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Fakulta jaderná a fyzikálně inženýrská

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Periodické magnetické hamiltoniány v Lobačevského rovině

Periodic magnetic Hamiltonians on the Lobachevsky plane

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Abstrakt: V této práci je analyzován periodický magnetický hamiltonián on Lobačevského rovině s nulovým skalárním potenciálem. Počas jeho analýzy jsou představené základy teorie direktního integrálu a vystrojených prostorů. Analýza hamiltoniánu vede ke spektrální analýze obyčejného diferenciálního operátoru s potenciálem Morseova typu, která je podrobně provedená. Pro tenhle důvod je zde prezentována modifikace teorému o úplném systému zobecněných vlastních vektorů daného obyčejného diferenciálního operátoru. Práce končí s jednoduchým příkladem rozkladu hamiltoniánu užitím direktních integrálů.

Klíčová slova: Lobačevského rovina, direktní integrál, Rieszova projekce, magnetický potenciál, zobecněné funkce

Title: **Periodic magnetic Hamiltonians on the Lobachevsky plane**

Author: Juraj Pohanka

Abstract: In this work a periodic magnetic Hamiltonian on the Lobachevsky plane with a zero scalar potential is analyzed. Through its analysis, the basics of theory of direct integrals and rigged Hilbert spaces are presented. The analysis of the Hamiltonian leads to the spectral analysis of ordinary differential operator with a Morse-type potential, which is intensively carried out. For this reason, a modification of a theorem regarding the complete system of generalized eigenvectors of the differential operator is presented. The work finishes with a simple example of a decomposition of the Hamiltonian using direct integrals.

Key words: Lobachevsky plane, direct integrl, Riesz projection, magnetic potential, generalized functions

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List of notations

Symbol	Description
A°	the interior of a set A
$\overline{A}$	the closure of a set A
$\mathbb{N}$	the set of natural numbers, $\{1, 2, 3, ...\}$
$\mathbb{N}_0$	the set $\mathbb{N} \cup \{0\}$
$\mathbb{Z}$	the set of integer numbers, $\{-2, -1, 0, 1, 2, ...\}$
$\mathbb{Z}^-$	the set of negative integers
$\mathbb{Z}_0^-$	the set $\mathbb{Z}^- \cup \{0\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers, $\{x \mid x > 0\}$
$\mathbb{R}_0^+$	the set $\mathbb{R}_+ \cup \{0\}$
$\mathbb{R}^-$	the set of negative real numbers, $\{x \mid x < 0\}$
$\mathbb{R}_0^-$	the set $\mathbb{R}_- \cup \{0\}$
$\mathbb{C}$	the set of complex numbers
i	the imaginary unit $i = \sqrt{-1}$
(a, b)	open interval $\{x; a < x < b\}$
$[a, b)$	semi-closed interval $\{x; a \leq x < b\}$
$[a, b]$	closed interval $\{x; a \leq x \leq b\}$
X'	the space of all bounded functionals on the vector space X, the dual space to X,
$\mathcal{L}(X, Y)$	the space of all linear mappings from X to Y, where X, Y are vector spaces
$\mathcal{B}(X, Y)$	the space of all continous linear mappings from X to Y, where X, Y are vector spaces
$\mathcal{L}(X)$	the space of all linear operators on X where X is a vector space
$\mathcal{B}(X)$	the space of all continous linear operators on X where X is a vector space
$\langle ., . \rangle$	scalar product(with antilinearity in the first and linearity in the second argument)
$\lVert . \rVert_2$	the Hilbert-Schmidt norm
$span(M)$	the linear hull of the set M
A^*	the adjoint operator of the operator A
$\mathcal{L}_{SA}(\mathcal{H})$	the space of all self-adjoint operators on the Hilbert space $\mathcal{H}$
$\mu - a.e$	stand for almost everywhere with respect to the measure μ

Symbol	Description
$L^p(M, d\mu(x))$	the Banacha space of μ-measurable functions on M with the L^p-norm $\|f\|_{L^p} = (\int_M; f(x)\|^p d\mu(x))^{1/p}$
$L^p(M, dx)$	the L^p space with the standard Lebesque measure
$L^p_{loc}(M, d\mu)$	the space of functions on M that are locally L^p-integrable
$AC(M)$	the space of absolutely continous functions on the set M
$AC_0(M)$	the space of absolutely continous functions on the set M with compact support in M
$C_0^\infty(M)$	the space of smooth functions on the set M with compact support in M
θ	the null vector in a vector space V(with the exception of Chapter 4)
Θ	the null operator on a vector space V
$\{\theta\}$	the null vector space V
R_T	the resolvent of the operator T at the point $z \in \mathbb{C}$, $R_T(z) = (T - zI)^{-1}$
$Dom(H)$	domain of the linear operator H
$Ran(H)$	range of the linear operator H
$\sigma(H)$	the spectrum of the linear operator H
$\rho(H)$	the resolvent set of the linear operator H
$N_+(H)$	number of positive eigenvalues of the linear operator H
$N_-(H)$	number of negative eigenvalues of the linear operator H
∂_i	partial derivative with respect to the i-th variable
Δ_B	Bochner Laplacian
Δ_{LB}	Laplace-Beltrami operator
H_L	Hamiltonian operator in the Lobachevski plane
H_{L0}	Hamiltonian operator in the Lobachevski plane without the potential
$\mathbb{H}$	the upper half-plane, $\{z \in \mathbb{C}; \Im z > 0$
$M_{\mathbb{H}}$	the Lobachevsky plane
$\int_M^\oplus$	the direct integral of vector spaces
$\Re z$	the real part of a complex number z
$\Im z$	the imaginary part of a complex number z

Chapter 1

Introduction

Quantum mechanics has evolved from, at that time, an obscure, paradoxical and counter-intuitive hypothesisin to a widely accepted, constantly verified(and verifying) theory, and nowadays, a standard field of physics. As the knowledge about the physical world has grown, so has grown the complexity of physical theories to incorporate all the known phenomena into a logical cohesion. Most of the current modern physical theories, that have the aim to broader our knowledge about the universe, use state-of-the-art mathematical tools, that may have the potential to unlock some missing pieces of Nature's puzzle.

One of the currently researched problem in quantum mechanics is the study of the effect of the geometry on a given system. This new feature introduces a whole new set of physical phenomena and a whole new category of mathematical problems than may dramatically increase the complexity of the problem, even thought, as we know from the history of physics, that even their 'euclidean counterparts' have presented, in many cases, a fearsome task.

This thesis will be concerned with studying a quantum system placed on a Riemannian manifold called the Lobachevsky plane, that represents a very important class of non-euclidean geometries, the so-called hyperbolic geometries. The aim of this thesis is to give a basic mathematical analysis of a self-adjoint operator, that is placed on the Lobachevsky plane, and which represents a simple quantum system of a spinless particle without charge. The particle will be under the influence of constant perpendicular magnetic field. Even thought, the geometry plays a crucial role in the problem, we will look at it from the point of functional analysis and we will here present only the absolute minimum of the geometrical aspect.

This thesis is organized as follows. In Chapter 2 we give a brief overview of the basic and advanced functional analysis, the minimal knowledge of hyperbolic geometry and the its groups of isometry, that will be used in the later chapters. Among the advanced topics is the basics of the theory of rigged spaces and the theory of direct integrals. Chapter 3 focuses a very special, yet important field of functional analysis- the theory of ordinary differential operators and its most important(for us) related topic-the spectral theory. Here we study intensively the spectral problem of a one-dimensional Schrödinger operator on the real line with a potential that, at one end, is diverging and at the other, decaying towards zero. Here we have to modify a theorem regarding the spectral analysis to our case. The final Chapter 4 concerns with the spectral problem of the Hamiltonian of the system. We present the work of few authors that studied this problem and even give a partial result of ours using a decomposition method by direct integrals which from a certain point of view mimics the more famous Bloch decomposition, even thought it is not directly it.

Chapter 2

Theoretical preliminaries

2.1 Basic functional analysis

2.1.1 The spectral representation theorem

Definition 2.1.1.1. *A* **spectral family** *(or* **spectral resolution***) on a Hilbert space $\mathcal{H}$ is a function $E : \mathbb{R} \to \mathcal{B}(\mathcal{H})$ having the following properties:*

1. *$E(t)$ is an orthonormal projector $\forall t \in \mathbb{R}$*
2. *$E(s) \leq E(t)$ for $s \leq t$ (monotonicity)*
3. *$s\text{-}\lim_{t \to t_0^+} E(t) = E(t_0)$(continuity from the right)*
4. *$s\text{-}\lim_{t \to -\infty} E(t) = \Theta$*
5. *$s\text{-}\lim_{t \to +\infty} E(t) = I$*

In the following we give two formulations of the Spectral representation theorem. The first can be found in Berezin-Shubin [4]. We will mainly use this formulation later in the text. The second one is the standard formulation, that could be found in most functional analysis textbooks. Here we follow the definitions and theorems in the classic monogram Dunford-Schwartz [15].

Theorem 2.1.1.2 (Spectral representation theorem)**.** *Let $\mathcal{H}$ be a separable Hilbert space. Let A be a self-adjoint operator on H. Then A can be represented as an operator of multiplication by a real-valued measurable almost-everywhere finite function $a(m)$ in the space $L^2(M, d\sigma) = \{f : M \to \mathbb{C}; \int_M |f|^2 d\sigma < +\infty\}$, where M is a measure space with positive measure $d\sigma$. More precisely, there exists a measure space M with a positive measure σ, a real-valued measurable function $a(m)$ defined on M and finite almost everywhere on M, and an isometry*

$$U : \mathcal{H} \to L^2(M, d\sigma)$$

of $\mathcal{H}$ onto $L^2(M, d\sigma)$ such that

$$f \in Dom(A) \iff f \in \mathcal{H} \wedge a(m)Uf \in L^2(M, d\sigma) \wedge (\forall m \in M)((UAf)(m) = a(m)(Uf)(m))$$

In other words,

$$Af = U^{-1} a U f$$

where a is the multiplication operator by the function $a = a(m)$, that is, A is equivalent to the multiplication operator by the function $a = a(m)$ in the space $L^2(M, d\sigma)$.

Definition 2.1.1.3. *Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$, A a non-empty set and let $\{\mu_\alpha\}_{\alpha\in A}$ be a family of finite positive measures defined on the Borel sets of the complex plane and vanishing on the complement of the spectrum of T. Let U be an isometrical isomorphism of $\mathcal{H}$ onto $\bigoplus_{\alpha\in A} L^2(\mathbb{R},\mu_\alpha)$. Let $V = UTU^{-1}$ be the corresponding self-adjoint operator on $\bigoplus_{\alpha\in A} L^2(\mathbb{R},\mu_\alpha)$. The transformation U is a* **spectral representation of $\mathcal{H}$ onto $\bigoplus_{\alpha\in A} L^2(\mathbb{R},\mu_\alpha)$ relative to** *T if the following conditions are satisfied:*

1. *for every Borel function F defined on the spectrum of T we have*

$$Dom(F(V)) = \{\xi \in \bigoplus_{\alpha\in A} L^2(\mathbb{R},\mu_\alpha); \xi = \sum_{\alpha\in A}\xi_\alpha, \sum_{\alpha\in A}\int_{\sigma(T)} |F(\lambda)\xi_\alpha(\lambda)|^2\mu_\alpha(d\lambda) < +\infty\}$$

2. *$(F(V)\xi_\alpha)(\lambda) = F(\lambda)\xi_\alpha(\lambda), \xi \in Dom(f(V))$ for μ_α-almost all λ.*

Theorem 2.1.1.4 (Spectral representation theorem)**.** *Every Hilbert space admits a spectral representation relative to an arbitrary self-adjoint operator defined in it.*

Definition 2.1.1.5. *Let μ be a positive measure defined on the family $\mathcal{B}$ of Borel sets of the complex plane and let $\{B_n\}_{n=1}^{+\infty}$ be a decreasing sequence of Borel sets whose first element B_1 is the entire plane. Let $\mu_n(B) = \mu(B\cap B_n)$ for $B \in \mathcal{B}$ and $n \in \mathbb{N}$. A spectral representation of a Hilbert space $\mathcal{H}$ onto $\sum_{n=1}^{+\infty} L^2(\mathbb{R},\mu_n)$ relative to a self-adjoint operator T in $\mathcal{H}$ is said to be an* **ordered representation of T relative to** *T. The measure μ is called the* **measure of the ordered representation***. The sets B_n will be called the* **multiplicity sets of the ordered representation***. If $\mu(B_k) > 0$ and $\mu(B_{k+1}) = 0$ then the ordered representation is said to have* **spectral multiplicity***(or shortly,* **multiplicity***) k. If $\mu(B_k) > 0$ for all k, the representation is said to have* **infinite spectral multiplicity***. Two ordered representations U and $\widetilde{U}$ of $\mathcal{H}$ relative to T and $\widetilde{T}$ respectively, with measures μ and $\widetilde{\mu}$, and multiplicity sets $\{B_n\}_{n=1}^{+\infty}$ and $\{\widetilde{B}_n\}_{n=1}^{+\infty}$ will be called* **equivalent** *if $\mu \simeq \widetilde{\mu}$(equivalence of measures) and $\mu(B_n\Delta\widetilde{B}_n) = \widetilde{\mu}(B_n\Delta\widetilde{B}_n) = 0$ for $n \in \mathbb{N}$.*

Remark 2.1.1.6. If the spectral multiplicity of a self-adjoint operator in a Hilbert space is equal to 1, it is sometimes said, that the operator has ***simple spectrum***.

Theorem 2.1.1.7. *A separable Hilbert space $\mathcal{H}$ has an ordered representation U relative to a given self-adjoint operator T in $\mathcal{H}$ and every ordered representation of $\mathcal{H}$ relative to T is equivalent to U. Moreover, two self-adjoint operators in $\mathcal{H}$ are unitarily equivalent if and only if the corresponding ordered representations of $\mathcal{H}$ relative to the operators are equivalent.*

2.1.2 Types of spectra

In this subsection we will briefly revise the basic definitions and theorems concerning the spectra of linear operators. We follow Weidmann [39] and Blank-Exner-Havlíček [7]. Our first classification is the most general classification of spectra of closed linear operators on Banach spaces.

Definition 2.1.2.1. *Let $\mathcal{H}$ be a Banach space, let T be a closed operator on $\mathcal{H}$.*
The **point spectrum** *is the set $\sigma_p(T) = \{\lambda \in \mathbb{C}; \ker(T-\lambda I) \neq \{\theta\}\}$.*
The **continuous spectrum** *is the set $\sigma_c(T) = \{\lambda \in \mathbb{C}; \overline{Ran(T-\lambda I)} = \mathcal{H}\}$.*
The **residual spectrum** *is the set $\sigma_r(T) = \{\lambda \in \mathbb{C}; \overline{Ran(T-\lambda I)} \neq \mathcal{H}\}$.*

Theorem 2.1.2.2. *Let $\mathcal{H}$ be a Banach space, let T be a closed operator on $\mathcal{H}$. Then*

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T) \tag{2.1}$$

Proof. Obvious. Since the first classification divides the spectrum into 2 disjoint sets of points in which the operator $T - \lambda I$ is not a injection and a surjection. And if $T - \lambda I$ is not a surjection, then it is further divided into disjoint sets whether the range of $T - \lambda I$ is or is not dense in $\mathcal{H}$. □

The proof of the following useful theorem can be found in Blank-Exner-Havlíček [7].

Theorem 2.1.2.3. *Let $\mathcal{H}$ be a Hilbert space, let T be a self-adjoint operator on $\mathcal{H}$. Then $\sigma_r(T) = \emptyset$.*

The geometrical structure of Hilbert spaces(induced by the scalar product) introduces another division of the spectrum.

Definition 2.1.2.4. *Let $\mathcal{H}$ be a Hilbert space, let T be a self-adjoint operator on $\mathcal{H}$.*

The **essential spectrum** *is the set $\sigma_{ess}(T)$ of those points of $\sigma(T)$ that are either accumulation points of $\sigma(T)$ or isolated eigenvalues of infinite multiplicity.*

The **discrete spectrum** *is the set $\sigma_d(T)$ defined as $\sigma_d(T) = \sigma(T) \setminus \sigma_{ess}(T)$ or in other words, the discrete spectrum of T is the set of those eigenvalues of finite multiplicity that are isolated points of $\sigma(T)$.*

We say that T has **pure discrete spectrum** *if $\sigma_{ess}(T) = \emptyset$.*

Theorem 2.1.2.5. *Let $\mathcal{H}$ be a Hilbert space, let T be a closed operator on $\mathcal{H}$. Then*

$$\sigma(T) = \sigma_p(T) \cup \sigma_r(T) \cup \sigma_{ess}(T) \tag{2.2}$$

$$\sigma_c(T) = \sigma_{ess}(T) \setminus (\sigma_p(T) \cup \sigma_r(T)) \tag{2.3}$$

The proof of the previous theorem can be found in Blank-Exner-Havlíček [7].

Now for the rest of the subsection, $\mathcal{H}$ will be a Hilbert space with the scalar product $\langle .,. \rangle$ and T be a self-adjoint operator on $\mathcal{H}$ and E_T its spectral family. The final classification, that we will use is based on the relationship between the Lebesque measure μ on $\mathbb{R}$ and the measure $\nu_x(.) = \langle x, E_T(.)x \rangle,\ x \in \mathcal{H}$.

Definition 2.1.2.6. *Let $\mathcal{B}$ be the Borel σ-algebra of $\mathbb{R}$. Denote by $\mathcal{B}_0$ the family of all sets N satisfying $\mu(N) = 0$. We define the following subsets of $\mathcal{H}$:*

- *$\mathcal{H}_p(T)$ is the closed linear hull of all eigenvectors of T, it will be called the* **discontinuous subspace of $\mathcal{H}$ with respect to T**.
- *$\mathcal{H}_c(T)$ is the orthogonal complement of $\mathcal{H}_p(T)$, it will be called the* **continuous subspace of $\mathcal{H}$ with respect to T**.
- *$\mathcal{H}_{sc}(T)$ is the set of those $x \in \mathcal{H}_c(T)$ for which there exists a set $N \in \mathcal{B}_0$ such that $E_T(N)x = x$, it will be called the* **singular continuous subspace of $\mathcal{H}$ with respect to T**.
- *$\mathcal{H}_{ac}(T)$ is the orthogonal complement of $\mathcal{H}_{sc}(T)$ relative to $\mathcal{H}_c(T)$($\mathcal{H}_c(T) = \mathcal{H}_{ac}(T) \oplus \mathcal{H}_{sc}(T)$), it will be called the* **absolute continuous subspace of $\mathcal{H}$ with respect to T**.

- *$\mathcal{H}_s(T)$ is defined as $\mathcal{H}_s(T) = \mathcal{H}_p(T) \oplus \mathcal{H}_{sc}(T)$, it will be called the* **singular subspace of $\mathcal{H}$ with respect to** *T.*

Lemma 2.1.2.7. *The space $\mathcal{H}_{sc}(T)$ in Definition 2.1.2.6 is closed.*

So from this lemma we have $\mathcal{H}_c(T) = \mathcal{H}_{ac}(T) \oplus \mathcal{H}_{sc}(T)$. The proofs of the following theorems can be found in Weidmann [39].

Theorem 2.1.2.8. *The following statements are true:*

1. *$\mathcal{H}_p(T)$ equals the set of those $x \in \mathcal{H}$ for which there exists an at most countable set $A \subset \mathbb{R}$ such that $\nu_x(\mathbb{R} \setminus A) = 0$, i.e., for which the measure ν_x is concentrated on(at most) countably many points.*

2. *$\mathcal{H}_c(T)$ equals the set of those $x \in \mathcal{H}$ for which $\nu_x(\{t\}) = 0$ for every $t \in \mathbb{R}$, i.e., for which the function $t \mapsto \langle x, E_T((-\infty, t])x \rangle$ is continuous.(For $x \in \mathcal{H}_c(T)$ we obviously have $\nu_x(A) = 0$ for every at most countable set $A \subset \mathbb{R}$.)*

3. *$\mathcal{H}_s(T)$ equals the set of those $x \in \mathcal{H}$ for which there exists a set $N \in \mathcal{B}_0$ such that $\nu_x(\mathbb{R} \setminus N) = 0$, i.e., for which ν_x is singular with respect to the Lebesque measure μ.*

4. *$\mathcal{H}_{ac}(T)$ equals the set of those $x \in \mathcal{H}$ for which $\nu_x(N) = 0$ for every $N \in \mathcal{B}_0$, i.e., for which ν_x is absolutely continuous with respect to the Lebesque measure μ.*

Definition 2.1.2.9. *Define the restrictions :*

- *$T_p \upharpoonright \mathcal{H}_p(T)$ is the* **discontinuous part** *of T.*
- *$T_c \upharpoonright \mathcal{H}_c(T)$ is the* **continuous part** *of T.*
- *$T_{sc} \upharpoonright \mathcal{H}_{sc}(T)$ is the* **singular continuous part** *of T.*
- *$T_{ac} \upharpoonright \mathcal{H}_{ac}(T)$ is the* **absolutely continuous part** *of T.*
- *$T_s \upharpoonright \mathcal{H}_s(T)$ is the* **singular part** *of T.*

Now we define the reduction of a operator by a closed subspace, which we will use later.

Definition 2.1.2.10. *Let A be a linear operator on $\mathcal{H}$. Let M be a closed subspace of $\mathcal{H}$, and let P be the orthogonal projection onto M. We say that M* **reduces** *the operator A if $PA \subset AP$. The formulae $Dom(A_M) = M \cap Dom(A)$ and $A_M x = Ax$ for $x \in Dom(A_M)$ define an operator on M. We say M is a* **reducing subspace of** *T if M reduces A.*

Theorem 2.1.2.11. *Let M be a reducing subspace of T. Then T_M and $T_{M^\perp}$ are self-adjoint on M and $M^\perp$, respectively. Also $\sigma(T) = \sigma(T_M) \cap \sigma(T_{M^\perp})$. The subspace M reduces T if and only if $PE_T(t) = E_T(t)P$ for every $t \in \mathbb{R}$, where P denotes the orthogonal projection onto M.*

Theorem 2.1.2.12. *We conclude:*

1. *The subspaces $\mathcal{H}_p(T), \mathcal{H}_c(T), \mathcal{H}_{sc}(T), \mathcal{H}_{ac}(T), \mathcal{H}_s(T)$ reduce the operator T.*

2. *The operators $T_p, T_c, T_{sc}, T_{ac}, T_s$ are self-adjoint.*

3. *$\sigma(T_p) = \overline{\sigma_p(T)}$.*

Thus we may re-define previously defined components of the spectrum $\sigma(T)$ in the language of operator reduction. The only exception is the point spectrum.

Definition 2.1.2.13. *Define the spectra :*

- $\sigma_c(T) = \sigma(T_c)$ *is the* **continuous spectrum** T.
- $\sigma_{sc}(T) = \sigma(T_{sc})$ *is the* **singular continuous spectrum** T.
- $\sigma_{ac}(T) = \sigma(T_{ac})$ *is the* **absolutely continuous spectrum** T.
- $\sigma_s(T) = \sigma(T_s)$ *is the* **singular spectrum** T.

We say that :

- *if* $\mathcal{H} = \mathcal{H}_p(T)$ *then* T *has a* **pure point spectrum** *and* $\sigma(T) = \overline{\sigma_p(T)}$.
- *if* $\mathcal{H} = \mathcal{H}_c(T)$ *then* T *has a* **pure continuous spectrum** *and* $\sigma(T) = \sigma_c(T)$.
- *if* $\mathcal{H} = \mathcal{H}_{sc}(T)$ *then* T *has a* **pure singular continuous spectrum** *and* $\sigma(T) = \sigma_{sc}(T)$.
- *if* $\mathcal{H} = \mathcal{H}_{ac}(T)$ *then* T *has a* **pure absolutely continuous spectrum** *and* $\sigma(T) = \sigma_{ac}(T)$.
- *if* $\mathcal{H} = \mathcal{H}_s(T)$ *then* T *has a* **pure singular spectrum** *and* $\sigma(T) = \sigma_s(T)$.

Finally, we arrive at the third classification of the spectrum of T.

Theorem 2.1.2.14. *From previous theorems we have the following equations:*

$$\sigma(T) = \overline{\sigma_p(T)} \cup \sigma_{sc}(T) \cup \sigma_{ac}(T). \tag{2.4}$$

$$\sigma(T) = \sigma_s(T) \cup \sigma_{ac}(T). \tag{2.5}$$

$$\sigma(T) = \overline{\sigma_p(T)} \cup \sigma_c(T). \tag{2.6}$$

2.1.3 The Riesz projection

In this subsection we introduce a particularly important operator, that we will use often in later sections. This operator is generally crucial in operator calculus, especially in the case of meromorphic operators. We use the terminology and theorems from Gohberg-Goldberg-Kaashoek [18] and from Berezin-Shubin [4]. First, we define some more terminology.

Definition 2.1.3.1. *A* **Cauchy domain** *is a disjoint union of a finite number of non-empty open connected sets* $\Delta_1, ..., \Delta_r \subset \mathbb{C}$, *such that* $\overline{\Delta_i} \cap \overline{\Delta_j} = \emptyset$ *for* $i \neq j$ *and for each* j *the boundary of* Δ_j *consists of a finite number of non-intersecting closed rectifiable Jordan curves which are oriented in such a way that* Δ_j *belongs to the inner domains of the curves.*

A contour Γ *is call a* **Cauchy contour** *if* Γ *is the oriented boundary of a bounded Cauchy domain in* $\mathbb{C}$.

Definition 2.1.3.2. *Let $\mathcal{B}$ be a Banach space, let A be a closed operator on $\mathcal{B}$. Assume, that the spectrum of A is the disjoint union of two non-empty closed subsets σ and τ. Let Γ be a Cauchy contour laying in the resolvent set of A such that σ belongs to the inner domain of Γ and τ to the outer domain of Γ. Then the operator*

$$P_\sigma = \frac{i}{2\pi} \int_\Gamma R_A(\lambda) d\lambda \tag{2.7}$$

is called the **Riesz projection**.

Remark 2.1.3.3. Here the integral in (2.7) is understood in the Bochner sense. For a basic preview of the Bochner integral, one can seek Blank-Exner-Havlíček [7].

Now we sum up the most important properties for closed operators. The proof of the next theorem is in Gohberg-Goldberg-Kaashoek [18].

Theorem 2.1.3.4. *Let $\mathcal{H}$ be a Banach space and let A be a closed operator on $\mathcal{H}$ with the spectrum $\sigma(A) = \sigma \cup \tau$, where σ is contained in a bounded Cauchy domain Δ such that $\overline{\Delta} \cap \tau = \emptyset$. Let Γ be the (oriented) boundary of Δ. Then*

- $P_\sigma = \frac{i}{2\pi} \int_\Gamma R_A(\lambda) d\lambda$ *is a projector,*
- *the subspaces $M = Ran(P_\sigma)$ and $N = Ker(P_\sigma)$ are A-invariant,*
- *the subspace M is contained in $Dom(A)$ and $A|M$ is bounded,*
- $\sigma(A|M) = \sigma$ *and* $\sigma(A|N) = \tau$.

Now we will examine the consequences of the previous theorem in a very special case, which will be used later.

Corollary 2.1.3.5. *Let $\mathcal{H}$ be a Hilbert space and let A be a self-adjoint operator on $\mathcal{H}$, let $\sigma = \{\lambda_1, ..., \lambda_n\} \subset \sigma_p(A)$ with each λ_j having finite multiplicity, let σ be contained in a bounded Cauchy domain Δ such that $\overline{\Delta} \cap (\sigma(A) \setminus \sigma) = \emptyset$ and let Γ be the (oriented) boundary of Δ. Then*

$$P_\sigma = \frac{i}{2\pi} \int_\Gamma R_A(\lambda) d\lambda = \sum_{j=1}^{n} P_j$$

where P_j is the projector on the subspace generated by the eigenfunctions corresponding to the eigenvalue λ_j.

Proof. Denote by $m(\lambda_j)$ the multiplicity of the eigenvalue λ_j and $M = Ran(P_\sigma)$. Let $\Lambda_j = span\{e_j^1, ..., e_j^{m(\lambda_j)}\}$ be the eigenspace of λ_j and let $\Lambda = \bigoplus_{l=1}^n \Lambda_j$. It is sufficient to prove that $Ran(P_\sigma) = \Lambda$. Define the operator $B = A|_M$.

The inclusion $\Lambda \subset Ran(P_\sigma)$ is trivial. We will prove $Ran(P_\sigma) \subset \Lambda$. Let $M = M' \oplus \Lambda$ for some $M' \neq \{\theta\}$. Ten for all $y \in M$ we have $y = u + t$, where $u \in M'$ and $t \in \Lambda$. Since the spectral decomposition theorem implies

$$B = \sum_{j=1}^{n} \lambda_j P_j$$

we have

$$Bu = \sum_{j=1}^{n} \lambda_j P_j u = \theta$$

since $Ran(P_\sigma)$ is closed, therefore a Hilbert space and therefore M' is the orthogonal complement of the space Λ and thus $P_j u = \theta$ for all $j = 1...n$. Thus $u \in Ker(B)$. We consider the two cases

- $0 \in \sigma \Longrightarrow Ker(B) = \Lambda_0 \Longrightarrow$ contradiction with $y \notin \Lambda$.
- $0 \notin \sigma \Longrightarrow Ker(B) = \{\theta\} \Longrightarrow$ contradiction with $M' \neq \{\theta\}$.

Hence $y \in \Lambda$. □

The following theorem tells us, when we can use the Riesz projection for unbounded regions of the spectrum. The proof can be found in Berezin-Shubin [4].

Lemma 2.1.3.6. *Let $a \in \mathbb{R}$, let there be a contour Γ_δ in the complex plane, consisting of the following three pieces:*

- *Γ_+ : $z(t) = t + i\gamma(t)$, t varies from $+\infty$ to 0, where $\gamma(t)$ is a continuous piecewise smooth function of $t \in [0, +\infty)$ such that $0 < \gamma(t) < \delta$ for all t;*
- *Γ_0 : is any continuous piecewise smooth curve starting from $i\gamma(0)$ and ending at $-i\gamma(0)$ and, apart from the end points, belonging to the half-plane $\Re z < 0$;*
- *Γ_- : $z(t) = t - i\gamma(t)$, t varies from 0 to $+\infty$;*

Denote by $\Gamma_{N,\delta}$ the piece of contour Γ_δ lying in the half-plane $\Re z < N$. Let $\gamma(t)$ be chosen, so that a lies inside Γ_δ.

Let A be an arbitrary self-adjoint operator in a Hilbert space $\mathcal{H}$ with $\sigma(A) \subset (a, +\infty)$. Then

$$\underset{N\to+\infty}{\text{s-lim}} \frac{i}{2\pi} \int_{\Gamma_{N,\delta}} R_z g dz = g \tag{2.8}$$

for any $g \in \mathcal{H}$ and $R_z = (A - zI)^{-1}$. Furthermore, the estimate

$$\left\| \frac{i}{2\pi} \int_{\Gamma_{N,\delta}} R_z g dz \right\| \leq \|g\|$$

is valid.

Finally, there exists in $\mathcal{H}$ a dense subspace $\mathcal{H}_A$ depending on the operator A such that for $g \in \mathcal{H}_A$ the limit in (2.8) is uniform for all admissible contours Γ_δ with $\delta < 1$.

Remark 2.1.3.7. Since the object

$$\underset{N\to+\infty}{\text{s-lim}} \frac{i}{2\pi} \int_{\Gamma_{N,\delta}} R_A(z) dz$$

is not a Riesz projection according to the definition, but it is the identity and hence a projector constructed as the strong limit of a sequence of operators, but which do not need to be projectors.

Corollary 2.1.3.8. *Let A be a self-adjoint operator with $\sigma(A) \subset (a, +\infty)$ for some $a \in \mathbb{R}$ and let the spectrum $\sigma(A)$ satisfy the inclusion:*

$$\sigma(a) \subset (a, b) \cup (c, d)$$

for $-\infty < a < b < c < d \leq +\infty$. Let Γ_1 be a Cauchy contour such that $(a, b) \subset \Gamma_1^\circ$ and define the mapping:

$$P_1 := \frac{i}{2\pi} \int_{\Gamma_1} R_A(z) dz \ , (a, b) \subset \Delta_1.$$

Denote by Γ_2 the contour having the same properties as Γ_δ in Theorem 2.1.3.6 except that Γ_2 encircles the set (c, d) and denote by $\Gamma_{2,N}$ a piece of Γ_2 lying the the half-plane $\Re z < N$. Define the mapping:

$$P_2 := \operatorname*{s-lim}_{N \to +\infty} \frac{i}{2\pi} \int_{\Gamma_{2,N}} R_A(z) dz \ ,$$

Then $P_1 + P_2 = I$.

Proof. Denote by $\Gamma_\delta, \Gamma_{N,\delta}$ the contours in Theorem 2.1.3.6. Take the strip $\{z \in \mathbb{C}; \Re z \in (b, c)\}$ and add two vertical lines in it, that intersect the countour $\Gamma_{N,\delta}$. Hence they divide $\Gamma_{N,\delta}$ into three parts. Denote these parts by $\Delta_1, \Delta_2, \Delta_3$.

Denote:

- by K the upper point of intersection of the left-most vertical line and $\Gamma_{N,\delta}$.
- by L the upper point of intersection of the right-most vertical line and $\Gamma_{N\delta}$.
- by M the lower point of intersection of the left-most vertical line and $\Gamma_{N,\delta}$.
- by N the lower point of intersection of the right-most vertical line and $\Gamma_{N,\delta}$.

Denote by γ_1 the contour formed by the part of $\Gamma_{N,\delta}$ between the points K, M and the part of the left-most vertical line which is between the points K, M. Obviously it is a closed contour encircling (a, b). And it is the Cauchy contour of Δ_1.

Denote by γ_2 the contour formed by the piece of $\Gamma_{N,\delta}$ between the points K, L, between the points M, N, the part of the left-most vertical line between the points K, M and the part of the right-most vertical line between the points L, N. Again, it is a closed contour. And it is the Cauchy contour of Δ_2.

Denote by γ_3 the contour formed by the part of the right-most vertical line between the points L, N and the rest of $\Gamma_{N,\delta}$ to the right from the right-most vertical line.

Thus we can write:

$$\frac{i}{2\pi} \int_{\Gamma_{N,\delta}} R_A(z) dz = \frac{i}{2\pi} \int_{\gamma_1} R_A(z) dz + \frac{i}{2\pi} \int_{\gamma_2} R_A(z) dz + \frac{i}{2\pi} \int_{\gamma_3} R_A(z) dz$$

The term $\frac{i}{2\pi} \int_{\gamma_2} R_A(z) dz$ is zero, sice Δ_2 lies in $\rho(A)$, thus from the theory of operator calculus, it is holomorphic. Hence we have two disjoint domains Δ_1, Δ_2. Since $\frac{i}{2\pi} \int_{\gamma_1} R_A(z) dz$ is a Riesz projection, we can identify it with P_1. And since $\mathbb{C} \setminus \mathbb{R}$ is the domain of holomorphicity of R_A, we identify $\frac{i}{2\pi} \int_{\gamma_3} R_A(z) dz$ with $\frac{i}{2\pi} \int_{\Gamma_{2,N}} R_A(z) dz$. Hence

$$\frac{i}{2\pi} \int_{\Gamma_{N,\delta}} R_A(z) dz - P_1 = \frac{i}{2\pi} \int_{\Gamma_{2,N}} R_A(z) dz$$

for every $N \in \mathbb{N}$, thus the limit of the right side exists and

$$I - P_1 = \underset{N\to+\infty}{\text{s-lim}} \left[\frac{i}{2\pi} \int_{\Gamma_{N,\delta}} R_A(z)dz - P_1 \right] = \underset{N\to+\infty}{\text{s-lim}} \frac{i}{2\pi} \int_{\Gamma_{2,N}} R_A(z)dz = P_2 \ .$$

so finally $P_1 + P_2 = I$. □

2.2 Advanced topics in functional anylysis

2.2.1 Rigged Hilbert spaces

Here we will briefly introduce the topic of rigged Hilbert spaces and its connection to the spectral analysis of self-adjoint operators. We use the definitions and theorems from Berezin-Shubin [4].

Definition 2.2.1.1. *Let $\mathcal{H}_+$ be a locally convex Hausdorff topological vector space over $\mathbb{C}$. Let $\mathcal{H}_-$ be the space of all continuous anti-linear functionals on $\mathcal{H}_+$. We endow the space $\mathcal{H}_-$ with the topology of the dual space $\mathcal{H}'_+$. The value of a functional $h \in \mathcal{H}_-$ on the vector $x \in \mathcal{H}_+$ is denoted by (h, x). The identities*

$$(h, \alpha x + y) = \overline{\alpha}(h, x) + (h, y)$$

$$(\alpha h_1 + h_2, x) = \alpha(h_1, x) + (h_2, x)$$

hold $\forall h, h_1, h_2 \in \mathcal{H}_-; \forall x, y \in \mathcal{H}_+; \forall \alpha \in \mathbb{C}$. Let $\mathcal{H}$ be a Hilbert space with the inner product $\langle ., . \rangle$ such that $\overline{\mathcal{H}_+} = \mathcal{H}$ and a topology of $\mathcal{H}_+$ is stronger than that of H, the ordered triplet $(\mathcal{H}_+, \mathcal{H}, \mathcal{H}_-)$ is called a **rigged Hilbert space** *or a* **Gel'fand triplet** *or a* **rigging of** $\mathcal{H}$. *We define the* **canonical inclusion** *$j : \mathcal{H}_+ \to \mathcal{H}_-$ by setting $\forall x \in \mathcal{H}_+$*

$$(j(x), y) = \langle x, y \rangle, \forall y \in \mathcal{H}_+.$$

Let $\mathcal{H}_+$ be a Banach space with the norm $\|.\|_+$. Then we have the norm $\|.\|_-$ for $h \in \mathcal{H}_-$:

$$\|h\|_- = \sup_{x\in\mathcal{H}_+} |(h, x)|,$$

since we use the topology of the dual space $\mathcal{H}'_+$.

Definition 2.2.1.2. *Let $\mathcal{H}$ be Hilbert space with the scalar product $\langle ., . \rangle$, let $K : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator such that $\ker K = \ker K^* = \{\theta\}$. Define $\mathcal{H}_+ = K\mathcal{H}$ and define the norm $\|.\|_+$ on $\mathcal{H}_+$ as*

$$\|h\|_+ = \sqrt{\langle K^{-1}h, K^{-1}h \rangle}, \ h \in \mathcal{H}_+ \ .$$

Then the rigging $(\mathcal{H}_+, \mathcal{H}, \mathcal{H}_-)$ is called a **rigging associated with** K.

Definition 2.2.1.3. *Let $\mathcal{H}$ be Hilbert space, let $K : \mathcal{H} \to \mathcal{H}$ be a Hilbert-Schmidt operator. Then the rigging $(\mathcal{H}_+, \mathcal{H}, \mathcal{H}_-)$ associated with K is called a* **Hilbert-Schmidt rigging**.

Definition 2.2.1.4. *Let $(M, \mathcal{A}, \sigma)$ be a measure space. Let $\mathcal{H}$ be a Hilbert space. Let A be a self-adjoint operator on $\mathcal{H}$. Let $(\mathcal{H}_+, \mathcal{H}, \mathcal{H}_-)$ a rigging of $\mathcal{H}$. Let Φ be a vector function defined for almost all $m \in M$ with values in $\mathcal{H}_-$ and let Φ satisfy the following:*

1. *for every* $h \in \mathcal{H}_+$ *the function* $m \mapsto (\Phi(m), h)$ *on* M *belongs to* $L^2(M, d\sigma)$.

2. *the map* $h \mapsto (\Phi(.), h)$ *can be extended to a unitary operator* $U : \mathcal{H} \to L^2(M, d\sigma)$.

3. *there exists a function* $a : M \to \mathbb{R}$ *that is measurable and almost everywhere finite on* M *and such that* $A = U^{-1}\hat{a}U$, *where* $\hat{a}$ *is the multiplication operator by the function* a *in* $L^2(M, d\sigma)$.

Then Φ *is called a* **complete system of generalized eigenvectors** *of the operator* A *and a vector* $\Phi(m)$, $m \in M$ *is called a* **generalized eigenvector**.

Let $f \in \mathcal{H}_+$, *define the function* $\tilde{f} : M \to \mathbb{C}$, $\tilde{f}(m) = (\Phi(m), f)$. *The transformation* $f \mapsto \tilde{f}$ *is called the* **generalized Fourier transform**. *The equality*

$$\|f\|^2 = \int_M |\tilde{f}(m)|^2 d\sigma$$

is called the **generalized Parseval identity**.

Remark 2.2.1.5. Thus from Defnition 2.2.1.4 it apparent that the inversion formula

$$f = \int_M \tilde{f}(m)\Phi(m) d\sigma \ , f \in \mathcal{H}$$

holds in the weak sense, i.e.

$$\langle f, h\rangle = \int_M \tilde{f}(m)(\Phi(m), h) d\sigma \ .$$

Now we arrive to a very profound theorem on the importance of Hilbert-Schmidt rigging of Hilbert spaces. The proof can be found in Berezin-Shubin [4].

Theorem 2.2.1.6. *Let* $\mathcal{H}$ *be a Hilbert space, let* A *be a self-adjoint operator on* $\mathcal{H}$. *Let* $(\mathcal{H}_+, \mathcal{H}, \mathcal{H}_-)$ *be a Hilbert-Schmidt rigging of* $\mathcal{H}$. *Then there exists a complete orthonormal system of generalized eigenvectors of* A.

Remark 2.2.1.7. The opposite implication holds also. The reader can find more information on it in Berezanski [3], where it is also proven, that a generalized Parseval identity holds if and only if one considers a Hilbert-Schmidt rigging. Thus the importance of this type of rigging can be seen.

2.2.2 Direct integrals of Hilbert spaces

We begin with some preliminaries from the theory of direct integrals, that is from from Dixmier [13], a classic and extensive textbook on the topic, and Wils [41]. Here if we write 'Hilbert space', we will always mean a complex Hilbert space.

For this part of the subsection, (Z, Σ, μ) will denote a general measure space if not specified.

Definition 2.2.2.1. *We call a collection* $\mathcal{H}(z))_{z\in Z}$ *a* **field of Hilbert spaces over** Z. *Elements of* $\prod_{z\in Z} \mathcal{H}(z)$ *are called* **vector fields**. *If* ϕ *is a complex-valued, bounded and measurable on* Z *and if* f *is a vector field, then* $\phi f : Z \ni z \to \phi(z)f(z)$. *If* f *and* g *are vector fields, then we define*

$$\langle x, y\rangle : Z \ni z \to \langle x(z), y(z)\rangle_z \ ,$$

where $\langle ., .\rangle_z$ *is the scalar product in* $\mathcal{H}(z)$, *and*

$$|f| : Z \ni z \to \|f(z)\|_z.$$

Definition 2.2.2.2. *Let $(\mathcal{H}(z))_{z\in Z}$ be a field of Hilbert spaces over Z and let Γ be a subspace of $\prod_{z\in Z}\mathcal{H}(z)$. Then $((\mathcal{H}(z))_{z\in Z},\Gamma)$ is said to be an* **measurable field of Hilbert spaces over** *Z if it fulfills the following conditions:*

1. *For every $x \in \Gamma$, the function $z \to \|x(z)\|_z$ is μ-measurable.*
2. *If $g \in \prod_{z\in Z}\mathcal{H}(z)$ is such that for every $x \in \Gamma$, the complex-valued function $z \to \langle x(z), y(z)\rangle_z$ is μ-measurable.*
3. *There exists a set $\{(x_n)_{n\in\mathbb{N}} \subset \Gamma\}$ such that, for every $z \in Z$, the set $\{x_n(z))_{n\in\mathbb{N}}\}$ forms a total set in $\mathcal{H}(z)$.*

A vector field belonging to Γ is called a **μ-measurable vector field**. *A set $\{(x_n)_{n\in\mathbb{N}}$ satisfying condition 3 is called a* **fundamental set** *of μ-measurable vector fields.*

Remark 2.2.2.3. The condition 3 obviously implies, that for every $z \in Z$, the space $\mathcal{H}(z)$ is separable. If one omits this condition, then the generally non-separable case can cause a variety of complications. Wils [41] and Vesterstrom-Wils [42] make a generalization to the non-separable case to study problems in von Neumann algebras.

Definition 2.2.2.4. *Let $\mathcal{H}_0$ be a separable Hilbert space. A* **constant field of $\mathcal{H}_0$ over** *Z is a field $(\mathcal{H}(z))_{z\in Z}$ of Hilbert spaces over Z with the properties:*

1. *$\mathcal{H}(z) = \mathcal{H}_0$ for every $z \in Z$.*
2. *The μ-measurable vector fields are μ-measurable mappings of Z into $\mathcal{H}_0$.*

Definition 2.2.2.5. *Let $((\mathcal{H}(z))_{z\in Z},\Gamma)$ be a measurable field of Hilbert spaces over Z. A vector field $x \in ((\mathcal{H}(z))_{z\in Z},\Gamma)$ is said to be* **square-integrable** *if it is measurable and if*

$$\int_Z \|x(z)\|_z d\mu(z) < +\infty \ .$$

The proof of the following theorem is not difficult, but we will omit it. The reader can find it in Dixmier [13].

Theorem 2.2.2.6. *Let $((\mathcal{H}(z))_{z\in Z},\Gamma)$ be a measurable field of Hilbert spaces over Z. Denote by K the set of all square-integrable vector fields in $((\mathcal{H}(z))_{z\in Z},\Gamma)$. Define the mapping $\langle,\rangle : ((\mathcal{H}(z))_{z\in Z},\Gamma) \times ((\mathcal{H}(z))_{z\in Z},\Gamma) \to \mathbb{C}$:*

$$\int_Z \langle x(z), y(z)\rangle d\mu(z)$$

for every $x, y \in ((\mathcal{H}(z))_{z\in Z},\Gamma)$. Then K with the mapping $\langle,\rangle$ is a Hilbert space.

Definition 2.2.2.7. *The Hilbert space $\mathcal{H} = (K,\langle,\rangle)$ in Theorem 2.2.2.6, is called the* **direct integral** *of $((\mathcal{H}(z))_{z\in Z},\Gamma)$ and it is denoted by*

$$\int_Z^{\oplus} \mathcal{H}(z) d\mu(z) \ .$$

Remark 2.2.2.8. Lets return to the constant fields. Let $\mathcal{H}_0$ be a separable Hilbert space, and let $(\mathcal{H}(z))_{z\in Z}$ be the constant field of the Hilbert space $\mathcal{H}_0$ over Z. Then one may check, that the square-integrable vector fields are the square-integrable mappings of Z into $\mathcal{H}_0$ and thus

$$\int_Z^{\oplus} \mathcal{H}(z) d\mu(z) = L^2(Z, d\mu, \mathcal{H}_0) \ .$$

Definition 2.2.2.9. *Let $(\mathcal{H}(z))_{z\in Z}, (\mathcal{H}'(z))_{z\in Z}$ be two fields of Hilbert spaces.*

- *A mapping $T : Z \ni z \to \mathcal{L}(\mathcal{H}(z), \mathcal{H}'(z))$ is called a* **field of linear mappings over** *Z.*
- *A mapping $T : Z \ni z \to \mathcal{B}(\mathcal{H}(z), \mathcal{H}'(z))$ is called a* **field of continuous linear mappings over** *Z.*
- *A mapping $T : Z \ni z \to \mathcal{L}(\mathcal{H}(z))$ is called a* **field of operators over** *Z.*
- *A mapping $T : Z \ni z \to \mathcal{B}(\mathcal{H}(z))$ is called a* **field of continuous operators over** *Z.*
- *A field of continuous linear mappings $T : Z \ni z \to \mathcal{L}(\mathcal{H}(z), \mathcal{H}'(z))$ is called μ-* **measurable** *if for every μ-measurable vector field $x \in (\mathcal{H}(z))_{z\in Z}$, the vector field $(z \to T(z)x(z)) \in (\mathcal{H}'(z))_{z\in Z}$ is μ-measurable. By $\|.\|_{\mathcal{L}(\mathcal{H}(z))}$ we will denote the operator norm in the Hilbert space $\mathcal{H}(z)$.*

Definition 2.2.2.10. *Let $((\mathcal{H}(z))_{z\in Z}, \Gamma)$ be a μ-measurable field of Hilbert spaces over Z. Let $\mathcal{H} = \int_Z^{\oplus} \mathcal{H}(z)d\mu(z)$. A field of continuous operators $T : Z \ni z \to \mathcal{B}(\mathcal{H}(z))$ is said to be* **essentially bounded** *if* $\text{ess-sup}_{z\in Z} \|T\|_{\mathcal{L}(\mathcal{H}(z))} < +\infty$.

Theorem 2.2.2.11. *Let $((\mathcal{H}(z))_{z\in Z}, \Gamma)$ be a μ-measurable field of Hilbert spaces over Z. Let $\mathcal{H} = \int_Z^{\oplus} \mathcal{H}(z)d\mu(z)$ and let $T : Z \ni z \to \mathcal{B}(\mathcal{H}(z))$ be a field of continuous operators.*

- *Let* $\text{ess-sup}_{z\in Z} \|T\|_{\mathcal{L}(\mathcal{H}(z))} = \lambda < +\infty$. *Then $\|T\| = \lambda$.*
- *If two essentially bounded measurable fields of continuous operators define the same element of $\mathcal{L}(\mathcal{H})$, they are equal almost everywhere.*
- *Let $\mathcal{H}$ be separable. Then if two essentially bounded measurable fields of continuous operators are equal almost everywhere, they define the same element of $\mathcal{L}(\mathcal{H})$.*

Proof. The proof of the first two assertions can be found in Dixmier [13]. Here we proof the third by contradictions. Let $(x_n)_{n\in\mathbb{N}}$ be the orthonormal base of $\mathcal{H}$. Denote the two measurable fields of continuous operators by T_1, T_2. Let $\psi \in \mathcal{H}$ be such that $T_1\psi \neq T_2\psi$. From the assumptions we have that the mappings $Z \ni z \to \langle x_n(z), T_1(z)\rangle_z$ and $Z \ni z \to \langle x_n(z), T_1(z)\rangle_z$ are $\mu - equivalent$. Thus

$$\int_Z \langle x_n(z), T_1(z)\psi(z)\rangle_z d\mu(z) = \int_Z \langle x_n(z), T_2(z)\psi(z)\rangle_z d\mu(z)$$

or

$$\langle x_n, T_1\psi\rangle = \langle x_n, T_2\psi\rangle$$

for every $n \in \mathbb{N}$. Thus $\sum_{n\in\mathbb{N}}\langle x_n, T_1\psi\rangle x_n = \sum_{n\in\mathbb{N}}\langle x_n, T_2\psi\rangle x_n = \psi$ and hence a contradiction. □

The previous results lead us to define the following:

Definition 2.2.2.12. *Let* $((\mathcal{H}(z))_{z\in Z},\Gamma)$ *be a* μ*-measurable field of Hilbert spaces over* Z*. Let* $\mathcal{H}=\int_Z^{\oplus}\mathcal{H}(z)d\mu(z)$*. An operator* $\tilde{T}\in\mathcal{L}(\mathcal{H})$ *is said to be* **decomposable** *if it is defined by an essentially bounded measurable field* $T : Z\ni z\to\mathcal{L}(\mathcal{H}(z))$*. We then write*

$$\tilde{T}=\int_Z^{\oplus}T(z)d\mu(z).$$

Here we add new terms using Reed-Simon [31].

Definition 2.2.2.13. *Let* $(\mathcal{H}(z))_{z\in Z}$ *be a field of Hilbert spaces over* Z*.*

- *A mapping* $T : Z\ni z\to\mathcal{L}_{SA}(\mathcal{H}(z))$ *is called a* **field of self-adjoint operators over** Z*.*
- *A field* T *of self-adjoint operators over* Z *is called* μ-**measurable** *if the field* $Z\ni z\to(T(z)+i)^{-1}$ *of continuous operators over* Z *is* μ-**measurable***.*
- *Let* $T : Z\ni z\to\mathcal{L}_{SA}(\mathcal{H}(z))$ *be* μ*-measurable. Let* $((\mathcal{H}(z))_{z\in Z},\Gamma)$ *be a measurable field of Hilbert spaces over* Z *and let*

  $$\mathcal{H}=\int_Z^{\oplus}\mathcal{H}(z)d\mu(z)\ .$$

  We define an operator $\tilde{T}$ *on* $\mathcal{H}$ *with domain*

  $$Dom(\tilde{T})=\{\psi\in\mathcal{H};\psi(z)\in Dom(T(z))\ \mu-a.e.\wedge\int_Z\|T(z)\psi(z)\|_z d\mu(z)<+\infty\}\ ,$$

  by

  $$(\tilde{T}\psi)(z)=T(z)\psi(z).$$

  It is denoted by

  $$\tilde{T}=\int_Z^{\oplus}T(z)d\mu(z).$$

Before venturing further, we will revise few topological terms. We use definitions from Arveson [2],

Definition 2.2.2.14.

- *A* **Polish space** *is a topological space which is homeomorphic to a separable complete metric space.*
- *Let* P *be a Polish space. A* **Borel set** *in* P *is a member of the* σ*-algebra generated by the closed subsets of* P*.*
- *Let* (X,τ) *be a topological space. Denote by* $\mathcal{B}$ *the* σ*-algebra generated by the closed subsets of* (X,τ)*. Then the pair* $(X,\mathcal{B})$ *is called a* **Borel space***.*
- *A* **standard Borel space** *is a Borel space that is isomorphic to a Borel subset of a Polish space.*

Theorem 2.2.2.15. *Every uncountable standard Borel space is homeomorphic to the unit interval $[0,1]$ with the usual Borel structure(the usual topology and the usual σ-algebra generated by it).*

Definition 2.2.2.16. *Let (Z,Σ) be a Borel space. Let (Z,Σ,μ) be a measure space. The measure μ is said to be* **standard** *if there exists a μ-negligible set $N \subset Z$ such that the Borel space $(Z \setminus N, \Sigma')$ is standard.*

Remark 2.2.2.17. *Thus if one takes any interval, or the whole $\mathbb{R}$ and endows it with the standard topology, one gets a standard Borel space. Moreover, if one takes the Lebesque measure, then from Theorem 2.2.2.15 and the definition of standard measure its obvious that the Lebesque measure is standard.*

The following important theorem with its proof is in Dixmier [13].

Theorem 2.2.2.18. *If μ is standard, then $\mathcal{H}$ is separable.*

Remark 2.2.2.19. We need the terminology, that we have just defined, since the following theorems are heavily based on them.

Now, we will slightly modify a theorem in Reed-Simon [31], that will be important to us later. But we will need the the following lemma, proved in Lennon [27].

Lemma 2.2.2.20. *Let (Z,Σ) be a standard Borel space and let (Z,Σ,μ) be a measure space with μ being standard. Let $((\mathcal{H}(z))_{z\in Z},\Gamma)$ be a μ-measurable field of Hilbert spaces over Z and let $\mathcal{H} = \int_Z^{\oplus} \mathcal{H}(z)d\mu(z)$. Let $\tilde{T} = \int_Z^{\oplus} T(z)d\mu(z)$ be a self-adjoint operator in $\mathcal{H}$. Then*

- *$T(z)$ is self-adjoint for $\mu - a.e.\ z \in Z$.*
- *Denote by E the spectral family of $\tilde{T}$ and by E_z the spectral family of $T(z)$. Then for any Borel set B*

$$(E(B))(z) = E_z(B)\ ,\ for\ a.e.\ z \in Z\ .$$

And now the proof of the main theorem of this subsection.

Theorem 2.2.2.21. *Let (Z,Σ) be a standard Borel space and let (Z,Σ,μ) be a measure space with μ being standard. Let $((\mathcal{H}(z))_{z\in Z},\Gamma)$ be a μ-measurable field of Hilbert spaces over Z and let $\mathcal{H} = \int_Z^{\oplus} \mathcal{H}(z)d\mu(z)$. Let T be a μ-measurable field of self-adjoint operators over Z and let*

$$\tilde{T} = \int_Z^{\oplus} T(z)d\mu(z)\ .$$

Then

1. *The operator $\tilde{T}$ is self-adjoint.*
2. *$\lambda \in \sigma(\tilde{T})$ if and only if for all $\epsilon > 0$*

$$\mu(\{z \in Z; \sigma(T(z)) \cap (\lambda-\epsilon, \lambda+\epsilon) \neq \emptyset\}) > 0.$$

3. *λ is an eigenvalue of $\tilde{T}$ if and only if*

$$\mu(\{z \in Z; \lambda \in \sigma_p(T(z))\}) > 0.$$

Proof. 1). From functional analysis $\tilde{T}$ is symmetric if and only if $\langle y, \tilde{T}x\rangle = \langle \tilde{T}y, x\rangle$ for every $x, y \in Dom(\tilde{T})$. Hence using a straightforward calculation for any $x, y \in Dom(\tilde{T})$ we have:

$$\langle y, \tilde{T}x\rangle = \int_Z \langle y(z), (\tilde{T}x)(z)\rangle d\mu(z) = \int_Z \langle y(z), T(z)x(z)\rangle d\mu(z)$$

$$= \int_Z \langle T(z)y(z), x(z)\rangle d\mu(z) = \int_Z \langle (\tilde{T}y)(z), x(z)\rangle d\mu(z) = \langle \tilde{T}y, x\rangle \ ,$$

hence $\tilde{T} \subset \tilde{T}^*$. From general functional analysis, we only need to prove that $Ran(\tilde{T} \pm iI) = \mathcal{H}$. Let $C_\pm(z) = (T(z) \pm iI)^{-1}$. From our assumptions, $C_\pm(z)$ is bounded. Here we use some special relations from the spectral theory of self-adjoint operators and unitary propagators. Define the propagator $U_z(s) = e^{-iT(z)s}$ for every $z \in Z$. Then the following relation holds(can be found in Blank-Exner-Havlíček [7]) for every $x(z) \in \mathcal{H}(z)$:

$$R_T(\xi)x(z) = isgn(\Im\xi) \int_{J_\xi} e^{i\xi s} U(s)x(z) ds \ ,$$

where $J_\xi = (-\infty,)$ if $\Im\xi < 0$ and $J_\xi = (0, +\infty)$ if $\Im\xi > 0$ and the integral is understood in the Bochner sense. And from the theory of Bochner integral, we can make the estimate at the point i:

$$\|R_T(i)x(z)\|_z = \| \int_0^{+\infty} e^{-s} U(s)x(z) ds\|_z \leq \int_0^{+\infty} e^{-s} \|U(s)x(z)\|_z ds$$

$$= \|x(z)\|_z \int_0^{+\infty} e^{-s} ds = \|x(z)\|_z.$$

Analogously with $-i$ we have

$$\|R_T(-i)x(z)\|_z \leq \|x(z)\|_z \ ,$$

thus

$$\|C_\pm(z)\|_{\mathcal{L}(\mathcal{H}(z))} \leq 1 \ ,$$

where $\|.\|_{\mathcal{L}(\mathcal{H}(z))}$ is the operator norm in $\mathcal{H}(z)$. Hence by Theorem ... we can define

$$\tilde{C}_\pm = \int_Z^\oplus C_\pm(z) d\mu(z).$$

Let $\eta \in \mathcal{H}$ and let $\psi = \tilde{C}_\pm \eta$. Then for $\mu - a.e.\ z \in Z$, $\psi(z) \in Ran(C_\pm(z)) = Dom(T(z))$ and

$$\|T(z)\psi(z)\|_z = \|T(z)C_\pm(z)\eta(z)\|_z \leq \|\eta(z)\|_z \ ,$$

so $\psi \in Dom(T(z))$. But $(A \pm iI)\psi = \eta$, thus $Ran(A \pm iI) = \mathcal{H}$ and so $\tilde{T}$ is self-adjoint.

2)+3): Let E denote the spectral family of $\tilde{T}$ and let E_z denote the spectral family of $T(z)$. Hence from Lemma 2.2.2.20 we have for every Borel set B, the equality $(E(B))(z) = E_z(B)$ and hence from Theorem 2.2.2.11 the equality

$$E = \int_Z^\oplus E_z d\mu(z) \ .$$

Again, from the spectral theory of self-adjoint operators, for a number $\lambda \in \mathbb{R}$ we have the statements:

- $\lambda \in \sigma(\tilde{T})$ if and only if $E((\lambda - \epsilon, \lambda + \epsilon)) \neq \Theta$ for all $\epsilon > 0$.
- $\lambda \in \sigma(\tilde{T})$ if and only if $\sigma(\tilde{T}) \cap (\lambda - \epsilon, \lambda + \epsilon) \neq \emptyset$.
- $\lambda \in \sigma_P(\tilde{T})$ if and only if $E(\{\lambda\}) \neq \Theta$.

Since from Theorem 2.2.2.11 we have that

$$E(B) = \int_Z^{\oplus} E_z(B) d\mu(z) = \Theta \tag{2.9}$$

if and only if $E_z(B) = \Theta$ for $\mu - a.e.\ z \in Z$.
Thus we have proved, that if $\lambda \in \sigma(\tilde{T})$(resp. $\lambda \in \sigma_p(\tilde{T})$), the corresponding inequality must hold. The sufficiency of the conditions holds obviously since if there is a set $W \subset Z$, such that for every $z \in W$, $\lambda \in \sigma(T(z))$(resp. $\lambda \in \sigma_p(T(z))$) and such that W is of non-zero measure, we have again from (2.9) that $E((\lambda - \epsilon, \lambda + \epsilon)) \neq \Theta$(resp. $E(\{\lambda\}) \neq \Theta$) and thus $\lambda \in \sigma(\tilde{T})$(resp. $\lambda \in \sigma_p(\tilde{T})$) and the statements 2) and 3) are proved. □

Remark 2.2.2.22. We will be working exclusively with the Lebesque measure on a finite interval, hence we are liberated from many obstacles, that the general theory may cause.

2.3 Hyperbolic geometry and Fuchsian groups

We use definitions and theorems, that the reader can find in Katok [23], Borthwick [8] and Pasles [29]. Of course, the problematic of hyperbolic geometry, Fuchsian groups, modular theory, and other closely related topics is generally vast and we here present only the minimum that we will need later.

2.3.1 The Lobachevsky plane

From now on we will denote the upper half-plane of $\mathbb{C}$ as $\mathbb{H}$. In this subsection, we use the terminology and theorems that the reader can find in Katok [23].

Definition 2.3.1.1. *A* **hyperbolic surface** *is a smooth surface equipped with a complete Riemannian metric of constant Gaussian curvature -1.*

In differential geometry it is well known, that there is, up to isometry, a unique simply connected hyperbolic surface, called the ***hyperbolic plane***, for which there are several models. The two mostly recognized are the ***Lobachevsky plane***(or the ***upper half-plane***) and the ***Poincaré disc***.

Definition 2.3.1.2. *The* **Lobachevsky plane model***, denoted by $M_{\mathbb{H}} = (\mathbb{H}, g_{\mathbb{H}})$, is the Riemannian manifold with the carrier set $\mathbb{H}$ and the metric*

$$g_{\mathbb{H}} = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{a^2}{y^2} \end{pmatrix} , \qquad z = x + iy \ .$$

The Lobachevsky plane model with the curvature parameter $a \neq 0$, denoted by $M_{\mathbb{H}}(a) = (\mathbb{H}, g_{\mathbb{H}}(a))$, is the Riemannian manifold with the carrier set $\mathbb{H}$ and the metric

$$g_{\mathbb{H}}(a) = \begin{pmatrix} \frac{a^2}{y^2} & 0 \\ 0 & \frac{a^2}{y^2} \end{pmatrix} , \qquad z = x + iy \ .$$

Definition 2.3.1.3. *The* **Poincaré disc model**, *denoted by $M_{\mathbb{B}} = (\mathbb{B}, g_{\mathbb{B}})$, is the Riemannian manifold with the carrier set $\mathbb{B} = \{z \in \mathbb{C}; |z| < 1\}$ and the metric*

$$g_{\mathbb{B}} = \begin{pmatrix} \frac{4}{(1-x^2-y^2)^2} & 0 \\ 0 & \frac{4}{(1-x^2-y^2)^2} \end{pmatrix} , \qquad z = x + iy \ .$$

Definition 2.3.1.4. *Let $I = [0,1]$, and $\gamma : I \to \mathbb{H}$ be a piece-wise differentiable path:*

$$\gamma = \{x(t) + iy(t); t \in I\}.$$

Then the **hyperbolic length** *is defined as*

$$h(\gamma) = \int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)} dt \ .$$

The **hyperbolic metric** *$\rho(z, w)$ is defined as*

$$\rho(z, w) = \inf\{h(\gamma); z, w \in \gamma\} \ .$$

for $z, w \in \mathbb{H}$.

Theorem 2.3.1.5. *The topology on $\mathbb{H}$ induced by the hyperbolic metric is the same as the topology induced by the Euclidean metric.*

Definition 2.3.1.6. *The group of all isometries of $\mathbb{H}$ is denoted by $Isom(\mathbb{H})$.*

Now we will express the metric $\rho(z, w)$ in a more appropriate form.

Theorem 2.3.1.7. *For $z, w \in \mathbb{H}$ we have:*

$$\rho(z, w) = \ln \frac{|z - \overline{w}| + |z - w|}{|z - \overline{w}| - |z - w|} \ .$$

Theorem 2.3.1.8. *Let $z_0 \in \mathbb{H}$. Define the mapping $f : \mathbb{H} \to \mathbb{C}$:*

$$f(z) = i\frac{z - z_0}{z + z_0}.$$

Then f is bijection between $\mathbb{H}$ a $\mathbb{B}$. Moreover, if we denote the metric on $(M_{\mathbb{H}}, g_{\mathbb{H}})$ by $\rho_{\mathbb{H}}$ and the metric on $(M_{\mathbb{B}}, g_{\mathbb{B}})$ by $\rho_{\mathbb{B}}$, then we have the following:

$$\rho_{\mathbb{B}}(x, y) = \rho_{\mathbb{H}}(f^{-1}(x), f^{-1}(y)) \ , \quad x, y \in \mathbb{B}.$$

Definition 2.3.1.9. *Let R be a number ring, denote by $SL(2, R)$ the group*

$$SL(2, R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; ac - bd = 1 \wedge a, b, c, d \in R \right\} \ .$$

Then

- *For $R = \mathbb{R}, \mathbb{C}$, the group $SL(2, R)$ is called the* **special linear group**.
- *For $R = \mathbb{Z}$, the group $SL(2, \mathbb{Z})$ is called the* **modular group**.

Also, the group $PSL(2, R)$ is defined as

$$PSL(2, R) = SL(2, R)/\{I_2, -I_2\},$$

where I_2 is the 2×2 identity matrix.

Definition 2.3.1.10. *The mapping $f : \mathbb{C} \to \mathbb{C}$ defined as*

$$f(z) = \frac{az+b}{cz+d}\ , \quad ac - bd = 1\ , \quad a, b, c, d \in \mathbb{C},$$

is called the **Möbius transformation**. *The set of all Möbius transformations is denoted by $A(\hat{\mathbb{C}})$, where $\hat{\mathbb{C}}$ is the extended complex plane, $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$.*

Theorem 2.3.1.11. *$A(\hat{\mathbb{C}})$ is isomorphic to $SL(2, \mathbb{C})$.*

Proof. Simple verification by composition of Möbius transformations and checking the group axioms. □

Theorem 2.3.1.12. *The group $Isom(\mathbb{H})$ is generated by $PSL(2, \mathbb{R})$ together with the transformation $z \to -\overline{z}$ for all $z \in \mathbb{H}$.*

Denote by $\mathcal{B}(\mathbb{H})$ the Borel σ-algebra on $\mathbb{H}$.

Definition 2.3.1.13. *For every $A \in \mathcal{B}(\mathbb{H})$ the measure μ on $\mathcal{B}(\mathbb{H})$ defined as*

$$\mu(A) = \int_A \frac{dxdy}{y^2},$$

is called the **hyperbolic area of** *A.*

Theorem 2.3.1.14. *The hyperbolic area is invariant under all transformations in $PSL(2, \mathbb{R})$, i.e. for $A \in \mathcal{B}(\mathbb{H})$ and $T \in PSL(2, \mathbb{R})$ the equality $\mu(T(A)) = \mu(A)$ holds.*

2.3.2 Fuchsian groups

In this subsection, we will take a closer look at some special subgroups of $PSL(2, \mathbb{R})$. Our tour will take us through the very basics of the theory of Fuchsian groups and then we will focus on a special type of subgroups of $PSL(2, \mathbb{R})$. Again, we follow Katok [23].

The following is a profound classification of elements in $PSL(2, \mathbb{R})$.

Definition 2.3.2.1. *Let $T \in PSL(2, \mathbb{R})$, denote by $Tr(T)$ the trace of the transformation T. Then*

- *If $Tr(T) < 2$, T is called* **elliptic**.
- *If $Tr(T) = 2$, T is called* **parabolic**.
- *If $Tr(T) > 2$, T is called* **hyperbolic**.

Let G be a subgroup of $PSL(2, \mathbb{R})$

- *If G contains only elliptic transformations, G is called an* **elliptic subgroup**.
- *If G contains only parabolic transformations, G is called an* **parabolic subgroup**.

- *If G contains only hyperbolic transformations, G is called an* **hyperbolic subgroup**.

We finally arrive at the definition of a Fuchsian group.

Definition 2.3.2.2. *A discrete subgroup G of $Isom(\mathbb{H})$ is called a* **Fuchsian group** *if it is a subgroup of $PSL(2,\mathbb{R})$.*

Theorem 2.3.2.3. *The following statements are true:*

1. *All hyperbolic and parabolic cyclic subgroups of $PSL(2,\mathbb{R})$ are Fuchsian group.*
2. *An elliptic cyclic subgroup of $PSL(2,\mathbb{R})$ is Fuchsian if and only if it is finite.*

Definition 2.3.2.4. *Let (X,ρ) be a metric space and let G be a group of homeomorphisms of (X,ρ). Let $\circ$ denote the action of G on X. Let $x \in X$. The* **stabilizer subgroup** *of x, denoted by G_x, is the set $G_x = \{g \in G; g \circ x = x\}$. A family $\{M_\alpha \subset X; \alpha \in A\}$ is called* **locally finite** *if for any compact subset $K \subset X$, $M_\alpha \cap K = \emptyset$ for only finitely many $\alpha \in A$. For $x \in X$, a family $Gx = \{g \circ x; g \in G\}$ is called the* **G-orbit of the point** *x. We say that G acts* **properly discontinuously** *on X if the G-orbit of any point $x \in X$ is locally finite.*

Theorem 2.3.2.5. *Let Γ be a subgroup of $PSL(2,\mathbb{R})$. Then Γ is a Fuchsian group if and only if Γ act properly discontinuously on $\mathbb{H}$.*

Theorem 2.3.2.6. *Every abelian Fuchsian group is cyclic.*

Definition 2.3.2.7. *Let (X,ρ) be a metric space and let G be a group of homeomorphisms acting properly discontinuously on (X,ρ). Let F be closed and $F^\circ \neq \emptyset$. Then F is called a* **fundamental region of** *G if*

1. $\displaystyle\bigcup_{T \in G} T(F) = X$.
2. *$F^\circ \cap T(F^\circ) = \emptyset$ for all $T \in G \setminus \{id_G\}$.*

Moreover, the family $\{T(F); T \in G\}$ is called the **tessellation** *of X.*

Theorem 2.3.2.8. *Define the group $P(\mathbb{Z}) = (P(\mathbb{Z})^\bullet, .)$ where set $P(\mathbb{Z})^\bullet$ is the set*

$$P(\mathbb{Z})^\bullet = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}; x \in \mathbb{Z} \right\} ,$$

and where . denotes the standart matrix multiplication. The following statements are true.

1. *$P(\mathbb{Z})$ is a Fuchsian group.*
2. *The action of $P(\mathbb{Z})$ is properly discontinuous.*
3. *$P(\mathbb{Z})$ is isomorphic with the group $(\mathbb{Z},+)$.*

Proof. 1). Since $P(\mathbb{Z})$ is a parabolic subgroup of $PSL(2,\mathbb{R})$ and it is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, Theorem 2.3.2.3 implies that $P(\mathbb{Z})$ is a Fuchian group. 2). Since $P(\mathbb{Z})$ is a Fuchian group, Theorem 2.3.2.5 implies that the action of $P(\mathbb{Z})$ is properly discontinuous. 3). Define the mapping $f : (\mathbb{Z},+) \to P(\mathbb{Z})$:

$$f(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

Hence verifying that f is a isomorphism the proof is finished. $\square$

Theorem 2.3.2.9. *The set*

$$F(P(\mathbb{Z})) = \{x + iy; x \in [0,1] \wedge y > 0\}$$

is a fundamental region of the group $P(\mathbb{Z})$.

Proof. Simple verification of the axioms of a fundamental region. One must also keep in mind that our metric space is the Lobachevsky plane, and the set $\{x+iy; x \in [0,1] \wedge y > 0\}$ is closed in the topology of the Lobachevsky plane. □

2.3.3 Multiplier systems

Here we briefly introduce the concept of a multiplier systems for the subgroups of $SL(2,\mathbb{R})$. We use the terminology from Roelcke [32], Elstrod [16] and Berndt-Knopp [6]. There have been made some generalizations in Pasles [29].

Definition 2.3.3.1. *Let* $k \in \mathbb{R}$ *and* G *is a discrete subgroup of* $SL(2,\mathbb{R})$. *Let there be a mapping* $w : G \times G \to \{-1, 0, 1\}$ *satisfying:*

$$w(M_1, M_2) + w(M_1M_2, M_3) = w(M_1, M_2M_3) + w(M_2, M_3),$$

where $M_i \in G, j = 1, 2, 3$. *Define the function* $\sigma_k(T,S) : G \times G \to \mathbb{R}$*:*

$$\sigma_k(T,S) = e^{2\pi ikw(T,S)} \ ,$$

which is known as the **factor system of** G. *Let the function* $v : G \to \mathbb{C}$ *satisfy the following:*

- $|v(M)| = 1$ *for every* $M \in G$.
- $v(-I) = e^{-\pi ik}$.
- $v(MN) = \sigma_k(M,N)v(M)v(N)$ *for every* $M, N \in G$.

Then v *is called a* **multiplier system of weight** k.

Remark 2.3.3.2. Of course, the term factor system is not tied only to the subgroups of $SL(2,\mathbb{R})$, but play an important role in general algebra algebra.

Remark 2.3.3.3. It is obvious from the definition that if one considers $k \in \mathbb{Z}$, then $\sigma_k(T,S) = 1$ for any $T, S \in G$ and any multiplier system is reduced to a group homomorphism from G to $\mathbb{C}$.

Now we will construct some multiplier systems on a special subgroup of $SL(2,\mathbb{R})$, that will be important to us later.

Definition 2.3.3.4. *Define the group* $\tilde{P}(\mathbb{Z}) = (\tilde{P}(\mathbb{Z})^{\bullet}, .)$, *where* . *the standard matrix multiplication and*

$$\tilde{P}(\mathbb{Z})^{\bullet} = \left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}; x \in \mathbb{Z}\right\} \bigcup \left\{\begin{pmatrix} -1 & x \\ 0 & -1 \end{pmatrix}; x \in \mathbb{Z}\right\} \ .$$

Theorem 2.3.3.5. *Let* $\theta \in [0, 2\pi)$, *let* $k \in \mathbb{Z}$. *Define the function* $v : \tilde{P}(\mathbb{Z}) \to \mathbb{C}$*:*

$$v\left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\right) = e^{i\theta n}.$$

Then v *is a multiplier system.*

Proof. Simple verification of the axioms of a multiplier system and the use of Remark 2.3.3.3. □

Chapter 3

The ordinary Schrödinger operator

3.1 Ordinary differential operators

In this section the basic notions of the spectral analysis of differential expressions will be presented. We will follow Weidmann [40] and Dunford-Schwartz [15], where all the definition and theorems are stated in a more general form than they are stated here. For the sake of simplicity we will work with L^2-spaces with Lebesque measure. In [40] the spaces are of the form $L^2(I, \rho(x)dx)$, where $\rho(x)$ is an appropriate function.

Remark 3.1.0.6. Throughout the whole section the interval $I \subset \mathbb{R}$ is considered arbitrary if not specified.

3.1.1 The minimal and the maximal operator

Definition 3.1.1.1. *Let $I \subset \mathbb{R}$ be an interval. Let $a_i(x) \in C^\infty(I), 0 \leq i \leq n, n \in \mathbb{N}$ and be function $a_n(x) \neq 0$ for all x in I. Then the following expression*

$$\tau = \sum_{j=0}^{n} a_j(x) \left(\frac{d}{dx} \right)^j \tag{3.1}$$

is called a **formal differential expression of order n**. *The functions $a_i(x)$ are called the* **coefficients** *and $a_n(x)$ is called the* **leading coefficient**.

Remark 3.1.1.2. The infinite differentiability condition of the functions $a_i(x), 0 \leq i \leq n, n \in \mathbb{N}$ could be weakened to a finite differentiability depending on the specific use of the formal differential expression. In our task the smoothness of the functions will suffice.

Definition 3.1.1.3. *Let τ be a formal differential expression of order n defined on the interval I. The formal differential expression*

$$\tau^* = \sum_{j=0}^{n} b_j(x) \left(\frac{d}{dx} \right)^j \tag{3.2}$$

where

$$b_j(x) = \sum_{k=j}^{n} (-1)^k \binom{k}{k} \left(\frac{d}{dx} \right)^{k-j} \overline{a_k(x)}$$

is called the **formal adjoint** *of the formal differential expression* τ. *If* $\tau = \tau^*$, *the* τ *is said to be* **formally self-adjoint** *or* **formally symmetric**. *If all the coefficient* $a_i(x)$ *are real, then* τ *is said to be* **real**.

Theorem 3.1.1.4. *Any formally self-adjoint formal differential expression of order n can be written in the unique form:*

$$\tau = \sum_{j=0}^{\left[\frac{n}{2}\right]}(-1)^j \left(\frac{d}{dx}\right)^j a_j(x) \left(\frac{d}{dx}\right)^j + i \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \left(\frac{d}{dx}\right)^j \left[\left(\frac{d}{dx}\right) b_j(x) + b_j(x) \left(\frac{d}{dx}\right)\right] \left(\frac{d}{dx}\right)^j$$

Conversely every formal differential expression in such form is formally self-adjoint.

Theorem 3.1.1.5. *Let* τ *be a real formal differntial expression of the form* (3.1). *Then* τ *is of the form*

$$\tau = \sum_{j=0}^{\left[\frac{n}{2}\right]}(-1)^j \left(\frac{d}{dx}\right)^j a_j(x) \left(\frac{d}{dx}\right)^j \tag{3.3}$$

Corollary 3.1.1.6. *A real formally self-adjoint formal differential expression defined on an interval I is of even order.*

Definition 3.1.1.7. *Let* τ *be a real formally self-adjoint formal differential expression of order 2 defined on an interval I. Then* τ *is a* **Sturm-Liouville differential expression** *and is of the form*

$$\tau = -\left(\frac{d}{dx}\right) p(x) \left(\frac{d}{dx}\right) + q(x). \tag{3.4}$$

Definition 3.1.1.8. *Let* τ *be a formal differential expression of order n on an interval I. Let* $T : Dom(T) \subset L^2(I, dx) \longrightarrow Ran(T) \subset L^2(I, dx)$ *be a linear operator defined as*

$$Tf = \tau f \ , \ f \in Dom(T). \tag{3.5}$$

Then the operator T is said to be **generated by** τ, *written T(τ).*

Remark 3.1.1.9. One can always choose a dense domain for an operator T on $L^2(I, dx)$ generated by a differential expression τ on an interval I. This is possible , for expample, for $Dom(T) = C_0^\infty(I)$, obviously because $\overline{C_0^\infty(I)} = L^2(I, dx)$.

Definition 3.1.1.10. *Let* T_0' *be a linear operator generated by the formal differential expression of order n on an interval I.* τ. *Let*

$$Dom(T_0') = \{f \in L^2(I, dx); f^{(i)} \in AC_0(I), 0 \le i \le n-1; \tau f \in L^2(I, dx)\}$$

Then the operator T_0' *is called the* **minimal operator** *generated by* τ, *written* $T_0'(\tau)$.

Definition 3.1.1.11. *Let T be a linear operator generated by the formal differential expression of order n on an interval I.* τ. *Let*

$$Dom(T) = \{f \in L^2(I, d\mu(x)); f^{(i)} \in AC(I), 0 \le i \le n-1; \tau f \in L^2(I, d\mu(x))\}.$$

Then the operator T is called the **maximal operator** *generated by* τ, *written* $T(\tau)$.

Theorem 3.1.1.12. *Let τ be a formal differntial expression of order n defined on an interval I. Then*

$$T(\tau) = T_0'(\tau^*)^*.$$

Corollary 3.1.1.13. *If τ is a formally self-adjoint formal differntial expression of order n defined on an interval I, then the operator $T_0'(\tau)$ is symmetric.*

Thus it makes sense to define the following:

Definition 3.1.1.14. *If τ is a formally self-adjoint formal differntial expression of order n defined on an interval I, then we define the operator $T_0(\tau) = \overline{T_0'(\tau)}$.*

The last two theorems can be found in Weidmann [40].

Theorem 3.1.1.15. *Let τ be a formally self-adjoint formal differntial expression of order n defined on an interval I. Then for $f \in Dom(T_0'^*(\tau))$ and $g \in Dom(T^*(\tau))$ we have*

$$(T_0'(\tau)f, g) = (f, T(\tau)g)$$

and henceforth $T(\tau) \subset T_0'^*(\tau)$.

Theorem 3.1.1.16. *If τ is a formally self-adjoint formal differntial expression of order n defined on an interval I. Then $\overline{Dom(T_0'(\tau))} = L^2(I, d\mu(x))$ and $T_0'^*(\tau) \subset T(\tau)$. From the previous theorem it follows that $T(\tau) = T_0'^*(\tau) = T_0^*(\tau)$.*

Now we arrive at a very important classification of differential expressions.

Definition 3.1.1.17. *Let τ be a formal differential expression of the form (3.1) defined on the interval $I = (a, b)$. If $a > -\infty$ and $a_i \in L^2_{loc}([a, b))$, then τ is said to be* **regular at a**. *If $b < +\infty$ and $a_i \in L^2_{loc}((a, b])$, then τ is said to be* **regular at b**. *If τ is regular at both endpoints, we say that τ is* **regular**. *If τ is regular at most at one boundary point, then τ is said to be* **singular**.

3.1.2 The limit point and the limit circle case

In this subsection we assume that τ is a formally self-adjoint formal differential expression of order n defined on the interval I if not specified.

Definition 3.1.2.1. *Let f be a measurable function on an interval I with endpoints a, b. If for all $c \in (a, b)$ $f \in L^2((a, c), dx)$, f is said to* **lie left**. *If for all $c \in (a, b)$ $f \in L^2((c, b), dx)$, f is said to* **lie right**.

Theorem 3.1.2.2. *Let τ be defined on the interval I with endpoints a, b.If for some $\lambda_0 \in \mathbb{C}$ all solutions of $(\tau - \lambda I)u = 0$ and of $(\tau - \overline{\lambda} I)u = 0$ lie right in $L^2(I, dx)$, then this holds $\forall \lambda \in \mathbb{C}$. An analogous statement holds for left case.*

Definition 3.1.2.3. *Let τ be defined on the interval I with endpoints a, b. We say that τ is* **quasi-regular at b** *if for some $\lambda \in \mathbb{C}$ all solutions of $(\tau - \overline{\lambda} I)u = 0$ lie right in $L^2(I, dx)$. We say that τ is* **quasi-regular at a** *if for some $\lambda \in \mathbb{C}$ all solutions of $(\tau - \overline{\lambda} I)u = 0$ lie left in $L^2(I, dx)$. We say that τ is* **quasi-regular** *if it is quasi regular at a and b.*

Here we arrive at a very profound theorem in the spectral theory of ordinary differential operators, which proof the reader can find in Weidmann [40].

Theorem 3.1.2.4 (Weyl's alternative). *Let τ be real and of order 2 defined on the interval I. Then exactly one of the following statements hold:*

1. *For every $\lambda \in \mathbb{C}$ all solutions of $(\tau - \lambda I)u = 0$ lie right in $L^2(I, d\mu(x))$.*
2. *For every $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exists a unique(up to a multiplicative constant) solution u of $(\tau - \lambda I)u = 0$ which lies right in $L^2(I, dx)$.*

An analogous statement for the left side holds.

Definition 3.1.2.5. *Let τ be real and of order 2 defined on the interval I with endpoints a, b. We say that:*

1. *τ is in the* **limit circle case(l.c.c)** *at a, if for every $\lambda \in \mathbb{C}$ all solutions of $(\tau - \lambda I)u = 0$ lie left in $L^2(I, dx)$.*
2. *τ is in the* **limit point case(l.p.c)** *at a, if for every $\lambda \in \mathbb{C}$ there is at least one solution of $(\tau - \lambda I)u = 0$, which does not lie left in $L^2(I, dx)$.*
3. *τ is in the* **limit circle case(l.c.c)** *at b, if for every $\lambda \in \mathbb{C}$ all solutions of $(\tau - \lambda I)u = 0$ lie right in $L^2(I, dx)$.*
4. *τ is in the* **limit point case(l.p.c)** *at b, if for every $\lambda \in \mathbb{C}$ there is at least one solution of $(\tau - \lambda I)u = 0$, which does not lie right in $L^2(I, dx)$.*

The following theorem, which's proof can be found in Weidmann [40], will be of a great importance for us later.

Theorem 3.1.2.6. *Let τ be real and of order 2 defined on the interval I with endpoints a, b. Let τ be in the limit point case at both endpoints. Then*

1. *The deficiency indices of the operator $T(\tau)$ are $(0, 0)$.*
2. *$Dom(T(\tau)) = Dom(T_0(\tau))$.*
3. *$T(\tau) = T_0(\tau)$ is the only self-adjoint extension of $T_0(\tau)$.*

3.1.3 The limit point-limit circle criteria

We present here one criterion for the limit point case that will be sufficient for our needs. The reader can find more criteria for special cases in Dunford-Schwartz [15] and few in Weidmann [40].

Theorem 3.1.3.1. *Let τ be a Sturm-Lioville differential expression of the form (3.4) defined on the interval $I = (a, b)$. For some $c \in (a, b)$ assume that $p > 0$ a.e. in (c, b) and define*

$$g(x) := \int_c^x \frac{1}{p(x)} dx$$

for $x > c$. If $g \notin L^2((c, b), dx)$ and

$$\liminf_{x \to b} q(x) > -\infty$$

then τ is in the limit point case at b. The analogous result holds for the boundary point a.

3.1.4 The resolvent

Here we briefly give two useful theorems concerning the resolvent of a self-adjoint operator generated by a differential expression. The proofs of the theorems and a far more broader theory can be found in [15].

Theorem 3.1.4.1. *Let T be a self-adjoint operator derived from a Sturm-Liouville differential expression τ on the interval I with endpoinst a, b by the imposition of a separated symmetric set of boundary conditions. Let $\Im\lambda \neq 0$. Then the boundary conditions are real, and there is exactly one solution $\phi(t,\lambda)$ of $(\tau-\lambda)\sigma = 0$ square-integrable at a and satisfying the boundary conditions at a, and exactly one solution $\psi(t,\lambda)$ of $(\tau-\lambda)\sigma = 0$ square-integrable at b and satisfying the boundary conditions at b. Also the resolvent $R_T(\lambda)$ is an integral operator whose kernel $K(t,s;\lambda)$ is in the form*

$$K(t,s;\lambda) = \frac{\psi(t,\lambda)\phi(s,\lambda)}{p(t)W_t(\phi(\lambda),\psi(\lambda))}, s < t$$

$$K(t,s;\lambda) = \frac{\phi(t,\lambda)\psi(s,\lambda)}{p(t)W_t(\phi(\lambda),\psi(\lambda))}, s > t$$

The next theorem tells us, when the resolvent is compact.

Theorem 3.1.4.2. *Let τ be a formally symmetric formal differential expression defined on an interval I. Let T be a self-adjoint extension of the symmetric operator $T_0(\tau)$. The resolvent $R_T(\lambda)$ is compact for every non-real λ if either the interval I is compact or the deficiency indices of $T_0(\tau)$ are equal to the order of the differential expression τ.*

Since we will later work mostly on $\mathbb{R}$ with operators having both deficiency indices zero, we will find use only for the first theorem.

3.1.5 Spectral measure, representation and the Kodaira-Weyl-Titchmarsch theory

Now we present a very profound theory in spectral analysis of ordinary differential operators. We use the definitions and theorems that one can find in Dunford-Schwartz [15].

Definition 3.1.5.1. *Let W be a measurable function defined on the product $S \times T$ of two measure spaces $(S, \mathcal{S}, \sigma)$ and $(T, \mathcal{T}, \tau)$, and let h be a σ-measurable function on S. We say, that the integral*

$$\int_S h(s)W(s,t)d\sigma(s)$$

exists in the mean square sense *in $L^2(T,\mathcal{T},\tau)$ if there is an increasing sequence $\{S_n\}$ of sets of finite σ-measure which covers S and such that for each $n \in \mathbb{N}$, $h(.)W(.,t)$ is σ-measurable for τ-almost all $t \in T$ and the function F_n defined by the equation*

$$F_n(t) = \int_{S_n} h(s)W(s,t)d\sigma(s)$$

is in $L^2(T,\mathcal{T},\tau)$ and converges in $L^2(T,\mathcal{T},\tau)$ as $n \to +\infty$. If $F_n \to F$ in $L^2(T,\mathcal{T},\tau)$, we write

$$F(t) = \int_S h(s)W(s,t)d\sigma(s) \ .$$

The first theorem is a rather general for operators on $L^2(\mathbb{R}, dx)$.

Theorem 3.1.5.2. *Let $(\mathbb{R}, \mathcal{B}, \mu)$ be a positive measure space with $\mathcal{B}$ being the Borel σ-algebra on $\mathbb{R}$, let T be a self-adjoint operator in $L^2(\mathbb{R}, d\mu)$ Let $\{S_n\}_{n=1}^{+\infty}$ be an increasing sequence of sets of finite measure μ covering $\mathbb{R}$. Let U be a spectral representation of $L^2(\mathbb{R}, d\mu)$ onto $\bigoplus_{\alpha \in A} L^2(\mathbb{R}, d\mu_\alpha)$. Let E be the spectral family for T and suppose that for each bounded Borel set B of real numbers the range of the projector $E(B)$ contains only functions which are μ-essentially bounded on each of the sets S_n. Then for each element $\alpha \in A$ there is a function W_α defined on $\mathbb{R}^2$ and having the properties*

1. *W_α is measurable with respect to the product measure $\mu \otimes \mu_\alpha$*
2. *for each bounded Borel set B on the real line we have*

$$\operatorname*{ess-sup}_{s \in S_n} \int_B |W_\alpha(s, \lambda)|^2 \mu_\alpha(d\lambda) < +\infty$$

3. *$(Uf)_\alpha(\lambda) = \int_S f(s)\overline{W_\alpha(s, \lambda)}\mu(ds)$, $f \in L^2(\mathbb{R}, \mu)$, the integral exists in the mean square sense $L^2(\mathbb{R}, \mu_\alpha)$.*

Moreover

$$f(s) = \sum_{\alpha \in A} \int_{-\infty}^{+\infty} (Uf)_\alpha(\lambda) W_\alpha(s, \lambda)\mu_\alpha(d\lambda)$$

for $f \in L^2(\mathbb{R}, d\mu)$, the the integrals exist in the mean square sense and the sum converges in L^2 norm.

In addition suppose that $L^2(\mathbb{R}, d\mu)$ is separable. Let ν be the measure of the ordered representation of $L^2(\mathbb{R}, d\mu)$ relative to T and B_n, $1 \le n \le k$, its multiplicity sets. Let W_n, $1 \le n \le k$ be the corresponding kernels defined above. Then for every $n \in \mathbb{N}$ which does not exceed the spectral multiplicity of the ordered representation, the set $\{W_1(\bullet, \lambda), ..., W_n(\bullet, \lambda)\}$ in the space of μ-measurable functions is linearly independent ν-almost everywhere on B_n.

But now we focus on its special variant for ordinary differential operators:

Theorem 3.1.5.3. *Let τ be a formally self-adjoint formal differential expression of order n on an interval I, and let T be a self-adjoint extension of $T_0(\tau)$. Let ν denotes the Lebesque measure on $\mathbb{R}$. Let U be an ordered representation of $L^2(I, d\nu)$ relative to T, with measure μ, multiplicity sets B_i, and spectral multiplicity m. Then $m \le n$. There exist kernels $W_i(t, \lambda)$, $i = 1, ..., m$, measurable with respect to $\nu \otimes \mu$, which vanish for λ in the complement of B_i, belong to $C^{+\infty}(I)$ for each fixed λ, and satisfy the differential equation $(\tau - \lambda)W_i(\bullet, \lambda) = 0$ for each fixed λ. Moreover the kernels W_i have the property that*

$$\nu - \operatorname*{ess-sup}_{t \in J} \int_B |W_i(t, \lambda)|^2 \mu(d\lambda) < +\infty$$

for each compact subinterval J of I and bounded Borel set B, and are such that

1. *$(Uf)_i(\lambda) = \int_I f(t)\overline{W_i(t, \lambda)}dt$, for $f \in L^2(\mathbb{R}, d\nu)$ and the integral existing in the mean square sence in $L^2(B_i, d\nu)$*

2. *for each Borel function* F,

$$U(Dom(F(T))) = \left\{ (f_i)_{i=1}^m \in \bigoplus_{k=1}^m L^2(B_k, d\mu); \sum_{k=1}^m \int_{-\infty}^{+\infty} |F(\lambda)|^2 |f_i(\lambda)|^2 \mu(d\lambda) < +\infty \right\}$$

and $(UF(T)g)_i(\lambda) = F(\lambda)(Ug)_i(\lambda)$, $g \in Dom(F(T))$, $-\infty < \lambda < +\infty$.

Moreover, for each $f \in L^2(I, d\nu)$ *we have*

$$f(t) = \lim_{N \to +\infty} \int_{-N}^{N} \sum_{i=1}^m (Uf)_i(\lambda) W_i(t, \lambda) \mu(d\lambda),$$

the limit existing in the mean square sense in $L^2(I, d\nu)$.

Definition 3.1.5.4. *Let* $\{\mu_{ij}\}_{i,j=1}^n$, $n \in \mathbb{N}$, *be a family of complex valued set functions defined on the bounded Borel subsets of the real line. The family* $\{\mu_{ij}\}_{i,j=1}^n$ *will be called an* n *by* n **positive matrix measure** *if*

1. *the matrix* $\{\mu_{ij}(B)\}_{i,j=1}^n$ *is Hermitian and positive semi-definite for each bounded Borel set* B.
2. *we have*

$$\mu_{ij}\left(\bigcup_{m=1}^{+\infty} B_n\right) = \sum_{m=1}^{+\infty} \mu_{ij}(B_m)$$

for each sequence of disjoint Borel sets with bounded union.

Theorem 3.1.5.5 (Weyl-Kodaira). *Let* τ *be a formally self-adjoint formal differential expression of order* n *defined on an interval* I *with end-points* a, b. *Let* T *be a self-adjoint extension of* $T_0(\tau)$. *Let* Λ *be an open interval of the real axis, and suppose that there is given a set* $\sigma_1, ..., \sigma_n$ *of functions, defined and continuous on* $I \times \Lambda$, *such that for each fixed* $\lambda \in \Lambda$, $\sigma_1(\bullet, \lambda), ..., \sigma_n(\bullet, \lambda)$ *forms a basis for the space of solutions of* $\tau\sigma = \lambda\sigma$. *Then there exists a positive* $n \times n$ *matrix measure* $\{\rho_{ij}\}$ *defined on* Λ, *such that*

1. *the limit*

$$(Vf)_i(\lambda) = \lim_{\substack{c \to a \\ d \to b}} \int_c^d f(t)\overline{\sigma_i(t, \lambda)} dt$$

exists in the topology of $L^2(\Lambda, \{\rho_{ij}\})$ *for each* $f \in L^2(I, dx)$ *and defines an isometric isomorphism* $V : E(\Lambda)L^2(I, dx) \to L^2(\Lambda, \{\rho_{ij}\})$

2. *for each Borel function* G *defined on the real line and vanishing outside* Λ,

$$(VG(T)f)_i(\lambda) = G(\lambda)(Vf)_i(\lambda)$$

for $i = 1, ..., n$, $\lambda \in \Lambda$, $f \in Dom(G(T))$.

Moreover, the positive matrix measure $\{\rho_{ij}\}$ *on* Λ *is unique.*

Theorem 3.1.5.6 (Titchmarsh-Kodaira). *Let τ be a formally self-adjoint formal differential expression of order n defined on an interval I with end-points a, b. Let T be a self-adjoint realization of τ. Let Λ be an open interval of the real axis and U be an open set in the complex plane containing Λ. Let $\sigma_1, ..., \sigma_n$ be a set of functions which form a basis for the solutions of the equations $(\tau - \lambda)\sigma = 0$, $\lambda \in U$, and which are continuous on $I \times U$ and analytically dependent on λ for $\lambda \in U$. Suppose that the kernel $K(t, s; \lambda)$ for the resolvent $R_T(\lambda)$ has the representation*

$$K(t, s; \lambda) = \sum_{i,j=1}^{n} \theta_{i,j}^{-} \sigma_i(t, \lambda) \overline{\sigma_j(s, \overline{\lambda})}, t < s$$

$$K(t, s; \lambda) = \sum_{i,j=1}^{n} \theta_{i,j}^{+} \sigma_i(t, \lambda) \overline{\sigma_j(s, \overline{\lambda})}, t > s$$

for all $\lambda \in \rho(T) \cap U$, and that $\{\rho_{ij}\}$ is a positive matrix measure on Λ associated with T as in Theorem 3.1.5.5. Then the functions $\theta_{ij}^{\pm}$ are analytic in $U \cap \rho(T)$, and given any bounded open interval $(\lambda_1, \lambda_2) \subseteq \Lambda$, we have for $1 \leq i, j \leq n$,

$$\begin{aligned} \rho_{ij}((\lambda_1, \lambda_2)) &= \lim_{\delta \to 0} \lim_{\epsilon \to 0_+} \frac{1}{2\pi i} \int_{\lambda_1 + \delta}^{\lambda_2 - \delta} \left[\theta_{ij}^{-}(\lambda - i\epsilon) - \theta_{ij}^{-}(\lambda + i\epsilon) \right] d\lambda \\ &= \lim_{\delta \to 0} \lim_{\epsilon \to 0_+} \frac{1}{2\pi i} \int_{\lambda_1 + \delta}^{\lambda_2 - \delta} \left[\theta_{ij}^{+}(\lambda - i\epsilon) - \theta_{ij}^{+}(\lambda + i\epsilon) \right] d\lambda \end{aligned}$$

If $\lambda_0 \in \Lambda \cap \sigma(T)$ is an isolated point, it is an isolated singularity of θ_{ij}^{+}(or, equivalently θ_{ij}^{-}). Moreover, $\rho_{ij}(\{\lambda_0\})$ is the residue at λ_0 of θ_{ij}^{+}(of θ_{ij}^{-}). If $\{\lambda_0\} = (a, b) \cap \sigma(T)$ and $C_{\delta,\epsilon}$ denotes the rectangle with corners $a + \epsilon + i\delta$, $a + \epsilon - i\delta$, $b - \epsilon - i\delta$, $b - \epsilon + i\delta$, then

$$\rho_{ij}(\{\lambda_0\}) = \lim_{\delta \to 0} \lim_{\epsilon \to 0_+} \frac{1}{2\pi i} \oint_{C_{\delta,\epsilon}} \theta_{ij}^{+}(\zeta) d\zeta$$

and similarly

$$\rho_{ij}(\{\lambda_0\}) = \lim_{\delta \to 0} \lim_{\epsilon \to 0_+} \frac{1}{2\pi i} \oint_{C_{\delta,\epsilon}} \theta_{ij}^{-}(\zeta) d\zeta$$

Theorem 3.1.5.7. *With the assumptions of Theorem 3.1.5.6, suppose that τ is a formal differential expression with real coefficients, that each of the functions σ_j is real for $t, \lambda \in \mathbb{R}$, and that the operator τ is defined by a set of real boundary conditions. Then we may write*

$$\rho_{ij}((\lambda_1, \lambda_2)) = \lim_{\delta \to 0} \lim_{\epsilon \to 0_+} \frac{1}{\pi} \int_{\lambda_1 + \delta}^{\lambda_2 - \delta} \Im \theta_{ij}^{-}(\lambda - i\epsilon) d\lambda = \lim_{\delta \to 0} \lim_{\epsilon \to 0_+} \frac{1}{\pi} \int_{\lambda_1 + \delta}^{\lambda_2 - \delta} \Im \theta_{ij}^{+}(\lambda - i\epsilon) d\lambda$$

Definition 3.1.5.8. *The matrix ρ_{ij} in Theorem 3.1.5.6 and Theorem 3.1.5.7 is called the* **spectral matrix** *and the matrix θ_{ij}^{+}(or θ_{ij}^{-}) is called the* **characteristic matrix**.

3.1.6 The spectral multiplicity

In this subsection we present few useful theorems regarding spectral multiplicity of ordinary differential operators. All the presented theorems with proofs can be found in Weidmann [40].

From Theorem 3.1.5.3 we have the following:

Corollary 3.1.6.1. *Let τ be a formal differential expression, let T be its self-adjoint realization. Then every point of the spectrum of T has finite spectral multiplicity.*

We can be little more specific:

Theorem 3.1.6.2. *Let τ be a formal differential expression on the interval (a, b), let T be its self-adjoint realization, (λ_1, λ_2) a real interval, Q an open set in $\mathbb{C}$ containing (λ_1, λ_2). Assume that $\{w_1(x, z), ..., w_p(x, z)\}$ is a fundamental system of $(\tau - zI)w = 0$ which is continuous in Q and such that for some $l, k \in \{0, ..., p\}$:*

- *if $\sum_{j=1}^{p} c_j w_j(x, z)$ lies left in $L^2((a, b), dx)$ and satisfies the boundary conditions at a, then $c_{k+1} = ... = c_l = 0$,*
- *if $\sum_{j=1}^{p} c_j w_j(x, z)$ lies right in $L^2((a, b), dx)$ and satisfies the boundary conditions at b, then $c_{l+1} = ... = c_p = 0$.*

Then the spectral multiplicity of T in (λ_1, λ_2) is at most k.

Now if we consider a Sturm-Liouville differential expression on the interval (a, b) being the limit point case at both endpoints, its obvious, that the spectral multiplicity of every eigenvalue is 1. We can make a similar, a more general statement:

Theorem 3.1.6.3. *Let τ be a Sturm-Liouville differential expression on the interval (a, b) and let τ be the limit point case at both endpoints and let T be its self-adjoint realization. Choose some $c \in (a, b)$. If the self-adjoint realizations of τ on (a, c) have discrete spectrum in some interval (λ_1, λ_2), then T has simple spectrum in (λ_1, λ_2).*

3.1.7 The singular spectrum of ordinary differential operators

The next theorem will give us some information about the singular continuous spectrum. The proof can be found in Weidmann [40].

Theorem 3.1.7.1. *Let τ be a formal differential expression, let T be its self-adjoint realization with separated boundary conditions. Let (λ_1, λ_2) be a real interval, Q an open subset of $\mathbb{C}$ such that $(\lambda_1, \lambda_2) \subset Q$. Assume that for $z \in Q$ there exist solutions $u_k(x, z), k = 1..p$ of $(\tau - zI)u = 0$ analytically dependent on z such that for $z \in Q_+ = \{z \in Q; \Im z > 0\}$*

- *$u_1, ..., u_k$ are linearly independent and lie left in $Dom(T)$.*
- *$u_{k+1}, ..., u_p$ are linearly independent and lie right in $Dom(T)$.*

Then $\sigma_p(T)$ has no accumulation points in (λ_1, λ_2) and $\sigma_{sc}(T) \cap (\lambda_1, \lambda_2) = \emptyset$.

3.2 Spectral theory of ordinary Schrödinger operators

3.2.1 The Schrödinger operator

Definition 3.2.1.1. *Let τ_S be a expression of the type (3.4) defined on an interval I with $p(x) = 1$ and $q(x) = V(x)$:*

$$\tau_S = -\left(\frac{d}{dx}\right)^2 + V(x). \tag{3.6}$$

Then τ_S is called a **Schrödinger differential expression**, *where $V(x)$ is a function called the* **potential** *of the Schrödinger differential expression.*

Definition 3.2.1.2. *Let τ_S be a Schrödinger differential expression defined on I with a potential $V(x)$. If there exist a constant $C > 0$ such that $\forall x \in I$,$V(x) \geq C$, then $V(x)$ is said to be* **bounded from below**.

Remark 3.2.1.3. *For the rest of this subsection, we will assume that $I \subset \mathbb{R}$ is a arbitrary interval unless specified otherwise.*

Definition 3.2.1.4. *Let τ_S be a Schrödinger differential expression defined on I. Let T_S be the operator generated by τ_S on some domain $Dom(T_S)$ dense in $L^2(I, dx)$. Then T_S is called a* **Schrödinger operator**. *The operator T'_{S0} defined according to 3.1.1.10 is called the* **minimal** *Schrödinger operator.The operator T_S defined according to 3.1.1.11 is called the* **maximal** *Schrödinger operator.*

The next theorem comes in handy, since it gives sufficient conditions for the essential self-adjointness by giving conditions on the potential $V(x)$. The proof is in Berezin-Shubin [4].

Theorem 3.2.1.5. *Let T_S be a Schrödinger operator on $I = \mathbb{R}$ with a potential $V(x)$ and with $Dom(T_S) = C_0^\infty(\mathbb{R})$. Let $Q(x)$ be an positive even function on $\mathbb{R}$ that is non-decreasing for $x \geq 0$, satisfies*

$$\int_{-\infty}^{+\infty} \frac{dx}{\sqrt{Q(2x)}} = +\infty. \tag{3.7}$$

and $\forall x \in \mathbb{R}$

$$V(x) \geq -Q(x). \tag{3.8}$$

Then T_S is essentially self-adjoint.

Now we will spend the remaining of this subsection for studying the asymptotic behavior of the solutions of a chrödinger operator with a potential $V(x)$. It will come very handy in proving few important theorems concerning the spectrum of an operator generated by a special ordinary differential expression. The proofs of the following theorems can by found in Berezin-Shubin [4].

Theorem 3.2.1.6. *Let T_S be a Schrödinger operator with a potential $V(x)$ on and interval I that is unbounded from the right. Let $a \in \mathbb{R}$ and $V(x) \geq \epsilon > 0$ for $x \geq a$. Then for any solution y of the equation*

$$\left[-\left(\frac{d}{dx}\right)^2 + V(x)\right] y(x) = 0 \tag{3.9}$$

one of the following limits holds

1. *$y(x) \to \pm\infty$ as $x \to +\infty$;*
2. *$y(x) \to 0$ as $x \to +\infty$.*

A solution satisfying 2. exists and is unique (up to a constant factor).

Theorem 3.2.1.7. *Let T_S be a Schrödinger operator with a potential $V(x)$ on and interval I that is unbounded from the right. Let $V(x) \to +\infty$ as $x \to +\infty$. Then for any solution y of equation* (3.9) *one and only one of the following two assertions is true:*

1. *for any $k > 0$ there exists $A \in \mathbb{R}$ such that $|y(x)| \geq e^{kx}$ if $x \geq A$;*

2. *for any* $k > 0$ *there exists* $A \in \mathbb{R}$ *such that* $|y(x)| \leq e^{-kx}$ *if* $x \geq A$.

Theorem 3.2.1.8 (Asymptotic behaviour of the solutions)**.** *Let* T_S *be a Schrödinger operator with a measurable, locally bounded potential* $V(x)$ *on an interval* I *that is unbounded from the right. Consider the three asymptotic behaviours of* $V(x)$ *towards* $+\infty$*:*

1. $V(x) \to +\infty$ *as* $x \to +\infty$
2. $V(x) \to 0$ *as* $x \to +\infty$
3. $V(x) \to -\infty$ *as* $x \to +\infty$

Simultaneously, let $V(x)$ *satisfy*

$$\int_{x_0}^{+\infty} \frac{|V'(x)|^2}{|V(x)|^{\frac{5}{2}}} dx < +\infty \ , \ \int_{x_0}^{+\infty} \frac{|V''(x)|^2}{|V(x)|^{\frac{3}{2}}} dx < +\infty \tag{3.10}$$

in the cases 1. and 3. and

$$\int_{x_0}^{+\infty} |V'(x)|^2 dx < +\infty \ , \ \int_{x_0}^{+\infty} |V''(x)| dx < +\infty \tag{3.11}$$

in the case 2. . The number x_0 *can be chosen arbitrarily large. Then, for the solutions of the equation*

$$T_S y = k^2 y \ , \ k \in \mathbb{C} \tag{3.12}$$

the following assertions about the asymptotic behaviour for $x \to +\infty$ *of the solutions of* (3.12) *hold:*

a) *if* $V(x)$ *satisfies 1. then*

$$y_\pm(x) \sim V(x)^{-1/4} \exp\left(\pm \int_{x_0}^{x} \sqrt{V(t) - k^2} dt\right)(1 + o(1)) \tag{3.13}$$

b) *if* $V(x)$ *satisfies 2. and* $k \neq 0$ *then*

$$y_\pm(x) \sim \exp\left(\pm ik \int_{x_0}^{x} \sqrt{1 - \frac{V(t)}{k^2}} dt\right)(1 + o(1)) \tag{3.14}$$

c) *if* $V(x)$ *satisfies 3. then*

$$y_\pm(x) \sim (-V(x))^{-1/4} \exp\left(\pm \int_{x_0}^{x} \sqrt{k^2 - V(t)} dt\right)(1 + o(1)) \tag{3.15}$$

Again, the number x_0 *can be chosen arbitrarily large.*

Remark 3.2.1.9. From this theorem, we will use the asymptotic behavior of the solutions of (3.9) with $V(x) \to 0$.

3.2.2 The spectrum of the Schrödinger operator

Lemma 3.2.2.1. *Let T_S be a self-adjoint Schrödinger operator in $L^2(\mathbb{R})$ with a potential $V \in C^2(\mathbb{R})$ on $\mathbb{R}$ such that at least one of the following conditions is fulfilled:*

1. $V(x) \to 0$ *as* $x \to -\infty$.
2. $V(x) \to 0$ *as* $x \to +\infty$.

Moreover, let V satisfy the conditions 3.11. Then $\sigma_p(T_S) \subset (-\infty, 0)$.

Proof. We will prove the case at $-\infty$. The case at $+\infty$ is treated analogously. From Theorem 3.2.1.8 we know the asymptotic behavior of the solutions of

$$T_S y = k^2 y \ , \qquad k^2 = l \in \mathbb{R}$$

and by this behavior they are uniquely determined. Let $k_0^2 = l_0 \in \sigma_p(T_S)$, so according to (3.14) it must decay exponentially, for, in absolute value, sufficiently large $x_0 < 0$, but this happens only in the case $ik_0 > 0$. Thus $k_0 = -ic$ for some $c > 0$ and so $l_0 < 0$ hence $\sigma_p(T_S) \subset (-\infty, 0)$. □

Theorem 3.2.2.2. *A self-adjoint operator A is bounded form below if and only if its spectrum is bounded from below. The greatest lower bound of A is equal to $\min \sigma(T)$.*

In the following theorems we can find sufficient conditions on the Schrödinger operator(mostly the potential V) that give us some important information about the location and the structure of the spectrum. We use theorems from Schechter [33], where many other cases are studied.

Theorem 3.2.2.3. *Let T_S be a self-adjoint Schrödinger operator on I with a potential $V(x)$, and let $V(x) \geq c \in \mathbb{R}$. Then*

$$(-\infty, c) \subset \rho(T_S)$$

Theorem 3.2.2.4. *Let T_S be a self-adjoint Schrödinger operator on $I = \mathbb{R}$ with a potential $V(x)$ and let $\sigma_{ess}(H) \subset [0, +\infty)$. If there is a number a such that*

$$\inf_{\alpha > 0} \alpha^{-1} \int_{-\infty}^{+\infty} V(x) e^{-\alpha(x-a)^2} dx < -\sqrt{\frac{\pi}{2}} \tag{3.16}$$

then T_S has at least one negative eigenvalue.

Theorem 3.2.2.5. *Let T_S be a self-adjoint Schrödinger operator on $I = \mathbb{R}$ with a potential $V(x)$. Assume that $\sup_\gamma \int_\gamma^{\gamma+1} V_-(x) dx < +\infty$ and (3.16) holds. If there is an N such that for $R > N$*

$$R \int_R^{+\infty} V_-(\pm x) dx \leq \frac{1}{4} \tag{3.17}$$

then T_S has at most a finite number of negative eigenvalues. In particular, this is true if for $|x| > N$

$$V(x) \geq -\frac{1}{4x^2}. \tag{3.18}$$

Theorem 3.2.2.6. *Let T_S be a self-adjoint Schrödinger operator on and interval $I = \mathbb{R}$ with a potential $V(x)$, which is locally bounded, bounded from below and has a continuous negative part. Then*

$$N_-(T_S) \leq 1 + \int_{-\infty}^{+\infty} |x||V_-(x)|dx \tag{3.19}$$

Theorem 3.2.2.7. *Let T_S be a self-adjoint Schrödinger operator on I with a potential $V(x)$.Define for $t > 0$ the following:*

$$\nu_t = \inf_\gamma t^{-1} \int_\gamma^{\gamma+t} V(x)dx,$$

$$\eta_t = \sup_\gamma t^{-1} \int_\gamma^{\gamma+t} V_-(x)dx,$$

where $V_-(x) = max\{-V(x), 0\}$. If $t^2\eta_t < \frac{\pi^2}{8}$ then

$$\sigma(T_S) \subset \left[\nu_t - \frac{4t^2(\nu_t + 2\eta_t)^2}{\pi^2 + 4t^2\nu_t}, +\infty\right). \tag{3.20}$$

Define

$$\nu_0 = \limsup_{t\to 0+} \nu_t$$

$$\eta_0 = \limsup_{t\to 0+} t\eta_t.$$

If $\nu_0 < +\infty$ then

$$\sigma(T_S) \subset \left[\nu_0 - \frac{16\eta_0^2}{\pi^2}, +\infty\right). \tag{3.21}$$

Theorem 3.2.2.8. *Let T_S be a self-adjoint Schrödinger operator on $I = \mathbb{R}$ with a potential $V(x)$. Suppose that $\sup_\gamma \int_\gamma^{\gamma+1} V_-(x)dx < +\infty$ and there is an N such that $\forall u \in Dom(T_S)$*

$$\int_{|x|>N} (T_S u)(x)\overline{u(x)}dx \geq \lambda_0 \int_{|x|>N} |u(x)|^2 dx \tag{3.22}$$

for some $\lambda_0 \in \mathbb{R}$. Then $\sigma_{ess}(T_S) \subset [\lambda_0, +\infty)$.

Theorem 3.2.2.9. *Let T_S be a self-adjoint Schrödinger operator on $I = \mathbb{R}$ with a potential $V(x)$. Define for $t > 0$*

$$\lambda_t = \liminf_{|\gamma|\to+\infty} t^{-1} \int_\gamma^{\gamma+t} V(x)dx$$

$$\omega_t = \limsup_{|\gamma|\to+\infty} t^{-1} \int_\gamma^{\gamma+t} V_-(x)dx$$

where $V_-(x) = max\{-V(x), 0\}$. If $t^2\omega_t < \frac{\pi^2}{8}$ then

$$\sigma_{ess}(T_S) \subset \left[\lambda_t - \frac{4t^2(\lambda_t + 2\omega_t)^2}{\pi^2 + 4t^2\lambda_t}, +\infty\right). \tag{3.23}$$

Define

$$\lambda_0 = \limsup_{t\to 0+} \lambda_t$$

$$\omega_0 = \limsup_{t \to 0+} t\omega_t.$$

If $\lambda_0 < +\infty$ then

$$\sigma_{ess}(T_S) \subset \left[\lambda_0 - \frac{16\omega_0^2}{\pi^2}, +\infty\right). \tag{3.24}$$

Theorem 3.2.2.10. *Let T_S be a self-adjoint Schrödinger operator on an unbounded interval I with a potential $V(x)$. Assume that $\sup_\gamma \int_\gamma^{\gamma+1} V_-(x)dx < +\infty$ and there is a sequence $\{I_n\}$ of intervals such that $|I_n| \to +\infty$ and for some $\mu \in \mathbb{R}$*

$$|I_n|^{-1} \int_{I_n} |V(x) - \mu| dx \to 0, \tag{3.25}$$

where $|I|$ is the length of I. Then $\sigma_{ess}(T_S) \supset [\mu, +\infty)$.

Now we will introduce some basic terminology used in the theory of differential equations. We will follow Kodaira's celebrated paper [25] and introduce the terminology used in there. For the rest of the subsection τ_S will be a Schrödinger differential expression on an I with endpoints a, b and with a potential $V(x)$, T_S will be its self-adjoint realization. Moreover, τ_S will be the limit point case at both endpoints a, b.

Definition 3.2.2.11. *By a* **system of fundamental solutions** *we shall mean the system of two solutions $s_1(x,l), s_2(x,l)$ of the equation $(\tau_S - lI)u = 0$, having the following properties:*

1. *$W(s_2, s_1) = 1$, W is the Wronskian of the system,*
2. *$s_k(x, \bar{l}) = \overline{s_k(x,l)}$, $\quad k = 1, 2,$*
3. *as functions of l, $s_k(x,l)$ and $\partial_x s_k(x,l)$ for $k = 1, 2$ are holomorphic in the whole complex plane.*

Remark 3.2.2.12. One can always find a system of fundamental solutions by, for example, solving the equation $(\tau_S - lI)u = 0$ under the boundary conditions

$$s_1(c) = s_2'(c) = 0 \ , \quad s_2(c) = s_1'(c) = 1 \ ,$$

for some $c \in I$.

Definition 3.2.2.13. *Let $\{s_1, s_2\}$ be a fundamental system of solutions. Define the functions*

$$m_a(l) = -\lim_{x \to a} \frac{s_2(x,l)}{s_1(x,l)} \ , \qquad m_b(l) = -\lim_{x \to b} \frac{s_2(x,l)}{s_1(x,l)}$$

for every $l \in \mathbb{C}$. The functions m_a, m_b are called the **Weyl-Titchmarsh functions***.*

Now we present a particularly useful theorem, which's proof can be found in [35].

Theorem 3.2.2.14. *The function m_b is for $\Im l \neq 0$ uniquely determined by the condition*

$$\int_c^b |s_2(x,l) + m_b(l) s_1(x,l)|^2 dx < +\infty \ ,$$

for some $c \in (a, b)$. If s_2 lies right then we may set $m_b(l) = 0$ for all $l \in \mathbb{C}$. If s_1 lies right then we may set $m_b(l) = \infty$ for all $l \in \mathbb{C}$.

The function m_a is uniquely determined by the condition

$$\int_a^c |s_2(x,l) + m_a(l)s_1(x,l)|^2 dx < +\infty \ ,$$

for some $c \in (a,b)$. If s_2 lies left then we may set $m_a(l) = 0$ for all $l \in \mathbb{C}$. If s_1 lies left then we may set $m_a(l) = \infty$ for all $l \in \mathbb{C}$. The symbol ∞ represents the complex infinity.

Definition 3.2.2.15. *Let $\{s_1, s_2\}$ be a fundamental system of solutions and let $m_b(l) = \infty$ for all $l \in \mathbb{C}$. Then the system is called* **normal**.

Theorem 3.2.2.16 (Weyl)**.** *The functions m_a, m_b are analytic functions of l which are meromorphic in $\Im l \neq 0$.*

Theorem 3.2.2.17. *The functions m_a, m_b are holomorphic in the resolvent set of T_S.*

The proof of the previous very useful theorem can be found in Weidmann [40]. The following theorem establishes the connection between the Weyl-Titchmarsh functions and the characteristic matrix defined in 3.1.5.8

Theorem 3.2.2.18 (Titchmarsh's spectral theorem)**.** *Let $\{s_1, s_2\}$ be a fundamental system of solutions, let m_a, m_b be corresponding the Weyl-Titchmarsh functions. Define the matrix $M = M(l)$ for $l \in \mathbb{C}$:*

$$M_{11}(l) = \frac{m_a(l)m_b(l)}{m_a(l) - m_b(l)} \ , \qquad M_{22}(l) = \frac{1}{m_a(l) - m_b(l)}$$

$$M_{12}(l) = M_{21}(l) = \frac{1}{2}\frac{m_a(l) + m_b(l)}{m_a(l) - m_b(l)}.$$

Then for every real number λ there exists the limit

$$\rho_{ij}(\{\lambda\}) = \lim_{\delta\to 0+}\lim_{\epsilon\to 0+}\frac{1}{\pi}\int_\delta^{\lambda+\delta} \Im M_{ij}(\zeta + i\epsilon)d\zeta \ , \qquad i,j = 1,2$$

where $(\rho_{ij})_{i,j=1,2}$ is the spectral matrix introduced in 3.1.5.8. Let

Remark 3.2.2.19. Thus we can identify the matrix M defined in Theorem 3.2.2.18 with the characteristic matrix defined in 3.1.5.8.

Theorem 3.2.2.20 (Special form of Titchmarsh's spectral theorem)**.** *Let $\{s_1, s_2\}, m_a, m_b$ be as in 3.2.2.18. Assume that $m_b(l) = \infty$ for all $l \in \mathbb{C}$. Then $m_a(l)$ is holomorphic for $\Im l \neq 0$. For every real number λ there exists the limit*

$$\rho(\{\lambda\}) = -\lim_{\delta\to 0+}\lim_{\epsilon\to 0+}\frac{1}{\pi}\int_\delta^{\lambda+\delta} \Im m_a(\zeta + i\epsilon)d\zeta = \lim_{\delta\to 0}\lim_{\epsilon\to 0_+}\frac{1}{2\pi i}\int_{C_{\lambda,\delta,\epsilon}} m_a(\zeta)d\zeta$$

where $C_{\lambda,\delta,\epsilon}$ is the rectangle with corners $\lambda+\epsilon+i\delta$, $\lambda+\epsilon-i\delta$, $\lambda-\epsilon-i\delta$, $\lambda-\epsilon+i\delta$. Moreover, every isolated singularity of m_a is a pole of order one.

Let E be the spectral family of the operator T_S. Denote $E(\Delta) = E(\lambda) - E(\mu)$ for every finite interval $\Delta = (\mu, \lambda]$. Then for arbitrary $u \in L^2(I, dx)$ we have

$$(E(\Delta)u)(x) = \int_a^b u(y)dy \int_\Delta s_1(x,\lambda)s_2(y,\lambda)d\rho(\lambda) \ , \tag{3.26}$$

where

$$\int_a^b dy |\int_\Delta s_1(x,\lambda)s_2(y,\lambda)d\rho(\lambda)|^2 < +\infty$$

and the integral in (3.26) converges absolutely.

The following lemma will help us find all the eigenvalues of T_S.

Lemma 3.2.2.21. *Let the interval I be $\mathbb{R}$. Let V satisfy the following:*

1. $V(x) \to 0$ *as* $x \to -\infty$,
2. $V(x) \to +\infty$ *as* $x \to +\infty$,
3. V *satisfies the conditions* (3.11) *for the point* $-\infty$.

Let the spectrum of T_S satisfy the following:

1. $\sigma_p(T_S)$ *is finite.*
2. $\sigma_c(T_S) \subset [0, +\infty)$.

Let $l = k^2 \in \mathbb{C}$, let $\{s_1(x,l), s_2(x,l)\}$ be a normal system of fundamental solutions for which s_1 lies right, let $u(x,k)$ be a solution uniquely(up to multiplication constant) determined by the asymptotic behavior towards $-\infty$ by the Theorem 3.2.1.7 and Theorem 3.2.1.8 and such that $u(x,k)$ is the corresponding eigenfunction to l for every $l \in \sigma_p(T_S)$. Then $\sigma_p(T_S)$ coincides with the set of all zeros of $W(u(k), s_1(k^2))$, where $W(u(k), s_1(k^2))$ is the Wronksian of the functions $u(x,k), s_1(x,k^2)$.

Proof. We will always assume $\Im k \geq 0$ and for now $l \neq 0$. From Lemma 3.2.2.1 it follows that every eigenvalue is negative. Set

$$u(x,k) = A(k)s_2(x,l) - B(k)s_1(x,l)$$

where $A(k), B(k)$ are uniquely determined since $\{s_1(x,l), s_2(x,l)\}$ is a system of fundamental solutions and since $u(x,l)$ is uniquely(up to multiplication constant) determined by Theorems 3.2.1.7 and 3.2.1.8. We can express them as

$$A(k) = W(u(k), s_1(k^2)) \ , \qquad B(k) = W(u(k), s_2(k^2)).$$

From Theorem 3.2.2.14 we have uniquely the function $m_{-\infty}(l)$ in the form

$$m_{-\infty}(k^2) = -\frac{B(k)}{A(k)}$$

Also, $u \neq 0$ for a fixed l, so $A(k)$ and $B(k)$ don't have a common zero point. Since $\{s_1(x,l), s_2(x,l)\}$ is a normal system of fundamental solutions, we have the characteristic matrix in the form

$$M(k^2) = \begin{pmatrix} -m_{-\infty}(k^2) & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$$

which implies $\rho(\lambda) = \rho_{11}(\lambda)$. Using Theorem 3.1.5.6 and taking a curve C_{λ_0} with only one point λ_0 in its interior we can write

$$\rho(\{\lambda_0\}) = \frac{1}{2\pi i}\int_{C_{\lambda_0}} m_{-\infty}(l)dl = -\frac{1}{\pi i}\int_{C_{\lambda_0}} \frac{B(k)}{A(k)}kdk$$

Since all u, s_1, s_2 are regular analytic, the coefficients $A(k), B(k)$ are regular analytic as well. Thus $\frac{1}{A(k)}$ has a singularity at $i\sqrt{-\lambda_0}$ and according to Theorem 3.2.2.20 this singularity is a pole of order one. Thus $W(u(i\sqrt{-\lambda_0}), s_1(\lambda_0)) = 0$. We can do this similarly for every eigenvalue. Now we will show they are the only zeros of $W(u(k), s_1(k^2))$. For a point $\lambda \in \rho(T_S)$ we use Theorem 3.2.2.17 which implies $A(\lambda^2) \neq 0$. So it remains to investigate $\sigma_c(T_S)$. But this is clear if we consider that for $k \geq 0$

$$u(x,k) = \overline{u(x,-k)} \tag{3.27}$$

which we know from the asymptotic behavior of the solution $u(x,k)$. So for nonzero k the solutions $u(x,k), u(x,-k)$ are linearly independent because of Theorem 3.2.1.7. Then from (3.27) and the properties of s_1, s_2 we have

$$A(-k) = \overline{A(k)} \ , \qquad B(-k) = \overline{B(k)}$$

So let there exit $l_0 > 0$ such that $A(\sqrt{l_0}) = \overline{A(-\sqrt{l_0})} = 0$ thus

$$u(x,\sqrt{l_0}) = B(\sqrt{l_0})s_1(x,l_0) = \overline{B(-\sqrt{l_0})}s_1(x,l_0)$$

hence $u(x,\sqrt{l_0})$ and $u(x,-\sqrt{l_0})$ are linearly dependent, but this contradicts the asymptotic behavior of $u(x,k)$ so $A(\sqrt{l_0}) \neq 0$.

Now we examine case $l = 0$.

If $0 \in \rho(T_S)$ then there is also a neighborhood of 0 in $\rho(T_S)$ but then $m_{-\infty}$ is holomorphic in 0 since Theorem 3.2.2.17.

If $0 \in \sigma_c(T_S)$ then for the contradiction lets assume that $m_{-\infty}$ has a singularity at 0 and so $\rho(0) > 0$. Let E be the spectral family of T_S Let $\Delta = (-\epsilon, 0]$ be an interval for some $\epsilon > 0$ not containing any eigenvalue of T_S so $E(\Delta) = \Theta$. Hence for every $u \in L^2(\mathbb{R}, dx)$ we can write the equality from Theorem 3.2.2.20 as

$$0 = \int_{-\infty}^{+\infty} u(y)dy \int_{\Delta} s_1(x,\lambda)s_2(y,\lambda)d\rho(\lambda) = s_1(x,0)\rho(0)\int_{-\infty}^{+\infty} u(y)s_2(y,0)dy \ , \tag{3.28}$$

which holds for every $x \in \mathbb{R}$. Since s_1 is a non-zero solution, we pick x_0 such that $s_1(x,0) \neq 0$. Also s_2 is non-zero, so we pick a bounded interval $(a,b) \subset \mathbb{R}$, on which $s_2(x,0) > \delta$ for some $\delta > 0$. Finally, lets take the function $u(x) = \chi_{(a,b)}(x)$ so

$$\rho(0)\int_{-\infty}^{+\infty} u(y)s_2(y,0)dy = \rho(0)\int_a^b s_2(y,0)dy > \rho(0)\delta > 0$$

hence $s_1(x,0)\rho(0)\int_{-\infty}^{+\infty} u(y)s_2(y,0) \neq 0$, but this contradicst equation (3.28). So from this we have $\rho(0) = 0$, so $m_{-\infty}$ is regular analytic at 0.

So finally we have proven, that the set of all zeros of $W(u(k), s_1(k^2))$ is the point spectrum $\sigma_p(T_S)$. □

3.2.3 The generalized Parseval identity

In this subsection let T_S be a self-adjoint Schrödinger operator on $I = \mathbb{R}$ with a potential $V : \mathbb{R} \to \mathbb{R}$, satisfy the following:

1. $V \in C^2(\mathbb{R})$,
2. $V(x) \to 0$ as $x \to -\infty$,
3. $V(x) \to +\infty$ as $x \to +\infty$,

4. V satisfies the conditions

$$\int_{-\infty}^{x_0} |V'(x)|^2 dx < +\infty \ , \qquad \int_{-\infty}^{x_0} |V''(x)| dx < +\infty \ ,$$

for some $x_0 \in \mathbb{R}$. Let the spectrum satisfy the conditions:

1. $\sigma_p(T_S)$ is a finite set,
2. $\sigma_c(T_S) = [0, +\infty)$.

Lemma 3.2.3.1. *The following assertions hold:*

1. *τ_S is the limit point case at both endpoints $\pm\infty$.*
2. *$\sigma_p(T_S) \subset (-\infty, 0)$*

Proof. Assertions 1 follows from the above assumptions Theorem 3.1.3.1 by setting $p(x) = 1,\ q(x) = V(x)$ and the interval $(a, b) = (-\infty, +\infty)$. Let $c \in \mathbb{R}$, then we have the function $g(x) = x - c$, which clearly is not in $L^2(\mathbb{R}, dx)$ and

$$\lim_{x \to +\infty} q(x) = \liminf_{x \to +\infty} q(x) = +\infty > -\infty$$

and

$$\lim_{x \to -\infty} q(x) = \liminf_{x \to -\infty} q(x) = 0 > -\infty$$

Hence τ_S is the limit point case at both endpoints.

Assertions 2 follows from Theorem 3.2.2.1. □

Define the set

$$N = \{k_n = i\sqrt{-E_n}; E_n \in \sigma_p(T_S)\}$$

Denote by l the spectral parameter. Let $\{s_1(t, l), s_2(t, l)\}$ be a normal system of fundamental solutions corresponding to the equation

$$(T_S - lI)\psi = 0 \tag{3.29}$$

Let the function $\phi(t, k)$ be the solution of (3.29) having the asymptotic behavior towards $-\infty$ according to Theorems 3.2.1.7 3.2.1.8 such that $\phi(t, k_n^2)$ is the n-th eigenfunction corresponding to the eigenvalue k_n^2. Then $\phi(t, k)$ can be written in the form

$$\phi(t, k) = A(k)s_2(t, k^2) - B(k)s_1(t, k^2) \ , l = k^2,$$

where $A(k) = W(\phi(t, k), s_1(t, k^2)),\ B(k) = W(\phi(t, k), s_2(t, k^2))$. We will use the following notation for the Wronskians:

$$W_{\pm}(k) = W(\phi(t, k), \phi(t, -k))$$

$$W(k) = W(\phi(t, k), s_1(t, k^2))$$

We set:

$$\Phi(x, k) = \begin{cases} \left|2k_n B(k_n) Res\left(\frac{1}{W(k)}, k_n\right)\right|^{1/2} s_1(x, k_n^2) & k_n = i\sqrt{E_n} \\ \left|\frac{1}{\pi}\frac{kW_{\pm}(k)}{|A(k)|^2}\right|^{1/2} s_1(x, k^2) & k \geq 0 \end{cases}$$

Let $M = \mathbb{R}_0^+ \cup N$. We will define the space $L^2(M)$ and the measure space $(M, \mathcal{A}, \tilde{\rho})$. Denote by $F(M)$ the set of all functions $f : M \to \mathbb{C}$ such that their restrictions $f|\mathbb{R}_0^+$ are Lebesque-measurable. Define the mapping $\langle ., . \rangle_M : M \times M \to \mathbb{C}$:

$$\langle f, g \rangle_M = \sum_{j=1}^{N_-(T_S)} \overline{f(k_j)} g(k_j) + \int_0^{+\infty} \overline{f(k)} g(k) dk \ , \qquad f, g \in F(M).$$

Clearly its a scalar product and we now define the space

$$L^2(M) = \{ f \in F(M) \ ; \ \langle f, f \rangle_M < +\infty \} \ .$$

And now we define the space $(M, \mathcal{A}, \tilde{\rho})$. The σ-algebra $\mathcal{A}$ is the standard Borel σ-algebra induced by the topology on $L^2(M)$ and the measure $\tilde{\rho}$ is defined as

$$\tilde{\rho}(A) = \mu(A \cap \mathbb{R}_0^+) + |A \cap N| \ , \qquad A \in \mathcal{A},$$

where $\mu(A \cap N)$ is the Lebesque measure of the subset $A \cap \mathbb{R}_0^+$ and $|A \cap N|$ is the cardinality of the subset $A \cap N$. Thus we may define the integral on $(M, \mathcal{A}, \tilde{\rho})$:

$$\int_M f d\tilde{\rho} = \sum_{j=1}^{N_-(T_S)} f(k_j) + \int_0^{+\infty} f(k) dk \ .$$

Now we define the mappings $U : L^2(\mathbb{R}, dx) \to L^2(M, d\tilde{\rho})$:

$$(Uf)(k) := \int_{\mathbb{R}} f(t) \Phi(t, k) dt \ , f \in Dom(U).$$

$V : L^2(M, d\tilde{\rho}) \to L^2(\mathbb{R}, dx)$:

$$(Va)(k) := \int_{\mathbb{R}} a(k) \Phi(t, k) dt \ , a \in Dom(V).$$

Let $f \in L^2(\mathbb{R}, dx)$ and $a \in L^2(M, d\tilde{\rho})$ be in a correspondence by the mapping U, resp. V. We will refer to the equality

$$\langle f, f \rangle = \int_M |g(k)|^2 d\tilde{\rho}(k) = \sum_{n=1}^{N_-(T_S)} |g(k_n)|^2 + \int_0^{+\infty} |g(k)|^2 dk \tag{3.30}$$

as the ***generalized Parseval equality***.

Lemma 3.2.3.2. *The set $C_0^\infty(\mathbb{R}^+)$ is dense in $L^2(\mathbb{R}_0^+, dx)$.*

Proof. Obvious. The $C_0^\infty(\mathbb{R}^+)$ is obviously dense in the set of all simple function in $L^2(\mathbb{R}_0^+, dx)$ except those having a support in the form $[0, c]$, $c \in \mathbb{R}^+$. Now if we take the characteristic function $\chi_{[0,c)}$, we can approximate it with sequence of functions, whose support is expanding towards 0 from the right and which converges is the L^2-topology to $\chi_{[0,c)}$. A good exaple is the function

$$f_n(x) = \exp\left[-\frac{1}{n(x - \epsilon_n)(x - c)} \right], \ x \in (\epsilon_n, c),$$

$$f_n(x) = 0, \ x \in \mathbb{R}^+ \setminus (\epsilon_n, c),$$

where $\epsilon_n \to 0+$. One can choose, for example $\epsilon_n = \frac{1}{n}$. Thus all simple functions can be approximated with the functions from $C_0^\infty(\mathbb{R}^+)$. The proof is complete. □

Theorem 3.2.3.3. *The following statements hold:*

1. *The mappings U, V are unitary and mutually inverse.*
2. *The operator T_S is unitarily equivalent to the multiplication operator k^2 on $L^2(M, d\tilde{\rho})$, where $k \in M$.*

Proof. Denote $\sigma = \sigma_p(T_S)$ and $\tau = \sigma_c(T_S)$. Let $N, \delta > 0$. Let $\Gamma_\delta, \Gamma_{N,\delta}$ denote the contours introduced in Lemma 2.1.3.6, define the contour $\Gamma_{\tau,\delta}$ having all properties of Γ_δ except that it encircles the continuous spectrum and denote by $\Gamma_{\tau,N,\delta}$ the part of $\Gamma_{\tau,\delta}$ laying in the half-plane $\Re z < N$. From there we know that

$$\operatorname*{s-lim}_{N\to+\infty} \frac{i}{2\pi} \int_{\Gamma_{N,\delta}} R_{T_S}(z)dz = I \ .$$

We will use the substitution

$$z = k^2. \tag{3.31}$$

With it, we need to represent the part of the spectrum and we need define new contours, that will encircle each part of the spectrum. Since the inverse of (3.31) ramifies, me choose a domain of its injectivity. We choose the domain $\{z \in \mathbb{C}; \Im z > 0\}$. Here, the eigenvalue E_n will be represented as $i\sqrt{|E_n|}$. Define the set $\sigma = \{i\sqrt{|E_n|}; n = 1, ..., N_-(T_S)\}$. We continue with representing the contour Γ_δ by an appropriate one through the transform (3.31). We do the trace of thought: Let $a = a_1 + ia_2$ a complex number and assume, that one wants to represent it through the inverse of (3.31) on the upper half-plane, one sets:

$$a_1 + ia_2 = (\pm b_1 + ib_2)^2 \ , b_2 > 0,$$

hence

$$b_1 = \frac{a_2}{\sqrt{2}\sqrt{|a| - a_1}} \ , \quad b_2 = \frac{1}{\sqrt{2}}\sqrt{|a| - a_1} \ . \tag{3.32}$$

Thus a suitable choice is to transform $\Gamma_\delta, \Gamma_{N,\delta}$ into new contours $\Pi_\delta, \Pi_{N,\delta}$ encircling the real axis and the few points on the upper imaginary axis and laying in the upper half-plane. Define $\tau = \{k \in \mathbb{R}; k^2 \geq 0\} = \mathbb{R}$. Thus we have $\sigma_p(T_S)$ and $\sigma_c(T_S)$ represented in $\mathbb{C}$ through the substitution $z = k^2$. Let Δ_σ be the Cauchy domain and let Δ_τ be a domain such that

- $\sigma \subset \Delta_\sigma^\circ$.
- $\tau \subset \Delta_\tau^\circ$.
- $\overline{\Delta_\sigma} \cap \overline{\Delta_\tau} = \emptyset$.

And now we define the corresponding contours through the transformation (3.31).

- Π_σ, be the Cauchy contour of Δ_σ.
- $\Pi_{\tau,\delta} \subset \Delta_\tau$, the transformation of $\Gamma_{\tau,\delta}$.
- $\Pi_{\tau,N,\delta} \subset \Delta_\tau$, the transformation of $\Gamma_{\tau,N,\delta}$.

Define the Riesz projection

$$P_\sigma = \frac{i}{\pi}\int_{\Pi_\sigma} kR_{T_S}(k^2)dk$$

and the operator

$$P_{\tau,\delta} = \operatorname*{s-lim}_{N\to+\infty} \frac{i}{\pi}\int_{\Pi_{\tau,N,\delta}} kR_{T_S}(k^2)dk$$

From Corollary 2.1.3.8 we know that $P_{\tau,\delta}$ is a projector and we have

$$P_\sigma + P_{\tau,\delta} = I$$

$$\operatorname*{s-lim}_{N\to+\infty} \frac{i}{\pi}\int_{\Pi_{N,\delta}} kR_{T_S}(k^2)dk = \frac{i}{\pi}\int_{\Pi_\sigma} kR_{T_S}(k^2)dk + \operatorname*{s-lim}_{N\to+\infty} \frac{i}{\pi}\int_{\Pi_{\tau,N,\delta}} kR_{T_S}(k^2)dk$$

To prove the generalized Parseval identity we will closely examine both P_σ and P_τ.

The projector P_σ:

Let $\phi_j = \phi(t, k_j^2)$ be the eigenfunction corresponding to the eigenvalue k_j^2 for $j = 1...N_-(T_S)$. Here we have from Corollary 2.1.3.5

$$P_\sigma g = \sum_{j=1}^{N_-(T_S)} \frac{\langle \phi_j, g\rangle}{\|\phi_j\|^2}\phi_j \tag{3.33}$$

for every $g \in L^2(\mathbb{R}, dx)$.

Now we will prove a very useful identity. From Theorem 3.1.4.1 we know that the resolvent of T_S is the operator

$$(R_{T_S}(k^2)g)(t) = \int_{\mathbb{R}} K(t,s,k^2)g(s)ds$$

with the kernel

$$K(t,s,k^2) = \frac{1}{W(k)}\left(\theta(t-s)s_1(t,k^2)\phi(s,k) + \theta(s-t)s_1(s,k^2)\phi(t,k)\right).$$

Let $\lambda_j = k_j^2$ be j-th eigenvalue of T_S. Let C_{k_j} be a Jordan curve encircling $k_j \in \sigma$, but no other point of σ. Thus from Corollary 2.1.3.5 we have for every $g \in L^2(\mathbb{R}, dx)$

$$P_{\{k_j\}}g = \frac{i}{\pi}\int_{C_{k_j}} kR_{T_S}(k^2)gdk = \frac{\langle \phi_j, g\rangle}{\|\phi_j\|^2}\phi_j\ , \tag{3.34}$$

hence

$$\frac{i}{\pi}\int_{C_{k_j}} k(R_{T_S}(k^2)g)(t)dk = \frac{i}{\pi}\int_{C_{k_j}} k\left(\int_{-\infty}^{+\infty} K(t,s,k^2)g(s)ds\right)dk$$

$$= \frac{i}{\pi}\int_{C_{k_j}}\int_{-\infty}^{+\infty} \frac{k}{W(k)}\left(\theta(t-s)s_1(t,k)\phi(s,k) + \theta(s-t)s_1(s,k^2)\phi(t,k)\right)g(t)dk$$

Now we will rearrange the double integral using a parametrization $\gamma : [0, 2\pi] \to \mathbb{C}$ of C_{k_j} and Fubini's theorem:

$$\frac{i}{\pi}\int_{C_{k_j}} k\left(\int_{-\infty}^{+\infty} K(t,s,k^2)g(s)ds\right)dk = \frac{i}{\pi}\int_0^{2\pi}\gamma(x)\left(\int_{-\infty}^{+\infty} K(t,s,\gamma(x)^2)g(s)ds\right)\gamma'(x)dx$$

$$= \frac{i}{\pi}\int_{-\infty}^{+\infty}\left(\int_0^{2\pi}\gamma(x)K(t,s,\gamma(x)^2)\gamma'(x)dx\right)g(s)ds = \int_{-\infty}^{+\infty}\left(\frac{i}{\pi}\int_{C_{k_j}} kK(t,s,k^2)dk\right)g(s)ds\,,$$

so now we will calculate the term $\frac{i}{\pi}\int_{C_{k_j}} kK(t,s,k^2)dk$ but this is easy thanks to Theorem 3.2.2.21 which gives us the following:

$$\frac{i}{\pi}\int_{C_{k_j}} kK(t,s,k^2)dk = \frac{i}{\pi}2\pi i Res(kK(t,s,k^2),k_j) = -2k_j Res(K(t,s,k^2),k_j)\ .$$

For further simplifications we use the fact that the singularity is contained in $\frac{1}{W(k)}$ and the fact that $\phi(t,k_j) = -B(k_j)s_1(t,k_j^2)$ thus leading us to

$$Res(K(t,s,k^2),k_j) =$$

$$= Res\left(\frac{1}{W(k)},k_j\right)\left[\theta(t-s)s_1(t,k_j^2)(-B(k_j))s_1(s,k_j^2) + \theta(s-t)s_1(s,k_j^2)(-B(k_j))s_1(t,k_j^2)\right]$$

$$= -B(k_j)Res\left(\frac{1}{W(k)},k_j\right)s_1(t,k_j^2)s_1(s,k_j^2)$$

so we write

$$\frac{i}{\pi}\int_{C_{k_j}} kK(t,s,k^2)dk = 2k_jB(k_j)Res\left(\frac{1}{W(k)},k_j\right)s_1(t,k_j^2)s_1(s,k_j^2)$$

and put the formula into the rearranged double integral and use the fact that for all $s \in \mathbb{R}\ ,\ s_1(s,k_j^2) \in \mathbb{R}$:

$$2k_jB(k_j)Res\left(\frac{1}{W(k)},k_j\right)s_1(t,k_j^2)\int_{-\infty}^{+\infty}\overline{s_1(s,k_j^2)}g(s)ds =$$

$$= 2k_jB(k_j)Res\left(\frac{1}{W(k)},k_j\right)\langle s_1(k_j^2),g\rangle s_1(t,k_j^2).$$

This is the Riesz projection $P_{\{k_j\}}$ so we have the equality

$$\frac{1}{\|s_1(k_j^2)\|^2}\langle s_1(k_j^2),g\rangle s_1(t,k_j^2) = 2k_jB(k_j)Res\left(\frac{1}{W(k)},k_j\right)\langle s_1(k_j^2),g\rangle s_1(t,k_j^2)$$

so finally we have the identity for the norm of the eigenvector $s_1(t,k_j)$

$$\|s_1(k_j^2)\|^2 = \frac{1}{2k_jB(k_j)Res\left(\frac{1}{W(k)},k_j\right)}\ ,$$

and the identity for the norm of $\phi(t,k_j)$ using the relation $A(k) = W(k)$

$$\|\phi(k_j^2)\|^2 = \left|\frac{B(k_j)}{2k_jRes\left(\frac{1}{A(k)},k_j\right)}\right|\ .$$

The projector P_τ.

The orientation of the contour $\Pi_{\tau,N,\delta}$ is from $+\infty$ to $-\infty$, which can be seen from (3.32). Let $\Pi_{\tau,N,\delta}$ be a straight line parallel with the real axis, passing the point $i\delta$. Then, minding the orientation, we have the operator

$$\frac{i}{\pi}\int_{\Pi_{\tau,N,\delta}} kR_{T_S}(k^2)dk = \frac{i}{\pi}\int_{N}^{-N}(m+i\delta)R_{T_S}((m+i\delta)^2)dm$$

$$= -\frac{i}{\pi}\int_{-N}^{N}(m+i\delta)R_{T_S}((m+i\delta)^2)dm\ ,$$

which we will examine closer. Let $g \in C_0^\infty(\mathbb{R})$. Define the operator

$$(Q_{N,\delta}g)(t) = -\frac{i}{\pi}\int_{-N}^{N}\int_{\mathbb{R}}(k+i\delta)K(t,s,(k+i\delta)^2)g(s)dsdk$$

with the kernel(here we can change the order of the integration by the theorem of Fubini)

$$q_{N,\delta}(t,s) = -\frac{i}{\pi}\int_{-N}^{N}(k+i\delta)K(t,s,(k+i\delta)^2)dk.$$

Using the fact that $W(k)$ has all zeros on $(-\infty,0)$ and Lemma 2.1.3.6, which implies $Q_N = \underset{\delta\to 0}{\text{u-lim}}\, Q_{N,\delta}$, we can set

$$q_N(t,s) = \lim_{\delta\to 0} q_{N,\delta}(t,s)$$

$$= -\frac{i}{\pi}\int_{-N}^{N}\frac{k}{W(k)}\left[\theta(t-s)s_1(t,k^2)\phi(s,k) + \theta(s-t)s_1(s,k^2)\phi(t,k)\right]dk$$

$$= J_1(t,s) + J_2(t,s)$$

uniformly for all $t, s \in \mathbb{R}$. Here

$$J_1(t,s) = -\frac{i}{\pi}\int_{-N}^{N}\frac{k}{W(k)}\theta(t-s)s_1(t,k^2)\phi(s,k)dk$$

and

$$J_2(t,s) = J_1(s,t)$$

Now we rewrite J_1:

$$J_1(t,s) = -\frac{i}{\pi}\theta(t-s)\left[\int_{-N}^{0}\frac{k}{W(k)}s_1(t,k^2)\phi(s,k)dk + \int_0^N\frac{k}{W(k)}s_1(t,k^2)\phi(s,k)dk\right]$$

$$= -\frac{i}{\pi}\theta(t-s)\int_0^N\frac{k}{W(k)W(-k)}\left[W(-k)\phi(s,k) - W(k)\phi(s,-k)\right]s_1(t,k^2)dk$$

and if we adjust the expressions :

$$W(-k)\phi(s,k) - W(k)\phi(s,-k)$$

$$= \left[\phi(s,-k)s_1'(s,k^2) - s_1(s,k^2)\phi'(s,-k)\right]\phi(s,k) - \left[\phi(s,k)s_1'(s,k^2) - s_1(s,k^2)\phi'(s,k)\right]\phi(s,-k)$$

$$= \left[\phi(s,-k)\phi'(s,k) - \phi(s,k)\phi'(s,-k)\right]s_1(s,k^2) = W_\pm(-k)s_1(s,k^2)$$

and

$$W(-k) = \phi(t,-k)s_1'(t,k^2) - s_1(t,k^2)\phi'(t,-k) = \overline{\phi(t,k)}s_1'(t,k^2) - s_1(t,k^2)\overline{\phi'(t,k)}$$
$$= \overline{\phi(t,k)s_1'(t,k^2) - s_1(t,k^2)\phi'(t,k)} = \overline{W(k)}$$

and finally

$$W_\pm(k) = \phi(t,k)\phi'(t,-k) - \phi(t,-k)\phi'(t,k) = \phi(t,k)\overline{\phi'(t,k)} - \overline{\phi(t,k)}\phi'(t,k)$$
$$= \phi(t,k)\overline{\phi'(t,k)} - \overline{\phi(t,k)\overline{\phi'(t,k)}} = 2i\Im(\phi(t,k)\overline{\phi'(t,k)}) \ .$$

Then J_1 becomes

$$J_1(t,s) = \frac{1}{\pi}\theta(t-s)\int_0^N \frac{k(iW_\pm(-k))}{|W(k)^2|}s_1(t,k^2)s_1(s,k^2)dk \ ,$$

which means

$$J(t,s) = J_1(t,s) + J_2(t,s) = \int_0^N \frac{1}{\pi}\frac{k(iW_\pm(-k))}{|W(k)^2|}s_1(t,k^2)s_1(s,k^2)dk \ .$$

Now if we define the function $\Phi(t,k)$ as

$$\Phi(t,k) = \frac{1}{\sqrt{\pi}}\frac{\sqrt{|kW_\pm(k)|}}{|A(k)|}s_1(t,k^2)$$

and since $\Im W_\pm(k)$ doesn't change sign in $[0,+\infty)$ we can always set the fundamental system such that $iW_\pm(k) > 0$, so it is obvious that $\Phi(t,k) = \overline{\Phi(t,k)}$ we can write $q_N(t,s)$ in the form :

$$q_N(t,s) = \int_0^N \Phi(t,k)\Phi(s,k)dk$$

and hence

$$(Q_N g)(t) = \int_{\mathbb{R}} q_N(t,s)g(s)ds = \int_{\mathbb{R}}\left(\int_0^N \Phi(t,k)\Phi(s,k)dk\right)g(s)ds$$
$$= \int_0^N \left(\int_{\mathbb{R}} \Phi(s,k)g(s)ds\right)\Phi(t,k)dk \ ,$$

where in the last step we used the Fubini's theorem. To finish the proof we define the functions $a : M \to \mathbb{C}$:

$$a(k_j) = \left(2k_jB(k_j)Res\left(\frac{1}{W(k)},k_j\right)\right)^{1/2}\langle s_1(k_j^2),g\rangle \ , \quad k_j \in N \tag{3.35}$$

$$a(k) = \frac{1}{\sqrt{\pi}}\frac{\sqrt{|kW_\pm(k)|}}{|A(k)|}\int_{\mathbb{R}} s_1(t,k^2)g(t)dt \tag{3.36}$$

now set the scalar product and again use the Fubini's theorem :

$$\langle g, Q_N g\rangle = \int_{\mathbb{R}}\left(\overline{g(t)}(Q_N g)(t)\right)dt =$$

$$= \int_{\mathbb{R}} \left(\overline{g(t)} \left(\int_0^N \left(\int_{\mathbb{R}} \Phi(s,k) g(s) ds \right) \Phi(t,k) dk \right) \right) dt$$

$$= \int_0^N \left(\int_{\mathbb{R}} \Phi(s,k) g(s) ds \right) \left(\overline{\int_{\mathbb{R}} \Phi(t,k) g(t) dt} \right) dk = \int_0^N |a(k)|^2 dk$$

From Corollary 2.1.3.8 we know that $P_\tau = \operatorname*{s-lim}_{N \to +\infty} Q_N = I - P_\sigma$ therefore

$$\lim_{N \to +\infty} \langle g, Q_N g \rangle = \int_0^{+\infty} |a(k)|^2 dk$$

and finaly we have the long-sought generalized Parseval identity for $g \in C_0^\infty(\mathbb{R})$:

$$\langle g, g \rangle = \sum_{j=1}^{N_-(T_S)} |a(k_j)|^2 + \int_0^{+\infty} |a(k)|^2 dk = \langle a, a \rangle_M. \tag{3.37}$$

Evidently $a \in L^2(M, d\tilde{\rho})$. Next we will extend this equality for the whole $L^2(\mathbb{R}, dx)$.

We will prove the unitarity of U and V.

Define the linear mapping $U : C_0^\infty(\mathbb{R}) \to L^2(M, d\tilde{\rho})$:

$$(Ug)(k) = a(k) \ , \quad g \in C_0^\infty(\mathbb{R})$$

where $a(k)$ is defined in (3.35) and (3.36). Since for every $g \in C_0^\infty(\mathbb{R})$ we have $\|g\| = \|a\|$, we can extend the mapping U on the whole $L^2(\mathbb{R}, dx)$. Obviously U is an bijection between the spaces $L^2(\mathbb{R}, dx)$ and $U(L^2(\mathbb{R}, dx))$. The inverse operator $V : U(L^2(\mathbb{R}, dx)) \to L^2(\mathbb{R}, dx)$ is defined as

$$f(t) = (Va)(t) = \int_M a(k) \Phi(t,k) d\tilde{\rho}(k) \ , a \in U(L^2(\mathbb{R}, dx))$$

which can be simply proved by checking the compositions UV, VU.

To prove the surjectivity, one must prove, that for a suitable dense set in $L^2(M, d\tilde{\rho})$ such that all functions in it have an L^2-integrable image via Φ. Then from the generalized Parseval identity V could be extended to the entire $L^2(M, d\tilde{\rho})$. Its obvious that it is sufficient to find a dense set in $L^2(\mathbb{R}_0^+, dk)$ such that for each element a in it, the function

$$\int_{\mathbb{R}_0^+} \Phi(t,k) a(k) dk$$

is square-integrable. Let $a \in C_0^\infty(\mathbb{R}^+)$. Obviously $\Phi(t,k)$ is $C^\infty(\mathbb{R} \times \mathbb{R}_0^+)$. From Theorem 3.2.1.7 we know the asymptotic behavior of $\Phi(t,k)$ for $t \to +\infty$ is:

$$|\Phi(t,k)| \le K_1 e^{-lt} \ , t > t_1$$

for some constant $K_1 > 0$, $l > 0$ and t_1 depends on l. We fix l, thus we have $t_1 > 0$. We make the following estimate for $t > t_1$:

$$f(t) = \int_{\mathbb{R}_0^+} \Phi(t,k) a(k) dk \le \int_{\mathbb{R}_0^+} |\Phi(t,k) a(k)| \le K_1 e^{-lt} \int_{\mathbb{R}_0^+} |a(k)| dk \le K_1 \tilde{C}_1 e^{-lt},$$

where $\tilde{C}_1 = \int_{supp\ a} |a(k)| dk$. Now we study the behavior towards $-\infty$. We choose $t_0 < 0$ such that for all $t < t_0$, $k^2 - V(t) > 0$. We can do that, since $supp\ a \subset [c, +\infty)$ for some

$c > 0$, thus we take $k \geq c$ and since $\lim_{t\to-\infty} V(t) = 0$, we can always find such t_0. From Theorem 3.2.1.8 we can write $\Phi(t,k)$ in the form :

$$\Phi(t,k) = \alpha_+(k)e^{ikt}\eta_+(t,k) + \alpha_-(k)e^{-ikt}\eta_-(t,k),$$

where $\alpha_\pm : \mathbb{R}_0^+ \to \mathbb{C}$ are in $C^\infty([c,+\infty))$(one can verify this by computing the corresponding Wronskians) and for a fixed $k > 0$, $\lim_{t\to-\infty} \eta_\pm(t,k) = 1$. We can make the estimate by applying integration by parts:

$$\left|\int_{\mathbb{R}_0^+} \alpha_\pm(k)e^{\pm ikt}\eta_\pm(t,k)a(k)dk\right| = \frac{1}{|t|}\left|\int_{supp\ a}\left(e^{\pm ikt}\frac{d}{dk}\left(\alpha_\pm(k)\eta_\pm(t,k)a(k)\right)\right)dk\right| \leq \frac{\tilde{C}_\pm}{|t|} ,$$

where $\tilde{C}_\pm = \displaystyle\int_{supp\ a}\left|\frac{d}{dk}\left(\alpha_\pm(k)\eta_\pm(t,k)a(k)\right)\right| dk < +\infty$ since $\eta(t,k)$ and $a(k)$ are C^∞. So far we have proven the following:

$$\int_{\mathbb{R}_0^+} \Phi(t,k)a(k)dk = \mathcal{O}(e^{-lt}) , \qquad t \to +\infty$$

$$\int_{\mathbb{R}_0^+} \Phi(t,k)a(k)dk = \mathcal{O}(t^{-1}) , \qquad t \to -\infty$$

Now since $\Phi(t,k), a(k)$ are C^∞, we can define

$$C = \sup_{t\in[t_0,t_1],k\in supp\ a} |\Phi(t,k)a(k)|.$$

Obviously $C < +\infty$. Define the function h:

$$h(t) = \int_{\mathbb{R}_0^+} \Phi(t,k)a(k)dk.$$

Now we make the estimate:

$$\begin{aligned}\|h\|^2 = \int_{\mathbb{R}} |h(t)|^2dt &= \int_{-\infty}^{t_0} |h(t)|^2dt + \int_{t_0}^{t_1} |h(t)|^2dt + \int_{t_1}^{+\infty} |h(t)|^2dt \\ &\leq \int_{-\infty}^{t_0} \mathcal{O}(t^{-2})dt + C\mu(supp\ a)(t_1 - t_0) + \int_{t_1}^{+\infty} \mathcal{O}(e^{-2lt})dt \\ &= \mathcal{O}(t_0^{-1}) + C\mu(supp\ a)(t_1 - t_0) + \mathcal{O}(e^{-2lt_1}),\end{aligned}$$

where $\mu(.)$ is the Lebesque measure. and since t_0, t_1 are finite, we get that $h \in L^2(\mathbb{R}, dx)$. Hence

$$\mathbb{C}^{N_-(T_S)} \oplus C_0^\infty(\mathbb{R}^+) \subset U(L^2(\mathbb{R}, dx))$$

and since $U(L^2(\mathbb{R}, dx))$ is closed and using Lemma 3.2.3.2 we find that $\mathbb{C}^{N_-(T_S)} \oplus C_0^\infty(\mathbb{R}^+)$ is dense in $L^2(M, d\tilde{\rho})$, V can be extended to the whole $L^2(M, d\tilde{\rho})$ and therefore U is surjective.

Define the set

$$S = \{f \in L^2(\mathbb{R}, dx); f, f^{(1)} \in AC_0(\mathbb{R}); \tau_S f \in L^2(\mathbb{R}, dx)\}.$$

For any $f \in S$ the equality

$$(UT_Sf)(k) = k^2(Uf)(k) \ , k \in M$$

holds. One can check it, by using the definition of U and the integration by parts. Thus we define the operator $\tilde{T}$ on $L^2(M, d\tilde{\rho})$ as

$$(\tilde{T}f)(k) = k^2 f(k)$$

on the domain $Dom(\tilde{T}) = U(S)$. Since $T_S|S$ is essentially self-adjoint according to Theorem 3.1.2.6, $\tilde{T}$ is essentially self-adjoint. Hence for every $f \in Dom(T_S)$

$$(UT_S f)(k) = k^2 (Uf)(k) \ , k \in M \ ,$$

thus T_S is unitarily equivalent with the operator T on $L^2(M, d\tilde{\rho})$ defined as:

$$(Tf)(k) = k^2 f(k) \ , \qquad f \in Dom(T) = U(Dom(T_S)).$$

□

Thus we have proven, that T_S on $L^2(\mathbb{R}, dx)$ is unitarily equivalent with the multiplication operator k^2 on $L^2(M, d\tilde{\rho})$ and that the functions $\Phi(t, k)$ form the complete system of generalized eigenvectors of T_S. But in order to be precise, we must find the space $\bigoplus_{\alpha \in I} L^2(\mathbb{R}, d\mu_\alpha)$, resp. $L^2(\mathbb{R}, d\{\rho_{ij}\})$. But this comes easily from the definition of the function $a(k)$: The space $L^2(\mathbb{R}, d\{\rho_{ij}\})$ is the space $L^2(\mathbb{R}, d\rho)$ where the spectral measure ρ is expressed as

$$\rho(\lambda) = \left| 2\sqrt{-\lambda} B(i\sqrt{-\lambda}) Res\left(\frac{1}{A(k)}, i\sqrt{-\lambda} \right) \right|^{1/2} \ , \qquad \lambda \in \sigma_p(T_S)$$

$$\rho(\lambda) = \sqrt{\frac{2}{\pi}} \frac{\sqrt{\sqrt{\lambda} |\Im(A(-\sqrt{\lambda}) B(\sqrt{\lambda}))|}}{|A(\sqrt{\lambda})|} \ , \qquad \lambda \in \sigma_c(T_S)$$

and $\{s_1(x, l)\}_{l \in \sigma(T_S)}$ forms the complete system of generalized eigenvectors of T_S and the operator T_S is unitarily equivalent with the multiplication operator λ on $L^2(\mathbb{R}, d\rho)$.

Chapter 4

Magnetic Hamiltonians in the Lobachevsky plane

4.1 Physical description

The particle, which is spinless and with no charge in this case, is confined to the Lobachevsky plane, on which an external perpendicular magnetic field is inpossed. Thus the magnetic field will have a profound influence on the spectrum, as well as the dynamics of the particle. In our case the periodic scalar electric potential will be zero. There are only a few explicitly solved cases with non-zero periodic fields.

4.2 The mathematical description

4.2.1 Schrödinger operators on Riemannian manifolds

We will construct the Schrödinger operator on a two-dimensional manifold according to Shubin [34] and Mine [28]. The following setup will incorporate the magnetic field in a geometrical sense.

Let (M, g) be a two-dimensional, oriented, connected complete C^∞-Riemannian manifold with the Riemannian metric g on M. Let (U, ϕ), $\phi = (x^1, x^2)$ be a local chart, denote an element of the metric tensor g as $g_{mn} = g(\partial_m, \partial_n)$ and an element of its inverse as g^{mn}. Denote $d\mu$ the canonical measure induced by the metric g, i.e. $d\mu = \sqrt{g} dx_1 dx_2$. Denote by $\Lambda^p_{(k)}(M)$ the set of all k-smooth complex-valued p-forms on M.

Let $A \in \Lambda^1_1(M)$. We will write $\Lambda^p(M)$ instead of $\Lambda^p_{(\infty)}(M)$. We will begin with the usual differential

$$d : C^\infty(M) \to \Lambda^1(M)$$

which will be modified into the form

$$d_A : C^\infty(M) \to \Lambda^1_{(1)}(M)$$

$$d_A f = df + ifA \ , \qquad f \in C_0^\infty(M) \tag{4.1}$$

and expressed in a local coordinate system

$$d_A f = (\partial_1 f + iA_1 f)dx^1 + (\partial_2 f + iA_2 f)dx^2 \ .$$

Let $u, v \in C_0^\infty(M)$, we define the inner product

$$\langle u, v\rangle_1 = \int_M u\bar{v}d\mu \ .$$

For $\alpha, \beta \in \Lambda^1(M)$ in local coordinates expressed in the form

$$\alpha = \alpha_1 dx^1 + \alpha_2 dx^2 \ , \qquad \beta = \beta_1 dx^1 + \beta_2 dx^2$$

we define the scalar product

$$\langle \alpha, \beta\rangle_2 = \sum_{m,n=1,2} g^{mn}\alpha_m\beta_n$$

so when α, β have compact support(their coefficients have compact support) we may write

$$\langle \alpha, \beta\rangle_1 = \int_M \langle \alpha, \overline{\beta}\rangle_2 d\mu$$

since $\langle \alpha, \beta\rangle_2$ is a scalar function on M. Let $\omega \in \Lambda^1_{(1)}(M)$, we define the linear mapping $d^*\omega$

$$d^* : \Lambda^1(M) \to C^\infty(M)$$

$$d^*\omega = -\frac{1}{\sqrt{|g|}} \sum_{m,n=1,2} \partial_m \left(\sqrt{|g|}g^{mn}\omega_n\right) \ , \qquad \omega = \omega_1 dx^1 + \omega_2 dx^2$$

which is the formall adjoint of the usual differential d satisfying

$$\langle du, \omega\rangle_1 = \langle u, d^*\omega\rangle_1 \ , \qquad u \in C_0^\infty(M) \ , \qquad \omega \in \Lambda^1(M) \ .$$

Now we define the formal adjoint to d_A as the mapping

$$d_A^* : \Lambda^1_{(1)}(M) \to C(M) \ ,$$

$$d_A^*\omega = d^*\omega - iA^*\omega \ , \qquad A^*\omega = \langle A, \omega\rangle$$

and in local coordinate system we have

$$d_A^*\omega = d^*\omega - i\sum_{m,n=1,2} g^{mn}A_m\omega_n = d^*\omega - \frac{1}{\sqrt{g}}\sum_{m,n=1,2}\sqrt{g}g^{mn}\left(iA_m\right)\omega_n$$

$$d_A^*\omega = -\frac{1}{\sqrt{g}}\sum_{m,n=1,2}\partial_m\left(\sqrt{g}g^{mn}\omega_n\right) - \frac{1}{\sqrt{g}}\sum_{m,n=1,2}\left(iA_m\right)\left(\sqrt{g}g^{mn}\omega_n\right)$$

$$= -\frac{1}{\sqrt{g}}\sum_{m,n=1,2}\left(\partial_m + iA_m\right)\left(\sqrt{g}g^{mn}\omega_n\right)$$

again, satisfying

$$\langle d_A u, \omega\rangle_1 = \langle u, d_A^*\omega\rangle_1 \ , \qquad u \in C_0^\infty(M) \ , \qquad \omega \in \Lambda^1_{(1)}(M) \ .$$

Now, we express localy the composition $(d_A^* d_A)$:

$$(d_A^* d_A)f = -\frac{1}{\sqrt{g}}\sum_{m,n=1,2}\left(\partial_m + iA_m\right)\sqrt{g}g^{mn}\left(\partial_n + iA_n\right)f) \ , \qquad f \in C_0^\infty(M)$$

Definition 4.2.1.1. *Let (M, g) be a two-dimensional, oriented, connected complete C^∞-Riemannian manifold with the Riemannian metric g on M. Then we define the* **magnetic Laplacian** Δ_A *as*

$$-\Delta_A = d_A^* d_A : C^\infty(M) \to C(M)$$

Definition 4.2.1.2. *Let (M, g) be a two-dimensional, oriented, connected complete C^∞-Riemannian manifold with the Riemannian metric g on M. Let $V \in L^2_{loc}(M)$, V is real-values and will be referred as the* **scalar potential**. *Let $A \in \Lambda^1_{(1)}(M)$ and it will be referred as the* **magnetic potential**. *Then we define the formal partial differential expression.*

$$\mathcal{L} = -\Delta_A + V \tag{4.2}$$

and the **magnetic Schrödinger operator** T_{min} *as*

$$T_{min} u = \mathcal{L} u \ , \qquad u \in Dom(T_{min}) = C_0^\infty(M)$$

Remark 4.2.1.3. We restrict ourselves to C^∞ 1-forms A. Of course, there are many interesting problems that consider a less restrictive or even a broader class of forms. For a good example, we the reader refer to Mine [28], where a system with δ-function type potentials is analyzed.

Remark 4.2.1.4. The reason, why we have chosen this rather lengthy derivation of the differential expression $\mathcal{L}$ is that we will later express the magnetic potential in our quantum system by a differential form and that this way also expresses one of the currently, mostly in mathematical physics, used mathematical interpretations of the magnetic field. This interpretation is encapsulated in the equation (4.1), which is also the choice of specific connection on our manifold(or in a physicist's words, our universe) expressed by the differential form d_A. For precise information about the significance of a connection, we refer the reader to Šťovíček-Kocábová [36].

Now we will present a very important theorem from Shubin [34] concerning the essential self-adjointness of the magnetic Laplacian Δ_A. The theorem is a special case of a more general theorem, that Shubin in [34] proves. It is also reformulated for our cause.

Theorem 4.2.1.5. *Let (M, g) be a two-dimensional, oriented, connected complete C^∞-Riemannian manifold. Let $A \in Lip_{loc}(M)$. Then $-\Delta_A = d_A^* d_A$ is essentially self-adjoint in $L^2(M, d\mu)$, where $d\mu$ is the canonical measure induced by the metric g.*

Remark 4.2.1.6. Generally the question of essential self-adjointess of Schrödinger operators on Riemannian manifolds is nontrivial and is still in active study. It has been answered for special classes of potentials V and magnetic fields A and the proofs often use heavily the theory of bundles and connection, as can be seen in Bravermann-Milatovic-Shubin [9].

Remark 4.2.1.7. The reason that we need only the special case is that we consider the periodic scalar potential $V = 0$.

4.3 Spectral analysis of the magnetic Hamiltonian

4.3.1 The Hamiltonian H_{L0} of the system

We will analyze a well-known problem of Comtet [11] ,[12]. Our manifold, or, more physically speaking, our universe will be the Lobachevsky plane model with the curvature parameter a defined in 2.3.1.2.

Let (U, ϕ) be a local chart. The magnetic field is expressed as a differential form $A \in C^\infty(\Lambda^1(M))$ which is written as

$$A = A_1 dx_1 + A_2 dx_2$$

in the coordinate neighborhood U. Hence the magnetic induction $B \in \Lambda^2(M)$ is given by

$$B = dA = (\partial_x A_2 - \partial_y A_1)\, dx_1 \wedge dx_2 \ .$$

We will examine the case where $B \propto \sqrt{g} dxdy$. In [11] ,[12] this is referred as the "constant magnetic field". The proportionality implies

$$B = \frac{ka^2}{y^2} dxdy$$

for some real non-zero k. For our case we will choose a particular gauge, specifically the Landau gauge

$$A_1(x_1, x_2) = \frac{b}{x_2} \quad , \qquad A_2(x_1, x_2) = 0 \quad , \qquad b = ka^2$$

From Definition 4.2.1.2 we have the formal pariatl differential expression

$$\mathcal{L} = -\frac{x_2^2}{a^2}\left((\partial_1 + iA_1)^2 + (\partial_2 + iA_2)^2\right)$$

$$= -\frac{x_2^2}{a^2}\left(\left(\partial_1 - \frac{ib}{x_2}\right)^2 + \partial_2^2\right) = -\frac{1}{a^2}\left(x_2^2\partial_1^2 + x_2^2\partial_2^2 + 2ibx_2\partial_1 - b^2\right)$$

and the operator

$$H_{L0,min} u := \mathcal{L} u \ , \qquad Dom(T_{min}) = C_0^\infty(\mathbb{H}) \tag{4.3}$$

Theorem 4.3.1.1. *$H_{L0,min}$ is essentially self-adjoint.*

Proof. Obvious. Let (U, ϕ) be a local chart. Since $A_1(x_1, x_2) = \frac{b}{x_2}$ and $A_2(x_1, x_2) = 0$, we have $A_1, A_2 \in Lip_{loc}(U)$, which implies $A_1, A_2 \in Lip_{loc}(M)$. So the self-adjointness follows from Theorem 4.2.1.5. □

Definition 4.3.1.2. *Denote by H_{L0} the closure of $H_{L0,min}$.*

4.3.2 The eigenvalue problem of automorphic forms on the hyperbolic plane

We will use the definitions, notations and theorems from the papers of Elstrod [16] and Roelcke [32].

Let G be a discrete subgroup of $SL(2, \mathbb{R})$ containing $-I$, and let B denote the fundamental region of G. Let $k \in \mathbb{R}$ and let v be a multiplier system of weight $2k$. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Now we take a function $f : \mathbb{H} \to \mathbb{C}$ and we define a new function $f|[M, k] : \mathbb{H} \to \mathbb{C}$:

$$f|[M, k](z) = e^{-2ik \arg(cz+d)} f(Mz) \ ,$$

where $Mz = \dfrac{az + b}{cz + d}$. By z we mean $z = x_1 + ix_2$.

Definition 4.3.2.1. *Let $\mathcal{H}_k(G, B, v)$ denote the Hilbert space of all the $\mu_{\mathbb{H}}$-measurable functions $f : \mathbb{H} \to \mathbb{C}$ satisfying:*

1. $f|[M, k] = v(M)f$ *for all* $M \in G$.
2. $\int_B |f|^2 d\mu_{\mathbb{H}}(z) < +\infty$.

The scalar product on $\mathcal{H}_k(G, B, v)$ is defined as

$$\langle f, g\rangle = \int_B \overline{f(z)}g(z)d\mu_{\mathbb{H}}(z).$$

Now we define the partial differential expression $\mathcal{L}_k$:

$$\mathcal{L}_k = x_2^2\left(\partial_1^2 + \partial_2^2\right) - 2ikx_2\partial_1 \ .$$

Definition 4.3.2.2. *Denote by $\mathcal{D}_k^2$ the set of all twice-differentiable functions $f : \mathbb{H} \to \mathbb{C}$ such that $\mathcal{L}_k f \in \mathcal{H}_k(G, B, v)$. Denote by $\mathcal{D}_k^\infty$ the set of all infinitely differentiable functions $f : \mathbb{H} \to \mathbb{C}$ having a support, that is modulo G compact(that is, supp f/G is compact).*

Thus obviously $\mathcal{D}_k^\infty = C_0^\infty(B)$.

Definition 4.3.2.3. *Define the operators Δ_k^2 and Δ_k^∞:*

$$\Delta_k^2 f = \mathcal{L}_k f \ , \qquad f \in Dom(\Delta_k^2) = \mathcal{D}_k^2.$$

$$\Delta_k^\infty f = \mathcal{L}_k f \ , \qquad f \in Dom(\Delta_k^\infty) = \mathcal{D}_k^\infty.$$

Now we come to a very important theorems, that is proved in [32].

Theorem 4.3.2.4 (Roelcke). *The following statements are true:*

1. *The operators $-\Delta_k^2$ and $-\Delta_k^\infty$ are essentially self-adjoint and have the same self-adjoint extension, denoted by $-\Delta_k$, with the domain $\mathcal{D}_k$.*
2. *Every twice-differentiable function from $\mathcal{D}_k$ lies in $\mathcal{D}_k^2$.*
3. *The spectrum of $-\Delta_k$ lies in $\left[\frac{|k|}{2}\left(1 - \frac{|k|}{2}\right), +\infty\right)$.*

Elstrod in [16] analyzed the case of the whole $\mathbb{H}$ by taking the group $G = \{I, -I\}$, choosing v to be trivial($v(I) = V(-I) = 1$) and using Roelcke's theorem he received the following results:

Theorem 4.3.2.5 (Elstrod). *Let $G = \{I, -I\}$, $v(I) = V(-I) = 1$ and $k \in \mathbb{R}$. Then:*

1. $\sigma_p(-\Delta_k) = \{(|k| - m)(1 - |k| + m); 0 \leq m < |k| - \frac{1}{2}, m \in \mathbb{Z}\}$.
2. $\sigma_c(-\Delta_k) = \left[\frac{1}{4}, +\infty\right)$.
3. *The spectrum $\sigma(-\Delta_k)$ has infinite spectral multiplicity.*

To calculate the corresponding eigenfunctions, Elstrod passed the problem to the Poincaré disc model used the connections between the operator Δ_k on the Poincaré disc and the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$. The results follow:

Theorem 4.3.2.6 (Elstrod)**.** *The eigenfunctions $g_{m,n} : \mathbb{B} \to \mathbb{C}$, defined on the Poincaré disc, corresponding to the eigenvalue $\lambda_m = (|k|-m)(1-|k|+m)$ are :*

$$g_{m,n}(re^{i\phi}) = r^{|n|}e^{in\phi}(1-r^2)^{|k|-m}\ {}_2F_1(|k|-m-k_n, |k|-m+k_n+|n|; |n|+1; r^2) \ ,$$

for $0 \le r < 0$, $0 \le \phi < 2\pi$, $0 \le m < |k| - \frac{1}{2}; m, n \in \mathbb{Z}$. Also, $k_n = \pm|k|$, where the sign is chosen to satisfy:

$$|k| - m - k_n \in \mathbb{Z}_0^- \vee |k| - m + k_n + |n| \in \mathbb{Z}_0^- \ .$$

To express the eigenfunctions $f_{m,n} : \mathbb{H} \to \mathbb{C}$ on the Lobachevsky plane, one can use the transformation

$$z = -\frac{r\sin\phi}{|1-re^{i\phi}|^2} + i\frac{1-r^2}{|1-re^{i\phi}|^2} \ .$$

4.3.3 The spectrum of the Hamiltonian H_{L0}

The two theorems of Elstrod, Theorem 4.3.2.5 and Theorem 4.3.2.6 elegantly solve the spectrum of H_{L0}. Thus we can summarize:

Corollary 4.3.3.1. *The following statements are true:*

1. $\sigma_p(H_{L0}) = \left\{ \frac{1}{a^2}(|b| + 2|b|m - m - m^2); 0 \le m < |b| - \frac{1}{2}, m \in \mathbb{Z} \right\}$.
2. $\sigma_c(H_{L0}) = \left[\frac{1}{a^2}\left(\frac{1}{4} + b^2\right), +\infty \right)$.
3. *$\sigma(H_{L0})$ has infinite spectral multiplicity.*
4. *The eigenfunctions, written in the Poincaré disc coordinates, corresponding to the eigenvalue $\frac{1}{a^2}(|b| + 2|b|m - m - m^2)$ are*

   $$g_{m,n}(re^{i\phi}) = r^{|n|}e^{in\phi}(1-r^2)^{|b|-m}\ {}_2F_1(|b|-m-b_n, |b|-m+b_n+|n|; |n|+1; r^2) \ ,$$

   for $0 \le r < 0$, $0 \le \phi < 2\pi$, $0 \le m < |b| - \frac{1}{2}; m, n \in \mathbb{Z}$. Also, $b_n = \pm|b|$, where the sign is chosen to satisfy:

   $$|b| - m - b_n \in \mathbb{Z}_0^- \vee |b| - m + b_n + |n| \in \mathbb{Z}_0^- \ .$$

Remark 4.3.3.2. But with the previous theorem our work has not finished. Next we will try to uncover some information about the spectrum by using an analysis that has many features of a Bloch decomposition, but it is not a Bloch decomposition in a true sense. We will then see, how much information from Theorem 4.3.3.1 can be verified by our method.

4.3.4 The decomposition of H_{L0} under the parabolic group $P(\mathbb{Z})$

In this subsection we will construct a decomposition of our operator H_{L0} under the group $P(\mathbb{Z})$. First, we present a few more definitions.

1. We here restrict ourselves to the case $b \in \mathbb{Z}$.
2. We shall operate with the fundamental region $F := F(P(\mathbb{Z}))$ defined in 2.3.2.9. We will also be working with the group $\tilde{P}(\mathbb{Z})$ which has the same fundamental region as $P(\mathbb{Z})$.

3. We use the multiplication system v_θ defined in Theorem 2.3.3.5 on the group $\tilde{P}(\mathbb{Z})$.
4. The norm in $L^2(\mathbb{H}, d\mu_{\mathbb{H}})$ will be denoted by $\|.\|$.
5. Define the Hilbert space $\mathcal{H}'(\theta) = \mathcal{H}_b(\tilde{P}(\mathbb{Z}), F, v_\theta)$ for $\theta \in [0, 2\pi)$, where $\mathcal{H}_k(G, B, v)$ is the Hilbert space from the Definition 4.3.2.1. We will denote the norm in $\mathcal{H}'(\theta)$ by $\|.\|_\theta$.

Now we define the direct integral

$$\mathcal{H} = \int_{[0,2\pi)}^{\oplus} \mathcal{H}'(\theta) \frac{d\theta}{2\pi} ,$$

with the norm in $\mathcal{H}$ denoted by $\|.\|_2$. Thus from the definition of $\mathcal{H}'(\theta)$ its obvious, that for every $f \in \mathcal{H}'(\theta)$ one has

$$e^{-2ib \arg(cz+d)} f(Mz) = v_\theta(M) f(z) , \quad M \in \tilde{P}(\mathbb{Z}) , \quad z \in \mathbb{H}.$$

Since for any $M \in \tilde{P}(\mathbb{Z})$ we have $c = 0$ and $d = \pm 1$, and we restrict to the case $b \in \mathbb{Z}$, the last equation reduces to

$$f(Mz) = v_\theta(M) f(z),$$

hence

$$f(z+n) = e^{in\theta} f(z) , \qquad z \in \mathbb{H}.$$

Define the mapping $U : L^2(\mathbb{H}, d\mu_{\mathbb{H}}) \to \mathcal{H}$:

$$(Uf)_\theta(z) = \sum_{n=-\infty}^{+\infty} e^{-i\theta n} f(z+n) , \tag{4.4}$$

for $\theta \in [0, 2\pi)$ and $z \in F$. For the following theorem we use the proof of a lemma in Reed-Simon [31, p. 289].

Theorem 4.3.4.1. *U is well defined for $C_0^\infty(\mathbb{H})$ and uniquely extendable to a unitary operator.*

Proof. First, we prove that for any $f \in C_0^\infty(\mathbb{H})$ the image Uf is in $\mathcal{H}$. For $f \in C_0^\infty(\mathbb{H})$ the sum (4.4) converges absolutely. Thus using the theorem of Fubini and the orthogonality of the functions $e^{in\theta}, n \in \mathbb{Z}$ on $[0, 2\pi]$ we have the following:

$$\int_0^{2\pi} \left(\int_{\mathbb{R}^+} \left(\int_0^1 \left| \sum_{n=-\infty}^{+\infty} e^{-in\theta} f(x+iy+n) \right|^2 dx \right) \frac{dy}{y^2} \right) \frac{d\theta}{2\pi}$$

$$= \int_{\mathbb{R}^+} \left(\int_0^1 \left[\left(\sum_{n=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} \overline{f(x+iy+n)} f(x+iy+j) \right) \int_0^{2\pi} e^{-i(j-n)\theta} \frac{d\theta}{2\pi} \right] dx \right) \frac{dy}{y^2}$$

$$= \int_{\mathbb{R}^+} \left(\int_0^1 \left(\sum_{n=-\infty}^{+\infty} |f(x+iy+n)|^2 \right) dx \right) \frac{dy}{y^2} = \int_{\mathbb{H}} |f(z)|^2 d\mu_{\mathbb{H}} ,$$

Hence U can be extended on the whole $L^2(\mathbb{H}, d\mu_{\mathbb{H}})$ to an isometry. Now we prove the surjectivity. For $g \in \mathcal{H}$, we define, for $z \in F, n \in \mathbb{Z}$ the mapping U^*:

$$(U^*g)(z+n) = \int_0^{2\pi} e^{in\theta} g_\theta(z) \frac{d\theta}{2\pi} \ .$$

If one makes the compositions UU^* and U^*U:

$$(U(U^*g))_\theta(z) = \sum_{n=-\infty}^{+\infty} e^{-i\theta n}(U^*g)(z+n) = \sum_{n=-\infty}^{+\infty} e^{-i\theta n} \int_0^{2\pi} e^{in\theta'} g_{\theta'}(z) \frac{d\theta'}{2\pi} = g_\theta(z) \ .$$

since the right-hand side is the Fourier series expansion and the for the other one:

$$(U^*(Uf))(z+n) = \int_0^{2\pi} e^{in\theta} \sum_{j=-\infty}^{+\infty} e^{-ij\theta} f(z+j) \frac{d\theta}{2\pi} = \sum_{j=-\infty}^{+\infty} f(z+j) \int_0^{2\pi} e^{-i(j-n)\theta} \frac{d\theta}{2\pi} = f(z+n),$$

since $e^{in\theta}, n \in \mathbb{Z}$ are orthogonal on $[0, 2\pi]$. Nex we have:

$$\|U^*g\|^2 = \int_{\mathbb{H}} |(U^*g)(z)|^2 d\mu_{\mathbb{H}} = \int_0^{+\infty} \left(\int_0^1 \left(\sum_{n=-\infty}^{+\infty} |(U^*g)(z+n)|^2 \right) dx \right) \frac{dy}{y^2}$$

$$= \int_0^{+\infty} \left(\int_0^1 \left(\sum_{n=-\infty}^{+\infty} \left| \int_0^{2\pi} e^{in\theta} g_\theta(z) \frac{d\theta}{2\pi} \right|^2 \right) dx \right) \frac{dy}{y^2}$$

Using the Parseval relation for the Fourier series:

$$\sum_{n=-\infty}^{+\infty} \left| \int_0^{2\pi} e^{in\theta} g_\theta(z) \frac{d\theta}{2\pi} \right|^2 = \int_0^{2\pi} |g_\theta(z)|^2 \frac{d\theta}{2\pi} \ ,$$

we finally get the equality

$$= \int_0^{+\infty} \left(\int_0^1 \left(\int_0^{2\pi} |g_\theta(z)|^2 \frac{d\theta}{2\pi} \right) dx \right) \frac{dy}{y^2} = \int_0^{+\infty} \left(\int_0^1 \|g_\theta(z)\|_\theta^2 dx \right) \frac{dy}{y^2}$$

$$= \|g\|_2^2.$$

Thus the equality:

$$\|U^*g\| = \|g\|_2$$

is proven along with the surjectivity. □

Now we define the operator $\tilde{H}_\theta$ using the operator Δ_k defined in 4.3.2.3:

$$Dom(\tilde{H}_\theta) := \mathcal{D}_b^\infty$$

where $\mathcal{D}_b^\infty$ is defined in 4.3.2.2 and we set:

$$\tilde{H}_\theta f = -\frac{1}{a^2}(\Delta_{-b} - b^2 I) f \ , \quad f \in Dom(\tilde{H}_\theta).$$

Denote the self-adjoint extension(according to Theorem 4.3.2.4 it is unique) of $\tilde{H}_\theta$ by H_θ. Here again, we use the lemma in Reed-Simon [31, p. 289].

Theorem 4.3.4.2. *The following equality holds:*

$$U(H_{L0})U^{-1} = \int_{[0,2\pi)}^{\oplus} H_\theta \frac{d\theta}{2\pi}. \tag{4.5}$$

Proof. We denote the right hand side of (4.5) as A. Since for all $\theta \in [0, 2\pi)$ H_θ is self-adjoint, we have from Theorem 2.2.2.21 that A is self adjoint. Denote the differential expression

$$\mathcal{L} = -\frac{1}{a^2}\left(x_2^2\partial_1^2 + x_2^2\partial_2^2 + 2ibx_2\partial_1 - b^2\right)$$

If we prove for $f \in C_0^\infty(\mathbb{H})$ that $Uf \in Dom(A)$ and the equation

$$U(\mathcal{L}f) = A(Uf), \tag{4.6}$$

holds, we will prove the equality $UH_{L0,min} = AU$ on the domain $C_0^\infty(\mathbb{H})$ and our job will be done, since then $UH_{L0,min}^* U^{-1} = A^*$, from Theorem 4.3.1.1 $H_{L0,min}$ is essentially self-adjoint, thus $H_{L0,min}^* = H_{L0}$ and $A = A^*$, hence leading us to the equality:

$$UH_{L0}U^{-1} = A.$$

So let $f \in C_0^\infty(\mathbb{H})$.

Let θ be fixed. From the definition of U we have:

$$(Uf)_\theta(z+n) = \sum_{k=-\infty}^{+\infty} e^{-i\theta k} f(z+n+k) = \sum_{l=-\infty}^{+\infty} e^{-i\theta l} e^{in\theta} f(z+l)$$

$$= e^{in\theta} \sum_{l=-\infty}^{+\infty} e^{-i\theta l} f(z+l) = e^{in\theta}(Uf)_\theta(z)\ .$$

One clearly observes, that $(Uf)_\theta \in C^\infty(\mathbb{H})$ and for any $p, q \in \mathbb{N}_0$ and $m, n = 1, 2$ the equality

$$x_m^p \partial_n^q (Uf)_\theta(z) = (U(x_m^p \partial_n^q f))_\theta(z)\ , \qquad z = x_1 + ix_2$$

holds. Since f has compact support in $\mathbb{H}$, there exist $R > 0$ such that for all $x_1 \in \mathbb{R}, x_2 > R$ we have $f(x_1 + ix_2) = 0$ and the topological structure of F implies, that $(Uf)_\theta$ has a modulo $\tilde{P}(\mathbb{Z})$ compact support. Hence $(Uf)_\theta \in Dom(H_\theta)$, moreover

$$A_\theta(Uf)_\theta = (U\mathcal{L}f)_\theta$$

To make the statement $Uf \in Dom(A)$ true it remains to prove

$$\int_{[0,2\pi)} \|A_\theta(Uf)_\theta\|_\theta^2 \frac{d\theta}{2\pi} < +\infty.$$

But again, since f has compact support in $\mathbb{H}$, there exists a rectangle $[-N, N] \times [c, R], N \in \mathbb{N}$ such that $supp f \subset [-N, N] \times [c, R]$, so we can make the estimate

$$\|A_\theta(Uf)_\theta\|_\theta = \|\ (U\mathcal{L}f)_\theta\ \|_\theta \le 2N\|\mathcal{L}f\|$$

for all $\theta \in [0, 2\pi)$, thus

$$\int_{[0,2\pi)} \|A_\theta(Uf)_\theta\|_\theta^2 \frac{d\theta}{2\pi} \le 2N\|\mathcal{L}f\| < +\infty$$

since $f \in Dom(H_{L0,min})$. Hence $Uf \in Dom(A)$ and $U(\mathcal{L}f) = A(Uf)$ and the equality 4.5 is proved. $\square$

4.3.5 The eigenvalue problem of H_θ

We are going to study the spectrum of H_θ and hence eigenvalue problem

$$-\frac{1}{a^2}\left(x_2^2\partial_1^2 + x_2^2\partial_2^2 + 2ibx_2\partial_1 - b^2\right)\psi = \lambda\psi \ . \tag{4.7}$$

We will use the following ansatz:

$$\psi(x_1, x_2) = e^{ilx_1}\phi(x_2) \ , \qquad l \in \mathbb{R}. \tag{4.8}$$

The equation (4.7) is reduced to

$$\phi'' + \left[-k^2 - \frac{2bl}{x_2} + \frac{a^2\lambda - b^2}{x_2^2}\right]\phi = 0$$

From the boundary conditions on the functions in $\mathcal{H}'_\theta$ we have the following restriction on the parameter l:

$$\phi(x_2) = e^{il+i\theta}\phi(x_2) \ , \qquad x_2 > 0 \ ,$$

thus

$$l = 2\pi n - \theta, \ n \in \mathbb{Z}.$$

After applying the substitution $z = 2|l|x_2$ and defining $v(z) = \phi(x_2)$ we have

$$\frac{d^2v}{dz^2} + \left[-\frac{1}{4} - sgn(l)\frac{b}{z} + \frac{a^2\lambda - b^2}{z^2}\right]v = 0$$

which is the famous Whittaker differential equation. Now define the formal differential expression τ_W

$$\tau_W = \frac{z^2}{a^2}\left(-\frac{d^2}{dz^2} + \frac{1}{4} + sgn(l)\frac{b}{z} + \frac{b^2}{z^2}\right)$$

so

$$\tau_W\psi = \lambda\psi$$

Now the corresponding minimal operator $T_{W,min}$:

$$T_{W,min}u = \tau_W u \ , \qquad u \in Dom(T_{W,min}) \tag{4.9}$$

$$Dom(T_{W,min}) = \{f \in L^2(\mathbb{R}_+, z^{-2}dz) \ ; \ f, f' \in AC_0(\mathbb{R}_+); \tau f \in L^2(\mathbb{R}_+, z^{-2}dz)\}$$

Lets take the substitution $z = e^t$ and define the mapping U_1:

$$U_1 : L^2(\mathbb{R}_+, z^{-2}dz) \to L^2(\mathbb{R}, e^{-t}dt)$$

$$U_1 : f(z) \to f(e^t)$$

Obviously U_1 is a bijection and

$$\int_0^{+\infty} |\psi(z)|^2 z^{-2}dz = \int_{-\infty}^{+\infty} |\psi(e^t)|e^{-t}dt$$

so its also unitary. Analogically we have the mapping U_2:

$$U_2 : L^2(\mathbb{R}, e^{-t}dt) \to L^2(\mathbb{R}, dt)$$

$$U_2 : g(t) \to g(t)e^{-t/2}$$

again a bijection and

$$\int_{-\infty}^{+\infty} |\psi(e^t)|e^{-t}dt = \int_{-\infty}^{+\infty} |\psi(e^t)e^{-t/2}|^2 dt$$

so again, unitary. Hence the composition

$$U_3 = U_2U_1,\ U_3 : L^2(\mathbb{R}_+, z^{-2}dz) \to L^2(\mathbb{R}, dx) \tag{4.10}$$

is unitary. After applying the transformations on τ_W we get :

$$\frac{1}{a^2}\left(-\frac{d^2}{dt^2} + \frac{1}{4}e^{2t} + sgn(l)be^t + \frac{1}{4} + b^2\right)$$

Let $U_3\psi = \xi$ So we get

$$\frac{1}{a^2}\left(-\frac{d^2}{dt^2} + \frac{1}{4}e^{2t} + sgn(l)be^t + \frac{1}{4} + b^2\right)\xi = \lambda\xi\ ,$$

or equivalently

$$\left(-\frac{d^2}{dt^2} + \frac{1}{4}e^{2t} + sgn(l)be^t\right)\xi = \mu\xi\ , \qquad \mu = a^2\lambda - \frac{1}{4} - b^2.$$

Define $\beta = -sgn(l)b$ and define the formal differential expression

$$\tau = \left(-\frac{d^2}{dt^2} + \frac{1}{4}e^{2t} - \beta e^t\right). \tag{4.11}$$

We will continue by analyzing the spectrum of a self-adjoint realization of τ.

4.3.6 The ordinary Schrödinger operator with a Morse potential

Through the solution ansatz and the unitary transformations U_1, U_2, U_3 we have transformed the 2-dimensional problem into a one-dimensional. In this subsection we will analyze the one-dimensional differential operators generated by the differential expression (4.11).

Definition 4.3.6.1. *A* **Morse potential** *is a function $V(x) : \mathbb{R} \to \mathbb{R}$ of the form*

$$V_M(x; a, c, \mu) = c\left(e^{-2a(x-\mu)} - 2e^{-a(x-\mu)}\right) \tag{4.12}$$

where $a, c, \mu \in \mathbb{R}$.

Remark 4.3.6.2. In our case we will study the Schrödinger operator with the potential

$$V(x) = \frac{1}{4}e^{2x} - \beta e^x \tag{4.13}$$

where $\beta \in \mathbb{R} \setminus \{0\}$. If $\beta > 0$, we have the special case of (4.12) :

$$V(x) = \frac{1}{4}e^{2x} - \beta e^x = V_M(x; -1, \beta^2, \ln 2\beta)$$

with parameters $a = -1$, $c = \beta^2$, $\mu = \ln 2\beta$.

For the rest of this subsection, we will use the following notations. By $V(x)$ we will always denote the potential (4.13). Define τ_M to be the Schrödinger differential expression on $\mathbb{R}$:

$$\tau_M = -\left(\frac{d}{dx}\right)^2 + \frac{1}{4}e^{2x} - \beta e^x \tag{4.14}$$

Define T'_{M0} and T_M to be the minimal, resp. maximal operators on $L^2(\mathbb{R}, dx)$ with their domains

$$Dom(T'_{M0}) = \{f \in L^2(\mathbb{R}, dx) | f, f' \in AC_0(\mathbb{R}); \tau f \in L^2(\mathbb{R}, dx)\} \tag{4.15}$$

$$Dom(T_M) = \{f \in L^2(\mathbb{R}, dx) | f, f' \in AC(\mathbb{R}); \tau f \in L^2(\mathbb{R}, dx)\}. \tag{4.16}$$

Since according to Corollary 3.1.1.13, T'_{M0} is symmetric, we may define $T_{M0} = \overline{T'_{M0}}$. Now we will take a closer look at $V(x)$. We set $V_-(x) = max\{-V(x), 0\}$. We have the following observations:

1. $V(x)$ has its global minimum at $-\beta^2$.
2. If $\beta < 0$ then $V(x)$ is non-negative on $\mathbb{R}$.
3. If $\beta > 0$ then :
   (a) The potential $V(x)$ is negative on $(-\infty, \ln 4\beta)$.
   (b)
   $$\int_{\mathbb{R}} V_-(x)dx = 2\beta^2. \tag{4.17}$$
   (c) Let $R_1 > \ln 4\beta$, then
   $$R_1 \int_{R_1}^{+\infty} V_-(x)dx = 0 < \frac{1}{4}. \tag{4.18}$$
   (d) Let $R_2 > -\ln 4\beta$, then
   $$R_2 \int_{R_2}^{+\infty} V_-(-x)dx = R_2 e^{-R_2}(\beta - \frac{1}{8}e^{-R_2}), \tag{4.19}$$
   which decays exponentially as $R_2 \to +\infty$ so we can find an $N > 0$ such that
   $$R_2 e^{-R_2}(\beta - \frac{1}{8}e^{-R_2}) < \frac{1}{4}. \tag{4.20}$$

And now we return to τ.

Lemma 4.3.6.3. *The following statements hold:*

1. τ is a singular differential expression.

2. τ is the limit point case at both $\pm\infty$.

Proof. 1) Obviously from the definition, since all coefficients are locally integrable and the endpoints of the interval are $\pm\infty$.

2) Here we use Theorem 3.1.3.1. We have the Sturm-Liouville differential expression with $p(x) = 1, q(x) = V(x)$ on the interval $(a, b) = (-\infty, +\infty)$. Let $c \in \mathbb{R}$ and define the function $g(x)$:

$$g(x) = \int_c^x \frac{1}{p(x)}dx = x - c,$$

so $g \notin L^2(\mathbb{R}, dx)$. Also

$$\liminf_{x\to+\infty} q(x) = \liminf_{x\to+\infty} V(x) = \lim_{x\to+\infty} V(x) = +\infty > -\infty,$$

hence τ is the limit point case at $+\infty$. The endpoint $-\infty$ is treated analogously. □

Lemma 4.3.6.4. *The operator T_M is essentially self-adjoint.*

Proof. This fact comes from Theorem 3.2.1.5. Define the operator T:

$$Dom(T) = C_0^\infty(\mathbb{R}) \ , \qquad Tf = \tau f \ , \ f \in Dom(T).$$

From the properties of our potential $V(x)$ we have $V(x) \geq -\beta^2$. By setting $Q(x) = \beta^2$ for all $x \in \mathbb{R}$ we have

$$V(x) \geq -Q(x), \forall x \in \mathbb{R} \quad , \qquad \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{Q(2x)}} = +\infty$$

So the requirements of the theorem are fulfilled and thus T is essentially self-adjoint and since $T \subset T_M$, the operator T_M is essentially self-adjoint. □

But we can prove even more, hence the following:

Lemma 4.3.6.5. *The operator T_M is self-adjoint.*

Proof. We first use Theorem 3.1.1.16, from which we have $T_M = T_{M0}^*$, then Theorem 3.1.2.6, since τ is the limit point case at both ends. Thus we have $T_{M0}^* = T_M = T_{M0}$. So T_M is self-adjoint. □

In this part of subsection we will examine the spectrum of $\sigma(T_M)$. First we start with its location.

Lemma 4.3.6.6. $\sigma(T_M) \subset \big[-\beta^2, +\infty\big)$.

Proof. The proof follows from Theorem 3.2.2.3 since for all $x \in \mathbb{R}$ $V(x) \geq -\beta^2$. □

One could also apply Theorem 3.2.2.7 but after a simple straightforward calculation the function ν_t yields:

$$\nu_t = -\frac{2\beta^2}{t}\frac{e^t - 1}{e^t + 1} \ , \ t > 0$$

with $\lim_{t\to 0+} \nu_t = -\beta^2$. And if we use the estimate 3.21, we get

$$\sigma(T_S) \subset \big[-\beta^2, +\infty\big) \subset \left[\nu_0 - \frac{16\eta_0^2}{\pi^2}, +\infty\right) ,$$

so our estimate will not improve. Hence Theorem 3.2.2.7 is not useful in our case. We continue with the essential spectrum:

Lemma 4.3.6.7. $\sigma_{ess}(T_M) = [0, +\infty)$.

Proof. 1) $\sigma_{ess}(T_M) \supset [0,+\infty)$. We apply Theorem 3.2.2.10 since $\int_{\mathbb{R}} V_-(x)dx < +\infty$. We set $I_n = (-n, \ln 4|\beta|)$ for every $n \in \mathbb{N}$ and $\mu = 0$. We can write

$$\int_{I_n} |V(x)|dx \leq \int_{-\infty}^{\ln 4|\beta|} |V(x)|dx \ , \ \forall n \in \mathbb{N},$$

hence the inequalities

$$0 \leq |I_n|^{-1} \int_{I_n} |V(x)|dx \leq |I_n|^{-1} \int_{-\infty}^{\ln 4|\beta|} |V(x)|dx$$

hold and since $\lim_{n\to+\infty} |I_n| = +\infty$ we have

$$\lim_{n\to+\infty} |I_n|^{-1} \int_{I_n} |V(x)|dx = 0$$

thus the inclusion is proved.

2) $\sigma_{ess}(T_M) \subset [0,+\infty)$. Now we put Theorem 3.2.2.9 to work. So the functions λ_t, ω_t for $t > 0$ with our potential $V(x)$ are

$$\lambda_t = \liminf_{|\gamma|\to+\infty} t^{-1} \int_{\gamma}^{\gamma+t} V(x)dx = 0$$

since $V(x)$ exponentially diverges to $+\infty$ when $x \to +\infty$ and decays exponentially when $x \to -\infty$ and

$$\omega_t = \limsup_{|\gamma|\to+\infty} t^{-1} \int_{\gamma}^{\gamma+t} V_-(x)dx = 0$$

since for $\beta < 0$ is $V_-(x) = 0$ for all $x \in \mathbb{R}$, and for $\beta > 0$ is $V_-(x)$ nonzero only on $(-\infty, \ln 4\beta)$ and also decays exponentially when $x \to -\infty$. Hence

$$\sigma_{ess}(T_S) \subset \left[\lambda_0 - \frac{16\omega_0^2}{\pi^2}, +\infty\right) = [0,+\infty) \, .$$

So $\sigma_{ess}(T_S) = [0,+\infty)$. □

And now we examine the point spectrum:

Lemma 4.3.6.8. *The following statements hold:*

1. *T_M has at most finite number of eigenvalues.*

2. *Every eigenvalue of T_M is negative.*

3. *The bound on the number of eigenvalues*

$$N_-(T_M) \leq \frac{7}{8} + 2\beta - 3\beta^2 + 2\beta^2 \ln 4\beta$$

valid for every $\beta > 0$.

Proof. 1) Since $\int_{\mathbb{R}} V_-(x)dx$ is finite and from the properties of $V(x)$ mentioned earlier we can use Theorem 3.2.2.5.

2) Obvious consequence from Theorem 3.2.2.1.

3) We use Theorem 3.2.2.6 and have the estimate for $\beta > 0$

$$N_-(T_M) \leq \frac{7}{8} + 2\beta - 3\beta^2 + 2\beta^2 \ln 4\beta.$$

□

So far, the point spectrum is an isolated finite set and the essential spectrum is the set $[0, +\infty)$, we may write:

Lemma 4.3.6.9. *For every $\beta \in \mathbb{R} \setminus \{0\}$, $\sigma_c(T_M) = \sigma_{ess}(T_M) = [0, +\infty)$.*

Proof. Obvious. From Theorem 2.1.2.5 we have

$$\sigma_c(T_M) = \sigma_{ess}(T_M) \setminus \sigma_p(T_M) = \sigma_{ess}(T_M),$$

since the point spectrum and the essential spectrum are disjoint, because every eigenvalue is of finite multiplicity ≤ 2, otherwise τ_M would have a fundamental system of at least three solutions, which is a contradiction with the theory of linear differential equations. □

Now, since we know that T_M has a finite number of eigenvalues and that $\sigma_c(T_M) = \sigma_{ess}(T_M) = [0, +\infty)$, all that remains is to locate the eigenvalues with their corresponding eigenfunctions and investigate the absolutely continuous and singular continuous spectrum. This will be done in the following. But first, we return for a little while to the potential $V(x)$. In order to use the asymptotic expressions for the solutions of the equation $\tau_M\psi = k^2\psi$ with the spectral parameter k^2, we must check, if $V(x)$ satisfies the conditions 3.11 for the endpoint $-\infty$. But that is simple computation and estimation: For $x_0 < \min\{0, \ln 2|\beta|\}$ we have:

$$\int_{-\infty}^{x_0} |V'(x)|^2 dx = \int_{-\infty}^{x_0} e^x \left|\frac{1}{2}e^x - \beta\right|^2 dx \leq \int_{-\infty}^{x_0} e^x(1+|\beta|)^2 dx = (1+|\beta|)^2 e^{x_0} < +\infty.$$

For $x_0 < \min\{0, \ln|\beta|\}$ we have:

$$\int_{-\infty}^{x_0} |V''(x)| dx = \int_{-\infty}^{x_0} e^x|e^x - \beta| \leq \int_{-\infty}^{x_0} e^x(1+|\beta|)dx = (1+|\beta|)e^{x_0} < +\infty.$$

One might want an asymptotic behavior towards $+\infty$, but to our misfortune $V(x)$ does not satisfy the second of the conditions in 3.11. But we are armed with Theorem 3.2.1.7 which will gratefully satisfy our needs. We will always assume $\Im k \geq 0$. The one solution of the equation $\tau_M\psi = k^2\psi$ is:

$$\psi(x,k) = e^{-\frac{x}{2}} W_{\beta,ik}(e^x) = e^{-\frac{x}{2}} W_{\beta,-ik}(e^x), \tag{4.21}$$

where $W_{k,m}(z)$ is the Whittaker function. We now give the asymptotic behavior and both endpoints for $2ik \notin \mathbb{Z}$.

$$\psi(x,k) = e^{ikx}\left(\frac{\Gamma(-2ik)}{\Gamma(\frac{1}{2}-\beta-ik)} + \frac{\beta\Gamma(-1-2ik)e^x}{\Gamma(\frac{1}{2}-\beta-ik)} + \mathcal{O}\left(e^{2x}\right)\right)$$

$$+e^{-ikx}\left(\frac{\Gamma(2ik)}{\Gamma(\frac{1}{2}-\beta+ik)}+\frac{\beta\Gamma(-1+2ik)e^{x}}{\Gamma(\frac{1}{2}-\beta+ik)}+\mathcal{O}\left(e^{2x}\right)\right)\ ,\ for\ x\to-\infty \tag{4.22}$$

$$\psi(x,k)=e^{-\frac{e^x}{2}}e^{\beta x}\left(e^{-x/2}+\mathcal{O}\left(e^{-\frac{3x}{2}}\right)\right)\ ,\ for\ x\to+\infty. \tag{4.23}$$

From the asymptotic behavior towards $+\infty$ we can see, that $\psi(x,k)$ is faster then exponentially decaying. The solution is real for real $k^2\in\mathbb{R}$, thus ψ is a suitable choice for $s_1(x,k^2)$. The exist an another independent real solution $\phi(x,k)$ as the other solution $s_2(x,k^2)$ for which we define $\frac{\phi(x,k)}{W(\phi(x,k),\psi(x,k))}$, where $W(\phi(x,k),\psi(x,k))$ denotes the Wronskian of the two solutions. But we only need $s_1(x,k^2)$ to find the eigenvalues.

Thus we have the system of fundamental solution according to Definition 3.2.2.11:

$$s_1(x,k^2)=\psi(x,k)\quad,\quad s_2(x,k^2)=\frac{\phi(x,k)}{W(\phi(x,k),\psi(x,k))}.$$

By $u(x,k)$ we will denote the unique solution having the asymptotic behavior towards $-\infty$ determined by Theorem 3.2.1.8 and chosen such that for k^2 being an eigenvalue it is exponentially decaying. Then from the same theorem we have an another, $u(x,-k)$, being divergent at $-\infty$, thus having a second system of independent solutions. Hence there exist unique functions $\alpha(k),\beta(k)$ satisfying

$$\psi(x,k)=\alpha(k)u(x,k)+\beta(k)u(x,-k),$$

but since for real k^2, $\psi(x,k)$ is real, we can write the equality in the form

$$\psi(x,k)=\alpha(k)u(x,k)+\alpha(-k)u(x,-k), \tag{4.24}$$

and obviously $\alpha(k)=\overline{\alpha(-k)}$.

So the asymptotic behavior of $\psi(x,k)$ will be a superposition of the asymptotic behavior of $u(x,k),u(x,-k)$:

$$\psi(x,k)\sim\alpha(k)\left[\exp\left(ik\int_x^{x_0}\sqrt{1-\frac{V(t)}{k^2}}dt\right)(1+o(1))\right]$$

$$+\alpha(-k)\left[\exp\left(-ik\int_x^{x_0}\sqrt{1-\frac{V(t)}{k^2}}dt\right)(1+o(1))\right].$$

But if we slightly modify (4.22), we get

$$\psi(x,k)=\frac{\Gamma(-2ik)}{\Gamma(\frac{1}{2}-\beta-ik)}e^{ikx}+o(e^{2x+ikx})+\frac{\Gamma(2ik)}{\Gamma(\frac{1}{2}-\beta+ik)}e^{-ikx}+o(e^{2x-ikx})=$$

$$=\frac{\Gamma(-2ik)}{\Gamma(\frac{1}{2}-\beta-ik)}e^{ikx}+o(e^{ikx})+\frac{\Gamma(2ik)}{\Gamma(\frac{1}{2}-\beta+ik)}e^{-ikx}+o(e^{-ikx}). \tag{4.25}$$

Here we use elementary rules from asymptotic analysis. For given functions f,g satisfying the relation $f(x)\sim g(x)(1+o(1))$ we have

$$f(x)=g(x)(1+o(1))+o(g(x)(1+o(1)))=g(x)+o(g(x))+o(g(x)+o(g(x)))$$

$$=g(x)+o(g(x))+o(g(x))+o(o(g(x)))=g(x)+o(g(x))+o(g(x))+o(g(x))=g(x)+o(g(x)).$$

So for $f_i(x) \sim g_i(x)(1+o(1))$, $i = 1, 2$, and for some $a_i \in \mathbb{C}$, $i = 1, 2$ such that

$$a_1 f_1(x) + a_2 f_2(x) \sim a_1 g_1(x)(1+o(1)) + a_2 g_2(x)(1+o(1)),$$

we have

$$a_1 f_1(x) + a_2 f_2(x) = a_1 g_1(x) + o(g_1(x)) + a_2 g_2(x) + o(g_2(x)).$$

Now if we apply this trace of thought on equations (4.24) and (4.25) and compare them, we find the coefficient $\alpha(k)$:

$$\alpha(k) = \frac{\Gamma(2ik)}{\Gamma(\frac{1}{2} - \beta + ik)}. \tag{4.26}$$

The Wronskian of the system $s_1(x, k^2), u(x, k)$ is

$$W(s_1(x, k^2), u(x, k)) = \frac{\Gamma(-2ik)}{\Gamma(\frac{1}{2} - \beta - ik)} W(u(x, -k), u(x, k)), \tag{4.27}$$

which is exactly what we need to find all the eigenvalues of T_M. The Wronskian $W(u(x, -k), u(x, k))$ will not cause any trouble, because its nonzero for $k \neq 0$, and holomorphic in $\mathbb{C}$. The function $\alpha(k)$ has zeros at the poles of the Gamma function, which is when

$$\frac{1}{2} - \beta - ik_n = -n \ , n \in \mathbb{N}_0 \ ,$$

thus, applying Theorem 3.2.2.21, we have the eigenvalues of T_M:

$$E_n = k_n^2 = \left(-i\left(\beta - \frac{1}{2} - n\right)\right)^2 = -\left(\beta - \frac{1}{2} - n\right)^2$$

with n satisfying $0 \leq n < \beta - \frac{1}{2}$, because $k_n = i|k_n|$ and so $-ik_n > 0$ thus $-n = \frac{1}{2} - \beta - ik_n > \frac{1}{2} - b$ and $n < \beta - \frac{1}{2}$. Hence eigenvalues exist only for $\beta > 0$.

And the fact, that we used the asymptotic behavior with the restriction $2ik \notin \mathbb{Z}$, will not cause a loss of generality since we are interested in the zero points of the Wronskian, not its singularity points.

Now we focus on the spectral measure. From Theorem 5.1 in Kodaira's paper [25] the following holds:

$$W(u(x, -k), u(x, k)) = -2ik, \tag{4.28}$$

so if we write $u(x, k)$ as the linear combination of $s_1(x, k^2)$ and $s_2(x, k^2)$

$$u(x, k) = A(k) s_2(x, k^2) - B(k) s_1(x, k^2),$$

we have:

$$A(k) = W(u(x, k), s_1(x, k^2)) = \frac{2ik\Gamma(-2ik)}{\Gamma(\frac{1}{2} - \beta - ik)}, \tag{4.29}$$

and here we use the expression for the spectral measure from Theorem 3.2.3.3

$$d\rho(k^2) = \frac{1}{\sqrt{\pi}} \frac{\sqrt{|kW(u(x, -k), u(x, k))|}}{|A(k)|} dk \ , k \geq 0 \tag{4.30}$$

since our $u(x, k)$ is the function $\phi(x, k)$ in Theorem 3.2.3.3. Thus

$$d\rho(k^2) = \frac{1}{\pi} \sqrt{k \sinh(2\pi k)} \left| \Gamma\left(\frac{1}{2} - \beta - ik\right) \right| dk \ , k \geq 0. \tag{4.31}$$

The function on the right side of (4.31) is well-defined for every $k > 0$(the only critical point may be $k = 0$ when the the argument in the gamma function is a negative integer, but this is a removable singularity). Hence this answers the question about the absolute continuity of the spectrum $\sigma_c(T)$, which can be summarized in the following:

Corollary 4.3.6.10. *The following assertions hold:*

1. $\sigma_{ac}(T_M) = \sigma_c(T_M) = [0, +\infty)$.
2. $\sigma_{sc}(T) = \emptyset$.

Remark 4.3.6.11. The second assertion of the corollary can be proven via Theorem 3.1.7.1.

Returning to the case $\beta > 0$ and the eigenvectors, they can be written in the form:

$$s_1(x, E_n) = e^{-\frac{x}{2}} W_{\beta,|k_n|}(e^x) = e^{-\frac{e^x}{2}} e^{(\beta-n)x} L_n^{2\beta-2n-1}(e^x)\ , \qquad E_n = k_n^2 = -\left(\beta - \frac{1}{2} - n\right)^2 ,$$

where $L_n^{(\alpha)}(x)$ is the generalized Laguerre polynomial. Here we used Corollary 6.1.2.9 and the relation

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} U(-n, \alpha+1, x)$$

with the Tricomi confluent hypergeometric function. Now we will normalize them using the equality

$$\int_0^{+\infty} e^{-z} z^{\alpha} (L^{(\alpha)} n(z))^2 dz = \alpha \int_0^{+\infty} e^{-z} z^{\alpha-1} (L^{(\alpha)} n(z))^2 dz = \frac{\Gamma(\alpha+n+1)}{n!}$$

from Gradshteyn-Ryzhik [19] and the substitutions $z = e^x,\ \alpha = 2\beta - 2n - 1$. We have the relation

$$\int_{\mathbb{R}} s_1(x, E_n) = \frac{\Gamma(2\beta - n)}{n!(2\beta - 2n - 1)},$$

which is well-defined, sice $n < \beta - \frac{1}{2}$.

Finally, using Theorem 3.2.3.3, which conditions our case satisfies, we have the complete system of generalized eigenvectors for the operator T_M:

- Case $b < 0$:

$$\Psi(x, k) = \frac{1}{\pi} \sqrt{k \sinh(2\pi k)} \left| \Gamma\left(\frac{1}{2} - \beta - ik\right) \right| e^{-\frac{x}{2}} W_{\beta,ik}(e^x)\ , \qquad k \geq 0 \tag{4.32}$$

- Case $b > 0$:

$$\Psi(x, k) = \left\{ \begin{array}{ll} \sqrt{\frac{n!(2\beta-2n-1)}{\Gamma(2\beta-n)}} e^{-\frac{e^x}{2}} e^{(\beta-n)x} L_n^{(2\beta-2n-1)}(e^x) & , k = i\sqrt{E_n} \\ \frac{1}{\pi}\sqrt{k \sinh(2\pi k)} \left|\Gamma\left(\frac{1}{2} - \beta - ik\right)\right| e^{-\frac{x}{2}} W_{\beta,ik}(e^x) & , k \geq 0 \end{array} \right\}, \tag{4.33}$$

4.3.7 The spectrum of H_θ

First we return to the operator $T_{W,min}$ defined in (4.9). By the transformation U_3, defined in (4.10), we know that

$$T_{W,min} = U_3^{-1}\left[\frac{1}{a^2}\left((T'_{M0} + \left(\frac{1}{4}+b^2\right)I\right)\right]U_3.$$

From Theorem 4.3.6.5 we have the essential self-adjointness of T'_{M0}, so $T_W := T^*_{W,min}$ is self-adjoint. Applying the unitary transformation, we get the spectrum and the complete system of generalized eigenvectors for the operator T_W defined as

$$T_W f = \frac{1}{a^2}\left(-\frac{d^2}{dt^2}+\frac{1}{4}e^{2t}+sgn(l)be^t+\frac{1}{4}+b^2\right)f\ , \qquad f\in Dom(T_W)=U_3^{-1}(Dom(T_M)).$$

The spectrum:

- The continuous spectrum : $\sigma_c(H_\theta)=\left[\frac{1}{a^2}\left(\frac{1}{4}+b^2\right),+\infty\right)$.
- The point spectrum : $\sigma_p(H_\theta)=\left\{\frac{1}{a^2}\left(\frac{1}{4}+b^2-\left(|b|-\frac{1}{2}-n\right)^2\right);0\le n<|b|-\frac{1}{2},\ z\in\mathbb{Z}\right\}$.

The generalized eigenvectors, $l\neq 0$:

- The continuous spectrum:

$$\Psi_W(t,k,l)=\sqrt{\frac{k\sinh(2\pi k)}{2\pi^2|l|}}\left|\Gamma\left(\frac{1}{2}+sgn(l)b-ik\right)\right|W_{sgn(l)b,ik}(2|l|t)\ ,k\ge 0 \quad (4.34)$$

- The point spectrum: $k=\sqrt{\frac{1}{a^2}\left(\frac{1}{4}+b^2-\left(|b|-\frac{1}{2}-n\right)^2\right)}$

$$\Psi_W(t,k,l)=\sqrt{\frac{n!(|b|-2n-1)}{2|l|\Gamma(2|b|-n)}}e^{-|l|t}(2|l|t)^{|b|-n}L_n^{(|b|-2n-1)}(2|l|t) \quad (4.35)$$

where we chosen to 'normalize' even the generalized eigenvectors for the continuous.

And now we analyze the operator H_θ. Here we keep in mind the fact, that the eigenvalues exist only in the case $\beta=-sgn(l)b>0$, or equivalently $lb<0$.

The generalized eigenvectors:

- The continuous spectrum: $k\ge 0,\ l_m=2\pi m-\theta,\ m\in\mathbb{Z}$.

$$\Phi(t,k,l_m)=\sqrt{\frac{k\sinh(2\pi k)}{4\pi^3|l_m|}}\left|\Gamma\left(\frac{1}{2}+sgn(l_m)b-ik\right)\right|e^{il_mx_1}W_{sgn(l_m)b,ik}(2|l_m|x_2) \quad (4.36)$$

- The point spectrum : eigenvalue $\frac{1}{a^2}\left(\frac{1}{4}+b^2-\left(|b|-\frac{1}{2}-n\right)^2\right)$: With the condition $l_mb<0$ we have the two cases:
  - Case $b>0,\theta=0$: $l_m=2\pi m-\theta,\ m\in\mathbb{Z}^-$
  - Case $b>0,\theta\neq 0$: $l_m=2\pi m-\theta,\ m\in\mathbb{Z}_0^-$

– Case $b < 0$: $l_m = 2\pi m - \theta,\ m \in \mathbb{N}$

$$\Phi_n(t,k,l_m) = \sqrt{\frac{n!(|b|-2n-1)}{4\pi|l_m|\Gamma(2|b|-n)}}e^{il_m x_1}e^{-|l_m|t}(2|l_m|x_2)^{|b|-n}L_n^{(|b|-2n-1)}(2|l_m|x_2) \tag{4.37}$$

Finally, we can make the summary:

Corollary 4.3.7.1. *The following statements are true:*

1. *every element of the set*

$$\left\{\frac{1}{a^2}\left(\frac{1}{4}+b^2-\left(|b|-\frac{1}{2}-n\right)^2\right); 0 \le n < |b|-\frac{1}{2},\ z \in \mathbb{Z}\right\}$$

   is an element of the point spectrum of the operator H_θ having infinite spectral multiplicity.

2. *every element of the set* $\left[\frac{1}{a^2}\left(\frac{1}{4}+b^2\right), +\infty\right)$ *is an element of the continuous spectrum of the operator H_θ having infinite spectral multiplicity.*

Now we return to the operator H_{L0}.

Theorem 4.3.7.2. *The following statements are true:*

1. *every element of the set*

$$\left\{\frac{1}{a^2}\left(\frac{1}{4}+b^2-\left(|b|-\frac{1}{2}-n\right)^2\right); 0 \le n < |b|-\frac{1}{2},\ z \in \mathbb{Z}\right\}$$

   is an element of the point spectrum of the operator H_{L0}.

2. *every element of the set* $\left[\frac{1}{a^2}\left(\frac{1}{4}+b^2\right), +\infty\right)$ *is an element of the continuous spectrum of the operator H_{L0}.*

Proof. To proof both assertions one uses Theorem 2.2.2.21 and Corollary 4.3.7.1. □

Chapter 5

Summary of the problem

5.0.8 Evaluation of the analysis of H_{L0}

With Theorem 4.3.7.2 we have partially verified Theorem 4.3.3.1. However, if one wants to complete this task, it is necessary to prove that the generalized eigenvectors mentioned above is the complete system of generalized eigenvectors of the operator H_θ. And if not, one must find the remaining generalized eigenvectors. This investigation will not be carried out here.

The reason, why we did not carry out the decomposition of $L^2(\mathbb{H}, d\mu_{\mathbb{H}})$ into the direct integral $\displaystyle\int_{[0,2\pi)}^{\oplus} L^2\left([0,1]\times\mathbb{R}^+, d\mu_{\mathbb{H}}\right)\frac{d\theta}{2\pi}$ (and thus making the Bloch decomposition), is that we would be forced to undertake the spectral analysis of the operator H_θ on a more complicated domain. It was not, however, our aim to take such endeavors deep into the field of partial differential equations.

The problem concerning periodic magnetic Hamiltonians on the Lobachevsky plane is far from being solved. One of the most challenging problems is to carry a Bloch decomposition with a general non-commutative group of isometries of the Lobachevsky plane, which produces non-compact, infinite-area cusps in the plane, where the rigorous analysis has produced even fewer results. Another reason for investigating the problem is its deep connection with number theory, especially the Eisenstein series and the Riemann zeta fuction. These are just few of many reasons, why the topic of quantum systems on hyperbolic geometries is worth studying.

Chapter 6

Appendix

6.1 The Whittaker differential equation

Through this entire section we will follow the monograms Whittaker-Watson [43] and Andrews-Askey-Roy[1].

6.1.1 The differential equation

Definition 6.1.1.1. *Let $k, m \in \mathbb{C}$. The differential equation*

$$\frac{d^2W}{dz^2} + \left(-\frac{1}{4} + \frac{k}{z} + \frac{1/4 - m^2}{z^2}\right) W = 0 \tag{6.1}$$

is called the **Whittaker differential equation**. *Its solutions are called the* **Whittaker functions**.

Remark 6.1.1.2. From the theory of second-order differential equations we know, that the Whittaker differential equation is a special case of the more general Riemann differential equation with singularities.

6.1.2 The Whittaker functions

Definition 6.1.2.1. *Let $M_{k,m}(z)$be a function defined as*

$$M_{k,m}(z) = e^{-z/2} x^{1/2+m} \ {}_1F_1\left(\frac{1}{2} + m - k, 1 + 2m; z\right)$$

where ${}_1F_1(a, b; x)$ is the confluent hypergeometric function.

Theorem 6.1.2.2. *For $2m \notin \mathbb{Z}$ the functions $M_{k,m}(z), M_{k,-m}(z)$ form the system of fundamental solutions of the equation* (6.1).

Theorem 6.1.2.3 (Kummer's first formula)**.** *For $2m \notin \mathbb{Z}^-$ the equality*

$$z^{-1/2-m} M_{k,m}(z) = (-z)^{-1/2-m} M_{-k,m}(-z)$$

holds.

As pointed out in Whittaker-Watson [43] the functions $M_{k,m}(z), M_{k,-m}(z)$ are not the most suitable system of fundamental solutions. This was the motivation to define an another system of fundamental solutions containing the function $W_{k,m}(z)$, which is defined next.

Definition 6.1.2.4. *Let $W_{k,m}(z)$ be a function defined by the integral*

$$W_{k,m}(z) = -\frac{1}{2\pi i}\Gamma\left(k+\frac{1}{2}-m\right)e^{-\frac{1}{2}z}z^k\int_{\infty}^{(0+)}(-t)^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m}e^{-t}dt$$

such that $\arg z$ *has its principal value and the contour(the path of integration from the complex infinity* ∞ *to* $0+$*) is so chosen that the point* $t=-z$ *is outside it. We take* $|arg(-t)| \leq \pi$ *and when* $t \to 0$ *along the contour,* $arg(1+\frac{t}{z}) \to 0$*. So we have a single-valued integral.*

Theorem 6.1.2.5. *The function $W_{k,m}(z)$ can be transformed into the form*

$$W_{k,m}(z) = \frac{e^{-\frac{1}{2}z}z^k}{\Gamma\left(\frac{1}{2}-k+m\right)}\int_0^{+\infty}t^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m}e^{-t}dt$$

and so $W_{k,m}(z)$ *is defined for all* $k,m \in \mathbb{C}$ *and for all* $z \in \mathbb{C}\setminus\mathbb{R}^-$ *under the condition* $\Re(k-\frac{1}{2}-m) \leq 0$*.*

Remark 6.1.2.6. When z is real and negative, $W_{k,n}(z)$ may be defined to be either $W_{k,n}(z+0i)$ or $W_{k,n}(z-0i)$, whichever is more convenient.

In the following theorem, we find the fundamental system containing $W_{k,n}()z$.

Theorem 6.1.2.7 (Asymptotic properties and the second solution for $W_{k,n}$). *When* $|\arg(z)| < \pi$ *then*

$$W_{k,m}(z) = e^{-\frac{1}{2}z}z^k\left(1+O(z^{-1})\right)$$

whereas, when $|\arg(-z)| < \pi$ *then*

$$W_{-k,m}(-z) = e^{\frac{1}{2}z}(-z)^{-k}\left(1+O(z^{-1})\right).$$

A more precise formula for $|\arg(z)| < \pi$ *is the following:*

$$W_{k,m}(z) \sim e^{-\frac{1}{2}z}z^k\left(1+\sum_{n=1}^{+\infty}\frac{\prod_{l=1}^{n-1}(m^2-(k-l-\frac{1}{2})^2)}{n!z^n}\right), \quad z\to\infty.$$

Moreover, the functions $W_{k,n}(z), W_{-k,m}(-z)$ *form a fundamental system of solutions of the Whittaker differential equations.*

The last theorem gives useful relations between$W_{k,n}$ and $M_{k,n}$

Theorem 6.1.2.8. *If* $|\arg(z)| < \frac{3}{2}\pi$ *and* $2m \notin \mathbb{Z}$*, then*

$$W_{k,m}(z) = \frac{\Gamma(-2m)}{\Gamma\left(\frac{1}{2}-m-k\right)}M_{k,m}(z) + \frac{\Gamma(2m)}{\Gamma\left(\frac{1}{2}+m-k\right)}M_{k,-m}(z)$$

Furthermore, when $|\arg(-z)| < \frac{3}{2}\pi$ *and* $2m \notin \mathbb{Z}$*, then*

$$W_{-k,m}(-z) = \frac{\Gamma(-2m)}{\Gamma\left(\frac{1}{2}-m+k\right)}M_{-k,m}(-z) + \frac{\Gamma(2m)}{\Gamma\left(\frac{1}{2}+m+k\right)}M_{-k,-m}(-z)$$

and when $-\frac{1}{2}\pi\arg(z) < \frac{3}{2}\pi$ *and* $-\frac{3}{2}\pi\arg(-z) < \frac{1}{2}\pi$*, then*

$$M_{k,m}(z) = \frac{\Gamma(2m+1)}{\Gamma\left(\frac{1}{2}+m-k\right)}e^{ik\pi}W_{-k,m}(-z) + \frac{\Gamma(2m+1)}{\Gamma\left(\frac{1}{2}+m+k\right)}e^{i(\frac{1}{2}+m+k)\pi}W_{k,m}(z).$$

Corollary 6.1.2.9. *For* $2m \notin \mathbb{Z}\setminus\{0\}$ *the function* $W_{k,m}(z)$ *can be expressed as*

$$W_{k,m}(z) = e^{-z/2}x^{1/2+m}\ U\left(\frac{1}{2}+m-k, 1+2m; z\right),$$

where $U(a,b,z)$ *is the Tricomi confluent hypergeometric function.*

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