

$$\begin{aligned} b_0 &= -\frac{1}{2i\pi} \oint_{|E-E_0|=\epsilon} (\Omega_0, (H_0 - E)^{-1}\Omega_0) dE = -\frac{1}{2i\pi} \oint_{|E-E_0|=\epsilon} (E_0 - E)^{-1} dE = 1 \\ b_1 &= \frac{1}{2i\pi} \oint_{|E-E_0|=\epsilon} (\Omega_0, (H_0 - E)^{-1} V (H_0 - E)^{-1}\Omega_0) dE \\ &= \frac{1}{2i\pi} \oint_{|E-E_0|=\epsilon} ((H_0 - \bar{E})^{-1}\Omega_0, V(H_0 - E)^{-1}\Omega_0) dE \\ &= \frac{1}{2i\pi} \oint_{|E-E_0|=\epsilon} V_{00}(E_0 - E)^{-2} dE = 0 \\ b_2 &= -\frac{1}{2i\pi} \oint_{|E-E_0|=\epsilon} (\Omega_0, (H_0 - E)^{-1}[V(H_0 - E)^{-1}]^2\Omega_0) dE \\ &= -\frac{1}{2i\pi} \oint_{|E-E_0|=\epsilon} (E_0 - E)^{-2} (\Omega_0, V(H_0 - E)^{-1} V\Omega_0) dE \\ &= -\frac{1}{2i\pi} \oint_{|E-E_0|=\epsilon} (E_0 - E)^{-2} \sum_{j=0}^{\infty} (V^*\Omega_0, \Omega_j)(\Omega_j, (H_0 - E)^{-1} V\Omega_0) dE \\ &= -\frac{1}{2i\pi} \oint_{|E-E_0|=\epsilon} (E_0 - E)^{-2} \sum_{j=0}^{\infty} (E_j - E)^{-1} (\Omega_0, V\Omega_j)(\Omega_j, V\Omega_0) dE \end{aligned}$$

Since

$$\frac{1}{2i\pi} \oint_{|E-E_0|=\epsilon} (E_0 - E)^{-3} dE = 0$$

and (for $j \neq 0$)

$$\frac{1}{2i\pi} \oint_{|E-E_0|=\epsilon} (E_0 - E)^{-2} (E_j - E)^{-1} dE = (E_j - E_0)^{-2}$$

we finally get

$$b_2 = -\sum_{j\neq 0}(E_j - E_0)^{-2}V_{0j}V_{j0}$$

Similarly,

$$\begin{aligned}
b_3 &= \sum_{i\neq 0}\sum_{j\neq 0}(E_i - 2E_0 + E_j)(E_i - E_0)^{-2}(E_j - E_0)^{-2}V_{0i}V_{ij}V_{j0} \\
&\quad - 2\sum_{i\neq 0}(E_i - E_0)^{-3}V_{0i}V_{i0}V_{00} \\
a_0 &= V_{00} \\
a_1 &= -\sum_{i\neq 0}(E_i - E_0)^{-1}V_{0i}V_{i0} \\
a_2 &= \sum_{i\neq 0}\sum_{j\neq 0}(E_i - E_0)^{-1}(E_j - E_0)^{-1}V_{0i}V_{ij}V_{j0} - 2\sum_{i\neq 0}(E_i - E_0)^{-2}V_{0i}V_{i0}V_{00} \\
a_3 &= -\sum_{i\neq 0}\sum_{j\neq 0}\sum_{k\neq 0}(E_i - E_0)^{-1}(E_j - E_0)^{-1}(E_k - E_0)^{-1}V_{0i}V_{ij}V_{jk}V_{k0} \\
&\quad + 2\sum_{i\neq 0}\sum_{j\neq 0}(E_i - 2E_0 + E_j)(E_i - E_0)^{-2}(E_j - E_0)^{-2}V_{00}V_{0i}V_{ij}V_{j0} \\
&\quad + 2\sum_{i\neq 0}\sum_{j\neq 0}(E_i - E_0)^{-2}(E_j - E_0)^{-1}V_{0i}V_{i0}V_{0j}V_{j0} - 3\sum_{i\neq 0}(E_i - E_0)^{-3}V_{0i}V_{i0}V_{00}^2
\end{aligned}$$

Thus, if we write $E(\beta) = E_0 + \sum_{n=1}^{\infty}\alpha_n\beta^n$, we have computed:

$$\begin{aligned}
\alpha_1 &= b_0^{-1}a_0 = V_{00} \\
\alpha_2 &= b_0^{-1}(a_1 - b_1\alpha_1) = -\sum_{i\neq 0}(E_i - E_0)^{-1}V_{0i}V_{i0} \\
\alpha_3 &= b_0^{-1}(a_2 - b_1\alpha_2 - b_2\alpha_1) \\
&= \sum_{i\neq 0}\sum_{j\neq 0}(E_i - E_0)^{-1}(E_j - E_0)^{-1}V_{0i}V_{ij}V_{j0} - \sum_{i\neq 0}(E_i - E_0)^{-2}V_{0i}V_{i0}V_{00} \\
\alpha_4 &= b_0^{-1}(a_3 - b_1\alpha_3 - b_2\alpha_2 - b_3\alpha_1) \\
&= -\sum_{i\neq 0}\sum_{j\neq 0}\sum_{k\neq 0}(E_i - E_0)^{-1}(E_j - E_0)^{-1}(E_k - E_0)^{-1}V_{0i}V_{ij}V_{jk}V_{k0} \\
&\quad + \sum_{i\neq 0}\sum_{j\neq 0}(E_i - 2E_0 + E_j)(E_i - E_0)^{-2}(E_j - E_0)^{-2}V_{00}V_{0i}V_{ij}V_{j0} \\
&\quad + \sum_{i\neq 0}\sum_{j\neq 0}(E_i - E_0)^{-2}(E_j - E_0)^{-1}V_{0i}V_{i0}V_{0j}V_{j0} - \sum_{i\neq 0}(E_i - E_0)^{-3}V_{0i}V_{i0}V_{00}^2
\end{aligned}$$

Remark 3. The nth Rayleigh-Schrödinger coefficient α_n is

$$(-1)^{n+1}\sum_{i_1\neq 0, i_2\neq 0, \ldots, i_{n-1}\neq 0}\prod_{j=1}^{n-1}(E_{i_j} - E_0)^{-1}V_{0i_1}V_{i_1i_2}\ldots V_{i_{n-1}0}$$

2.3 A little more on perturbations

Definition 6. Let $H(\beta)$ be a closed operator in a Hilbert space $\mathcal{H}$ and $H(\beta)$ is holomorphic for β in a domain D of the complex plane symmetric with respect to the real axis, $H(\beta)$ is densely defined for each β and that $H(\beta)^* = H(\bar{\beta})$. Then we call $H(\beta)$ a *self-adjoint holomorphic family.*

Remark 4. It is clear that $H(\beta)$ is self-adjoint for each real $\beta \in D$.

Theorem 10. Let $H(\beta)$ be a self-adjoint holomorphic family of type (A) defined for β in a neighborhood of an interval I_0 of the real axis. Furthermore, let $H(\beta)$ have compact resolvent. Then there is a sequence of scalar-valued functions $\mu_n(\beta)$ and a sequence of vector-valued functions $\varphi_n(\beta)$, all holomorphic on I_0, such that for $\beta \in I_0$, the $\mu_n(\beta)$ represent all the repeated eigenvalues of $H(\beta)$ and the $\varphi_n(\beta)$ form a complete orthonormal family of the associated eigenvectors of $H(\beta)$.

The proof can be found in [1] chap. VII §3.

Chapter 3

Infinite-dimensional Jacobi matrix

3.1 Introduction and general results

Definition 7. A matrix $A \in \mathbb{C}^{d\times d}$ is called *Jacobi matrix* when A has the following tridiagonal form:

$$A = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & \dots & 0 & 0 & 0 \\ \gamma_1 & \alpha_2 & \beta_2 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \dots & \gamma_{d-2} & \alpha_{d-1} & \beta_{d-1} \\ 0 & 0 & 0 & \dots & 0 & \gamma_{d-1} & \alpha_d \end{pmatrix}$$

Consider first a semi-infinite Jacobi matrix $K \in \ell^2(\mathbb{N})$ in the form:

$$K = \begin{pmatrix} \lambda_1 & w_1 & 0 & 0 & \dots \\ \tilde{w_1} & \lambda_2 & w_2 & 0 & \dots \\ 0 & \tilde{w_2} & \lambda_3 & w_3 & \dots \\ 0 & 0 & \tilde{w_3} & \lambda_4 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

and assume that $0 < \lambda_1 \leqq \lambda_2 \leqq \dots$ and $\lim_{k\to\infty} \lambda_k = \infty$ and $\tilde{w_k} = \overline{w_k}$ for $\forall k$. Assume also that the sequence $\{w_k\}_{k=1}^{\infty}$ is bounded. K can be decomposed as

$$K = K_0 + W + W^*$$

where

$$K_0 = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & 0 & \dots \\ 0 & 0 & \lambda_3 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad W = \begin{pmatrix} 0 & w_1 & 0 & 0 & \dots \\ 0 & 0 & w_2 & 0 & \dots \\ 0 & 0 & 0 & w_3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Then the matrix K is Hermitian. W is bounded, i.e. $W \in \mathcal{B}(\ell^2(\mathbb{N}))$ and $\|W\| = \|WW^*\|^{1/2} = \sup\{|w_k| \big| k \in \mathbb{N}\}$. Since $\sigma_{ess}(K_0) = \emptyset$, K_0 has a compact resolvent and $(W+W^*)R_i(K_0)$ is compact too. Finally we can claim that $(W+W^*)$ is K_0-compact self-adjoint perturbation and use Weyl theorem. Thus K has discrete spectrum.
Denote by $\{\tilde{\lambda}_k\}_{k=1}^{\infty}$ the sequence of eigenvalues of K ordered increasingly ($\tilde{\lambda}_k \leq \tilde{\lambda}_{k+1}$). By the min-max principle,

$$\tilde{\lambda}_k = \sup_{dimV<k} \inf\{(\psi, K\psi) \big| \psi \in D(K) \cap V^{\perp},\ \|\psi\| = 1\}$$

(V is a subspace of ℓ^2).
Using the Schwarz inequality, considering $\|\psi\| = 1$, we have

$$|(\psi, K\psi) - (\psi, K_0\psi)| = |(\psi, (W + W^*)\psi)| \leq \|W + W^*\|$$

$$(\psi, K_0\psi) - \|W + W^*\| \leq (\psi, K\psi) \leq (\psi, K_0\psi) + \|W + W^*\|$$

and using the min-max principle we get

$$|\tilde{\lambda}_k - \lambda_k| \leq \|W + W^*\| \leq 2\|W\|$$

From now let us assume that the sequence $\{\lambda_k\}_{k=1}^{\infty}$ is strictly increasing and that $w_k = w \in \mathbb{R}$ is a constant sequence (then $\|W\| = |w|$). In this case let us write $\lambda_k(w)$ instead of $\tilde{\lambda}_k$. It is easy to see that $\lambda_k(0) = \lambda_k$ and by the regular perturbation theory it follows that $\{\lambda_k(w)\}_{k=1}^{\infty}$ is strictly incresing. Since

$$\begin{pmatrix} \lambda_1 & -w & 0 & 0 & \dots \\ -w & \lambda_2 & -w & 0 & \dots \\ 0 & -w & \lambda_3 & -w & \dots \\ 0 & 0 & -w & \lambda_4 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & -1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} \lambda_1 & w & 0 & 0 & \dots \\ w & \lambda_2 & w & 0 & \dots \\ 0 & w & \lambda_3 & w & \dots \\ 0 & 0 & w & \lambda_4 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & -1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

it is easy to see that $\lambda_k(w)$ is an even function.
From the min-max principle it follows that

$$|\lambda_k(w) - \lambda_k(z)| \leq 2|w - z|$$

so that all $\lambda_k(w)$ are continuous functions. In particular, for $\forall k$

$$|\lambda_k(w) - \lambda_k| \leq 2|w|$$

But one can claim much more. Since W is bounded, W is defined everywhere, thus, the domain $D(K) = D(K_0)$ is independent of w and, obviously, for any $\psi \in D(K_0)$, the vector-valued function $K\psi$ is holomorphic in w (even linear).Thus, the family $K = K(w)$ is holomorphic of type (A). Moreover, the family $K(w)$ is self-adjoint, i.e., $K(w)^* = K(\bar{w})$, and by the above discussion, the resolvent of $K(w)$ is compact. According to the theory introduced above, the functions $\lambda_k(w)$ are real-holomorphic everywhere on the real line.

3.2 Perturbation series of an eigenvalue

Now we will use the theory introduced in the previous chapter and compute first four terms of the Rayleigh-Schrödinger serie of an eigenvalue $\lambda_n(w)$. Denote $e_j = \{\delta_{ij}\}_{i=1}^{\infty}$, the family $\{e_1, e_2, \dots e_j, \dots\}$ is an orthonormal basis of ℓ^2 and $K_0 e_j = \lambda_j e_j$ for $\forall j \in \mathbb{N}$.

$$K = K_0 + W + W^* = K_0 + w(T + T^*) = K_0 + wV$$

where

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad V = T + T^*$$

First let us simplify the expression for V_{ij},

$$V_{ij} = (e_i, V_j e_j) = (e_i, e_{j+1} + e_{j-1}) = \delta_{i,j+1} + \delta_{i,j-1}$$

Then

$$\lambda_n(w) = \lambda_n + \alpha_1^{(n)} w + \alpha_2^{(n)} w^2 + \alpha_3^{(n)} w^3 + \alpha_4^{(n)} w^4 + o(w^4) \tag{3.1}$$

where

$$\alpha_1^{(n)} = V_{nn} = \delta_{n,n+1} + \delta_{n,n-1} = 0$$

$$\alpha_2^{(n)} = -\sum_{i=1, i\neq n}^{\infty} (\lambda_i - \lambda_n)^{-1} V_{ni} V_{in} = -\sum_{i=1, i\neq n}^{\infty} (\lambda_i - \lambda_n)^{-1} (\delta_{n,i+1} + \delta_{n,i-1})(\delta_{i,n+1} + \delta_{i,n-1})$$

$$= -\sum_{i=1, i\neq n}^{\infty} (\lambda_i - \lambda_n)^{-1} (\delta_{n,i+1} + \delta_{n,i-1}) = -\frac{1}{\lambda_{n-1} - \lambda_n} - \frac{1}{\lambda_{n+1} - \lambda_n}$$

$$\alpha_3^{(n)} = \sum_{i=1, i\neq n}^{\infty} \sum_{j=1, j\neq n}^{\infty} (\lambda_i - \lambda_n)^{-1} (\lambda_j - \lambda_n)^{-1} V_{ni} V_{ij} V_{jn} - \sum_{i=1, i\neq n}^{\infty} (\lambda_i - \lambda_n)^{-2} V_{ni} V_{in} V_{nn}$$

$$= \sum_{i=1, i\neq n}^{\infty} \sum_{j=1, j\neq n}^{\infty} (\lambda_i - \lambda_n)^{-1} (\lambda_j - \lambda_n)^{-1} (\delta_{n,i+1} + \delta_{n,i-1})(\delta_{i,j+1} + \delta_{i,j-1})(\delta_{j,n+1} + \delta_{j,n-1})$$

$$= \sum_{i=1, i\neq n}^{\infty} \sum_{j=1, j\neq n}^{\infty} (\lambda_i - \lambda_n)^{-1} (\lambda_j - \lambda_n)^{-1} (\delta_{n-1,i} + \delta_{n+1,i})(\delta_{i,n+2} + 2\delta_{i,n} + \delta_{i,n-2}) = 0$$

Similarly, we shall get

$$\alpha_4^{(n)} = -\frac{1}{(\lambda_{n-1} - \lambda_n)^2 (\lambda_{n-2} - \lambda_n)} - \frac{1}{(\lambda_{n+1} - \lambda_n)^2 (\lambda_{n+2} - \lambda_n)} + \left(\frac{1}{(\lambda_{n-1} - \lambda_n)^2} + \frac{1}{(\lambda_{n+1} - \lambda_n)^2} \right) \left(\frac{1}{(\lambda_{n-1} - \lambda_n)} + \frac{1}{(\lambda_{n+1} - \lambda_n)} \right)$$

Now we would like to find a domain where the expression (3.1) holds. To do this, we can simply use theorem 9. Since $\|(T + T^*)\varphi\| \leq 2\|\varphi\|$, $a = 0$ and $b = 2$ and $\epsilon = \frac{1}{2}\operatorname{dist}(\lambda_n, \sigma(K_0) \setminus \lambda_n) = \frac{1}{2}\min\{|\lambda_{n+1} - \lambda_n|, |\lambda_n - \lambda_{n-1}|\}$. Then we have

$$r = r_n = \frac{1}{4}\min\{|\lambda_{n+1} - \lambda_n|, |\lambda_n - \lambda_{n-1}|\}$$

and the expression (3.1) for $\lambda_n(w)$ holds at least in the circle of radius r.

Interesting case of the problem is when spectral gaps of K are shrinking when approaching infinity. As an example we can take $\lambda_n \sim n^\alpha$ where $0 < \alpha < 1$. Since

$$r = r_n = \frac{1}{4}(\lambda_{n+1} - \lambda_n) \sim \frac{\alpha}{4} n^{\alpha - 1} = \frac{\alpha}{4} \frac{1}{n^{1-\alpha}}$$

the radius of convergence r_n goes to zero when $n \to \infty$. Thus the regular perturbation theory works only locally. A description of global perturbations, i.e. perturbations of K_0 as a whole, need not be available, however, because of the shrinking gaps.

Chapter 4

Finite-dimensional Jacobi matrix

Now we will investigate some properties of a finite-dimensional Jacobi matrix. We are interested in a Jacobi matrix $K \in \mathbb{C}^{(2d+1)\times(2d+1)}$ in the form:

$$K := \begin{pmatrix} -\lambda_d^- & w & & & & & & & & \\ w & -\lambda_{d-1}^- & w & & & & & & & \\ & \ddots & \ddots & \ddots & & & & & & \\ & & w & -\lambda_1^- & w & & & & & \\ & & & w & 0 & w & & & & \\ & & & & w & \lambda_1^+ & w & & & \\ & & & & & \ddots & \ddots & \ddots & & \\ & & & & & & w & \lambda_{d-1}^+ & w \\ & & & & & & & w & \lambda_d^+ \end{pmatrix} \tag{4.1}$$

(all unspecified elements are zero)
Denote $\chi(z)$ the characteristic polynomial of the matrix K, that is

$$\chi(z) := \det(zI - K).$$

If we know an element of an eigenvector of K we can compute others recursively. It follows that the <u>spectrum of K is simple</u>.

4.1 Function $\mathfrak{F}(x)$

Definition 8. Define $\mathfrak{F} : D \to \mathbb{C}$,

$$\mathfrak{F}(x) := 1 + \sum_{m=1}^{\infty}(-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1}x_{k_1+1}x_{k_2}x_{k_2+1}\ldots x_{k_m}x_{k_m+1}$$

where

$$D = \left\{ x = \{x_k\}_{k=1}^{\infty} \middle| \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}$$

Remark 5. (a) Note that the summation indices satisfy $k_j \geq 2j - 1$.
(b) Obviously, if all but finitely many elements of x are zeroes then $\mathfrak{F}(x)$ reduces to a finite sum. For a finite number of variables we often will write $\mathfrak{F}(x_1, x_2, \ldots, x_k)$ instead of $\mathfrak{F}(x)$ where $x = (x_1, x_2, \ldots, x_k, 0, 0, 0, \ldots)$.

Definition 9. The operator T_1 defined on the space of all sequences indexed by $\mathbb{N}$ such that

$$T_1(\{x_k\}_{k=1}^{\infty}) := \{x_{k+1}\}_{k=1}^{\infty}$$

in called the operator of *truncation from the left*. Next, set $T_n := (T_1)^n$, $n = 0, 1, 2, \ldots$, hence

$$T_n(\{x_k\}_{k=1}^{\infty}) = \{x_{k+n}\}_{k=1}^{\infty}$$

In particular, T_0 is the identity.

Proposition 1. It holds

$$\mathfrak{F}(T_n x) - \mathfrak{F}(T_{n+1} x) + x_{n+1} x_{n+2} \mathfrak{F}(T_{n+2} x) = 0, \quad n = 0, 1, 2, \ldots \tag{4.2}$$

Proof. To verify this identity, note that after the substitution $x' = T_n x$ one can restrict oneself to the particular case $n = 0$. Consider that

$$\mathfrak{F}(T_n x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=n+1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \ldots x_{k_m} x_{k_m+1}$$

And so

$$\begin{aligned} \mathfrak{F}(x) - \mathfrak{F}(T_1 x) &= -x_1 x_2 + \sum_{m=2}^{\infty} (-1)^m \sum_{k_2=3}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_1 x_2 x_{k_2} x_{k_2+1} \ldots x_{k_m} x_{k_m+1} \\ &= -x_1 x_2 \mathfrak{F}(T_2 x) \end{aligned}$$

□

Proposition 2. It holds

$$\mathfrak{F}\left(\left\{\frac{z}{k+n}\right\}_{k=1}^{\infty}\right) = n! z^{-n} J_n(2z), \quad z \in \mathbb{C}, \; n = 0, 1, 2, \ldots \tag{4.3}$$

where $J_n(z)$ is the Bessel function of the first kind.

Proof. This can be verified with the aid of the identity:

$$\sum_{k=n}^{\infty} \frac{1}{k(k+1)\ldots(k+m)} = \frac{1}{mn(n+1)\ldots(n+m-1)}$$

whose derivation is immediate if we consider that the summands can be written as

$$\frac{1}{k(k+1)\ldots(k+m)} = \frac{1}{m}\left(\frac{1}{k(k+1)\ldots(k+m-1)} - \frac{1}{(k+1)\ldots(k+m)}\right)$$

By a (finite) mathematical induction in $j = 0, 1 \dots, m-1$, it is easy to show, with the aid of the above identity, that

$$\sum_{k_1=n+1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} \frac{1}{k_1(k_1+1)k_2(k_2+1)\dots k_m(k_m+1)}$$

$$= \frac{1}{j!} \sum_{k_1=n+1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_{m-j}=k_{m-j-1}+2}^{\infty} \frac{1}{k_1(k_1+1)\dots k_{m-j}(k_{m-j}+1)\dots(k_{m-j}+j+1)}$$

In particular, for $j = m-1$ we get

$$\sum_{k_1=n+1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} \frac{1}{k_1(k_1+1)k_2(k_2+1)\dots k_m(k_m+1)}$$
$$= \frac{1}{(m-1)!} \sum_{k_1=n+1}^{\infty} \frac{1}{k_1(k_1+1)\dots(k_1+m)} = \frac{1}{m!(n+1)\dots(n+m)}$$
$$= \frac{n!}{m!(n+m)!}$$

From the definition of J_n it follows that

$$\sum_{j=0}^{\infty} (-1)^j \frac{z^{2j}}{j!(n+j)!} = z^{-n} J_n(2z)$$

Finally, we have

$$\mathfrak{F}\left(\left\{\frac{z}{k+n}\right\}_{k=1}^{\infty}\right) =$$
$$= 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=n+1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} \frac{z^{2m}}{k_1(k_2+1)k_1(k_2+1)\dots k_m(k_m+1)}$$
$$= 1 + \sum_{m=1}^{\infty} (-1)^m z^{2m} \frac{n!}{m!(n+m)!} = n! z_{-n} J_n(2z)$$

□

Corollary 1. It holds

$$zJ_n(z) - 2(n+1)J_{n+1}(z) + zJ_{n+2}(z) = 0 \tag{4.4}$$

Proof. Take $x = \left\{\frac{z}{k}\right\}_{k=1}^{\infty}$, then $T_n(x) = \left\{\frac{z}{k+n}\right\}_{k=1}^{\infty}$ and use (4.2) and (4.3). □

4.2 Kernel of K in the symmetric case

The case, when elements λ_k^- in K are equal to λ_k^+ (denote $\lambda_k := \lambda_k^- = \lambda_k^+$), will be called the *symmetric case* (although the word *antisymmetric* would be better to use). Now, suppose the symmetric case.
In the symmetric case the characteristic polynomial is an odd function:
Let us define matrices $S, T \in \mathbb{R}^{(2d+1)\times(2d+1)}$

$$S := \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \quad T := \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Then $STKTS = -K$, $STITS = I$ and so

$$\begin{aligned} \chi(z) &= \det(zI - K) = \det(ST(zI-K)TS) = \det(zI + K) \\ &= \det(-((-z)I - K)) = -\det((-z)I - K) = -\chi(-z) \end{aligned}$$

Since the characteristic polynomial is an odd function in the symmetric case, 0 is always an eigenvalue. Lets see the eigenvector with eigenvalue 0. We denote $a = (\alpha_{-d}, \ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_d)$ the unknown eigenvector and we choose $\alpha_0 = 1$.

Proposition 3. It holds

$$\alpha_{-k} = (-1)^k \alpha_k, \quad k = 0, 1 \ldots, d \tag{4.5}$$

Proof. Since

$$Ka = 0$$

we have

$$\begin{aligned} w\alpha_{-k-1} - \lambda_k \alpha_{-k} + w\alpha_{-k+1} &= 0 \qquad k = 0, \ldots, d \quad (\alpha_{-d-1} := 0) \qquad (4.6) \\ w\alpha_{k-1} + \lambda_k \alpha_k + w\alpha_{k+1} &= 0 \qquad k = 0, \ldots, d \quad (\alpha_{d+1} := 0) \qquad (4.7) \end{aligned}$$

The statement will be proved by a (finite) mathematical induction in $k = 0, \ldots, d$.The statement is trivial for $k = 0$. For $k = 1$, (4.6) gives $\alpha_{-1} = -\alpha_1$.
The induction step $k \to k+1$: let the equation

$$\alpha_{-j} = (-1)^j \alpha_j$$

holds for all $j = 0, \ldots, k$. Using the induction presumption in (4.6) we get

$$\begin{aligned} w\alpha_{-(k+1)} - \lambda_k(-1)^k \alpha_k + w(-1)^{k-1}\alpha_{k-1} &= 0 \qquad |(-1)^{k+1} \\ w(-1)^{k+1}\alpha_{-(k+1)} + \lambda_k \alpha_k + w\alpha_{k-1} &= 0 \qquad (4.8) \end{aligned}$$

and by subtracting (4.8) from (4.7) we get

$$\alpha_{-(k+1)} = (-1)^{k+1}\alpha_{k+1}$$

□

Proposition 4. vector $a = (\alpha_{-d}, \ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_d)$, where $\alpha_{-k} = (-1)^k\alpha_k$ and

$$\alpha_k = (-1)^k \frac{w^k}{\prod_{j=1}^k \lambda_j} \mathfrak{F}(wT_k\kappa) \quad k = 0, 1, \ldots, d \tag{4.9}$$

with

$$\kappa = (\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_d}, 0, 0, 0, \ldots)$$

is an eigenvector with eigenvalue 0 in the symmetric case.

Proof. We have to verify that

$$w\alpha_{k-1} + \lambda_k\alpha_k + w\alpha_{k+1} = 0 \quad k = 1, \ldots, d.$$

Putting $\alpha_k = (-1)^k \frac{w^k}{\prod_{j=1}^k \lambda_j}\mathfrak{F}(wT_k\kappa)$ in the equation we have to show that

$$\frac{(-1)^{k-1}w^k}{\prod_{j=1}^{k-1}\lambda_j}\mathfrak{F}(wT_{k-1}\kappa) + \lambda_k \frac{(-1)^k w^k}{\prod_{j=1}^{k}\lambda_j}\mathfrak{F}(wT_k\kappa) + \frac{(-1)^k w^{k+2}}{\prod_{j=1}^{k+1}\lambda_j}\mathfrak{F}(wT_{k+1}\kappa) = 0.$$

Multiplying the equation by $(-1)^{k+1}w^{-k}\prod_{j=1}^{k-1}\lambda_j$ we get

$$\mathfrak{F}(wT_{k-1}\kappa) - \mathfrak{F}(wT_k\kappa) + \frac{w}{\lambda_k}\frac{w}{\lambda_{k+1}}\mathfrak{F}(wT_{k+1}\kappa) = 0$$

and from (4.2) it follows that the last equation holds. □

4.3 The infinite linear and symmetric case

Now let us consider the infinite case again. Considering the limit $d \to \infty$ in (4.1) we arrive at a Jacobi matrix $K \in \ell^2(\mathbb{Z})$ infinite in both directions. Furthermore let us suppose the linear and symmetric case, i.e., the diagonal entries of K depend linearly on the index, $\lambda_k^- = \lambda_k^+ = k\Delta$ (Δ is a constant). The entries just above and below the diagonal are all equal to a $w > 0$. Let $\mathcal{T}$ be the shift operator in $\ell^2(\mathbb{Z})$,

$$(\mathcal{T}\psi)_k = \psi_{k+1} \quad \psi \in \ell^2(\mathbb{Z})$$

it is clear that $\mathcal{T}$ is unitary. Then

$$K = \Lambda + W + W^*$$

where

$$(\Lambda\psi)_k = k\Delta\psi_k, \qquad W = w\mathcal{T}$$

It can be easily verified that

$$\Lambda\mathcal{T} = \mathcal{T}\Lambda - \Delta\mathcal{T}, \qquad K\mathcal{T} = \mathcal{T}K - \Delta\mathcal{T}$$

and similarly

$$K\mathcal{T}^{-1} = \mathcal{T}^{-1}K + \Delta\mathcal{T}^{-1} \qquad (\text{since } \mathcal{T}^{-1} = \mathcal{T}^*)$$

Consequently, if ψ is an eigenvector of K, $K\psi = \xi\psi$, then

$$K\mathcal{T}\psi = \mathcal{T}K\psi - \Delta\mathcal{T}\psi = (\xi - \Delta)\psi$$

thus, $\mathcal{T}\psi$ is also an eigenvector with a shifted eigenvalue.
Let us now investigate a kernel of K. For $d < \infty$ we already have shown the eigenvector with eigenvalue 0 in Proposition 4. Sending d to infinity in (4.9) we obtain a vector $\psi_0 \in Ker(K) \subset \ell^2(\mathbb{Z})$,

$$(\psi_0)_k = \alpha_k = (-1)^k J_k\left(2\frac{w}{\Delta}\right), \quad k \in \mathbb{Z} \tag{4.10}$$

Verification: verification of the fact that $\psi_0 \in Ker(K)$ can be done by the same way as in the proof of Proposition 4.9. (4.3) implies

$$\begin{aligned}(\psi_0)_k &= (-1)^k \frac{w^k}{\prod_{j=1}^k j\Delta}\mathfrak{F}\left(wT_k\left\{\frac{1}{j\Delta}\right\}_{j=1}^{\infty}\right) = (-1)^k\frac{w^k}{\Delta^k k!}k!\left(\frac{\Delta}{w}\right)^k J_k\left(2\frac{w}{\Delta}\right)\\ &= (-1)^k J_k\left(2\frac{w}{\Delta}\right)\end{aligned}$$

Since $J_{-k}(z) = J_k(-z) = (-1)^k J_k(z)$, the identity $(\psi_0)_{-k} = (-1)^k(\psi_0)_k$ holds.
Next set $\psi_l := \mathcal{T}^{-l}\psi_0$, $l \in \mathbb{Z}$. Then

$$(\psi_l)_k = \alpha_{k-l}$$

and

$$K\psi_l = K\mathcal{T}^{-l}\psi_0 = (\mathcal{T}^{-l}K + l\Delta\mathcal{T}^{-l})\psi_0 = l\Delta\psi_l$$

Therefore ψ_l is an eigenvector of K with the eigenvalue $l\Delta$ and since the identity

$$\sum_{k=-\infty}^{\infty} J_k(z)J_{k-l}(z) = \delta_{0l}$$

holds, the eigenvalue ψ_l is already normalized.

From the regular perturbation theory it follows that the eigenvalues of K are simple for sufficiently small w. By the above computation, the eigenvalues do not depend on w. Hence the eigenvalues of K are all simple for any w (considering $K = K_0 + w(W + W^*) = (K_0 + w_0(W + W^*)) + (w - w_0)(W + W^*)$) and

$$\sigma(K) = \sigma_p(K) = \Delta\mathbb{Z}$$

4.4 The characteristic function in the symmetric case

Definition 10. Let $f_1(z), \ldots, f_d(z)$ be a sequence of complex functions of one complex variable z. Define $\mathcal{D}(f_1(z), \ldots, f_d(z))$ as the determinant of a Jacobi matrix in the form:

$$\begin{pmatrix}
-f_d(-z) & w & & & & & & & \\
w & -f_{d-1}(-z) & w & & & & & & \\
 & \ddots & \ddots & \ddots & & & & & \\
 & & w & -f_1(-z) & w & & & & \\
 & & & w & -z & w & & & \\
 & & & & w & f_1(z) & w & & \\
 & & & & & \ddots & \ddots & \ddots & \\
 & & & & & & w & f_{d-1}(z) & w \\
 & & & & & & & w & f_d(z)
\end{pmatrix}$$

Remark 6. Note that for $f_k(z) = \lambda_k - z$, $\mathcal{D}(f_1(z), \ldots, f_d(z)) = \det(K - zI)$ is the characteristic polynomial in the symmetric case.

Definition 11. Define recursively a sequence of functions $\mathcal{N}_k(y_1, \ldots, y_k)$, $k = 0, 1, 2 \ldots$. The k^{th} function depends on k variables, $\mathcal{N}_0$ is a constant.

$$\begin{aligned}
\mathcal{N}_0 &= 1 \\
\mathcal{N}_1(y_1) &= y_1 \\
\mathcal{N}_{k+1}(y_1, y_2, \ldots, y_{k+1}) &= y_1 \mathcal{N}_k(y_2, \ldots, y_{k+1}) - w^2 \mathcal{N}_{k-1}(y_3, \ldots, y_{k+1}) \quad \text{for } k \geq 1
\end{aligned}$$

Lemma 3. For all $n \in \mathbb{N}$, it holds

$$\mathcal{N}_n(f_1, \ldots, f_n) = f_n \mathcal{N}_{n-1}\left(f_1, \ldots, f_{n-2}, f_{n-1} - \frac{w^2}{f_n}\right)$$

Proof. For $n = 1$ the statement is clear. Suppose that the equation holds for all integers smaller or equal to n. Then

$$\mathcal{N}_{n+1}(f_1, \ldots, f_{n+1}) = f_1 \mathcal{N}_n(f_2, \ldots, f_{n+1}) - w^2 \mathcal{N}_{n-1}(f_3, \ldots, f_{n+1})$$

by using the induction presumption we get

$$\begin{aligned}
&= f_1 f_{n+1} \mathcal{N}_{n-1}\left(f_2, \ldots, f_n - \frac{w^2}{f_{n+1}}\right) - w^2 f_{n+1} \mathcal{N}_{n-2}\left(f_3, \ldots, f_n - \frac{w^2}{f_{n+1}}\right) \\
&= f_{n+1} \mathcal{N}_n\left(f_1, \ldots, f_{n-1}, f_n - \frac{w^2}{f_{n+1}}\right)
\end{aligned}$$

□

Lemma 4. Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{n\times m}$, $A \in \mathbb{D}^{m\times m}$ and let A^{-1} exists. Then

$$\det \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \det(A)\det(D - BA^{-1}C)$$

Proof. Since

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} I & C \\ BA^{-1} & D \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

and

$$\begin{pmatrix} I & 0 \\ BA^{-1} & I \end{pmatrix}\begin{pmatrix} I & C \\ BA^{-1} & D \end{pmatrix} = \begin{pmatrix} I & C \\ 0 & D - BA^{-1}C \end{pmatrix}$$

we have

$$\begin{aligned} \det \begin{pmatrix} A & C \\ B & D \end{pmatrix} &= \det \begin{pmatrix} I & C \\ BA^{-1} & D \end{pmatrix} \det(A) \\ &= \det(D - BA^{-1}C)\det(A) \end{aligned}$$

□

Proposition 5. It holds

$$\begin{aligned} \mathcal{D}(f_1(z), \ldots, f_d(z)) = (-1)^{d+1}\big[& z\mathcal{N}_d(f_1(z), \ldots, f_d(z))\mathcal{N}_d(f_1(-z), \ldots, f_d(-z)) \\ & - w^2\mathcal{N}_d(f_1(z), \ldots, f_d(z))\mathcal{N}_{d-1}(f_2(-z), \ldots, f_d(-z)) \\ & + w^2\mathcal{N}_{d-1}(f_2(z), \ldots, f_d(z))\mathcal{N}_d(f_1(-z), \ldots, f_d(-z))\big] \end{aligned}$$

Proof. The statement will be proved by a mathematical induction in $d = 1, 2, \ldots$.Let $d = 1$, then

$$\begin{aligned} \mathcal{D}(f_1(z)) = \det \begin{pmatrix} -f_1(-z) & w & 0 \\ w & -z & w \\ 0 & w & f_1(z) \end{pmatrix} &= zf_1(z)f_1(-z) - w^2 f_1(z) + w^2 f_1(-z) \\ = z\mathcal{N}_1(f_1(z))\mathcal{N}_1(f_1(-z)) - w^2\mathcal{N}_1(f_1(z))\mathcal{N}_0 &+ w^2\mathcal{N}_0\mathcal{N}_1(f_1(-z)) \end{aligned}$$

Now let $d \geq 2$.

$$\mathcal{D}(f_1(z), \ldots, f_d(z)) = \det \begin{pmatrix} -f_d(-z) & w & & & & & \\ w & -f_{d-1}(-z) & w & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & w & -z & w & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & w & f_{d-1}(z) & w \\ & & & & & w & f_d(z) \end{pmatrix}$$

Shifting the last column of the matrix to the place of the first column:

$$= \det \begin{pmatrix} 0 & -f_d(-z) & w & \dots & 0 \\ 0 & w & -f_{d-1}(-z) & w \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ w & 0 & \dots & w & f_{d-1}(z) \\ f_d(z) & 0 & \dots & & w \end{pmatrix}$$

and similarly by shifting the last row to the place of the first one, we get

$$= \det \begin{pmatrix} f_d(z) & 0 & 0 & 0 & \dots & 0 & w \\ 0 & -f_d(-z) & w & 0 & \dots & 0 & 0 \\ 0 & w & -f_{d-1}(-z) & w & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & w & f_{d-2}(z) & w \\ w & 0 & 0 & \dots & & w & f_{d-1}(z) \end{pmatrix}$$

Now, Lemma 4, where

$$A = \begin{pmatrix} f_d(z) & 0 \\ 0 & -f_d(-z) \end{pmatrix}, \qquad D = \begin{pmatrix} -f_{d-1}(-z) & w & & & \\ w & \ddots & & & \\ & \ddots & \ddots & \ddots & \\ & & & \ddots & w \\ & & & w & f_{d-1}(z) \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & \dots & 0 & w \\ w & 0 & \dots & 0 & 0 \end{pmatrix} = B^T,$$

follows

$$\mathcal{D}(f_1(z), \dots, f_d(z)) = \det(A)\det(D - BA^{-1}C)$$

$$= -f_d(z)f_d(-z)\det \begin{pmatrix} -f_{d-1}(-z) + \frac{w^2}{f_d(-z)} & w & & & \\ w & -f_{d-2}(-z) & & & \\ & \ddots & \ddots & \ddots & \\ & & & f_{d-2}(z) & w \\ & & & w & f_{d-1}(z) - \frac{w^2}{f_d(z)} \end{pmatrix}$$

We arrive at the expression:

$$\mathcal{D}(f_1(z), \dots, f_d(z)) = -f_d(z)f_d(-z)\mathcal{D}(f_1(z), \dots, f_{d-2}(z), f_{d-1}(z) - \frac{w^2}{f_d(z)})$$

using the induction presumption:

$$\begin{aligned}
&\mathcal{D}(f_1(z),\dots,f_d(z)) = -f_d(z)f_d(-z)(-1)^d \\
&\quad\Big[z\mathcal{N}_{d-1}(f_1(z),\dots,f_{d-1}(z)-\frac{w^2}{f_d(z)})\mathcal{N}_{d-1}(f_1(-z),\dots,f_{d-1}(-z)-\frac{w^2}{f_d(-z)}) \\
&\quad - w^2\mathcal{N}_{d-1}(f_1(z),\dots,f_{d-1}(z)-\frac{w^2}{f_d(z)})\mathcal{N}_{d-2}(f_2(-z),\dots,f_{d-1}(-z)-\frac{w^2}{f_d(-z)}) \\
&\quad + w^2\mathcal{N}_{d-2}(f_2(z),\dots,f_{d-1}(z)-\frac{w^2}{f_d(z)})\mathcal{N}_{d-1}(f_1(-z),\dots,f_{d-1}(-z)-\frac{w^2}{f_d(-z)})\Big]
\end{aligned}$$

Using Lemma 3 we finally arrive at the expression:

$$\begin{aligned}
\mathcal{D}(f_1(z),\dots,f_d(z)) = (-1)^{d+1}\Big[&z\mathcal{N}_d(f_1(z),\dots,f_d(z))\mathcal{N}_d(f_1(-z),\dots,f_d(-z)) \\
&- w^2\mathcal{N}_d(f_1(z),\dots,f_d(z))\mathcal{N}_{d-1}(f_2(-z),\dots,f_d(-z)) \\
&+ w^2\mathcal{N}_{d-1}(f_2(z),\dots,f_d(z))\mathcal{N}_d(f_1(-z),\dots,f_d(-z))\Big]
\end{aligned}$$

□

Proposition 6. It holds

$$\mathcal{N}_d(f_1,\dots,f_d) = f_1\dots f_d\mathfrak{F}\left(\frac{w}{f_1},\dots,\frac{w}{f_d}\right)$$

Proof. We must verify whether the function $f_1\dots f_d\mathfrak{F}\left(\frac{w}{f_1},\dots,\frac{w}{f_d}\right)$ satisfies the recursive relation in the definition of $\mathcal{N}$. It is easy to see that

$$\mathcal{N}_0 = \mathfrak{F}(0) = 1$$

and

$$\mathcal{N}_1(f_1) = f_1\mathfrak{F}\left(\frac{w}{f_1}\right) = f_1$$

Next for $d \geq 2$ we should show that

$$\mathcal{N}_{d+1}(f_1,\dots,f_{d+1}) = f_1\mathcal{N}_d(f_2,\dots,f_{d+1}) - w^2\mathcal{N}_{d-1}(f_3,\dots,f_{d+1})$$

by putting the function $f_1\dots f_d\mathfrak{F}\left(\frac{w}{f_1},\dots,\frac{w}{f_d}\right)$ in we get

$$\mathfrak{F}\left(\frac{w}{f_1},\dots,\frac{w}{f_{d+1}}\right) = \mathfrak{F}\left(\frac{w}{f_2},\dots,\frac{w}{f_{d+1}}\right) - \frac{w}{f_1}\frac{w}{f_2}\mathfrak{F}\left(\frac{w}{f_3},\dots,\frac{w}{f_{d+1}}\right)$$

The last equation holds due to (4.2). □

Corollary 2.

$$\begin{aligned}\mathcal{D}_d(f_1(z),\dots,f_d(z)) = (-1)^{d+1}&\left(\prod_{k=1}^{d} f_k(z)f_k(-z)\right)\\ &\left[z\mathfrak{F}\left(\frac{w}{f_1(z)},\dots,\frac{w}{f_d(z)}\right)\mathfrak{F}\left(\frac{w}{f_1(-z)},\dots,\frac{w}{f_d(-z)}\right)\right.\\ &-\frac{w^2}{f_1(-z)}\mathfrak{F}\left(\frac{w}{f_1(z)},\dots,\frac{w}{f_d(z)}\right)\mathfrak{F}\left(\frac{w}{f_2(-z)},\dots,\frac{w}{f_d(-z)}\right)\\ &\left.+\frac{w^2}{f_1(z)}\mathfrak{F}\left(\frac{w}{f_2(z)},\dots,\frac{w}{f_d(z)}\right)\mathfrak{F}\left(\frac{w}{f_1(-z)},\dots,\frac{w}{f_d(-z)}\right)\right]\end{aligned}$$

Let us denote

$$K_S := \begin{pmatrix} -\lambda_d & w & 0 & & & & & & \\ w & -\lambda_{d-1} & w & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & w & -\lambda_1 & w & & & & \\ & & & w & 0 & w & & & \\ & & & & w & \lambda_1 & w & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & w & \lambda_{d-1} & w \\ & & & & & & & w & \lambda_d \end{pmatrix}$$

and set

$$\chi_S(z) := \det(K_S - zI)$$

Proposition 7. It holds

$$\begin{aligned}(-1)^{d+1}\frac{1}{z}\chi_S(z) = &\left(\prod_{k=1}^{d}(\lambda_k^2-z^2)\right)\mathfrak{F}\left(\frac{w}{\lambda_1-z},\dots,\frac{w}{\lambda_d-z}\right)\mathfrak{F}\left(\frac{w}{\lambda_1+z},\dots,\frac{w}{\lambda_d+z}\right)\\ +2\sum_{j=1}^{d}w^{2j}&\left(\prod_{k=j+1}^{d}(\lambda_k^2-z^2)\right)\mathfrak{F}\left(\frac{w}{\lambda_{j+1}-z},\dots,\frac{w}{\lambda_d-z}\right)\mathfrak{F}\left(\frac{w}{\lambda_{j+1}+z},\dots,\frac{w}{\lambda_d+z}\right)\end{aligned}$$

Proof. $\chi_S(z) = \mathcal{D}(f_1(z),\dots,f_d(z))$ for $f_k(z) = \lambda_k - z$. If we use Corollary 2, we can see that to show this proposition it is enough to show that:

$$\begin{aligned}2\sum_{j=0}^{d}w^{2j}&\left(\prod_{k=1}^{j}\frac{1}{\lambda_k^2-z^2}\right)\mathfrak{F}\left(\frac{w}{\lambda_{j+1}-z},\dots,\frac{w}{\lambda_d-z}\right)\mathfrak{F}\left(\frac{w}{\lambda_{j+1}+z},\dots,\frac{w}{\lambda_d+z}\right)\\ &=\frac{w^2}{z(\lambda_1-z)}\mathfrak{F}\left(\frac{w}{\lambda_2-z},\dots,\frac{w}{\lambda_d-z}\right)\mathfrak{F}\left(\frac{w}{\lambda_1+z},\dots,\frac{w}{\lambda_d+z}\right)\\ &-\frac{w^2}{z(\lambda_1+z)}\mathfrak{F}\left(\frac{w}{\lambda_1-z},\dots,\frac{w}{\lambda_d-z}\right)\mathfrak{F}\left(\frac{w}{\lambda_2+z},\dots,\frac{w}{\lambda_d+z}\right)\end{aligned}\tag{4.11}$$

Let us denote

$$A_j^+ := \mathfrak{F}\left(\frac{w}{\lambda_j + z}, \ldots, \frac{w}{\lambda_d + z}\right), \quad A_j^- := \mathfrak{F}\left(\frac{w}{\lambda_j - z}, \ldots, \frac{w}{\lambda_d - z}\right)$$

$$A_j := A_j^- A_j^+ = A_j^+ A_j^-, \quad a_j^+ := \frac{w}{\lambda_j + z}, \quad a_j^- := \frac{w}{\lambda_j - z}, \quad a_j := a_j^+ a_j^-$$

Note that

$$A_d^\pm = A_{d+1}^\pm = 1, \quad a_j = \frac{w^2}{\lambda_j^2 - z^2}$$

and from identity (4.2) it follows

$$A_j^\pm = A_{j+1}^\pm - a_j^\pm a_{j+1}^\pm A_{j+2}^\pm \qquad \text{for } j = 1, \ldots, d-1$$

Then

$$\begin{aligned} A_1^- A_2^+ &= A_2 - a_1^- a_2^- A_3^- A_2^+ = A_2 - a_1^- a_2^- A_3 + a_1^- a_2 a_3^+ A_3^- A_4^+ = \cdots = \\ &= A_2 - a_1^- a_2^- A_3 + a_1^- a_2 a_3^+ A_4 + \cdots + (-1)^{d-1} a_1^- a_2 \ldots a_{d-2}^\pm A_{d-1} \\ &+ (-1)^d a_1^- a_2 \ldots a_{d-2} a_{d-1}^\mp A_{d-1}^\pm A_d^\mp \end{aligned}$$

Note that $A_{d-1}^\pm A_d^\mp = 1 = A_{d-1}$, then we can write:

$$a_1^+ A_1^- A_2^+ = \sum_{k=2}^{d+1} a_1 a_2 \ldots a_{k-1}^{(-1)^k} A_k \tag{4.12}$$

where the exponent in $a_{k-1}^{(-1)^k}$ has only symbolic meaning ($a_{k-1}^{(-1)^k} = a_{k-1}^+$ for even k and $a_{k-1}^{(-1)^k} = a_{k-1}^-$ for odd k). Similarly we can get

$$a_1^- A_2^- A_1^+ = \sum_{k=2}^{d+1} a_1 a_2 \ldots a_{k-1}^{(-1)^{k-1}} A_k \tag{4.13}$$

Next

$$\begin{aligned} a_{k-1}^{(-1)^k} - a_{k-1}^{(-1)^{k-1}} &= \frac{w}{\lambda_{k-1} + (-1)^k z} - \frac{w}{\lambda_{k-1} + (-1)^{k-1} z} = 2(-1)^{k-1} z \frac{w}{\lambda_{k-1} - z^2} \\ &= 2(-1)^{k-1} \frac{z}{w} a_{k-1} \end{aligned} \tag{4.14}$$

Finally, we can verify the above identity (4.11) with the aid of (4.12),(4.13) and (4.14).

$$\begin{aligned} RHS &= \frac{w^2}{z(\lambda_1 - z)} A_2^- A_1^+ - \frac{w^2}{z(\lambda_1 + z)} A_1^- A_2^+ = \frac{w}{z}(a_1^- A_2^- A_1^+ - a_1^+ A_1^- A_2^+) \\ &= \frac{w}{z}\left(\sum_{k=2}^{d+1} a_1 a_2 \ldots a_{k-1}^{(-1)^{k-1}} A_k - \sum_{k=2}^{d+1} a_1 a_2 \ldots a_{k-1}^{(-1)^k} A_k\right) \\ &= 2\sum_{k=2}^{d+1} a_1 a_2 \ldots a_{k-1} A_k = 2\sum_{k=1}^{d} a_1 a_2 \ldots a_k A_{k+1} = LHS \end{aligned}$$

□

Conclusion

In this paper we dealt with the description of the spectrum of an operator with the Jacobi matrix. With the aid of some results from functional analysis we presented some basic properties of the spectrum of the Jacobi matrix in the whole. Next we introduced the perturbation theory and then we applied it to these operators. We pointed out some problems, which appear while trying to describe the spectrum with the aid of the perturbation theory, first of all this is the possibility of only local description of the spectrum. In the end we also presented some features of finite-dimensional Jacobi matrices.

The aim of this paper was to describe the basic properties of the operator with the Jacobi matrix and its spectrum, saying it informally: "to state about the spectrum what can be stated". Partly it is also a concept of my future work, in which I'm going to try to derive a global description of the spectrum of the operator with the Jacobi matrix, i.e., the description of the spectrum of the operator as a whole. In this afford I will use also the knowledge of the spectrum of finite-dimensional Jacobi matrices.

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Prohlášení

Prohlašuji, že jsem svou bakalářskou práci vypracoval samostatně a použil jsem pouze podklady uvedené v přiloženém seznamu.

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Praha, July 21, 2008

František Štampach