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Time-periodic Quantum Systems

Ph.D. Thesis

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Abstract

This work deals with the theory of stability of time-periodic quantum systems. We discuss three different notions of stability and their interrelationship. Further, we consider three important methods: the quantum version of the KAM method, so called adiabatic and the anti-adiabatic method, and their use in investigation of stability of systems described by a Hamiltonian of perturbative type $H(t) = H_0 + V(t)$. We suppose that the spectrum of H_0 is pure point and that T-periodic perturbation $V(t)$ is small in a certain sense.

The knowledge of the asymptotic behaviour of the matrix entries of $V(t)$ in the eigenbasis of H_0 is important for possible applications of these methods. We present results of analysis of one-dimensional models with $H_0 = -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + |x|^\alpha$, for $\alpha > 0$, in the high energy and semiclassical regime.

Next model we study is the harmonic oscillator in the so called resonant regime. We show that it is stable under a large class of non-localised perturbations. The last part of the thesis consist of four articles in which I participated as the co-author.

Dedicated to V. & V.

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Contents

List of Symbols

$\mathrm{ad}_A(B) = AB - BA$	Commutator	
AP	Set of almost periodic functions on $\mathbb{R}$	
$D = -\imath\partial_t$	Operator on $L^2[0,T]$ with periodic b.c.	
$\mathrm{Dom}\, A$	Domain of definition of operator A	
$\mathscr{H}$	A separable Hilbert space	
$\mathscr{H}^f$	Set of propagating states	Section 2.3
$\mathscr{H}^p$	Set of bound states	Section 2.3
$\mathscr{H}^{pp}(U(T,0))$	Point spectral subspace of $U(T,0)$	Section 2.3
$\mathscr{H}^{cont}(U(T,0))$	Continuous spectral subspace of $U(T,0)$	Section 2.3
$K = -\imath\partial_t + H(t)$	Floquet Hamiltonian	Section 2.4
$\bar{K}$	Quasienergy	Section 2.4
$\mathscr{K} = L^2([0,T],\mathscr{H})$	Extended phase space in periodic case	Section 2.4
$\bar{\mathscr{K}} = L^2(\mathbb{R},\mathscr{H})$	Extended phase space	Section 2.4
$\mathbb{N}, \mathbb{R}, \mathbb{Z}$	Natural numbers ($0 \notin \mathbb{N}$), reals, integers	
$U(t,s)$	Propagator	Section 2.1
$U(T,0)$	Monodromy	Section 2.3
$\dot{A}(t) = \frac{\mathrm{d}}{\mathrm{d}t}A(t)$	Time derivative of $A(t)$	
$\mathcal{X}(p,\alpha)$	Howland's class	Definition 4.1.3
$\zeta(x) = \sum_{n\in\mathbb{N}} n^{-x}$	Riemann's zeta function	
$[\,\cdot\,]$	Integer part	
$\langle x\rangle = \max\{1,\lvert x\rvert\}$		
$\lVert\cdot\rVert_1$	Trace norm	
$\lVert\cdot\rVert_{p,\alpha}^H$	Norm in $\mathcal{X}(p,\alpha)$	Definition 4.1.3
$\lVert\cdot\rVert_{SH}$	Shur-Holmgren norm	Formula (3.7)

Chapter 1

Preface

Although the theory of non-relativistic quantum mechanics was formulated in a rigorous way long time ago, the theory of time-dependent Hamiltonians was for its difficulty developed at the second half of the 20^{th} century. They are usually used as an effective-theory approximation to more complicated systems. The key question is stability; one studies the dynamics of observables and trajectories generated by the Schrödinger equation. In the time-periodic case some geometrical properties of trajectories can be due to [EV] and [YK] characterised with the help of the spectral analysis of the monodromy - the evolution operator taken over one period $U(T, 0)$. This is a generalisation of the celebrated RAGE (Ruelle, Amrein, Georgescu, Enss) theorem well-known for time independent case. We distinguish two approaches to stability: a direct analysis of dynamics (dynamical stability) and the spectral analysis of the monodromy. Further, the spectral properties of the monodromy may be either investigated directly ([Co2], [Bo1], [BHG], [DGJKA], [Jo3], [DKGJA], [Bo2], [MM1], [MM2], $\ldots$) or determined by the spectral properties of so called Floquet Hamiltonian according to results [Ho1], [Ho2] or [Ya].

The first general method to study stability of integrable Hamiltonians under time-periodic perturbations was the quantum adaptation of the KAM (Kolmogorov-Arnold-Moser) method introduced by Bellissard [Be]. Since that time, time-periodic systems are thought as mechanisms which could describe the quantum chaos. Although this is a subject of great interest in mathematical physics, up to the present day there is no satisfactory definition of the quantum chaos. Further development of the quantum KAM method is due [Co1] and [DS] and others.

In late 80's Howland combined in paper [Ho3] adiabatic analysis with a result of the scattering theory to exclude absolutely continuous part of the spectrum of the monodromy. This method was further extended in [Jo1] and [Ne] and we shall call it adiabatic. It was Howland again, who introduced a further important method in [Ho4]. The idea of this method is roughly speaking the same as the one of the adiabatic method, but with the interchange of time and space. Therefore, we call this method anti-adiabatic.

The aim of this thesis is to study stability and instability of time-periodic quantum systems by spectral and dynamical methods. Let us describe the organisation of this work. After introducing some notions of stability and standard results of the theory (2.1-2.5), we draw our attention in Chapter 2 to three important methods, the adiabatic, anti-adiabatic and the quantum version of the KAM method. We illustrate how these three methods bear upon the stability theory and try to indicate their common background. Possible application of these methods is based on some knowledge of the behaviour of matrix entries, we present some results of the analysis of one-dimensional oscillators. The very last part of this chapter deals with the dynamical stability.

In Chapter 3, we consider non-localised perturbations of the resonant harmonic oscillator, a system which is a little bit exceptional in our context (constant gaps, non-dense point spectrum of the Floquet Hamiltonian). An extension of the anti-adiabatic method is introduced in Chapter 4. In Chapter 5 we summarise the results included in this thesis. The very last part of this thesis is formed by the reprints [DLSV1], [DLSV2], [LS] and [DLS]. The first two articles concern about a generalisation of the quantum KAM method. In the third one a semiclassical limit of some matrix entries is proved, and in the last article an upper bound of the energy growth of some time-periodic systems with shrinking gaps in the spectrum is introduced.

Chapter 2

Introduction

2.1 The Propagator

In the non-relativistic quantum mechanics the time evolution is described by the Schrödinger equation

$$\imath \frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) = H(t)\Psi(t),$$

where $H(t)$ is a family of self-adjoint operators acting on a Hilbert space of quantum states $\mathscr{H}$. Time evolution from time s to time t of an initial state Ψ_s is described with the help of the **propagator** $U(t,s)$

$$\Psi(t) = U(t,s)\Psi_s.$$

$U(t,s)$ forms a family of unitary operators jointly strongly continuous in $t, s \in \mathbb{R}$ and satisfying the Chapman-Kolmogorov chain rule

$$\begin{aligned} &U(t,r)U(r,s) = U(t,s) \\ &U(t,t) = 1. \end{aligned}$$

If $H(t)$ is independent of time the propagator is obtained by the functional calculus

$$U(t,s) = \exp\left(-\imath(t-s)H\right).$$

On the contrary, in the time-dependent case the existence of the propagator describing the time evolution according to the Schödinger equation is not an easy matter. By a solution of the Schödinger equation for propagator we mean a propagator, such that for all $s \in \mathbb{R}$ and $\Psi_s \in \operatorname{Dom} H(s)$ the function $t \mapsto U(t,s)\Psi_s$ takes the values in $\operatorname{Dom} H(t)$ and is continuously differentiable in the sense of the norm on $\mathscr{H}$. Moreover, for all $t \in \mathbb{R}$ and $\Psi_s \in \operatorname{Dom} H(s)$

$$\imath \frac{\mathrm{d}}{\mathrm{d}t}U(t,s)\Psi = H(t)U(t,s)\Psi$$

holds true. Let us mention a classical sufficient condition on the existence of the propagator (see [RS]) which goes originally to Kato.

Theorem 2.1.1 ([Ka]). *Let $H(t)$ be a family of self-adjoint operators such that*

(i) the domain $\operatorname{Dom} H$ of $H(t)$ is independent of t

(ii) the function

$$(t,s) \to \frac{1}{t-s}\left(H(s)-H(t)\right)\left(H(s)+\imath\right)^{-1}$$

extends to a strongly continuous bounded operator-valued function on $\mathbb{R}^2$.

Then there exists unique propagator $U(t,s)$ such that $U(t,s)\operatorname{Dom} H \subset \operatorname{Dom} H$ and for all $\Psi \in \operatorname{Dom} H$

$$\imath\frac{\mathrm{d}}{\mathrm{d}t}U(t,s)\Psi = H(t)U(t,s)\Psi$$

holds true. Moreover, if $H(t)$ is T periodic

$$H(t+T) = H(t) \ \text{ for all } t$$

then the propagator satisfies

$$\begin{aligned} U(t+T,s+T) &= U(t,s) \ \text{ for all } t,s \\ U(t+nT,s) &= U(t,s)\left[U(s+T,s)\right]^n \end{aligned} \tag{2.1}$$

For more general sufficient conditions one can consult [Kr]. We shall not be engaged in the problem of the existence of the propagator in details, since it is not the aim of this work.

2.2 Three notions of stability

The only question we are interested in is stability of periodic time-dependent system. There are several notions of it, we would like to introduce three of them. From the point of view of the scattering theory it is an interesting question to study whether the particle escapes to infinity or rests in a bounded region. This problem is solved by the celebrated RAGE theorem in the time-independent case. Fortunately, the RAGE theorem was further generalised to the time-periodic case by Enss and Veselić in [EV]. We summarise some results of this paper in Section 2.3 introducing bound and propagating states and illustrating the significance of the monodromy operator.

The second notion of stability deals with the spectral properties of so called Floquet Hamiltonian. The relationship between the Floquet Hamiltonian and the monodromy operator is described in Section 2.4. Further, in Section 2.5 we study stability of the point or absolutely continuous spectrum of some Floquet Hamiltonians with respect to time-dependent perturbations.

The problem of dynamical stability investigates long-time behaviour of expectation values of physical observables. We shall focus only on the energy expectation value to decide whether the external force can pump arbitrary amount of energy into the system or not. We shall address this question to Section 2.7.

2.3 Bound states and propagating states

In this section we introduce the first notion of stability, closely connected to the scattering theory. Probably the first contribution to this subject was the one of Hagedorn [Ha] in 1983, who treated the impact parameter approximation to three body scattering problem. In the same year, Enss and Veselić collected in the beautiful paper [EV] fundamental results of the scattering theory for time-dependent systems and generalised the results of the celebrated RAGE theorem to time-dependent case. By chance, in the same issue of the Annales de l'Institut Henri Poincaré as Enss and Veselić, Yajima and Kitada [YK] introduced the notion of bound states and scattering states in the context of time-dependent Schrödinger Hamiltonians. However the concept of the article [EV] is more abstract and powerful. Let us summarise some results of this paper, useful for our purpose.

Suppose that for a time dependent quantum system, which is described by the Hamiltonian $H(t)$, the propagator $U(t,s)$ exists. Then it is possible to define two closed orthogonal subspaces in $\mathscr{H}$, the spaces of bound and propagating states. The set of **bound states** is defined by

$$\mathscr{H}^p := \{\Psi \in \mathscr{H} |\ \text{trajectory with initial state } \Psi \text{ is precompact in } \mathscr{H}\},$$

whereas the set of **propagating states** is defined by the condition

$$\mathscr{H}^f := \{\Psi \in \mathscr{H} |\ \lim_{t\to\infty} \frac{1}{t}\int_0^t \|CU(s,0)\Psi\| \mathrm{d}s = 0\} \tag{2.2}$$

which must be fulfilled for any compact operator C. In other words, the evolution of a state from $\mathscr{H}^p$ is approximately finite dimensional, while the trajectory of a state from $\mathscr{H}^f$ will leave any compact subset of $\mathscr{H}$ in the time average.

If we moreover assume that our quantum system is T periodic, i.e.

$$H(t+T) = H(t) \ \text{ for } t \in \mathbb{R}.$$

then it holds true that (see [EV])

$$\begin{aligned}\mathscr{H}^{p} &= \mathscr{H}^{pp}\left(U(T,0)\right)\\ \mathscr{H}^{f} &= \mathscr{H}^{cont}\left(U(T,0)\right).\end{aligned}$$

By definition, $\mathscr{H}^{pp}\left(U(T,0)\right)$ is the closure of a subspace formed by the spanned eigenvectors of $U(T,0)$, $\mathscr{H}^{cont}\left(U(T,0)\right)$ its orthogonal complement. From this fact follows the decomposition of $\mathscr{H}$ into subspaces of bound and propagating states.

$$\mathscr{H} = \mathscr{H}^{p} \oplus \mathscr{H}^{f}.$$

The operator $U(T,0)$ is therefore very important and it is called the **monodromy** operator (or Floquet or period operator). If for example the spectrum of $U(T,0)$ is pure point, we conclude that all quantum states are bound.

It may seem that the definition of the monodromy depends on the choice of the origin of the time axis. It is not so since the operator $U(T,0)$ is unitary equivalent to $U(T+t,t)$ for any $t \in \mathbb{R}$. Thanks to (2.1) it holds true that

$$U(T+t,t) = U(T+t,T)U(T,0)U(0,t) = U(t,0)U(T,0)U^{*}(t,0)$$

At the end of this section let us remark that in [EV] a geometric definition of bound and propagating states is discussed. Since it is not useful for our purpose, we skip the details and refer the reader to [EV].

2.4 The Floquet theory for time-dependent Hamiltonians

In this part we will show how the spectral properties of the monodromy operator are related to the spectral properties of the Floquet Hamiltonian. This relation, useful for the theory of stability of time-periodic Hamiltonians, was for the first time proved by Kenji Yajima [Ya] in 1977. More abstract concept was later (in 1979) introduced by Howland [Ho2] and we mainly refer to this paper.

Following [Ho1] let us start with a procedure, inspired by reduction to an autonomous system in classical mechanics. Instead of the Hamilton function $h(p,q,t)$ depending explicitly on time we extend the phase space by new independent variables, time t and (the conjugate momentum) energy E, and define the new Hamiltonian function

$$k(p,E,q,t) := E + h(p,q,t).$$

The Hamilton equations with new parameter σ now read

$$\begin{aligned}\frac{\mathrm{d}q}{\mathrm{d}\sigma} &= \frac{\partial k}{\partial p}, \quad \frac{\mathrm{d}p}{\mathrm{d}\sigma} = -\frac{\partial k}{\partial q}\\ \frac{\mathrm{d}t}{\mathrm{d}\sigma} &= \frac{\partial k}{\partial E} = 1, \quad \frac{\mathrm{d}E}{\mathrm{d}\sigma} = -\frac{\partial k}{\partial t}\end{aligned}$$

and are equivalent to the standard ones. The new parameter is just shifted time $\sigma = t + \mathrm{const}$. Analogously in quantum mechanics we consider the new Hamiltonian

$$\bar{K} := -\imath\partial_t + H(t) \tag{2.3}$$

which is called **quasienergy** and acts on the extended Hilbert space $\bar{\mathscr{K}} := L^2(\mathbb{R}, \mathscr{H})$. So that for $t \in \mathbb{R},\ \Psi(t)$ is a vector from $\mathscr{H}$. Sometimes it is convenient to use the identification $\bar{\mathscr{K}} \simeq L^2(\mathbb{R}) \otimes \mathscr{H}$. Suppose that the unitary propagator $U(t, s)$ corresponding to $H(t)$ exists. For $\sigma \in \mathbb{R}$ we define operator $V(\sigma)$ on $\bar{\mathscr{K}}$

$$V(\sigma)\Psi(t) := U(t, t-\sigma)\Psi(t-\sigma). \tag{2.4}$$

$V(\sigma)$ forms a strongly continuous unitary group on $\bar{\mathscr{K}}$, which is a consequence of the definition of the propagator U. Thus, by the Stone theorem there exists a generator of this group. In [Ho2] Howland shows that $-\imath\bar{K}$ is formally equal to the generator. Hence

$$V(\sigma) = e^{-\imath\sigma\bar{K}}.$$

We define the unitary transformation in the spirit of the Bloch analysis into the space $\tilde{\mathscr{K}} := L^2\left([0, T], l^2(\mathscr{H})\right)$ formed by square-integrable functions of θ with values in $l^2(\mathscr{H})$

$$\mathcal{T} : \bar{\mathscr{K}} \to \tilde{\mathscr{K}} : \Psi \mapsto x_n(\theta) := \frac{1}{T}\int_{\mathbb{R}} e^{-\frac{\imath 2\pi t}{T^2}(Tn+\theta)}\Psi(t)\mathrm{d}t,$$

so that the inverse transform reads

$$\left(\mathcal{T}^* x\right)(t) = \Psi(t) := \frac{1}{T}\sum_{n\in\mathbb{Z}}\int_0^T e^{\frac{\imath 2\pi t}{T^2}(Tn+\theta)}x_n(\theta)\mathrm{d}\theta.$$

If the Hamiltonian is T-periodic (this is what we suppose in what follows) then the transformed quasienergy takes the form

$$\left(\mathcal{T}\bar{K}\mathcal{T}^* x\right)(\theta, n) = (Tn+\theta)x_n(\theta) + \sum_{m\in\mathbb{Z}}\hat{H}_{n-m}x_m(\theta),$$

with $\hat{H}_n := \frac{1}{T}\int_0^T H(t)e^{-\frac{\imath 2\pi t}{T}n}\mathrm{d}t$. $\tilde{\mathscr{K}}$ may be realised as $L^2[0, T] \otimes l^2(\mathscr{H})$, too, where the decomposition $\mathcal{T}\bar{K}\mathcal{T}^* = M(\theta) \otimes 1 + 1 \otimes \tilde{K}$ takes place with $M(\theta)$ being the multiplication by the identity function on $L^2[0, T]$ and

$$(\tilde{K}x)(n) = Tn\, x_n + \sum_{n\in\mathbb{Z}}\hat{H}_{n-m}x_m$$

for $x \in l^2(\mathscr{H})$. Now we take single $\tilde{K}$ and proceed further transformation $\mathcal{F}$ which takes the sequence $x \in l^2(\mathscr{H})$ into

$$(\mathcal{F}x)(t) = f(t) := \sum_{n\in\mathbb{Z}} e^{\frac{\imath 2\pi n}{T}t}x_n,$$

locally square-integrable $\mathscr{H}$-valued function of period T. The new operator

$$\hat{K} := \mathcal{F}\tilde{K}\mathcal{F}^* = -\imath\partial_t + H(t)$$

is formally the same as the original quasienergy (2.3), but it acts on periodic functions. The relationship (2.4) between the propagator $U(t,s)$ and $\hat{K}$ is reproduced

$$e^{-\imath\sigma\hat{K}}f(t) = U(t,t-\sigma)f(t-\sigma).$$

Using the property (2.1) and periodicity of f we obtain

$$e^{-\imath T\hat{K}}f(t) = U(t,t-T)f(t-T) = U(t+T,t)f(t).$$

Since $U(t+T,t) = U(t,0)U(T,0)U^*(t,0)$ we come to the desired formula

$$e^{-\imath T\hat{K}}f(t) = U(t,0)U(T,0)U^*(t,0)f(t)$$

which relates the monodromy operator to $\hat{K}$. Trivially, one can transform the operator $\hat{K}$ unitarily to

$$K := -\imath\partial_t + H(t) \tag{2.5}$$

acting on $\mathscr{K} := L^2([0,T],\mathscr{H})$ with periodic boundary condition in time, i.e. $\Psi(0) = \Psi(T)$. The very last operator we shall call the **Floquet Hamiltonian.** We have to bear in mind, that $K = -\imath\partial_t + H(t)$ is nothing, but the formal expression. In a rigour approach it is defined (up to the factor $-\imath$ and the transformation mentioned above) as the generator of the group $e^{-\imath t\hat{K}}$. One has to discuss the question of the domain of K. In the case of $H(t) = H_0 + V(t)$, with $V(t)$ uniformly bounded, which will be of our interest, it is more or less direct. Let us formulate the main result of this section which suits to this case. One can find the proof in [DSSV].

Theorem 2.4.1 ([DSSV])**.** *Suppose that a quantum system is driven by a T- periodic Hamiltonian $H(t) = H_0 + V(t)$ acting on a separable Hilbert space $\mathscr{H}$. Assume that $V(t)$ is uniformly bounded and that the propagator $U(t,s)$ exists. Define the Floquet Hamiltonian K as the closure of the operator $-\imath\partial_t + H(t)$ with the domain $\{f \in C^\infty[0,T] | f(0) = f(T)\} \otimes \mathrm{Dom}\, H_0$. Then the spectral properties of the monodromy operator $U(T,0)$ are the same as those of $e^{-\imath TK}$.*

Remark 2.4.2. *(i) Notice that the terminology in the Floquet theory is not unified. The operators (2.3) and (2.5) are of same action and differ only by the underlying spaces. Our notation uses bar not to get confused and distinguish between the quasienergy and the Floquet Hamiltonian. The latter lies in the centre of our interest, because of its relation to the monodromy operator.*

(ii) In all what follows we deal with Floquet Hamiltonians of the type $K = -\imath\partial_t + H_0 + V(t)$, with $V(t)$ uniformly bounded. We suppose implicitly that the domain of K is chosen as in Theorem 2.4.1.

2.5 Spectral stability of pure point Floquet Hamiltonians

Suppose that a quantum system is described by a Hamiltonian of perturbative type

$$H(t) = H_0 + V(t)$$

and acting on the Hilbert space $\mathscr{H}$. Here H_0 is assumed to be self-adjoint with pure point spectrum and the spectral decomposition

$$H_0 = \sum_{n=1}^{\infty} E_n P_n,$$

with eigen-values ordered increasingly $E_1 \leq E_2 \leq \cdots$. Let $V(t)$ be a self-adjoint perturbation T-periodic in time and bounded in a suitable norm. As we have shown, for the question of the existence or absence of propagating states it is important to know spectral properties of the monodromy operator $U(T, 0)$. From Theorem 2.4.1 it follows that this is equivalent to the spectral problem of the corresponding Floquet Hamiltonian

$$K = D + H_0 + V(t)$$

acting on $L^2([0, T], \mathscr{H})$ with periodic boundary condition in time. We use the notation $D := -\imath\partial_t$. The spectrum of the unperturbed Floquet Hamiltonian $K_0 := D + H_0$ is pure point with eigenvalues

$$\lambda_{k,n} = \frac{2\pi}{T}k + E_n$$

labelled by integers k and $n \geq 1$. Thus the set of propagating states $\mathscr{H}^f$ defined by the condition (2.2) is empty.

The **goal** of the perturbation problem is to prove that this holds true for the perturbed Floquet Hamiltonian K as well, if $V(t)$ is sufficiently ”smooth or small”. This question is not trivial at all, since for H_0 unbounded the point spectrum of $K_0 := D + H_0$ is dense in $\mathbb{R}$ in generic case. More precisely (see [DSV]) the set

$$\omega\mathbb{Z} + \{E_n\}_{n\in\mathbb{N}}$$

is dense in $\mathbb{R}$ for almost all $\omega \in \mathbb{R}$ provided $\sup E_n = +\infty$.

There are several methods which deal with the spectral problem of K, however we focus only on three of them: the quantum adaptation of the KAM method, and what we call adiabatic and anti-adiabatic method. The information about the spectrum of K is less precise in the case of latter two methods. On the contrary to KAM, which guarantees the pure point character of the spectrum of K, the adiabatic and the anti-adiabatic methods just excludes the absolutely continuous spectrum. Later, we introduce key ideas of these three methods, now we just say that for their applications the asymptotic behaviour of the gaps in the spectrum of H_0 i.e.

$$E_{n+1} - E_n$$

is crucial. We distinguish three classes of Hamiltonians H_0 according to behaviour of the gaps

- Constant gaps $E_n = n,$ e.g. the harmonic oscillator.
- Shrinking gaps, typically $E_{n+1} - E_n \simeq n^{-2\gamma}$, with $\gamma > 0.$
- Increasing gaps, $E_{n+1} - E_n \simeq n^{\alpha}$, with $\alpha > 0$ for example.

The case of constant gaps is quite special. In [Co1] Monique Combescure combined the KAM theory with the Nash-Moser trick and obtained a result about the stability of the pure point spectrum of the monodromy under some class of periodic perturbations, which does not include potentials however. Perturbations of the harmonic oscillator localised in space are discussed in [EV]; we extend their result to a class of non-localised potentials in Chapter 3. The KAM and the adiabatic method apply successfully in the case of increasing gaps, while the anti-adiabatic method in the case of shrinking gaps.

In the picture we present, these three methods may be viewed as variants of application of the following formula

$$e^A B e^{-A} = B + \sum_{j=1}^{\infty} \frac{1}{j!} \operatorname{ad}_A^j(B) \tag{2.6}$$

with a convenient choice of A, B. We use the notation $\operatorname{ad}_A(B) := [A, B] = AB - BA.$

Both adiabatic and anti-adiabatic methods use the following result of the scattering theory about the stability of the absolutely continuous spectrum of $U(T, 0)$ (or K, equivalently). See Schmidt [Sch] and the generalisation by Howland to trace-class perturbations. The statement presented here is, in fact a consequence of Theorem 5 in [Ho2].

Theorem 2.5.1 ([Ho2])**.** *Let $V(t)$ be a measurable self-adjoint trace-class-valued function. Assume that $V(t)$ is T-periodic and that*

$$\int_0^T \|V(t)\|_1 \mathrm{d}t < \infty.$$

Let H_0 be self-adjoint, and let $U(t, s)$ be the propagator associated with $H(t) := H_0 + V(t)$. Then

$$\sigma_{ac}(H_0) = \emptyset \implies \sigma_{ac}\left(U(T, 0)\right) = \emptyset.$$

Remark 2.5.2. *The absence of the absolutely continuous spectrum of the monodromy is weaker condition than the pure-pointness and does not imply the absence of propagating states.*

2.5.1 The adiabatic method

In 1989, James S. Howland introduced in paper [Ho3] a new method to treat stability of time-dependent systems. In agreement with some authors we shall call this method adiabatic. The idea is to combine adiabatic analysis with the result of the scattering theory (see Theorem 2.5.1) to exclude any absolutely continuous part in the spectrum of the monodromy. An essential assumption of this method is that the gaps in the spectrum of the unperturbed Hamiltonian have to grow. The adiabatic method was extended by Alain Joye [Jo1] and Georgiu Nenciu [Ne] to the case of growing multiplicities of the eigen-values of H_0.

To see better the similarities with the anti-adiabatic and the KAM method we present this method in an algebraic form used by Howland in paper [Ho3]. Let us deal with the Floquet Hamiltonian

$$K := D + H_0 + V(t)$$

acting on $\mathscr{K} := L^2([0,T],\mathscr{H})$ with periodic boundary condition imposed in time. Let the spectrum of H_0 be discrete and obeying the **growing gap condition**

$$\inf \frac{E_{n+1} - E_n}{n^\alpha} > 0 \tag{2.7}$$

for a given $\alpha > 0$. $V(t)$ is assumed to be T-periodic sufficiently differentiable function with values in the space of **hermitian** (i.e. bounded self adjoint) operators on $\mathscr{H}$. Suppose moreover that the monodromy operator $U(T,0)$ associated with $H_0 + V(t)$ exists. A typical result of the adiabatic method (extracted from [Ho3]) is

Theorem 2.5.3 ([Ho3]). *Assume that a T-periodic quantum system is described by the Hamiltonian $H_0 + V(t)$. Suppose that the spectrum of H_0 is pure-point and simple and obeys the growing gap condition (2.7) for $\alpha > 0$. If $V(t) \in C^r(\mathbb{R}, \mathscr{B}(\mathscr{H}))$, with $r \geq [\frac{1}{\alpha}] + 1$ then the monodromy $U(T,0)$ has no absolutely continuous spectrum.*

Let us explain the main idea of the proof. Suppose for simplicity that the diagonal part of the matrix of operator $V(t)$ in the eigen-basis of H_0 is zero. Define $W(t)$ as a solution of the commutator equation

$$[W(t), H_0] = -V(t).$$

Such a solution is not unique, we can add to the given one any operator commuting with H_0 and obtain a new solution. Let us choose $W(t)$, such that its matrix entries in the eigen-basis of H_0 take the form

$$\begin{aligned} W_{m,n}(t) &:= \frac{-V_{m,n}(t)}{E_m - E_n} \quad \text{for } m \neq n \\ &:= 0 \quad \text{else.} \end{aligned}$$

Observe that $W(t)$ is anti-symmetric, therefore $e^{W(t)}$ is formally unitary. Define the adiabatic transformation of K by

$$K_1 := e^{W(t)} K e^{-W(t)};$$

hence we have obtained a new Floquet Hamiltonian which is unitary equivalent to the original one. Notice that $[W(t), D] = \imath \dot{W}(t) := \imath \frac{\mathrm{d}}{\mathrm{d}t} W(t)$. Expanding the right hand side due to (2.6) yields

$$\begin{aligned} K_1 &= D + H_0 + V(t) + [W(t), D + H_0 + V(t)] + \dots \\ &= D + H_0 + \imath \dot{W}(t) + [W(t), V(t)] + \dots \end{aligned} \tag{2.8}$$

Using the adiabatic transform we have replaced effectively $V(t)$ by $\imath \dot{W}(t)$. It is technically difficult to prove that the remainder in (2.8), i.e. $[W(t), V(t)] + \dots$ is “less important”. For the matrix entries we have

$$\dot{W}_{m,n}(t) = \frac{-\dot{V}_{m,n}(t)}{E_m - E_n} \text{ for } m \neq n$$

Notice that the diagonal of $\dot{W}(t)$ vanishes as well as the diagonal of $W(t)$. The growing gap condition (2.7) implies

$$\inf_{m \neq n} \frac{|E_m - E_n|}{|m^{\alpha+1} - n^{\alpha+1}|} > 0.$$

Since one can easily prove that $|m^{\alpha+1} - n^{\alpha+1}| \geq (mn)^{\frac{\alpha}{2}} |m - n|$, we may estimate

$$\left| \dot{W}_{m,n}(t) \right| \leq \text{const} \frac{|\dot{V}_{m,n}(t)|}{(mn)^{\frac{\alpha}{2}} |m - n|} \text{ for } m \neq n.$$

Thus we observe that the new perturbation has matrix elements with better decay properties than original $V(t)$. We remark that the decay of entries is improved both in the direction parallel to the diagonal (due to the presence of $(mn)^{\frac{\alpha}{2}}$) and perpendicular to the diagonal (term $|m - n|$). By repeating the adiabatic transform r times we come to a new Floquet Hamiltonian, unitary equivalent to the original one

$$K_r = D + H_0 + B(t),$$

with $B(t)$ in trace class uniformly. The following estimate on the trace norm $\|A\|_1 \leq \sum_{m,n} |A_{m,n}|$ is used. By Theorem 2.5.1 we conclude that the absolutely continuous spectrum of K_r is the same as $\sigma_{ac}(K_0) = \emptyset$. Consequently that the same holds for K and the monodromy $U(T, 0)$ corresponding to $H_0 + V(t)$.

2.5.2 The anti-adiabatic method

To our knowledge the only general method to deal with spectral stability of systems with shrinking gaps is the one introduced by Howland in [Ho4]. We shall see that the core of this method is very similar to the essence of the adiabatic method. From the discussion it hopefully clear up why we call this method anti-adiabatic. Consider again the Floquet Hamiltonian

$$K := D + H_0 + V(t)$$

on $\mathscr{K} := L^2([0,T],\mathscr{H})$ with periodic b.c. The following theorem is our improvement of the result in [Ho4].

Theorem 2.5.4. *Let the spectrum of H_0 be pure point. Suppose that the eigen-values $E_1 \leq E_2 \leq \cdots$ obey the* **shrinking gap condition**

$$\sup n^{2\gamma}(E_{n+1} - E_n) < \infty \tag{2.9}$$

for a given $\gamma \in]0, 1/2[$. Let $V(t)$ be a T-periodic function with values in the space of hermitian operators on $\mathscr{H}$. Suppose that the monodromy $U(T,0)$ corresponding to $H_0 + V(t)$ exists. If the matrix entries of $V(t)$ in the eigen-basis of H_0 are measurable functions of t and satisfy

$$|V_{m,n}(t)| \leq \mathrm{const}\frac{1}{(mn)^\gamma \langle m-n\rangle^r} \tag{2.10}$$

uniformly, with $r > 1 + \frac{2}{\gamma}$, then the monodromy operator $U(T,0)$ has no absolutely continuous spectrum.

Remark 2.5.5. *(i) We use the notation $\langle k\rangle := \max\{1, |k|\}$.*

(ii) The shrinking gap condition (2.9) is equivalent to

$$|E_m - E_n| \leq \mathrm{const}\frac{|m-n|}{(mn)^\gamma} \quad \text{for } m,n \in \mathbb{N} \tag{2.11}$$

used by Howland.

(iii) We take $\gamma < 1/2$ since for $\gamma > 1/2$ condition (2.11) implies that the eigen-values E_n are bounded, what is not of our interest. We also exclude the case $\gamma = 1/2$ which corresponds to $E_{k_n} \sim^{n\to\infty} \log k_n$ for a sub-sequence (k_n).

The proof and the discussion can be found in Chapter 4, here we just sketch the main idea. Set $\bar{V} := \frac{1}{T}\int_0^T V(t)\mathrm{d}t$ and denote $\tilde{V}(t) := V(t) - \bar{V}$. We define the anti-adiabatic transformation of K by the same formula as in the adiabatic case

$$K_1 := e^{W(t)}Ke^{-W(t)} = D + H_0 + V(t) + [W(t), D + H_0 + V(t)] + \dots,$$

but now we choose $W(t)$ so that $\tilde{V}(t)$ and $[W(t), D]$ cancel each other, effectively replacing $\tilde{V}(t)$ by $[W(t), H_0]$. Recall that in the adiabatic case we have chosen $W(t)$ so that $V(t)$ and $[W(t), H_0]$ cancel each other, effectively replacing $V(t)$ by $[W(t), D]$. The regularity of the perturbation $V(t)$ in time was used to improve the decay of matrix entries. In the anti-adiabatic method the regularity in space (the part of $\mathscr{K}$ corresponding to $\mathscr{H}$) is used. Both methods are based on the same type of transformation, however the roles of time and space are interchanged.

Notice that after the anti-adiabatic transformation the average part of the original perturbation $\bar{V}$ remains, but as we shall see it does not represent an obstacle, since this operator does not depend on time. Using assumption (2.10) one can observe that $\bar{V}$ is compact. Let us define

$$W(t) := \imath \int_0^t \tilde{V}(s)\mathrm{d}s$$

so that $W(t)$ is T-periodic and anti-hermitian for every t. Since $[W(t), D] = \imath\dot{W}(t) = -\tilde{V}(t)$ we have

$$K_1 = D + H_0 + \bar{V} + [W(t), H_0] + [W(t), V(t)] + \dots .$$

As already noticed, $\bar{V}$ is compact and does not depend on time. One has to overcome some technical points to prove that $[W(t), V(t)] + \dots$ is less important than the term $[W(t), H_0]$. If we look at the matrix entries (in the eigen-basis of H_0) of the commutator

$$[W(t), H_0]_{m,n} = W_{m,n}(t)\,(E_m - E_n)$$

and use condition (2.11) we obtain the estimate

$$|[W(t), H_0]_{m,n}| \leq \frac{|W(t)_{m,n}(m-n)|}{(mn)^\gamma}.$$

Thus, using the anti-adiabatic transformation we can improve the decay of the perturbation along the main diagonal at the expense of the decay in the direction perpendicular to the diagonal. Applying the anti-adiabatic machinery $l := [1/2\gamma]$ times we come to

$$K_l = D + H_0 + A + B(t)$$

with A compact time-independent and $B(t)$ in the trace class uniformly. Recall that the spectrum of H_0 is discrete. Using the Weyl's theorem we conclude that the spectrum of $H_0 + A$ is discrete too. The statement then follows from Theorem 2.5.1.

2.5.3 The KAM theory

The Kolmogorov-Arnold-Moser method well known in the classical mechanics was adopted to the quantum case by Bellissard [Be]. Further development was due to Combescure [Co1], who used the Nash-Moser trick which consist in a splitting of the perturbation into parts. Each part of the perturbation is added and the KAM-type diagonalisation procedure is applied on this partially perturbed Floquet Hamiltonian. This process is repeated infinitely many times until the whole perturbation is added. Later, Duclos and Šťovíček combined the KAM method with the adiabatic one in [DS]. The Nash-Moser trick was involved into the diagonalisation procedure; in each step of the algorithm a part of the perturbation is added. In contrast to [Co1] the diagonalisation is applied only once. Further versions of the KAM-type theorem are presented in the articles [DLSV1] and [DLSV2] which are included in this thesis. The adiabatic part is omitted and the splitting of the perturbation is done in an appropriate form. The algorithm of the diagonalisation is described in an abstract form with the help of an inductive limit of Banach spaces. A convenient choice of the norms in these spaces led to a weakening of the regularity assumptions imposed on the perturbation. See also Section 10 in [DLSV1]. The result of [DLSV2] include even a class of unbounded perturbations. An abstract concept of the algebraic background of the KAM theory is described in [Vi]. Remark that a KAM-type algorithm was also used in time quasi-periodic case in articles [BG], [Ge1] and [Ge2].

Let us describe the settings. Consider Floquet Hamiltonian

$$K_\omega := D + H_0 + V(\omega t)$$

on $\mathscr{K} := L^2([0,T],\mathscr{H})$ with periodic b.c. V is now supposed to be 2π-periodic and we denote $\omega := 2\pi/T$. From reasons which become clear later we treat the problem as depending on the parameter ω lying in a compact interval Ω. The KAM method is iterative; one tries to diagonalise the operator K_ω by constructing a sequence of operators K_ω^s which converges (in an appropriate sense) to a diagonal operator unitarily equivalent to K_ω. The statement of the theorem is not able to guarantee pure-point spectrum of K_ω for all frequencies ω, one excludes a small (in the Lebesgue sense) set of resonant frequencies to prevent so called small divisors problem. Unlike to the adiabatic and anti-adiabatic method, V is also supposed to be sufficiently small. At this place we reproduce a rough version of a KAM-type theorem.

Theorem 2.5.6 ([DLSV1]). *Let Ω be a fixed compact interval, H_0 acting on a separable Hilbert $\mathscr{H}$ have a simple discrete spectrum $\{E_1 < E_2 < \ldots\}$ obeying the growing gap condition*

$$\inf \frac{E_{n+1} - E_n}{n^\alpha} > 0$$

for $\alpha > 0$. Let $V(t)$ be 2π-periodic strongly continuously function with values in hermitian operators on $\mathscr{H}$. Denote $V_{k,m,n} := \int_0^{2\pi} e^{-\imath kt} V_{m,n}(t)\mathrm{d}t$, the k-th Fourier coefficient of the *matrix element of $V(t)$ in the eigen-basis of H_0. Then there exists $p(\alpha) > 0$ such that for every $r > p(\alpha)$, $\exists\delta(\alpha, r) > 0$ and $C(\alpha, r)$, such that*

$$\|V\|_r := \sup_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{Z}} |V_{k,m,n}| \langle k\rangle^r < \delta(\alpha, r)$$

implies $\exists\Omega_{res} \subset \Omega$ with $|\Omega_{res}| \leq C(\alpha, r)\|V\|_r|\Omega|$ so that the Floquet Hamiltonian

$$K_\omega = -\imath\partial_t + H_0 + V(\omega t)$$

is pure point for all $\omega \in \Omega \setminus \Omega_{res}$. Here $|\Omega|$ stands for the Lebesgue measure of Ω.

Remark 2.5.7. *The theorem is generalised to the case of degenerate eigen-values in [DLSV1], and further to a class of unbounded perturbations in [DLSV2].*

In the following we describe inductively the algorithm which diagonalise K_ω in the limit. For brevity we skip the labelling of the dependence on ω and t. At first, we split the operator V into a sum

$$V = \sum_{s=0}^{\infty} V^{(s)}.$$

At this place we do not specify how this splitting is defined, just remark that $V^{(0)}$ is chosen so that it commutes with H_0. For details we refer to [DLSV1]. According to KAM algorithm 4 sequences of operators $\{K_s\}, \{G_s\}, \{V_s\}$ and $\{W_s\}$ are constructed. We work with matrices in the eigen-basis of $D + H_0$. Notice that the matrices are labelled by 4-tuple of indices k, l, m, n. The first two indices are integers, the latter two natural numbers. We shall denote by $\operatorname{diag} A$ and $\operatorname{offdiag} A$ the diagonal and off diagonal part of an operator A with respect to the eigen-basis of $D + H_0$. Remark that G_s will be diagonal for every s while V_s and W_s will be always off diagonal. The sequences are defined recursively by the following rules

1. $G_0 := V^{(0)}, V_0 := 0$

2. Provided G_s, V_s were already determined, define W_s as the solution of

$$[D + H_0 + G_s, W_s] = V_s \quad \text{and} \quad \operatorname{diag} W_s = 0. \tag{2.12}$$

   Hence we have for off diagonal entries

$$(W_s)_{k,l,m,n} = \frac{(V_s)_{k,l,m,n}}{\omega(k - l) + E_m - E_n + (G_s)_{k,l,m,m} - (G_s)_{k,l,m,n}} \tag{2.13}$$

3. Set inductively,

$$U_0 := 1, \quad U_s := \exp W_s \cdots \exp W_1.$$

Then

$$K_{s+1} := e^{W_s}(D + H_0 + G_s + V_s)e^{-W_s} + U_s V^{(s+1)} U_s^* \tag{2.14}$$

and

$$G_{s+1} := \operatorname{diag} K_{s+1} - D - H_0, \quad V_{s+1} := \operatorname{offdiag} K_{s+1} \tag{2.15}$$

By induction one shows that from (2.14) and (2.15) follows

$$K_s = D + H_0 + G_s + V_s = U_{s-1}\left(D + H_0 + \sum_{j=0}^{s} V^{(j)}\right) U_{s-1}^*.$$

The goal of this method is to prove the existence of the limits

$$G_s \to G,\ V_s \to 0,\ U_s \to U.$$

Then we would obtain for the Floquet Hamiltonian

$$D + H_0 + G = U\,(D + H_0 + V)\,U^* = UKU^*$$

with G diagonal and we would be done. Let us outline some points of the proof. W_s is chosen as the solution of the commutator equation (2.12). The main problem of the algorithm is to control the small divisors in (2.13), or in other words to keep the size of W_s comparable to the size of V_s. However, this is not possible for all frequencies $\omega \in \Omega$, one has to exclude so called resonant ones. In each step s, one defines the set of the resonant frequencies $\Omega_{res}^s \supset \Omega_{res}^{s-1}$ as the set of $\omega \in \Omega$ for which the expression $\omega(k-l) + E_m - E_n + (G_s)_{k,l,m,m} - (G_s)_{k,l,m,n}$ is "small". The final set Ω_{res} is defined as the union of these Ω_{res}^s. It turns out that Ω_{res} is not so large and it is possible to yield an estimate of its Lebesgue measure proportional to $\|V\|_r$.

Overcoming this obstruction we are able to control the size of W_s comparable to V_s. Expanding (2.14) according to formula (2.6) and using the definition of W_s one obtains

$$K_{s+1} = D + H_0 + G_s + \mathcal{O}\left((V_s)^2\right) + U_s V^{(s+1)} U_s^*$$

It is evident that $V^{(s)} \to 0$, since the sum of all $V^{(s)}$ gives V. The fact that $V_s \to 0$ demonstrates the progress of the diagonalisation procedure and is, in fact, a consequence of the finiteness of the norm $\|V\|_r$ (which implies power decay of the matrix entries of V.) If the off diagonal part of V is small, K is transformed into K_1 in the first step, with the off diagonal part V_1 smaller. Further, as s is increasing, V_s is shrinking. Clearly, for the convergence of U_s the convergence of the series $\sum_{s=0}^{\infty} \|W_s\|$ is sufficient. By a convenient choice of the splitting of V it is possible to satisfy this condition, too.

2.6 Asymptotic behaviour of matrix entries

In Section 2.5 we have presented the KAM, adiabatic, and anti-adiabatic methods to study time-periodic Hamiltonians of perturbative type

$$H(t) = H_0 + V(t),$$

with H_0 pure point. The problem was reformulated in the matrix representation and therefore the assumptions of the theorems requires some knowledge of the matrix entries of the perturbation $V(t)$ in the eigen-basis of H_0, see Theorems 2.5.4 and 2.5.6. The adiabatic method (Theorem 2.5.3) is a little bit exceptional from this point of view. However, the differentiability property is transformed, in fact, into the decay properties of the matrix entries of the new perturbation. For possible applications of these theorems it is sufficient to know the asymptotic behaviour of the matrix entries of $V(t)$ in the eigen-basis of H_0, i.e.

$$\langle m|V(t)|n\rangle,$$

where $|n\rangle$ denotes the n-th eigen-state of H_0. This fact has motivated our study of one-dimensional models described by the Hamiltonian

$$H_0 = -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + |x|^\alpha,$$

acting on $L^2(\mathbb{R})$ with $\alpha > 0$. Using standard methods one concludes that H_0 is positive with simple pure point spectrum. Let E_n denote the n-th eigen-value and $|n\rangle$ the corresponding eigen-function, so that $H_0|n\rangle = E_n|n\rangle$. Remark that both eigen-values and eigen-functions depend on $\hbar$. For simplicity, we suppose that time-dependence of $V(t)$ is factorised, i.e. $V(t) = f(t)v$, with f being a T-periodic continuously differentiable real-valued function and v a hermitian operator on $L^2(\mathbb{R})$. The question may be formulated as follows: What are the asymptotic properties of the transition amplitudes of an observable v

$$\langle m|v|n\rangle$$

for m, n large. In fact, we were investigating two regimes, the high energy regime and the semiclassical one.

2.6.1 High energy regime

In the high energy regime we treat the case when $m, n \to \infty$ while the Planck constant $\hbar$ is fixed. By a convenient choice of units we may suppose that $\hbar = 1$. Let us present a theorem which deals with entries close to the main diagonal and observables v depending only on the position, i.e. potentials. In fact, it is a generalisation of some results of the diploma thesis [Lev].

Theorem 2.6.1. *Let $\alpha > 0$, v be an even real-valued differentiable function, such that $v' \in L^1(\mathbb{R})$ and for some $r < 1$*

$$v(x) \leq \frac{\text{const}}{1 + |x|^r},$$

holds true for every $x \in \mathbb{R}$. Denote by $|n\rangle$ the n-th normalised eigen-function of the oscillator $H_0 := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + |x|^\alpha$ acting on $L^2(\mathbb{R})$. The phase of $|n\rangle$ is fixed by the condition that $|n\rangle$ is positive on a neighbourhood of $+\infty$ and the enumeration of eigen-functions starts from the index $n = 0$. Then for every $k \in \mathbb{Z}$ fixed

$$\langle n|v(x)|n + 2k\rangle \sim C_\alpha (-1)^k n^{-\frac{2}{\alpha+2}} \int_{\mathbb{R}} v(x)\mathrm{d}x, \quad \text{for } n \to \infty \tag{2.16}$$

holds true with an explicitly known constant $C_\alpha > 0$.

Remark 2.6.2. *(i) Since H_0 commutes with the parity operator, the parity of $|n\rangle$ corresponds to the parity of its quantum number n. Therefore $\langle n|v|m\rangle = 0$ if the difference $n - m$ is odd.*

(ii) Using the Bohr-Sommerfeld quantisation condition one deduces the asymptotics of the energy levels

$$E_n \sim K_\alpha n^{\frac{2\alpha}{\alpha+2}}, \quad \text{as } n \to \infty$$

with a constant K_α known explicitly. Thus for $\alpha > 2$ the gaps $E_{n+1} - E_n$ between the eigen-values are growing, for $\alpha = 2$ they are constant, and finally for a parameter $\alpha \in]0, 2[$ the gaps are shrinking.

(iii) In the proof of the theorem the approximated eigen-functions are constructed using the WKB analysis and further studied by asymptotic methods. The proof is rather technical, therefore we skip it and refer the reader to [Lev].

For applications one can ask whether v lies in a class $\mathcal{X}(p, \delta)$ defined in Definition 4.1.3. From the theorem it follows that $|\langle m|v|n\rangle| \leq \text{const}(mn)^{-\frac{1}{\alpha+2}}$ since one can split v into the positive and the negative part and apply the Schwartz inequality

$$\langle m|A|n\rangle \leq \sqrt{\langle m|A|m\rangle\langle n|A|n\rangle}$$

to both of them. Hence the rapidity of the decay along the main diagonal is greater than or equal to $\frac{1}{\alpha+2}$. Since formula (2.16) gives the asymptotics $\langle m|v|m\rangle \sim \text{const}\, m^{-\frac{2}{\alpha+2}}$ for $n \to \infty$, we observe that this estimate is optimal.

If non-constant, v can not lie in any class $\mathcal{X}(p, \delta)$ with $p > 1, \delta > 0$. In that case v would be compact self-adjoint and therefore pure point. This is possible only in the case of

$v(x) = \mathrm{const}$, since an operator which acts as the multiplication by a non-constant continuous function has always nontrivial continuous spectrum.

Comparing this knowledge with the assumptions of Theorems 2.5.3, 2.5.4 and 2.5.6 one concludes that it is possible to apply the KAM and adiabatic method to the model described by

$$H(t) := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + |x|^\alpha + f(t)v,$$

with $\alpha > 2$, f continuously differentiable (sufficiently many times) and T-periodic and v satisfying the assumptions of Theorem 2.6.1. the anti-adiabatic method is not directly applicable to this model. To assure that $v \in \mathcal{X}(p,\delta)$ wit $p > 1, \delta > 0$ one has to treat the case of more general observables depending non-trivially on the momentum. This is an open question.

2.6.2 Semiclassical regime

The semiclassical limit is defined by the conditions $n \to \infty$ and $\hbar \to 0$ in such a way that $E_n = E$ is constant. In fact, one has to choose an appropriate sequence $\hbar_n \to 0$ so that the Bohr-Sommerfeld quantisation condition relating $n, \hbar$ and E is satisfied. Let us present the following theorem.

Theorem 2.6.3. *Let $E > 0$ and $k \in \mathbb{Z}$ be fixed. Assume that v is a real-valued bounded and continuously differentiable function of x. Then there exist a sequence of positive numbers $\hbar_n$ and a sequence of real-valued $L^2(\mathbb{R})$-normalised functions $|n\rangle$, such that $\hbar_n \to 0$ and $|n\rangle$ is the n-th eigen-function of*

$$H_n := -\hbar_n^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + |x|^\alpha$$

corresponding to the eigen-value E. The enumeration of eigen-functions starts from the index $n = 0$ and their phase is fixed by the condition that $|n\rangle$ is positive on a neighbourhood of $+\infty$. Moreover, in the semiclassical limit, i.e. $n \to \infty, \hbar_n \to 0$, we get

$$\langle n|v(x)|n+k\rangle \to \frac{1}{T}\int_0^T v\,(q(t))\, e^{\imath k\omega t}\mathrm{d}t$$

where $(q(t), p(t))\,, t \in [0,T]$, is the classical trajectory in the phase space at the energy E and the initial point chosen as $p(0) := 0$, and $q(0) := E^{\frac{1}{\alpha}}$ (the right turning point). T is the period of the classical motion and $\omega = 2\pi/T$ is the frequency.

The theorem is further generalised in the preprint [LS] which is included in this thesis. Not to get confused in the notation we remark that in [LS] we treat more general oscillators $H_0 := -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$, while the perturbation is denoted by W. So that the matrix $\langle n|W|n+k\rangle$ is investigated. The semiclassical regime is not directly applicable to the theory of time-dependent systems, however it gives a qualitative information. Its asymptotic analysis is less difficult, so it was possible to obtain more general results than in the high energy regime.

2.7 Dynamical stability

Stability properties of a quantum system may be determined by the behaviour of the expectation values of quantum observables, for example energy. On the contrary to time independent systems where the energy is conserved during the time evolution, it is an important question to study behaviour of expectation values of the energy in the time-dependent case, not necessarily periodic. Mathematically, one investigates asymptotic properties of the function

$$\langle U(t,0)\Psi, H(t)U(t,0)\Psi\rangle$$

in the long-time regime for a Ψ from some convenient dense set in $\mathscr{H}$. In the case of $H(t) = H_0 + V(t)$ and $V(t)$ bounded uniformly one can choose for example $\operatorname{Dom} H_0$. Provided that $H(t) = H_0 + V(t)$, with V in $C^1(\mathbb{R}, \mathscr{B}(\mathscr{H}))$, there exists a trivial bound due to Nenciu [Ne], which does not depend on the spectral properties of H_0. Since

$$\partial_t \langle U(t,0)\Psi, H(t)U(t,0)\Psi\rangle = \langle U(t,0)\Psi, \dot{V}(t)U(t,0)\Psi\rangle,$$

where $\dot{V}(s)$ denotes the time-derivative in the sense of the operator norm, we get

$$\langle U(t,0)\Psi, H(t)U(t,0)\Psi\rangle = \mathcal{O}(t) \tag{2.17}$$

for any Ψ from $\operatorname{Dom} H_0$.

One can ask, whether the energy expectation value remains even bounded during the time evolution of a given state Ψ form $\operatorname{Dom} H_0$. Physically, this means that the amount of the energy absorbed from the neighbourhood is finite. It was observed by de Oliveira in [deO] that such a condition implies for time-periodic systems that $\mathscr{H}^{pp}(U(T,0)) = \mathscr{H}$. From the Floquet theory (see Section 2.4) it further follows that the spectrum of K is pure point. Hence this condition is stronger then the pure-pointness of the spectrum of K. The boundedness of the energy expectation values was for the first time proved by Asch, Duclos and Exner in [ADE] using the KAM theory. They have treated the case of $H(t) = H_0 + V(\omega t)$, with V being 2π-periodic and C^∞. In 2005, their theorem was improved by Duclos, Šťovíček, Soccorsi and Vittot in [DSSV]. The result is

Theorem 2.7.1. *Let the assumptions of Theorem 2.5.6 be satisfied. Then the propagator* $U(t,s)$ *associated with* $H(t) := H_0 + V(\omega t)$ *exists and for every* $\Psi \in \operatorname{Dom} H_0$

$$\sup_{t\in\mathbb{R}} \langle U(t,0)\Psi, H(t)U(t,0)\Psi\rangle < \infty$$

holds true provided ω *is taken from* $\Omega \setminus \Omega_{res}$.

The statement of the theorem is restricted to time-periodic systems, non-resonant frequencies and applies only in the case of growing gaps. All of these conditions were relaxed in the papers [Ne], [Jo2], and [BJ], but with a weaker estimate

$$\langle U(t,0)\Psi, H(t)U(t,0)\Psi\rangle = \mathcal{O}(t^\beta),$$

with $\beta \in]0,1[$. The authors applied three different methods; Nenciu used the adiabatic machinery supposing that the spectrum of H_0 is formed by bands which draw apart. Joye availed the assumption of fast decay of the matrix entries of $V(t)$ along the diagonal. Finally, Barbaroux and Joye have developed another trick using the Dyson expansion under the hypothesis that the elements decay fast in the direction perpendicular to the main diagonal.

In [DLS] we give an upper bound on the energy growth using a new method, based on the application of the anti-adiabatic method, and the so called progressive diagonalisation. The procedure requires periodicity and smallness of the perturbation, sufficient decay of the matrix entries of the perturbation in the direction perpendicular to the main diagonal, and, also, certain small decay along the diagonal. On the other hand, compared to [BJ] the method turns out to be more efficient in the case of shrinking gaps, the diffusive exponent β obtained by our method is typically smaller in comparison with the one in [BJ]. Also, the set from which the initial conditions are taken is larger then the one in [BJ] (see [DLS]) for details). The paper [Jo2] does not give a better diffusive exponent, since it profits from the diagonal decay only. The adiabatic method used in [Ne] is not applicable in this case since the gaps in the spectrum do not grow.

Chapter 3

Non-localised perturbations of the harmonic oscillator

3.1 Introduction and the result

Stability of the harmonic oscillator under time-dependent perturbations is in general a difficult question. The first result goes already to [EV] where the Stark effect and some time-periodic perturbations in the resonant regime and localised in space are analysed. A general quadratic time-dependent Hamiltonian was analysed in [HLS] using the solubility of the system. A combination of KAM-type technique with the Nash-Moser trick was applied in [Co1] to prove the stability with respect to a class of time-periodic operators at least for non-resonant frequencies. Notice that this class of perturbation however does not contain potentials. Stability with respect to a large class of perturbations which involves decaying potentials is proved in the work by Duclos and Vittot which have not been published yet.

In this part, we extend some results by Enss & Veselić presented in [EV], part V.II where the stability of the harmonic oscillator under some special time-dependent perturbation is treated. New result include some non-localised potentials, e.g. any finite linear combination of $\cos x$ multiplied by one fixed time-dependent function $f(t)$. Let us specify the settings. Let

$$H_0 := \frac{1}{2}P^2 + \frac{\omega^2}{2}Q^2,$$

be the Hamiltonian of the harmonic oscillator, acting on $L^2(\mathbb{R})$. Further, let

$$V(t,x) := f(t)v(x),$$

with f being T-periodic $C^1(\mathbb{R})$ function, and v a bounded potential on the real line. We assume the resonant case, i.e. $\omega T/2\pi$ rational. Remark that in [Co1] the frequency $1/T$ is assumed to be Diophantine what is far from to be resonant. Denote by $U(t,s)$ the unitary propagator of

$$H(t) := H_0 + V(t).$$

Under previous assumptions the propagator exists and is unique. We shall prove that the set of propagating states which equals $\mathscr{H}^{cont}(U(T,0))$ is empty. In the unperturbed case when $V(t,x) = 0$ the monodromy is equal to e^{-iH_0T} and is pure-point. The resonant case of the harmonic oscillator is a little bit exceptional, as the point spectrum of its Floquet Hamiltonian is not dense in $\mathbb{R}$, however it contains some eigen-values of infinite multiplicity. We look for perturbations $V(t,x)$ such that the pure-point character of the spectrum of the monodromy is conserved. A sufficient condition is formulated in the following theorem.

Theorem 3.1.1. *The set of the propagating states of T-periodic one dimensional quantum system driven by*

$$H(t) := \frac{1}{2}P^2 + \frac{\omega^2}{2}Q^2 + f(t)v(Q)$$

is empty, provided f is a continuously differentiable T-periodic function, v is a real-valued almost periodic function, and $\omega T/2\pi$ is rational. In other words, under previous assumptions the monodromy associated with $H(t)$ is pure point.

Using the Fourier transform one deduces

Corollary 3.1.2. *The set of the propagating states of a quantum system described by the Hamiltonian*

$$H(t) := \frac{1}{2}P^2 + \frac{\omega^2}{2}Q^2 + f(t)A,$$

is empty, if f is a T-periodic $C^1(\mathbb{R})$ function, $\frac{\omega T}{2\pi}$ is rational and A is any linear combination of $\cos(aP)$ with $a \in \mathbb{R}$.

Notice that the action of such A may be non-localised since

$$\cos(aP)\Psi(x) = \frac{1}{2}\left(\Psi(x+a) + \Psi(x-a)\right).$$

Remark 3.1.3. *(i) The set of almost periodic functions is studied in monographs [Bs] and [Le]. Basic properties are summarised in [DSw] Chapter XI. Remark that if v is continuous with $\lim_{x\to\pm\infty} v(x) = 0$ then v is almost periodic. Thus any potential localised in space is included in the set of AP functions.*

(ii) If $\omega T/2\pi$ is rational, the monodromy corresponding to

$$H(t) := \frac{1}{2}P^2 + \frac{\omega^2}{2}Q^2 + Q\sin(\frac{2\pi}{T}t)$$

is purely absolutely continuous (see [EV]). Obviously the perturbation is not bounded in this case, thus one can ask where the border of the stability lies.

Proof. If $v(x)$ is constant, we can split the function f into the time-average $\bar{f}$, and periodic part $\tilde{f}(t)$. With the help of a convenient unitary transform, one can get rid of $\tilde{f}(t)v$. The rest, $\bar{f}v$, is nothing but a constant which does not change the spectral properties of the monodromy. Set

$$M := \operatorname{span}\left\{e^{\imath ax} | a \in \mathbb{R}, a \neq 0\right\}.$$

The set of all almost periodic functions may by written as the sum of the sets

$$AP = \operatorname{span}\{1\} + \overline{M}^{\infty},$$

where $\overline{M}^{\infty}$ denotes the closure of M with respect to the supreme norm. Suppose for the rest of the proof that v is real and from $\overline{M}^{\infty}$. Due to Theorem 5.2 in [EV] the difference

$$e^{-iH_0T} - U(T,0)$$

is compact if this is true for

$$W_V(t,s) := \int_s^t e^{iH_\omega\sigma}V(\sigma)e^{-iH_\omega\sigma}d\sigma \tag{3.1}$$

for all $s,t \in [0,T]$. In Section 3.2 we show that $W_{f\cos}$ and $W_{f\sin}$ are compact. By the approximation argument W_{fv} is compact too. Since we are in the resonant case, the spectrum of e^{-iH_0T} contains at most finite number of accumulation points which form the essential spectrum. Using the Weyl's theorem we claim that the spectrum of the monodromy $U(T,0)$ has the same property, since the difference $e^{-iH_0T} - U(T,0)$ is compact . Thus the spectrum of the monodromy is pure-point and therefore the set of propagating states is empty. ☐

3.2 Potential $\cos(bx)$ and $\sin(bx)$

Without loss of generality we assume $b > 0$. The goal is to prove that the operator

$$W_{f\cos}(t,s) := \int_s^t e^{iH_\omega\sigma}f(\sigma)\cos(bX)e^{-iH_\omega\sigma}d\sigma$$

is compact for all $s, t \in [0, T]$ and the same property follows for $W_{f\sin}$. Lemma 5.4 in [EV] states that a sufficient condition for the compactness of $W_{f\cos}$ is the same property of matrix O with entries

$$O_{m,n} := \frac{\langle m|\cos(bX)|n\rangle}{\langle m-n\rangle}. \tag{3.2}$$

Here $|n\rangle$ stands for the n-th eigen-state of the harmonic oscillator $\langle x|n\rangle = \frac{1}{\sqrt[4]{\pi}\sqrt{2^n n!}} e^{-\frac{x^2}{2}} H_n(x)$ and $\langle k\rangle := \max\{1, |k|\}$. Because of the parity of the Hermite functions $O_{m,n}$ vanishes if the difference $m-n$ is odd. In the following, we deal with matrix entries $O_{m,m+2u}$ with $m, u \in \mathbb{N}$. This is possible, since the matrix is symmetric. To prove that O represents a compact operator we exploit the fact that integrals of the Hermite functions with sine and cosine can be computed explicitly. Further we study the result of integration by asymptotic methods. For brevity we skip some lengthy computations and estimates. We believe however that it is straightforward to reconstruct these steps for a reader.
Due to [GR], formula 7.388 (7)

$$\int_0^\infty e^{-x^2}\cos(bx)H_m(x)H_{m+2u}(x)\mathrm{d}x = 2^{m-\frac{1}{2}}\sqrt{\frac{\pi}{2}}m!(-1)^u b^{2u} e^{-\frac{b^2}{4}} L(m, 2u, \frac{b^2}{2})$$

holds true for every $b > 0$. Hence

$$\langle m|\cos(bX)|m+2u\rangle = \frac{(-\frac{1}{2})^u b^{2u} e^{-\frac{b^2}{4}} m! L(m, 2u, \frac{b^2}{2})}{2\sqrt{m!(m+2u)!}}$$

It is well known (see for example the book of Olver [Ol]) that there is a relationship between the Laguerre polynomials L and the function U

$$L(m, \alpha, x) = L_m^{(\alpha)}(x) = \frac{(-1)^m}{m!}U(-m, \alpha+1, x),$$

and that one can express the asymptotics of U in terms of the function Gamma

$$\begin{aligned} U(a, b, x) &= \Gamma(\frac{b}{2} - a + \frac{1}{4})\pi^{-\frac{1}{2}} e^{\frac{x}{2}} x^{\frac{1}{4}-\frac{b}{2}} \cos\left(\sqrt{(2b-4a)x} - (\frac{b}{2} - a - \frac{1}{4})\pi\right) \\ &\quad \times \left(1 + \mathcal{O}(|\frac{b}{2} - a|^{-\frac{1}{2}})\right). \end{aligned}$$

If we apply the Stirling's formula after some algebraic manipulations and estimates we come to

$$|\langle m|\cos(bX)|m+2u\rangle| \le \frac{C}{\sqrt{b}} A_{m,u} B_{m,u}, \tag{3.3}$$

with an universal constant C, and A, B defined by

$$A_{m,u} := \frac{(m+u)^{m+u}}{m^{\frac{m}{2}}(m+2u)^{\frac{m}{2}+u}}, \quad B_{m,u} := \frac{\left(1 + \frac{3}{4(m+u)}\right)^{m+u}(m+u+\frac{3}{4})^{\frac{1}{4}}}{m^{\frac{1}{4}}(m+2u)^{\frac{1}{4}}}$$

To estimate $B_{m,u}$ we simply use the fact that $(1+x)^{\frac{1}{x}} \leq e$, for $x > 0$. Thus one gets

$$B_{m,u} \leq \frac{e}{\sqrt[4]{m}}. \tag{3.4}$$

It is convenient to rewrite $\log A_{m,u} = mg(\frac{u}{m})$ with the help of auxiliary function $g(x) := (1+x)\log(1+x) - \frac{1}{2}(1+2x)\log(1+2x)$ defined for $x > 0$. Using some estimates of the derivative of g we obtain $g(x) \leq 0, \text{ for x} \in [0,2)$ and $g(x) \leq x\log\frac{3}{4}, \text{ for x} \geq 2$. Applying this knowledge we estimate A by

$$\begin{aligned} A_{m,u} &\leq 1, \text{ for u} \leq \text{2m} \\ &\leq \left(\frac{3}{4}\right)^{u}, \text{ for u} \geq \text{2m} \end{aligned} \tag{3.5}$$

3.2.1 Compactness

Using the symmetry, (3.4) and (3.5) together in (3.3) we get for a universal constant C, independent of m, n, b

$$\begin{aligned} |\langle m|\cos(bX)|n\rangle| &\leq \frac{C}{\sqrt{b}\sqrt[4]{\min(m,n)}} \text{ for } [\frac{\text{n}}{5}] \leq \text{m} \leq \text{5n} \\ &\leq \frac{C}{\sqrt{b}\sqrt[4]{\min(m,n)}}\left(\frac{\sqrt{3}}{2}\right)^{|m-n|} \text{ for n} \geq \text{5m, or m} \geq \text{5n.} \end{aligned} \tag{3.6}$$

$[a]$ denotes the integer part of a real number a. Now we are ready to prove that the matrix O defined by (3.2) represents a compact operator. We write down O as the product YZY of a compact operator Y and a bounded one Z, with

$$\begin{aligned} Y_{k,m} &:= \frac{\delta_{k,m}}{\log m} \\ Z_{m,n} &:= \frac{\log m\,\langle m|\cos(bX)|n\rangle\log n}{\langle m-n\rangle}. \end{aligned}$$

Clearly, Y is compact since it is diagonal and the limit of diagonal entries is 0. In Proposition 3.2.1 we show that Z is Shur-Holmgren. Then since $\|Z\| \leq \|Z\|_{SH}$ holds true for any operator, we are done. Recall the definition of the Shur-Holmgren norm

$$\|Z\|_{SH} := max\left\{\sup_{n\in\mathbb{N}}\sum_{m=1}^{\infty}|Z_{m,n}|\,,\sup_{m\in\mathbb{N}}\sum_{n=1}^{\infty}|Z_{m,n}|\right\}. \tag{3.7}$$

Proposition 3.2.1. *Z is Shur-Holmgren.*

Proof. Since Z is symmetric, it is sufficient to verify that $\sup_{n\in\mathbb{N}}\sum_{m=1}^{\infty}|Z_{m,n}|<\infty$. It is not difficult to prove using (3.6) that

$$\sum_{m=1}^{\infty}|Z_{m,n}|=\frac{1}{\sqrt{b}}\mathcal{O}(n^{-\frac{1}{5}})$$

and this completes the proof. ☐

3.2.2 Potential $\sin(bX)$

Quite similar computation may be applied in the case the potential $\sin(bX)$. Due to [GR], formula 7.388 (6)

$$\int_0^{\infty}e^{-x^2}\sin(bx)H_m(x)H_{m+2u+1}(x)\mathrm{d}x=2^m\sqrt{\frac{\pi}{2}}m!(-1)^u b^{2u}e^{-\frac{b^2}{4}}L(m,2u+1,\frac{b^2}{2})$$

holds true for every $b>0$. Hence

$$\langle m|\sin(bX)|m+2u+1\rangle=\frac{(-\frac{1}{2})^u b^{2u}e^{-\frac{b^2}{4}}m!L(m,2u+1,\frac{b^2}{2})}{2\sqrt{m!(m+2u+1)!}}$$

Following the asymptotics of L and U and using the Stirling's formula again one obtain

$$\begin{aligned}|\langle m|\sin(bX)|m+2u+1\rangle| &\le \frac{K}{b^{\frac{3}{4}}}\frac{(m+u)^{m+u}}{m^{\frac{m}{2}}(m+2u)^{\frac{m}{2}+u}}\frac{\left(1+\frac{5}{4(m+u)}\right)^{m+u}(m+u+\frac{5}{4})^{\frac{3}{4}}}{(1+\frac{1}{m+2u})^{\frac{m}{2}+u}(m+2u+1)^{\frac{3}{4}}m^{\frac{1}{4}}}\\ &\le \frac{K}{b^{\frac{3}{4}}\sqrt[4]{m}}\frac{(m+u)^{m+u}}{m^{\frac{m}{2}}(m+2u)^{\frac{m}{2}+u}}.\end{aligned}$$

The rest is exactly the same as in the cosine case.

Chapter 4

The anti-adiabatic method

4.1 Introduction

The anti-adiabatic method was invented by J. S. Howland in [Ho4]. The idea is to apply a gauge-type transformation to the Floquet Hamiltonian. On the contrary to the adiabatic transform where the application of an inverse commutator improves the behaviour of the perturbation, this method relies on application of the commutator with H_0. This is why we call it anti-adiabatic. We would like to introduce an improvement of the results by Howland. By a more careful analysis, we are able to depress the assumption $V(t) \in \mathcal{X}(\infty, \gamma)$ to $V(t) \in \mathcal{X}(r, \gamma)$ with $r > 1 + \frac{1}{2\gamma}$. We have to say that Howland remarks at the end of the article that it is possible to do it, however it does not seem to him to be worthwhile to do it. This is not our case; this extension of the anti-adiabatic method is an important ingredient of [DLS]. Remark that a slightly modified definition of the Howland's classes is established there, see [DLS] for details.

Let us begin with the result of this chapter.

Theorem 4.1.1. *Let the spectrum of H_0 be discrete with eigen-values $E_1 \leq E_2 \leq \cdots$ obeying*

$$|E_n - E_m| \leq \mathrm{const}|n - m|(nm)^{-\gamma}, \tag{4.1}$$

for a given $\gamma \in]0, \frac{1}{2}[$. Let $V(t)$ be a T-periodic function with values in the space of hermitian operators on $\mathscr{H}$. Suppose that the monodromy $U(T, 0)$ associated with $H_0 + V(t)$ exists. If the matrix entries of $V(t)$ in the eigen-basis of H_0 are measurable functions of t and satisfy

$$|V_{m,n}(t)| \leq \mathrm{const}\frac{1}{(mn)^{\gamma}\langle m - n\rangle^{r}} \tag{4.2}$$

uniformly in time, with $r > 1 + \frac{1}{2\gamma}$, then the monodromy $U(T, 0)$ has no absolutely continuous spectrum.

Remark 4.1.2. *(i) Shrinking gap condition (2.11) is equivalent to this one, with $\beta = 2\gamma$.*

(ii) If H_0 is unbounded and the time-dependence of $V(t)$ is factorised, i.e. $V(t) = f(t)v$ then the theorem holds true provided f is T-periodic and integrable over $[0, T]$, and the matrix entries of v satisfy

$$|v_{m,n}| \leq \mathrm{const}\frac{1}{\langle m-n\rangle^p},$$

with $p > 2 + \frac{1}{2\gamma} + \frac{\gamma}{2}$.

The proof consist in the repeated application of the anti-adiabatic transform. Before we start with this procedure, let us bring out some notions and useful facts. At first, we recall the definition of the Howland's classes $\mathcal{X}(p, \alpha)$. Not to get confused with [DLS] where another classes and norms are defined, we denote the Howland's original norm by $\|\cdot\|^H_{p,\alpha}$.

Definition 4.1.3. Let $p > 1, \alpha \geq 0$. We say, that an infinite matrix depending on $t \in [0, T]$

$$A(t) = \{A_{n,m}(t)\}_{n,m\in\mathbb{N}}$$

is in class $\mathcal{X}(p, \alpha)$ if and only if

$$\|A\|^H_{p,\alpha} := \sup_{t\in[0,T]} \sup\{|A_{n,m}(t)|(nm)^\alpha \langle n-m\rangle^p : n, m \geq 1\} < \infty,$$

with $\langle n-m\rangle := \max\{1, |n-m|\}$. We say that an operator-valued function B(t) on $\mathscr{H}$ is in class $\mathcal{X}(p, \alpha)$ if and only if its matrix in the eigen-basis of H_0 lies in the class $\mathcal{X}(p, \alpha)$.

Notice that $\mathcal{X}(p, \alpha)$ is a Banach space equipped with the norm $\|\cdot\|^H_{p,\alpha}$. $\mathcal{X}(p, \alpha)$ is a subset of the set of bounded operators on $l^2(\mathbb{N})$, since for the Shur-Holmgren norm (see 3.7)

$$\|A\|_{SH} \leq (1 + 2\zeta(p))\, \|A\|^H_{p,\alpha}$$

holds true. $\zeta(p) := \sum_{k=1}^\infty k^{-p}$ denotes the Riemann's zeta function. Further, $A \in \mathcal{X}(p, \alpha)$ is compact if $\alpha > 0$.

We will often use inequalities (holding for every $n, m, k \geq 1$)

$$\frac{m}{k} \leq 2\langle m-k\rangle, \quad \langle n-m\rangle \leq 2\langle n-k\rangle\langle k-m\rangle, \tag{4.3}$$

in fact consequences of $a, b \geq 1 \implies a + b \leq 2ab$. From the definition of the norm $\|\cdot\|^H_{p,\alpha}$ one deduces using (4.1) the following lemma

Lemma 4.1.4. *Let $p > 2$. If $A \in \mathcal{X}(p, \alpha)$, then the commutator $[A, H_0]$ is in $\mathcal{X}(p-1, \alpha+\gamma)$ and*

$$\|[A, H_0]\|^H_{p-1,\alpha+\gamma} \leq \mathrm{const}\|A\|^H_{p,\alpha}$$

holds true.

4.2 The anti-adiabatic transform

As it was outlined in part 2.5.2, using the anti-adiabatic transformation, in fact applying the commutator with H_0, one can improve the decay of the matrix entries of the perturbation along the main diagonal at the expense of the decay in the direction perpendicular to this diagonal. With the help of the definition of the Howland's classes, the anti-adiabatic transform may be viewed as the passing from a perturbation $A(t) \in \mathcal{X}(p,\alpha)$ to a new one $\tilde{A}(t) \in \mathcal{X}(p-1,\alpha+\gamma)$, where γ is given by the shrinking gap condition (4.1), see Lemma 4.1.4. An important technical tool used in [Ho4] is the following lemma about the product of two Howland's classes.

Lemma 4.2.1. *If $A \in \mathcal{X}(p,\alpha)$ and $B \in \mathcal{X}(p,\beta)$, then the product AB is in $\mathcal{X}(r,\alpha+\beta)$ if*

$$1 < r < \min\{p - \frac{1}{2} - \frac{\alpha+\beta}{2}, p-\alpha, p-\beta\}$$

This lemma may be generalised into a result of the type

$$\mathcal{X}(p,\alpha)\mathcal{X}(r,\beta) \subset \mathcal{X}(q,\delta),$$

with a convenient choice of q and δ. For our purpose it is important that one can play with the choice of these parameters; sometimes we concentrate ourselves to obtain δ the best possible (it is $\alpha+\beta$ in fact), sometimes we do not need such a large δ, but we want to obtain better q. Let us present a new lemma about the product of two Howland's classes. It is stated for a special choice of the classes, suitable for the later use.

Lemma 4.2.2. *Let n be a natural number or zero, $p > n+1$, and $i \in \{0, \cdots, n+1\}$. Suppose that X lies in $\mathcal{X}(p-n, \gamma(n+1))$ and Y in $\mathcal{X}(p-i, \gamma(i+1))$. Then both of the products XY, YX are in $\mathcal{X}(p-n-1, \gamma(n+2))$.*

Let us postpone the proof of the lemma and formulate the main result of the anti-adiabatic method.

Theorem 4.2.3. *Let the spectrum of H_0 be discrete with eigen-values $E_1 \leq E_2 \leq \cdots$ obeying the condition (4.1). Further, let $V(t) \in \mathcal{X}(p,\gamma)$ be a measurable T-periodic and symmetric for every $t \in [0,T]$. Then for every natural number $l < p-1$ there exists a family of unitary operators J(t) on $\mathscr{H}$ such that*

$$K := D + H_0 + V(t) = J(t)\left(D + H_0 + W_l + \tilde{V}_l(t)\right)J(t)^*,$$

holds true with W_l symmetric compact and time-independent, and $\tilde{V}_l(t) \in \mathcal{X}(p-l,(l+1)\gamma)$ symmetric and T-periodic.

Proof. We proceed by induction in $n \in \{0, \dots, l\}$. Let us begin with $n = 0$. Set $V_0(t) := V(t)$ and decompose $V_0(t) := \bar{V}_0 + \tilde{V}_0(t)$ into the mean $\bar{V}_0 := \int_0^T V_0(t)\mathrm{d}t$ and the rest. Set $W_0 := \bar{V}_0$ and $J_0(t) := 1$. Then W_0 is compact, since it lies in $\mathcal{X}(p, \gamma)$ and the statement holds true.

Let us describe the induction step $n \to n + 1$. Consider the Floquet Hamiltonian

$$K_n = D + H_0 + W_n + \tilde{V}_n(t) = J_n(t)^* (D + H_0 + V(t)) J_n(t) \tag{4.4}$$

with $\tilde{V}_n(t)$ lying in $\mathcal{X}(p - n, (n + 1)\gamma)$ symmetric T-periodic and such that

$$\int_0^T \tilde{V}_n(t)\mathrm{d}t = 0.$$

W_n is supposed to be time-independent and compact. Set

$$G_{n+1}(t) := \int_0^t \tilde{V}_n(s)\mathrm{d}s,$$

so that $G_{n+1}(t)$ is symmetric T-periodic and in $\mathcal{X}(p - n, \gamma(n + 1))$. We define K_{n+1} by the gauge-type transformation

$$K_{n+1} := e^{\imath G_{n+1}(t)} K_n e^{-\imath G_{n+1}(t)} = D + H_0 + W_n + V_{n+1}(t), \tag{4.5}$$

with

$$V_{n+1}(t) = e^{\imath G_{n+1}(t)} \left(D + H_0 + W_n + \tilde{V}_n(t) \right) e^{-\imath G_{n+1}(t)} - (D + H_0 + W_n) . \tag{4.6}$$

Obviously, $V_{n+1}(t)$ is T-periodic. Later, we prove the following lemma

Lemma 4.2.4. *$V_{n+1}(t)$ lies in $\mathcal{X}(p - n - 1, \gamma(n + 2))$.*

Define $\bar{V}_{n+1} := \int_0^T V_{n+1}(t)\mathrm{d}t$ and $\tilde{V}_{n+1}(t) := V_{n+1}(t) - \bar{V}_{n+1}$. Set $W_{n+1} := W_{n+1} + \bar{V}_{n+1}$. Then it holds true that

$$K_{n+1} = D + H_0 + W_{n+1} + \tilde{V}_{n+1}(t)$$

with W_{n+1} compact symmetric and not depending on time, and $\tilde{V}_{n+1}(t) \in \mathcal{X}(p - n - 1, \gamma(n + 2))$ symmetric and T-periodic. Further, thanks to (4.4, 4.5) we get

$$K_{n+1} = J_{n+1}(t)^* (D + H_0 + V(t)) J_{n+1}(t),$$

with $J_{n+1}(t) = J_n(t)e^{-\imath G_{n+1}(t)}$. Since $G_{n+1}(t)$ is symmetric, $J_{n+1}(t)$ is unitary. This completes the induction step and the proof of the theorem.

Notice that from the proof it follows that

$$W_n = \sum_{i=0}^{n} \bar{V}_i,$$

with $\bar{V}_i \in \mathcal{X}(p - i, (i + 1)\gamma)$. $\square$

4.2.1 Proof of Lemma 4.2.2 and Lemma 4.2.4

Proof of Lemma 4.2.2. We treat the case XY, the opposite one is analogous. Using the definition of the norm $\|\cdot\|^H_{q,\alpha}$ we get with the help of inequalities (4.3)

$$\begin{aligned}\|XY\|_{p-n-1,(n+2)\gamma} &= \sup_{m,l\in\mathbb{N}} (ml)^{(n+2)\gamma}\langle m-l\rangle^{p-n-1}\sum_{k=1}^{\infty}|X_{mk}Y_{kl}| \\ &\le \|X\|_{p-n,(n+1)\gamma}\|Y\|_{p-i,(i+1)\gamma}\sup_{m,l\in\mathbb{N}}\sum_{k=1}^{\infty}\frac{(ml)^{(n+2)\gamma}\langle m-l\rangle^{p-n-1}}{(mk)^{(n+1)\gamma}(kl)^{(i+1)\gamma}\langle m-k\rangle^{p-n}\langle l-k\rangle^{p-i}} \\ &\le 2^{p-n-1+(n-i+2)\gamma}\|X\|_{p-n,(n+1)\gamma}\|Y\|_{p-i,(i+1)\gamma}L_{n,i,\gamma},\end{aligned}$$

with

$$L_{n,i,\gamma} := \sup_{m,l\in\mathbb{N}}\sum_{k=1}^{\infty}k^{-2i\gamma}\langle m-k\rangle^{\gamma-1}\langle k-l\rangle^{(\gamma-1)(n-i+1)}. \tag{4.7}$$

Let $i\neq 0$. From the Hölder inequality (with $s=\frac{1+\gamma}{1-\gamma}$, $r=\frac{1+\gamma}{2\gamma}$) it follows that

$$\begin{aligned}L_{n,i,\gamma} &\le \sup_{m\in\mathbb{N}}\sum_{k=1}^{\infty}k^{-2\gamma}\langle m-k\rangle^{\gamma-1} \\ &\le \left(\sum_{k=1}^{\infty}k^{-2r\gamma}\right)^{\frac{1}{r}}\sup_{m\in\mathbb{N}}\left(1+\sum_{k=1}^{\infty}k^{-s(1-\gamma)}+\sum_{k=1}^{m-1}k^{-s(1-\gamma)}\right)^{\frac{1}{s}} \\ &= \zeta^{\frac{2\gamma}{1+\gamma}}(1+\gamma)\left(1+2\zeta(1+\gamma)\right)^{\frac{1-\gamma}{1+\gamma}} \le \left(1+2\zeta(1+\gamma)\right).\end{aligned}$$

The same expression estimates $L_{n,0,\gamma}$ for $n\ge 1$ since $\frac{2\gamma}{1-\gamma}<2$. Finally, one obtains directly $L_{0,0,\gamma}\le\left(1+2\zeta(2-2\gamma)\right)$. The lemma is proved. □

Proof of Lemma 4.2.4. Since $\operatorname{ad}_{G_{n+1}(t)} D = \imath\dot{G}_{n+1}(t) = \imath\tilde{V}_n(t)$, we get by expanding the right-hand side of (4.6) due to formula (2.6)

$$\begin{aligned}V_{n+1}(t) &= \sum_{j=1}^{\infty}\frac{\imath^j}{j!}\operatorname{ad}^{j-1}_{G_{n+1}(t)}\left(\imath\tilde{V}_n(t)+\left[G_{n+1}(t),H_0+W_n+\tilde{V}_n(t)\right]\right)+\tilde{V}_n(t) \\ &= \sum_{j=1}^{\infty}\frac{\imath^j}{j!}\operatorname{ad}^{j-1}_{G_{n+1}(t)}B_{n,j}(t),\end{aligned} \tag{4.8}$$

with

$$B_{n,j}(t) := \operatorname{ad}_{G_{n+1}(t)}\left(H_0+W_n+\frac{j}{j+1}\tilde{V}_n(t)\right).$$

Recall that from the proof Theorem 4.2.3 we have

$$W_n = \sum_{i=0}^{n} \bar{V}_i,$$

with $\bar{V}_i \in \mathcal{X}(p-i,(i+1)\gamma)$.

By Lemma 4.1.4 $\operatorname{ad}_{G_{n+1}} H_0 \in \mathcal{X}(p-n-1,(n+2)\gamma)$ and using Lemma 4.2.2 the same holds true for $\operatorname{ad}_{G_{n+1}} W_n$ and $\operatorname{ad}_{G_{n+1}} \tilde{V}_n$. Thus we conclude that $B_{n,j}(t)$ is in $\mathcal{X}(p-n-1,(n+2)\gamma)$. Applying Lemma 4.2.2 again we get that in the same class lies also $\operatorname{ad}^{j-1}_{G_{n+1}(t)} B_{n,j}(t)$ and moreover

$$\| \operatorname{ad}^{j-1}_{G_{n+1}(t)} B_{n,j}(t)\|^H_{p-n-1,(n+2)\gamma} \leq C_{p,n,\gamma,H} \left(C_{p,n,\gamma,H}\|G_{n+1}\|^H_{p-n,(n+1)\gamma}\right)^{j-1} \|B_{n,j}\|^H_{p-n-1,(n+2)\gamma},$$

with a constant $C_{p,n,\gamma,H}$. Then due to the presence of the factor $\frac{1}{j!}$ it is easy to conclude that $V_{n+1}(t)$ lies in $\mathcal{X}\left(p-n-1,\gamma(n+2)\right)$ since such a class forms a Banach space. The proof of Lemma 4.2.4 is complete. □

4.3 Proof of Theorem 4.1.1

Proof. We apply Theorem 4.2.3 with $l := [\frac{1}{2\gamma}]$, the integer part of $\frac{1}{2\gamma}$. Therefore $K = D + H_0 + V(t)$ is unitarily equivalent to $D + H_0 + W_l + \tilde{V}_l(t)$, with W_l compact and $\tilde{V}_l(t) \in \mathcal{X}(r-l,(l+1)\gamma)$. It is easy to see that

$$1 < r-l,\ \frac{1}{2} < (l+1)\gamma < 1.$$

$\tilde{V}_l(t)$ is trace-class uniformly since its trace norm may be estimate by

$$\|\tilde{V}_l(t)\|_1 \leq \sum_{i,j=1}^{\infty} |\tilde{V}_l(t)_{i,j}| \leq \|\tilde{V}_l\|^H_{r-l,(l+1)\gamma} \sum_{i,j=1}^{\infty} \frac{1}{(ij)^{(l+1)\gamma}\langle i-j\rangle^{r-l}} < \infty.$$

It is not difficult to check that the last sum is convergent. By the Weyl's theorem the spectrum of $H_0 + W_l$ is discrete, since the same is true for the spectrum of H_0 and W_l is compact. To finish the proof it suffices to apply Theorem 2.5.1. □

Chapter 5

The results of the thesis

During the study of time-periodic quantum systems I have focused myself on two directions: the first one was the development of the methods, the second one the analysis of some models. According to this, we can organise the objectives of my work into two groups:

- **The methods:** Try to improve general methods for time-periodic systems
- **The models:** Analyse models with H_0 being (a general) oscillator

Let us discuss the results.

5.1 The KAM method

We begin with the KAM method which deals with the spectral stability of the Floquet Hamiltonian $K_0 := -\imath\partial_t + H_0$ (acting on $L^2([0,T],\mathscr{H})$ with periodic b.c.) with respect to T-periodic perturbations $V(t)$. Since H_0 is supposed to have discrete spectrum, the spectrum of K_0 is pure point. By the spectral stability we mean the property that the spectrum of $K_0 + V(t)$ is pure point too.

As we explain in Section 2.5, the KAM theory is applicable in that case when the gaps in the spectrum of H_0 are growing. Remark that in [Co1] it is modified for constant gaps, too. The statement guarantees that the spectrum of $K_0 + V(t)$ is pure point for a set of periods T of non-zero Lebesgue measure. In the papers [DLSV1] and [DLSV2], the theory was generalised to the case of growing multiplicities of the eigen-values of H_0 and a class of unbounded perturbations $V(t)$. An important assumption of the theorem is that $V(t)$ is sufficiently differentiable with respect to time. Let us remark that some results of these papers were already included in my Diploma thesis [Lev].

Since the theory is most developed in the case of the growing gaps, it may seem that it is "nice and solved". The spectral result of the KAM-like theorem is however non-trivial. In the paper [Bo2], Bourget showed the evidence of purely singular spectrum (for almost all T) for the system corresponding formally to

$$K = -\imath\partial_t + H_0 + |\phi\rangle\langle\phi| \sum_{n\in\mathbb{Z}} \delta(t - nT),$$

with H_0 having pure point spectrum with the eigen-values E_n obeying the condition

$$E_{n+1} - E_n \geq Cn^{-2\gamma},$$

for some $\gamma > 0$. The vector ϕ is assumed to be cyclic with respect to H_0 and satisfying some additional properties. Notice that the time-dependence of such a perturbation is singular, on the other hand the perturbation is of the rank one.

5.2 The anti-adiabatic method

The new result of the anti-adiabatic method is formulated in Theorem 4.1.1. Provided that $V(t)$ is T-periodic and in class $\mathcal{X}(r,\gamma)$, with $r > 1 + \frac{1}{2\gamma}$, the statement excludes any absolutely continuous part from the spectrum of $K = -\imath\partial_t + H_0 + V(t)$, where H_0 is supposed to have discrete spectrum, such that the gaps in the spectrum diminish (see 4.1).

In fact, already Howland remarked in [Ho4] that it is possible to weak the assumption $V(t) \in \mathcal{X}(\infty,\gamma)$ to $V(t) \in \mathcal{X}(r,\gamma)$ (see Definition 4.1.3), with an $r > 1$ finite . We think that this generalisation is remarkable by itself, anyway it is an important ingredient of [DLS] where we combine the anti-adiabatic method with the progressive diagonalisation and some results of [Jo2]. Without the generalisation to r finite this work would not be possible.

Similarly to the KAM theory, the spectral result of the anti-adiabatic theory is not trivial, since the example of Bourget (see the previous section) involves the case of the shrinking gaps, too. Notice that there is no restrictive condition on the multiplicities of the eigenvalues of H_0 in Theorem 4.1.1, they can grow arbitrarily with n. Aside from the question of the optimality of r, we do not see any possible extension of this idea.

5.3 Asymptotic behaviour of matrix entries

Inspired by the methods mentioned above, we have started to analyse a family of one-dimensional models described by

$$H_0 = -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + |x|^\alpha,$$

with $\alpha > 0$. It is not difficult to prove that H_0 has simple pure point spectrum. Depending on the choice of α, the gaps in the spectrum are growing ($\alpha > 2$), constant ($\alpha = 2$) or shrinking ($\alpha \in]0, 2[$). Denote by $|n\rangle$ the n-th eigen-vector of H_0.

Choosing an observable A, we are interested in the asymptotic behaviour of the matrix entries

$$\langle m|A|n\rangle,$$

for m, n large. We introduce two results corresponding to the high energy regime ($\hbar = 1$, $n \to \infty, E_n \to \infty$) and the semiclassical regime ($\hbar \to 0, n \to \infty, E_n = E = \mathrm{const}$). In the first case, we prove for a localised even potential v (see Theorem 2.6.1) that

$$\lim_{n\to\infty} \langle n|v|n+2k\rangle = C_\alpha (-1)^k n^{-\frac{2}{\alpha+2}} \int_{\mathbb{R}} v(x)\mathrm{d}x$$

holds true for every $k \in \mathbb{Z}$ fixed. This is a generalisation of the result of [Lev]. It turns out that the KAM and the adiabatic theories are applicable to this model for $\alpha > 2$, however, for possible application of the anti-adiabatic method it is necessary to treat more general observables depending also on the momentum.

In the semiclassical limit (see Theorem 2.6.3), we have for a more general v

$$\langle n|v|n+k\rangle \to \frac{1}{T} \int_0^T v\left(q(t)\right) e^{\imath k\omega t} \mathrm{d}t$$

where $\left(q(t), p(t)\right), t \in [0, T]$, is the classical trajectory in the phase space at the energy E and the initial point chosen as $p(0) := 0$, and $q(0) := E^{\frac{1}{\alpha}}$. T is the period of the classical motion and $\omega = 2\pi/T$. For the harmonic oscillator a result of this type is known for a long time in the physical literature (see [LL]). We extend its validity in [LS] to a class of general oscillators $H_0 = -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$ and prove it rigorously.

5.4 Non-localised perturbations of the harmonic oscillator

The Harmonic oscillator in the resonant regime is described by the Floquet Hamiltonian

$$K_0 = -\imath\partial_t + \frac{1}{2}P^2 + \frac{\omega^2}{2}Q^2,$$

which acts on $L^2[0, T] \otimes L^2(\mathbb{R})$, with periodic b. c. in time. The resonance is defined by the condition that $\omega T/2\pi$ is rational. This is an example of a system with constant gaps. Treating the spectral stability of K_0, we profit from the fact, that its point spectrum is not dense in $\mathbb{R}$. It turns out that for a generic Floquet Hamiltonian the opposite is true (see Section 2.5).

In Chapter 3, we prove that the spectrum of the Floquet Hamiltonian K_0 is stable with respect to some large class of non-localised perturbations $V(t)$. The statement of Theorem 3.1.1 may be reformulated into the following shape.

The spectrum of the Floquet Hamiltonian

$$K := -\imath\partial_t + \frac{1}{2}P^2 + \frac{\omega^2}{2}Q^2 + f(t)v(Q)$$

acting on $L^2[0,T] \otimes L^2(\mathbb{R})$ is pure point, provided f is a continuously differentiable T-periodic function, v is a real-valued almost periodic function, and $\omega T/2\pi$ is rational. This is a generalisation of the result in [EV] where v is assumed to be compactly supported.

Applying the Fourier transform one concludes that the same theorem is valid for

$$K := -\imath\partial_t + \frac{1}{2}P^2 + \frac{\omega^2}{2}Q^2 + f(t)v(P).$$

Since it is well-known (see [EV]) that the spectrum of

$$-\imath\partial_t + \frac{1}{2}P^2 + \frac{\omega^2}{2}Q^2 + Q\sin(\frac{2\pi}{T}t)$$

is purely absolutely continuous is is natural to ask, whether there is a condition which relates the growth of the perturbation in the phase space to the spectral properties of K.

5.5 Energy growth of some systems with the shrinking gaps

In the preprint [DLS] we introduce an upper bound of the energy growth of some periodically driven quantum systems with shrinking gaps in the spectrum. Let us describe briefly the result, see [DLS] for more details. Assume that a quantum system is described by the Hamiltonian $H(t) := H_0 + V(t)$, where H_0 has pure point spectrum with eigen-values $E_1 < E_2 < \ldots$ obeying the shrinking gap condition

$$c_H \frac{|m-n|}{\max\{m,n\}^{2\gamma}} \leq |E_m - E_n| \leq C_H \frac{|m-n|}{\max\{m,n\}^{2\gamma}},$$

with a $\gamma \in]0,1/2[$. Remark that this condition is fulfilled for $E_n = n^\alpha$, where $\alpha \in]0,1[$ and $\gamma = (1-2\alpha)/2$.

Assume that $V(t)$ is T-periodic, strongly C^1 and sufficiently small in the norm

$$\|V\|_{p,\gamma} := \sup_{t\in[0,T]} \sup_{m,n\in\mathbb{N}} \langle m-n\rangle^p \max\{m,n\}^{2\gamma} \|V(t)_{m,n}\|.$$

If p is sufficiently large then for any Ψ from the form domain of H_0 it holds true that

$$\langle U(t,0)\Psi, H_0\, U(t,0)\Psi\rangle = \mathcal{O}(t^\beta),$$

with $\beta = \frac{\alpha}{2[p]\gamma - 1/2}$, where $U(t,s)$ is the unitary propagator associated with $H(t)$. We observe that for p large enough, one can obtain arbitrarily small β. To our knowledge, the result of this type is the first one in the case of shrinking gaps and a small decay of entries of $V(t)$ along the main diagonal.

List of publications

- Duclos P., Lev O., Šťovíček P., Vittot M., *Progressive diagonalization and applications*, Proceedings of ”Operator Algebras & Mathematical Physics”, Constanta, Romania (2001)

- Duclos P., Lev O., Šťovíček P., Vittot M., *Weakly regular Floquet Hamiltonians with pure point spectrum*, Rev. Math. Phys. **14** (2002) 531-568.

- Lev O., Šťovíček P., *A semiclassical formula for non-diagonal matrix elements,* (2006), accepted for publication in Int. J. Theor. Phys.

- Duclos P., Lev O., Šťovíček P., *On the energy growth of some periodically driven quantum systems with shrinking gaps in the spectrum,* preprint (2007), submitted to J. Stat. Phys.

List of citations

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1. Bourget O., Howland J.S., Joye A., *Spectral analysis of unitary band matrices*, Commun. Math. Phys. **234** (2003) 191-227

2. Gentile G., *Quasi-periodic solutions for two-level systems*, Commun. Math. Phys. **242** (2003) 221-250

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Reprints

WEAKLY REGULAR FLOQUET HAMILTONIANS WITH PURE POINT SPECTRUM

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Abstract. We study the Floquet Hamiltonian $-i\partial_t + H + V(\omega t)$, acting in $L^2([0,T],\mathcal{H},dt)$, as depending on the parameter $\omega = 2\pi/T$. We assume that the spectrum of H in $\mathcal{H}$ is discrete, $\mathrm{Spec}(H) = \{h_m\}_{m=1}^{\infty}$, but possibly degenerate, and that $t \mapsto V(t) \in \mathcal{B}(\mathcal{H})$ is a 2π-periodic function with values in the space of Hermitian operators on $\mathcal{H}$. Let $J > 0$ and set $\Omega_0 = [\frac{8}{9}J, \frac{9}{8}J]$. Suppose that for some $\sigma > 0$ it holds true that $\sum_{h_m > h_n} \mu_{mn}(h_m - h_n)^{-\sigma} < \infty$ where $\mu_{mn} = (\min\{M_m, M_n\})^{1/2} M_m M_n$ and M_m is the multiplicity of h_m. We show that in that case there exist a suitable norm to measure the regularity of V, denoted ϵ_V, and positive constants, $\epsilon_\star$ and $\delta_\star$, with the property: if $\epsilon_V < \epsilon_\star$ then there exists a measurable subset $\Omega_\infty \subset \Omega_0$ such that its Lebesgue measure fulfills $|\Omega_\infty| \geq |\Omega_0| - \delta_\star \epsilon_V$ and the Floquet Hamiltonian has a pure point spectrum for all $\omega \in \Omega_\infty$.

1. Introduction

The problem we address in this paper concerns spectral analysis of so called Floquet Hamiltonians. The study of stability of non autonomous quantum dynamical systems is an effective tool to understand most of quantum problems which involve a small number of particles. When these systems are time-periodic the spectral analysis of the evolution operator over one period can give a fairly good information on this stability, see e.g. [1]. In fact this type of result generalises the celebrated RAGE theorem concerned with time-independent systems (one can consult [2] for a summary). As shown in [3] and [4] the spectral analysis of the evolution operator over one period (so called monodromy operator or Floquet operator) is equivalent to the spectral analysis of the corresponding Floquet Hamiltonian (sometimes called operator of quasi-energy). This is also what we are aiming for in this article. More precisely, we analyse time-periodic quantum systems which are weakly regular in time and "space" in the sense of an appropriately chosen norm, and give sufficient conditions to insure that the Floquet Hamiltonians has a pure point spectrum.

Such a program is not new. In the pioneering work [5] Bellissard has considered the so called pulsed rotor which is analytic in time and space, using a KAM type algorithm. Then Combescure [6] was able to treat harmonic oscillators driven by sufficiently smooth perturbations by adapting to quantum mechanics the well known Nash-Moser trick (c.f. [7] and [8]). Later on these ideas have been extended to a wider class of systems in [9]; it was even possible to require no regularity in space by using the so called adiabatic regularisation, originally proposed in [10] and further extended in [11], [12]. However none of these papers can be considered as optimal in the sense of having found the minimal value of regularity in time below which the Floquet Hamiltonian ceases to be pure point.

Though it is impossible to mention all the relevant contributions to the study of stability of time-dependent quantum systems we would like to mention the following ones. Perturbation theory for a fixed eigenvalue has been extended, in [13], to Floquet Hamiltonians which generically have a dense point spectrum. Bounded quasi-periodic time dependent perturbations of two level systems are considered in [14] whereas the case of unbounded perturbation of one dimensional oscillators are studied in [15]. Averaging methods combined with KAM techniques were described in [16] and [17].

In the present paper we attempt to further improve the KAM algorithm, particularly having in mind more optimal assumptions as far as the regularity in time is concerned. As a thorough analysis of the algorithm has shown this is possible owing to the fact that the algorithm contains several free parameters (for example the choice of norms in auxiliary Banach spaces that are constructed during the algorithm) which may be adjusted. This type of improvements is also illustrated on an example following Theorem 1 in Section 2. A more detailed discussion of this topic is postponed to concluding remarks in Section 10.

Another generalisation is that in the present result (Theorem 1) we allow degenerate eigenvalues of the unperturbed Hamilton operator (denoted H in what follows). The degeneracy of eigenvalues h_m of H can grow arbitrarily fast with m provided the time-dependent perturbation is sufficiently regular. To our knowledge this is a new feature in this context. Previously two conditions were usually imposed, namely bounded degeneracy and a growing gap condition on eigenvalues h_m, reducing this way the scope of applications of this theory to one dimensional confined systems. Owing to the generalisation to degenerate eigenvalues we are able to consider also some models in higher dimensions, for example the N-dimensional quantum top, i.e., the N-dimensional version of the pulsed rotor. A short description of this model is given, too, in Section 2 after Theorem 1.

The article is organised as follows. In Section 2 we introduce the notation and formulate the main theorem. The basic idea of the KAM-type algorithm is outlined in

Section 3. The algorithm consists in an iterative procedure resulting in diagonalisation of the Floquet Hamiltonian. For this sake one constructs an auxiliary sequence of Banach spaces which form in fact a directed sequence. The procedure itself may formally be formulated in terms of an inductive limit. Sections 4–8 contain some additional results needed for the proof, particularly the details of the construction of the auxiliary Banach spaces and how they are related to Hermitian operators in the given Hilbert space, and a construction of the set of "non-resonant" frequencies for which the Floquet Hamiltonian has a pure point spectrum (the frequency is considered as a parameter). Section 9 is devoted to the proof of Theorem 1. In Section 10 we conclude our presentation with several remarks concerning comparison of the result stated in Theorem 1 with some previous ones.

2. Main theorem

The central object we wish to study in this paper is a self-adjoint operator of the form $\mathbf{K}+\mathbf{V}$ acting in the Hilbert space

$$\mathcal{K} = L^2([\,0,T\,],\mathrm{dt})\otimes\mathcal{H}\cong L^2([\,0,T\,],\mathcal{H},\mathrm{dt})$$

where $T=2\pi/\omega$, ω is a positive number (a frequency) and $\mathcal{H}$ is a fixed separable Hilbert space. The operator $\mathbf{K}$ is self-adjoint and has the form

$$\mathbf{K} = -i\,\partial_t\otimes 1 + 1\otimes H$$

where the differential operator $-i\partial_t$ acts in $L^2([\,0,T\,],\mathrm{dt})$ and represents the self-adjoint operator characterised by periodic boundary conditions. This means that the eigenvalues of $-i\partial_t$ are $k\omega$, $k\in\mathbb{Z}$, and the corresponding normalised eigenvectors are $\chi_k(t) = T^{-1/2}\exp(ik\omega t)$. H is a self-adjoint operator in $\mathcal{H}$ and is supposed to have a discrete spectrum. Finally, $\mathbf{V}$ is a bounded Hermitian operator in $\mathcal{K}$ determined by a measurable operator-valued function $t\mapsto V(\omega t)\in\mathcal{B}(\mathcal{H})$ such that $\sup_{t\in\mathbb{R}}\|V(t)\|<\infty$, $V(t)$ is 2π-periodic, and for almost all $t\in\mathbb{R}$, $V(t)^* = V(t)$. Naturally, $(\mathbf{V}\psi)(t) = V(\omega t)\psi(t)$ in $\mathcal{K}\cong L^2([\,0,T\,],\mathcal{H},\mathrm{dt})$.

Let

$$\sum_{k\in\mathbb{Z}} k\omega\,P_k$$

be the spectral decomposition of $-i\partial_t$ in $L^2([\,0,T\,],\mathrm{dt})$ and let

$$H = \sum_{m\in\mathbb{N}} h_m\,Q_m$$

be the spectral decomposition of H in $\mathcal{H}$. Thus we can write

$$\mathcal{H} = \sum_{m\in\mathbb{N}}^{\oplus}\mathcal{H}_m$$

where $\mathcal{H}_m = \operatorname{Ran} Q_m$ are the eigenspaces. We suppose that the multiplicities are finite,

$$M_m = \dim\mathcal{H}_m < \infty,\ \ \forall m\in\mathbb{N}.$$

Hence the spectrum of $\mathbf{K}$ is pure point and its spectral decomposition reads

$$\mathbf{K} = \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{N}} (k\omega + h_m)\, P_k \otimes Q_m, \tag{1}$$

implying a decomposition of $\mathcal{K}$ into a direct sum,

$$\mathcal{K} = \sum_{(k,m)\in\mathbb{Z}\times\mathbb{N}}^{\oplus} \mathrm{Ran}(P_k \otimes Q_m)\,.$$

Here is some additional notation. Set

$$V_{knm} = \frac{1}{T}\int_0^T e^{-ik\omega t}\, Q_n V(\omega t)\, Q_m\, \mathrm{dt} = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}\, Q_n V(t)\, Q_m\, \mathrm{dt} \in \mathcal{B}(\mathcal{H}_m, \mathcal{H}_n)\,. \tag{2}$$

Further,

$$\Delta_{mn} = h_m - h_n,$$

and

$$\Delta_0 = \inf_{m\neq n} |\Delta_{mn}|.$$

Finally we set

$$\mu_{mn} = (\min\{M_m, M_n\})^{1/2}\, M_m M_n.$$

Now we are able to formulate our main result. Though not indicated explicitly in the notation the operator $\mathbf{K}+\mathbf{V}$ is considered as depending on the parameter ω.

Theorem 1. *Fix $J>0$ and set $\Omega_0 := [\,\frac{8}{9}J, \frac{9}{8}J\,]$. Assume that $\Delta_0 > 0$ and that there exists $\sigma > 0$ such that*

$$\Delta_\sigma(J) := J^\sigma \sum_{\substack{m,n\in\mathbb{N}\\ \Delta_{mn}>J/2}} \frac{\mu_{mn}}{(\Delta_{mn})^\sigma} < \infty\,.$$

Then for every $r > \sigma + \frac{1}{2}$ there exist positive constants (depending, as indicated, on σ, r, Δ_0 and J but independent of V), $\epsilon_\star(r, \Delta_0, J)$ and $\delta_\star(\sigma, r, J)$, with the property: if

$$\epsilon_V := \sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{N}} \|V_{knm}\|\, \max\{|k|^r, 1\} < \min\left\{\epsilon_\star(r, \Delta_0, J), \frac{|\Omega_0|}{\delta_\star(\sigma, r, J)}\right\}$$

(here $|\Omega_*|$ stands for the Lebesgue measure of Ω_*) then there exists a measurable subset $\Omega_\infty \subset \Omega_0$ such that

$$|\Omega_\infty| \geq |\Omega_0| - \delta_\star(\sigma, r, J)\, \epsilon_V \tag{3}$$

and the operator $\mathbf{K}+\mathbf{V}$ has a pure point spectrum for all $\omega \in \Omega_\infty$

Remarks. 1) In the course of the proof we shall show even more. Namely, for all $\omega \in \Omega_\infty$ and any eigenvalue of $\mathbf{K}+\mathbf{V}$ the corresponding eigen-projector P belongs to the Banach algebra with the norm

$$\|P\| = \sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{N}} \|P_{knm}\| \, \max\{|k|^{r-\sigma-\frac{1}{2}}, 1\}.$$

This shows that P is $(r-\sigma-1/2)$-differentiable as a map from $[\,0,T\,]$ to the space of bounded operators in $\mathcal{H}$

2) The constants $\epsilon_\star(r,\Delta_0,J)$ and $\delta_\star(\sigma,r,J)$ are in fact known quite explicitly and are given by formulae (70), (71), (77) and (78). Setting $\alpha=2$ and $q^r=e^2$ in these formulae (this is a possible choice) we get

$$\epsilon_\star(r,\Delta_0,J) = \min\left\{\frac{2}{135\,e^3}\,\Delta_0, \frac{1}{270\,e^3}\,J\right\},$$

and

$$\begin{aligned}
\delta_\star(\sigma,r,J) &= 1440\,e^5 2^\sigma \left(\frac{2\sigma+1}{\left(1-e^{-\frac{2}{r}}\right)e}\right)^{\sigma+\frac{1}{2}} \left(\sum_{s=1}^\infty s^2 e^{-\frac{2}{r}(r-\sigma-\frac{1}{2})s}\right) \Delta_\sigma(J) \\
&= 1440 \left(\frac{2\sigma+1}{\left(1-e^{-\frac{2}{r}}\right)e}\right)^{\sigma+\frac{1}{2}} 2^\sigma e^{3+\frac{2}{r}\left(\sigma+\frac{1}{2}\right)} \frac{1+e^{-2+\frac{2}{r}\left(\sigma+\frac{1}{2}\right)}}{\left(1-e^{-2+\frac{2}{r}\left(\sigma+\frac{1}{2}\right)}\right)^3} \Delta_\sigma(J)
\end{aligned}$$

3) The formulae for $\epsilon_\star$ and $\delta_\star$ can be further simplified if we assume that r is not too big, more precisely under the assumption that $r \le \frac{7}{8}\,(2\sigma+1)$ (if this is not the case we can always replace r by a smaller value but still requiring that $r > \sigma+\frac{1}{2}$). A better choice than that made in the previous remark is $\alpha=2$ and $q=e^{4/(2\sigma+1)}$. We get (c.f. (71))

$$\epsilon_\star(r,\Delta_0,J) = \frac{\min\{4\,\Delta_0, J\}}{270\,e}\, e^{-4r/(2\sigma+1)} \ge \frac{\min\{4\,\Delta_0, J\}}{270\,e^{9/2}}$$

and (c.f. (77) and (78))

$$\delta_\star(\sigma,r,J) = 1440\,e\,2^\sigma \left(\frac{2\sigma+1}{\left(1-e^{-\frac{4}{2\sigma+1}}\right)e}\right)^{\sigma+\frac{1}{2}} e^{\frac{8r}{2\sigma+1}} \left(\sum_{s=1}^\infty s^2 e^{-2\frac{2r-2\sigma-1}{2\sigma+1}s}\right) \Delta_\sigma(J)\,.$$

Using the estimate

$$\sum_{s=1}^\infty s^2 e^{-2xs} = \frac{\cosh(x)}{4\,\sinh(x)^3} \le \frac{1}{4\,x^3}$$

we finally obtain

$$\delta_\star(\sigma,r,J) \;\le\; 45\,e\,2^\sigma \left(\frac{2\sigma+1}{\left(1-e^{-\frac{4}{2\sigma+1}}\right)e}\right)^{\sigma+\frac{1}{2}} e^{\frac{8r}{2\sigma+1}} \left(\frac{2\sigma+1}{r-\sigma-\frac{1}{2}}\right)^3 \Delta_\sigma(J)\,.$$

We conclude this section with a brief description of two models illustrating the effectiveness of Theorem 1. In the first model we set $\mathcal{H} = L^2([0,1], \mathrm{dx})$, $H = -\partial_x^2$ with Dirichlet boundary conditions, and $V(t) = z(t)x^2$ where $z(t)$ is a sufficiently regular 2π-periodic function. As shown in [18] the spectral analysis of this simple model is essentially equivalent to the analysis of the so called quantum Fermi accelerator. The particularity of the latter model is that the underlying Hilbert space itself is time-dependent, $\mathcal{H}_t = L^2([0, a(t)], \mathrm{dx})$ where $a(t)$ is a strictly positive periodic function. The time-dependent Hamiltonian is $-\partial_x^2$ with Dirichlet boundary conditions. Using a convenient transformation one can pass from the Fermi accelerator to the former model getting the function $z(t)$ expressed in terms of $a(t)$, $a'(t)$ and $a''(t)$. But let us return to the analysis of our model. Eigenvalues of H are non-degenerate, $h_m = m^2\pi^2$ for $m \in \mathbb{N}$, with normalised eigenfunctions equal to $\sqrt{2}\sin(m\pi x)$. Note that in the notation we are using in the present paper $0 \notin \mathbb{N}$. A straightforward calculation gives

$$V_{knm} = z_k \times \begin{cases} \frac{8(-1)^{m+n}mn}{(m^2-n^2)^2\pi^2} & \text{if } m \neq n, \\ \\ \frac{1}{3} - \frac{1}{2m^2\pi^2} & \text{if } m = n, \end{cases}$$

where $z_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt} z(t)\, dt$ is the Fourier coefficient of $z(t)$. Hence one derives that

$$\epsilon_V = \sup_{n\in\mathbb{N}} \left(\frac{1}{3} + \frac{2}{n^2\pi^2} + \frac{4}{\pi^2}\sum_{j=1}^{n-1}\frac{1}{j^2} \right) \sum_{k\in\mathbb{Z}} |z_k| \max\{|k|^r, 1\} = \sum_{k\in\mathbb{Z}} |z_k| \max\{|k|^r, 1\}.$$

For any $J > 0$, $\Delta_\sigma(J)$ is finite if and only if $\sigma > 1$. On the other hand, to have ϵ_V finite it is sufficient that $z(t) \in C^s$ where $s > r + 1 > \sigma + \frac{1}{2} + 1 > \frac{5}{2}$. So $z(t) \in C^3$ suffices for the theory to be applicable. This may be compared to an older result in [9], §4.2, giving a much worse condition, namely $z(t) \in C^{17}$.

The second model is the pulsed rotator in N dimensions. In this case $\mathcal{H} = L^2(S^N, d\mu)$, with $S^N \subset \mathbb{R}^{N+1}$ being the N-dimensional unit sphere with the standard (rotationally invariant) Riemann metric and the induced normalised measure $d\mu$, and $H = -\Delta_{LB}$ is the Laplace-Beltrami operator on S^N. The spectrum of H is well known, $\mathrm{Spec}(H) = \{h_m\}_{m=0}^\infty$, where

$$h_m = m(m + N - 1)$$

and the multiplicities are

$$M_m = \binom{m+N}{N} - \binom{m+N-2}{N}.$$

The time-dependent operator $V(t)$ in $\mathcal{H}$ acts via multiplication, $(V(t)\varphi)(x) = v(t,x)\varphi(x)$, where $v(t,x)$ is a real measurable bounded function on $\mathbb{R} \times S^N$ which is 2π-periodic in the variable t. Consequently, $\mathcal{K} \cong L^2([0,T] \times S^N, dt\, d\mu)$ and $(\mathbf{V}\psi)(t,x) = v(\omega t, x)\psi(t,x)$. Note that the asymptotic behaviour of the eigenvalues and the multiplicities, as $m \to \infty$, is $h_m \sim m^2$, $M_m \sim (2/(N-1)!)\, m^{N-1}$. So $\Delta_\sigma(J)$ is finite, for any $J > 0$, if and only if

$$\sum_{m^2-n^2>J/2} \frac{n^{\frac{3}{2}(N-1)} m^{N-1}}{(m^2-n^2)^\sigma} < \infty\,.$$

To ensure this condition we require that $\sigma > \frac{5}{2}(N-1)+1$. Let us assume that there exist $s, u \in \mathbb{Z}_+$ such that, for any system of local (smooth) coordinates $(y_1, \dots, y_N)$ on S^N, the derivatives $\partial_t^\alpha \partial_{y_1}^{\beta_1} \dots \partial_{y_N}^{\beta_N} v(t, y_1, \dots, y_N)$ exist and are continuous for all α, β, $\alpha \leq s$ and $\beta_1 + \ldots + \beta_N \leq u$. If $u \geq 4$ then $[\,H, [\,H, V(t)\,]]$ is a well defined second order differential operator with continuous coefficient functions and the operator $[\,H, [\,H, V(t)\,]](1+H)^{-1}$ is bounded. Clearly,

$$\frac{(h_m-h_n)^2}{1+h_m}\, Q_n V(t) Q_m = Q_n[\,H, [\,H, V(t)\,]](1+H)^{-1} Q_m.$$

Using this relation one derives an estimate on V_{knm},

$$\|V_{knm}\| \leq \text{const}\, \frac{1+\min\{h_n, h_m\}}{|k|^s (h_m-h_n)^2}\,,$$

valid for $k \neq 0$ and $m \neq n$. The number

$$\sup_{n\in\mathbb{Z}_+} \sum_{m\in\mathbb{Z}_+,\, m\neq n} \frac{1+\min\{h_n, h_m\}}{(h_m-h_n)^2}$$

is finite. To see it one can employ the asymptotics of h_m and the fact that the sequence

$$a_n = \sum_{m\in\mathbb{Z}_+,\, m\neq n} \frac{1+\min\{n^2, m^2\}}{(m^2-n^2)^2} = \left(1+\frac{1}{n^2}\right)\frac{\pi^2}{12} - \frac{3}{16n^2} + \frac{5}{16n^4} - \frac{1}{2n}\sum_{m=1}^{2n-1}\frac{1}{m}\,,$$

$n = 1, 2, 3, \ldots$, is bounded. It follows that the norm ϵ_V is finite if $s > r+1 > \sigma + \frac{1}{2} + 1 > \frac{5}{2}(N-1) + 1 + \frac{3}{2} = \frac{5}{2}\,N$. Thus the theory is applicable provided $u \geq 4$ and $s > \frac{5}{2}N$. The same example has also been treated by adiabatic methods in [11]. In that case the assumptions are weaker. It suffices that $v(t,x)$ be $(N+1)$-times differentiable in t with all derivatives $\partial_t^\alpha v(t,x)$, $0 \leq \alpha \leq N+1$, uniformly bounded. However the conclusion is somewhat weaker as well. Under this assumption $\mathbf{K}+\mathbf{V}$ has no absolutely continuous spectrum but nothing is claimed about the singular continuous spectrum.

3. Formal limit procedure

Suppose there is given a directed sequence of real or complex Banach spaces, $\{\mathfrak{X}_s\}_{s=0}^\infty$, with linear mappings

$$\iota_{us} : \mathfrak{X}_s \to \mathfrak{X}_u \quad \text{if } s \leq u, \text{ with } \|\iota_{us}\| \leq 1,$$

(and ι_{ss} is the unite mapping in $\mathfrak{X}_s$) and such that

$$\iota_{vu}\iota_{us} = \iota_{vs} \quad \text{if } s \leq u \leq v\,.$$

To simplify the notation we set in what follows

$$\iota_s = \iota_{s+1,s}.$$

Denote by $\mathfrak{X}_\infty$ the norm inductive limit of $\{\mathfrak{X}_s,\ \iota_{us}\}$ in the sense of [19], §1.3.4 or [20], §1.23 (the algebraic inductive limit is endowed with a seminorm induced by $\limsup_s \|\cdot\|_s$, the kernel of this seminorm is divided out and the result is completed). $\mathfrak{X}_\infty$ is related to the original directed sequence via the mappings $\iota_{\infty s} : \mathfrak{X}_s \to \mathfrak{X}_\infty$ obeying $\|\iota_{\infty s}\| \leq 1$ and $\iota_{\infty u}\iota_{us} = \iota_{\infty s}$ if $s \leq u$. By the construction, the union $\bigcup_{s \geq s_0} \iota_{\infty s}(\mathfrak{X}_s)$ is dense in $\mathfrak{X}_\infty$ for any $s_0 \in \mathbb{Z}_+$.

If $\{A_s \in \mathcal{B}(\mathfrak{X}_s)\}$ is a family of bounded operators, defined for $s \geq s_0$ and such that

$$A_u\iota_{us} = \iota_{us}A_s \quad \text{if } s_0 \leq s \leq u, \text{ and } \sup_s \|A_s\| < \infty,$$

then $A_\infty \in \mathcal{B}(\mathfrak{X}_\infty)$ designates the inductive limit of this family characterised by the property $A_\infty\iota_{\infty s} = \iota_{\infty s}A_s$, $\forall s \geq s_0$.

Let $\mathcal{D}_\infty \in \mathcal{B}(\mathfrak{X}_\infty)$ be the inductive limit of a family of bounded operators $\{\mathcal{D}_s \in \mathcal{B}(\mathfrak{X}_s);\ s \geq 0\}$, with the property

$$\|\mathcal{D}_s\| \leq 1,\ \|1 - \mathcal{D}_s\| \leq 1, \quad \forall s. \tag{4}$$

We also suppose that there is given a sequence of one-dimensional spaces $\mathbf{k}K_s$, $s = 0, 1, \ldots, \infty$, where the K_s are distinguished basis elements. Here the field $\mathbf{k}$ is either $\mathbb{C}$ or $\mathbb{R}$ depending on whether the Banach spaces $\mathfrak{X}_s$ are complex or real. Set

$$\tilde{\mathfrak{X}}_s = \mathbf{k}K_s \oplus \mathfrak{X}_s, \quad s = 0, 1, \ldots, \infty.$$

Then $\{\tilde{\mathfrak{X}}_s\}_{s=0}^\infty$ becomes a directed sequence of vector spaces provided one defines $\tilde{\iota}_{us} : \tilde{\mathfrak{X}}_s \to \tilde{\mathfrak{X}}_u$ by

$$\tilde{\iota}_{us}|_{\mathfrak{X}_s} = \iota_{us} \text{ and } \tilde{\iota}_{us}(K_s) = K_u \quad \text{if } s \leq u.$$

Set

$$\phi(x) = \frac{1}{x}\left(e^x - \frac{e^x - 1}{x}\right) = \sum_{k=0}^\infty \frac{k+1}{(k+2)!}\, x^k\,. \tag{5}$$

Proposition 2. *Suppose that, in addition to the sequences $\{\mathfrak{X}_s\}_{s=0}^\infty$, $\{K_s\}_{s=0}^\infty$ and $\{\mathcal{D}_s\}_{s=0}^\infty$, there are given sequences $\{V_s\}_{s=0}^\infty$ and $\{\Theta_u^s\}_{u=s+1}^\infty$ such that $V_s \in \mathfrak{X}_s$, $\Theta_u^s \in \mathcal{B}(\mathfrak{X}_u)$, and*

$$\Theta_v^s\iota_{vu} = \iota_{vu}\Theta_u^s \quad \textit{if } s < u \leq v. \tag{6}$$

Set

$$T_s = e^{\Theta_s^{s-1}}e^{\Theta_s^{s-2}}\ldots e^{\Theta_s^0} \in \mathcal{B}(\mathfrak{X}_s) \quad \textit{for } s \geq 1\,. \tag{7}$$

Let $\{W_s\}_{s=0}^\infty$ be another sequence, with $W_s \in \mathfrak{X}_s$, defined recursively:

$$\begin{aligned} W_0 &= V_0, \\ W_{s+1} &= \iota_s(W_s) + T_{s+1}(V_{s+1} - \iota_s(V_s)) \\ &\quad + \Theta_{s+1}^s\phi(\Theta_{s+1}^s)\iota_s(1 - \mathcal{D}_s)(W_s - \iota_{s-1}(W_{s-1})), \end{aligned} \tag{8}$$

where we set, by convention, $\mathfrak{X}_{-1} = 0$, $W_{-1} = 0$. *Extend the mappings* Θ_u^s *to* $\tilde{\Theta}_u^s : \tilde{\mathfrak{X}}_u \to \tilde{\mathfrak{X}}_u$ *by*

$$\tilde{\Theta}_u^s(K_u) = -\Theta_u^s \mathcal{D}_u(\iota_{us}(W_s)) - (1-\mathcal{D}_u)(\iota_{us}(W_s) - \iota_{u,s-1}(W_{s-1})), \tag{9}$$

and consequently the mappings T_s *to* $\tilde{T}_s : \tilde{\mathfrak{X}}_s \to \tilde{\mathfrak{X}}_s$,

$$\tilde{T}_s = e^{\tilde{\Theta}_s^{s-1}} e^{\tilde{\Theta}_s^{s-2}} \dots e^{\tilde{\Theta}_s^0} \text{ for } s \geq 1,\ \tilde{T}_0 = 1.$$

Then it holds

$$\tilde{T}_s(K_s + V_s) = K_s + \mathcal{D}_s(W_s) + (1-\mathcal{D}_s)(W_s - \iota_{s-1}(W_{s-1})), \quad s = 0,1,2,\dots. \tag{10}$$

Remark. Since $\tilde{\Theta}_u^s(K_u) \in \mathfrak{X}_u$ it is easy to observe that

$$\tilde{T}_s(K_s) - K_s \in \mathfrak{X}_s.$$

Furthermore, note that (9) implies that $\tilde{\Theta}_v^s(K_v) = \iota_{vu}\tilde{\Theta}_u^s(K_u)$ if $0 \leq s < u \leq v$, and so the mappings $\tilde{\Theta}_u^s$ still satisfy

$$\tilde{\Theta}_v^s \tilde{\iota}_{vu} = \tilde{\iota}_{vu} \tilde{\Theta}_u^s \quad \text{if } s < u \leq v.$$

Proof. By induction in s. For $s = 0$ the claim is obvious. In the induction step $s \to s+1$ one may use the induction hypothesis and relations (9) and (8):

$$\begin{aligned}
\tilde{T}_{s+1}(K_{s+1} + V_{s+1}) &= \tilde{T}_{s+1}\tilde{\iota}_s(K_s + V_s) + T_{s+1}(V_{s+1} - \iota_s(V_s)) \\
&= e^{\tilde{\Theta}_{s+1}^s}\tilde{\iota}_s \tilde{T}_s(K_s + V_s) + T_{s+1}(V_{s+1} - \iota_s(V_s)) \\
&= e^{\tilde{\Theta}_{s+1}^s}\tilde{\iota}_s(K_s + \mathcal{D}_s(W_s) + (1-\mathcal{D}_s)(W_s - \iota_{s-1}(W_{s-1}))) \\
&\quad + T_{s+1}(V_{s+1} - \iota_s(V_s)) \\
&= K_{s+1} + \mathcal{D}_{s+1}(\iota_s(W_s)) + \frac{e^{\Theta_{s+1}^s} - 1}{\Theta_{s+1}^s}\tilde{\Theta}_{s+1}^s \tilde{\iota}_s(K_s + \mathcal{D}_s(W_s)) \\
&\quad + e^{\Theta_{s+1}^s}\iota_s(1-\mathcal{D}_s)(W_s - \iota_{s-1}(W_{s-1})) + T_{s+1}(V_{s+1} - \iota_s(V_s)) \\
&= K_{s+1} - (1-\mathcal{D}_{s+1})\iota_s(W_s) + \iota_s(W_s) + T_{s+1}(V_{s+1} - \iota_s(V_s)) \\
&\quad + \left(e^{\Theta_{s+1}^s} - \frac{e^{\Theta_{s+1}^s} - 1}{\Theta_{s+1}^s}\right)\iota_s(1-\mathcal{D}_s)(W_s - \iota_{s-1}(W_{s-1})) \\
&= K_{s+1} - (1-\mathcal{D}_{s+1})\iota_s(W_s) + W_{s+1} \\
&= K_{s+1} + \mathcal{D}_{s+1}(W_{s+1}) + (1-\mathcal{D}_{s+1})(W_{s+1} - \iota_s(W_s))\,.
\end{aligned}$$

□

Proposition 3. *Assume that the sequences* $\{V_s\}_{s=0}^\infty$, $\{W_s\}_{s=0}^\infty$ *and* $\{\Theta_u^s\}_{u=s}^\infty$ *have the same meaning and obey the same assumptions as in Proposition 2. Denote*

$$w_s = \|W_s - \iota_{s-1}(W_{s-1})\|$$

(with $w_0 = \|W_0\|$*). Assume, in addition, that there exist a sequence of positive real numbers,* $\{F_s\}_{s=0}^\infty$, *such that*

$$\|\Theta_u^s\| \leq F_s w_s, \quad \forall s, u,\ u > s, \tag{11}$$

a sequence of non-negative real numbers $\{v_s\}_{s=0}^{\infty}$ *such that*

$$\|V_s - \iota_{s-1}(V_{s-1})\| \leq v_s, \ \forall s,$$

(for $s = 0$ *this means* $\|V_0\| \leq v_0$*) and a constant* $A \geq 0$ *such that*

$$F_s v_s^2 \leq A\, v_{s+1}, \ \forall s, \tag{12}$$

and that it holds true

$$B = \sum_{s=0}^{\infty} F_s v_s < \infty. \tag{13}$$

Denote

$$C = \sup_s F_s v_s. \tag{14}$$

If $d > 0$ *obeys*

$$e^{dB} + A\phi(dC)\, d^2 \leq d \tag{15}$$

then

$$w_s \leq d\, v_s, \ \forall s. \tag{16}$$

Proof. We shall proceed by induction in s. If $s = 0$ then $v_0 = w_0 = \|V_0\|$ and (16) holds true since (15) implies that $d \geq 1$. The induction step $s \to s+1$: according to (8), (7), (4) and (15), and owing to the fact that $\phi(x)$ is monotone, we have

$$\begin{aligned}
w_{s+1} &\leq \|T_{s+1}\|\, v_{s+1} + \|\Theta_{s+1}^s\|\, \phi(\|\Theta_{s+1}^s\|)\, w_s \\
&\leq \exp\left(\sum_{j=0}^{s} F_j w_j\right) v_{s+1} + \phi(F_s w_s) F_s w_s^2 \\
&\leq \exp\left(d \sum_{j=0}^{s} F_j v_j\right) v_{s+1} + \phi(d\, F_s v_s) F_s d^2 v_s^2 \\
&\leq e^{dB} v_{s+1} + \phi(dC) d^2 A\, v_{s+1} \\
&\leq d\, v_{s+1}.
\end{aligned}$$

□

Remark. If

$$B \leq \frac{1}{3}\ln 2 \quad \text{and} \quad A\,\phi(3C) \leq \frac{1}{9}$$

then (15) holds true with $d = 3$.

Recall that $\Theta_\infty^s \in \mathcal{B}(\mathfrak{X}_\infty)$ is the unique bounded operator on $\mathfrak{X}_\infty$ such that

$$\Theta_\infty^s \iota_{\infty u} = \iota_{\infty u} \Theta_u^s, \ \forall u > s.$$

If (11) is true then its norm is estimated by

$$\|\Theta_\infty^s\| \leq F_s w_s. \tag{17}$$

Corollary 4. *Under the same assumptions as in Proposition 3, if $d>0$ exists such that condition (15) is satisfied, and*

$$F_{\inf} = \inf_s F_s > 0 \tag{18}$$

then the limits

$$V_\infty = \lim_{s\to\infty} \iota_{\infty s}(V_s), \quad W_\infty = \lim_{s\to\infty} \iota_{\infty s}(W_s)$$

exist in $\mathfrak{X}_\infty$, the limit

$$T_\infty = \lim_{s\to\infty} e^{\Theta_\infty^{s-1}} \ldots e^{\Theta_\infty^0}$$

exists in $\mathcal{B}(\mathfrak{X}_\infty)$, and $T_\infty \in \mathcal{B}(\mathfrak{X}_\infty)$ can be extended to a linear mapping $\tilde{T}_\infty : \tilde{\mathfrak{X}}_\infty \to \tilde{\mathfrak{X}}_\infty$ by

$$\tilde{T}_\infty(K_\infty) - K_\infty = \lim_{s\to\infty} \iota_{\infty s}\left(\tilde{T}_s(K_s) - K_s\right), \tag{19}$$

with the limit existing in $\mathfrak{X}_\infty$. These objects obey the equality

$$\tilde{T}_\infty(K_\infty + V_\infty) = K_\infty + \mathcal{D}_\infty(W_\infty). \tag{20}$$

Proof. If $u \ge s$ then

$$\|\iota_{\infty u}(V_u) - \iota_{\infty s}(V_s)\| \;=\; \Big\| \sum_{j=s+1}^{u} \iota_{\infty j}(V_j - \iota_{j-1}(V_{j-1}))\Big\| \le \sum_{j=s+1}^{u} v_j.$$

Since

$$\sum_{s=0}^{\infty} v_s \le \frac{1}{F_{\inf}} \sum_{s=0}^{\infty} F_s v_s < \infty$$

the sequence $\{\iota_{\infty s}(V_s)\}$ is Cauchy in $\mathfrak{X}_\infty$ and so $V_\infty \in \mathfrak{X}_\infty$ exists. Under assumption (16) we can apply the same reasoning to the sequence $\{\iota_{\infty s}(W_s)\}$ to conclude that the limit $W_\infty = \lim_{s\to\infty} \iota_{\infty s}(W_s)$ exists in $\mathfrak{X}_\infty$. Set

$$\bar{T}_s = e^{\Theta_\infty^{s-1}} \ldots e^{\Theta_\infty^0} \quad \text{if } s \ge 1, \text{ and } \bar{T}_0 = 1.$$

If $u \ge s$ then, owing to (17) and (16), we have

$$\begin{aligned}
\|\bar{T}_u - \bar{T}_s\| \;&\le\; \left(\exp\left(\sum_{j=s}^{u-1} \|\Theta_\infty^j\|\right) - 1\right) \exp\left(\sum_{j=0}^{s-1} \|\Theta_\infty^j\|\right) \\
&\le\; \exp\left(d \sum_{j=0}^{u-1} F_j v_j\right) - \exp\left(d \sum_{j=0}^{s-1} F_j v_j\right).
\end{aligned}$$

Assumption (13) implies that $\{\bar{T}_s\}$ is a Cauchy sequence in $\mathcal{B}(\mathfrak{X}_\infty)$ and so $T_\infty \in \mathcal{B}(\mathfrak{X}_\infty)$ exists.

To show (19) let us first verify the inequality

$$\|e^{\tilde{\Theta}_u^s}(K_u) - K_u\| \le \frac{1+dB}{F_{\inf}} \left(e^{F_s w_s} - 1\right), \tag{21}$$

valid for all $u > s$. Actually, using definition (9) and assumption (11), we get

$$
\begin{aligned}
\|e^{\tilde{\Theta}_u^s}(K_u) - K_u\| &\leq \frac{e^{\|\Theta_u^s\|} - 1}{\|\Theta_u^s\|}\|\tilde{\Theta}_u^s(K_u)\| \\
&\leq \frac{e^{\|\Theta_u^s\|} - 1}{\|\Theta_u^s\|}\left(\|\Theta_u^s\|\|W_s\| + \|W_s - \iota_{s-1}(W_{s-1})\|\right) \\
&\leq \left(e^{F_s w_s} - 1\right)\left(\|W_s\| + \frac{1}{F_s}\right).
\end{aligned}
$$

To finish the estimate note that (13) and (16) imply

$$
\|W_s\| = \sum_{j=1}^{s}(\|W_j\| - \|W_{j-1}\|) + \|W_0\| \leq \sum_{j=0}^{\infty} dv_j \leq \frac{d}{F_{\inf}}\sum_{j=0}^{\infty} F_j v_j = \frac{dB}{F_{\inf}}.
$$

With the aid of an elementary identity,

$$
a_j \ldots a_0 - 1 = a_j \ldots a_1(a_0 - 1) + a_j \ldots a_2(a_1 - 1) + \cdots + (a_j - 1),
$$

we can derive from (21): if $0 \leq s \leq t < u$ then

$$
\begin{aligned}
\|e^{\tilde{\Theta}_u^t} \ldots e^{\tilde{\Theta}_u^s}(K_u) - K_u\| &\leq e^{\|\Theta_u^t\| + \cdots + \|\Theta_u^{s+1}\|}\|e^{\tilde{\Theta}_u^s}(K_u) - K_u\| \\
&\quad + e^{\|\Theta_u^t\| + \cdots + \|\Theta_u^{s+2}\|}\|e^{\tilde{\Theta}_u^{s+1}}(K_u) - K_u\| \\
&\quad + \cdots + \|e^{\tilde{\Theta}_u^t}(K_u) - K_u\| \\
&\leq \frac{1 + dB}{F_{\inf}}\left(e^{F_t w_t + \cdots + F_{s+1} w_{s+1}}\left(e^{F_s w_s} - 1\right)\right. \\
&\quad + e^{F_t w_t + \cdots + F_{s+2} w_{s+2}}\left(e^{F_{s+1} w_{s+1}} - 1\right) \\
&\quad \left. + \cdots + \left(e^{F_t w_t} - 1\right)\right) \\
&= \frac{1 + dB}{F_{\inf}}\left(e^{F_t w_t + \cdots + F_s w_s} - 1\right).
\end{aligned}
$$

Set temporarily in this proof

$$
\tau_s = \iota_{\infty s}(\tilde{T}_s(K_s) - K_s) \in \mathfrak{X}_\infty.
$$

If $t \geq s$ then

$$
\begin{aligned}
\tau_t - \tau_s &= \iota_{\infty t}\left(e^{\tilde{\Theta}_t^{t-1}} \ldots e^{\tilde{\Theta}_t^0}(K_t) - \iota_{ts} e^{\tilde{\Theta}_s^{s-1}} \ldots e^{\tilde{\Theta}_s^0}(K_s)\right) \\
&= \iota_{\infty t}\left(e^{\tilde{\Theta}_t^{t-1}} \ldots e^{\tilde{\Theta}_t^0}(K_t) - e^{\tilde{\Theta}_t^{s-1}} \ldots e^{\tilde{\Theta}_t^0}(K_t)\right) \\
&= \iota_{\infty t}\left(\left(e^{\Theta_t^{t-1}} \ldots e^{\Theta_t^s} - 1\right)\left(e^{\tilde{\Theta}_t^{s-1}} \ldots e^{\tilde{\Theta}_t^0}(K_t) - K_t\right)\right. \\
&\quad \left. + e^{\tilde{\Theta}_t^{t-1}} \ldots e^{\tilde{\Theta}_t^s}(K_t) - K_t\right).
\end{aligned}
$$

Hence

$$\begin{aligned}\|\tau_t - \tau_s\| &\leq \frac{1+dB}{F_{\inf}}\Big(\big(e^{F_{t-1}w_{t-1}+\cdots+F_s w_s}-1\big)\big(e^{F_{s-1}w_{s-1}+\cdots+F_0 w_0}-1\big)\\ &\quad + e^{F_{t-1}w_{t-1}+\cdots+F_s w_s}-1\Big)\\ &= \frac{1+dB}{F_{\inf}}\big(e^{F_{t-1}w_{t-1}+\cdots+F_0 w_0}-e^{F_{s-1}w_{s-1}+\cdots+F_0 w_0}\big).\end{aligned}$$

This shows that the sequence $\{\tau_s\}$ is Cauchy and thus the limit on the RHS of (19) exists.

We conclude that it holds true, in virtue of (10), that

$$\begin{aligned}\tilde{T}_\infty(K_\infty + V_\infty) &= K_\infty + \lim_{s\to\infty}\iota_{\infty s}(\tilde{T}_s(K_s)-K_s) + \lim_{s\to\infty}\bar{T}_s\iota_{\infty s}(V_s)\\ &= K_\infty + \lim_{s\to\infty}\iota_{\infty s}(\tilde{T}_s(K_s+V_s)-K_s)\\ &= K_\infty + \lim_{s\to\infty}\iota_{\infty s}\big(\mathcal{D}_s(W_s)+(1-\mathcal{D}_s)(W_s-\iota_{s-1}(W_{s-1}))\big)\\ &= K_\infty + \lim_{s\to\infty}\big(\mathcal{D}_\infty(\iota_{\infty s}(W_s))+(1-\mathcal{D}_\infty)(\iota_{\infty s}(W_s)-\iota_{\infty,s-1}(W_{s-1}))\big)\\ &= K_\infty + \mathcal{D}_\infty(W_\infty).\end{aligned}$$

So equality (20) has been verified as well. □

4. Convergence in a Hilbert space

Let $\{\mathfrak{X}_s, \iota_{us}\}$ be a directed sequence of real or complex Banach spaces, as introduced in Section 3. In this section it is sufficient to know that $\mathcal{K}$ is a separable complex Hilbert space and $\mathbf{K}$ is a closed (densely defined) operator in $\mathcal{K}$. Suppose that for each $s \in \mathbb{Z}_+$ there is given a bounded linear mapping,

$$\kappa_s : \mathfrak{X}_s \to \mathcal{B}(\mathcal{K}), \text{ with } \|\kappa_s\| \leq 1,$$

and such that

$$\forall s,u,\ 0\leq s\leq u,\quad \kappa_u\iota_{us} = \kappa_s.$$

If the Banach spaces $\mathfrak{X}_s$ are real then the mappings κ_s are supposed to be linear over $\mathbb{R}$ otherwise they are linear over $\mathbb{C}$. Then there exists a unique linear bounded mapping $\kappa_\infty : \mathfrak{X}_\infty \to \mathcal{B}(\mathcal{K})$ satisfying, $\forall s \in \mathbb{Z}_+$, $\kappa_\infty\iota_{\infty s} = \kappa_s$. Clearly, $\|\kappa_\infty\| \leq 1$. Extend the mappings κ_s to $\tilde{\kappa}_s : \tilde{\mathfrak{X}}_s = \mathbf{k}K_s + \mathfrak{X}_s \to \mathbb{C}\mathbf{K} + \mathcal{B}(\mathcal{K})$ by defining

$$\tilde{\kappa}_s(K_s) = \mathbf{K},\ \forall s \in \mathbb{Z}_+ \cup \{\infty\}.$$

So $\tilde{\kappa}_s(K_s + X) = \mathbf{K} + \kappa_s(X)$, with $X \in \mathfrak{X}_s$, is a closed operator in $\mathcal{K}$ with $\mathrm{Dom}(\mathbf{K} + \kappa_s(X)) = \mathrm{Dom}(\mathbf{K})$.

Suppose, in addition, that there exists $\mathbf{D} \in \mathcal{B}(\mathcal{B}(\mathcal{K}))$ such that

$$\forall s \in \mathbb{Z}_+,\quad \mathbf{D}\kappa_s = \kappa_s\mathcal{D}_s.$$

Then it holds true, $\forall s \in \mathbb{Z}_+$, $\forall X \in \mathfrak{X}_s$,

$$\kappa_\infty\mathcal{D}_\infty(\iota_{\infty s}X) = \kappa_\infty\iota_{\infty s}\mathcal{D}_s(X) = \kappa_s\mathcal{D}_s(X) = \mathbf{D}\kappa_s(X) = \mathbf{D}\kappa_\infty(\iota_{\infty s}X).$$

Since the set of vectors $\{\iota_{\infty s}(X);\ s \in \mathbb{Z}_+,\ X \in \mathfrak{X}_s\}$ is dense in $\mathfrak{X}_s$ we get $\kappa_\infty \mathcal{D}_\infty = \mathbf{D}\kappa_\infty$.

Proposition 5. *Under the assumptions of Corollary 4 and those introduced above in this section, let $\{\mathbf{A}_s\}_{s=0}^{\infty}$ be a sequence of bounded operators in $\mathcal{K}$ such that,*

$$\forall s, u,\ 0 \le s < u,\ \forall X \in \mathfrak{X}_u, \quad \kappa_u\big(\Theta_u^s(X)\big) = [\,\mathbf{A}_s, \kappa_u(X)\,], \tag{22}$$

$$\forall s \in \mathbb{Z}_+, \quad \mathbf{A}_s(\operatorname{Dom}\mathbf{K}) \subset \operatorname{Dom}\mathbf{K},$$

and

$$\forall s, u,\ 0 \le s < u, \quad [\,\mathbf{A}_s, \mathbf{K}\,] = \kappa_u(\tilde{\Theta}_u^s(K_u))\big|_{\operatorname{Dom}(\mathbf{K})}.$$

Moreover, assume that

$$\sum_{s=0}^{\infty} \|\mathbf{A}_s\| < \infty. \tag{23}$$

Set

$$\mathbf{V} = \kappa_\infty(V_\infty),\ \ \mathbf{W} = \kappa_\infty(W_\infty).$$

Then the limit

$$\mathbf{U} = \lim_{s\to\infty} e^{\mathbf{A}_{s-1}} \dots e^{\mathbf{A}_0} \tag{24}$$

exists in the operator norm, the element $\mathbf{U} \in \mathcal{B}(\mathcal{K})$ has a bounded inverse, and it holds true that

$$\mathbf{U}(\operatorname{Dom}\mathbf{K}) = \operatorname{Dom}\mathbf{K}$$

and

$$\mathbf{U}(\mathbf{K} + \mathbf{V})\mathbf{U}^{-1} = \mathbf{K} + \mathbf{D}(\mathbf{W}). \tag{25}$$

For the proof we shall need a lemma.

Lemma 6. *Assume that $\mathcal{H}$ is a Hilbert space, K is a closed operator in $\mathcal{H}$, $A, B \in \mathcal{B}(\mathcal{H})$,*

$$A(\operatorname{Dom}K) \subset \operatorname{Dom}K,$$

and

$$[\,A, K\,] = B\big|_{\operatorname{Dom}(K)}.$$

Then it holds, $\forall \lambda \in \mathbb{C}$,

$$e^{\lambda A}(\operatorname{Dom}K) = \operatorname{Dom}K \tag{26}$$

and

$$e^{-\lambda A} K e^{\lambda A} = K + \frac{e^{-\lambda\,\mathrm{ad}_A} - 1}{\mathrm{ad}_A}\, B.$$

Remark. Here and everywhere in what follows we use the standard notation: $\mathrm{ad}_A B = [\,A, B\,]$ and so $e^{\lambda\,\mathrm{ad}_A} B = e^{\lambda A} B\, e^{-\lambda A}$.

Proof. Choose an arbitrary vector $v \in \mathrm{Dom}(K)$ and set

$$\forall n \in \mathbb{Z}_+, \quad v_n = \sum_{k=0}^{n} \frac{\lambda^k}{k!} A^k v \,.$$

Then $v_n \in \mathrm{Dom}(K)$ and $v_n \to e^{\lambda A} v$ as $n \to \infty$. On the other hand,

$$\begin{aligned} K v_n &= \sum_{k=0}^{n} \frac{\lambda^k}{k!} (K\, A^k - A^k K) v + \sum_{k=0}^{n} \frac{\lambda^k}{k!} A^k K v \\ &= -\sum_{k=1}^{n} \frac{\lambda^k}{k!} \sum_{j=0}^{k-1} A^j B\, A^{k-1-j} v + \sum_{k=0}^{n} \frac{\lambda^k}{k!} A^k K v \,. \end{aligned}$$

So the limit $\lim_{n\to\infty} K v_n$ exists. Consequently, since K is closed, $e^{\lambda A}(\mathrm{Dom}\, K) \subset \mathrm{Dom}\, K$. But $(e^{\lambda A})^{-1} = e^{-\lambda A}$ has the same property and thus equality (26) follows. Furthermore, the above computation also shows that

$$K\, e^{\lambda A} = -\sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \sum_{j=0}^{k-1} A^j B\, A^{k-1-j} + e^{\lambda A} K \,.$$

Application of the following algebraic identity (easy to verify),

$$\sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \sum_{j=0}^{k-1} A^j B\, A^{k-1-j} = e^{\lambda A} \left(\frac{1 - e^{-\lambda\, \mathrm{ad}_A}}{\mathrm{ad}_A}\, B \right),$$

concludes the proof. □

Proof of Proposition 5. We use notation of Corollary 4. From (22) follows that, $\forall s, u,\ 0 \le s < u, \forall X \in \mathfrak{X}_u$,

$$\kappa_\infty \Theta^s_\infty(\iota_{\infty u} X) = \kappa_u \Theta^s_u(X) = [\, \mathbf{A}_s, \kappa_u(X)\,] = [\, \mathbf{A}_s, \kappa_\infty(\iota_{\infty u} X)\,].$$

Since the set of vectors $\{\iota_{\infty u}(X);\ s < u,\ X \in \mathfrak{X}_u\}$ is dense in $\mathfrak{X}_\infty$, we get, $\forall X \in \mathfrak{X}_\infty$, $\kappa_\infty \Theta^s_\infty(X) = [\, \mathbf{A}_s, \kappa_\infty(X)\,]$, and hence

$$\kappa_\infty \left(e^{\Theta^s_\infty}(X) \right) = e^{\mathbf{A}_s} \kappa_\infty(X)\, e^{-\mathbf{A}_s}.$$

Set

$$\mathbf{U}_s = e^{\mathbf{A}_{s-1}} \ldots e^{\mathbf{A}_0} \text{ for } s \ge 1,\ \mathbf{U}_0 = 1.$$

Assumption (23) implies that both sequences $\{\mathbf{U}_s\}$ and $\{\mathbf{U}_s^{-1}\}$ are Cauchy in $\mathcal{B}(\mathcal{K})$ and hence the limit (24) exists in the operator norm, with $\mathbf{U}^{-1} = \lim_{s\to\infty} \mathbf{U}_s^{-1} \in \mathcal{B}(\mathcal{K})$. Moreover, $\forall X \in \mathfrak{X}_\infty$,

$$\kappa_\infty T_\infty(X) = \kappa_\infty \left(\lim_{s\to\infty} e^{\Theta^{s-1}_\infty} \ldots e^{\Theta^0_\infty} X \right) = \lim_{s\to\infty} \mathbf{U}_s \kappa_\infty(X)\, \mathbf{U}_s^{-1}. \tag{27}$$

Next let us compute $\tilde{\kappa}_s \tilde{T}_s(K_s)$. For $0 \le s < u$, set $\mathbf{B}_s = \kappa_u(\tilde{\Theta}^s_u(K_u)) \in \mathcal{B}(\mathcal{K})$. $\mathbf{B}_s$ doesn't depend on $u > s$ since if $0 \le s < u \le v$ then

$$\kappa_u\big(\tilde{\Theta}^s_u(K_u)\big) = \kappa_v\big(\iota_{vu} \tilde{\Theta}^s_u(K_u)\big) = \kappa_v\big(\tilde{\Theta}^s_v(K_v)\big).$$

We can apply Lemma 6 to the operators $\mathbf{K}$, $\mathbf{A}_s$, $\mathbf{B}_s$ to conclude that $e^{-\mathbf{A}_s}(\mathrm{Dom}\,\mathbf{K}) = \mathrm{Dom}\,\mathbf{K}$ and

$$e^{\mathbf{A}_s}\mathbf{K}\,e^{-\mathbf{A}_s} = \mathbf{K} + \frac{e^{\mathrm{ad}_{\mathbf{A}_s}} - 1}{\mathrm{ad}_{\mathbf{A}_s}}\,\mathbf{B}_s. \tag{28}$$

On the other hand,

$$\tilde{\kappa}_u\left(e^{\tilde{\Theta}_u^s}(K_u)\right) = \tilde{\kappa}_u\left(K_u + \frac{e^{\Theta_u^s} - 1}{\Theta_u^s}\tilde{\Theta}_u^s(K_u)\right) = \mathbf{K} + \frac{e^{\mathrm{ad}_{\mathbf{A}_s}} - 1}{\mathrm{ad}_{\mathbf{A}_s}}\,\mathbf{B}_s.$$

Thus $\tilde{\kappa}_u\left(e^{\tilde{\Theta}_u^s}(K_u)\right) = e^{\mathbf{A}_s}\mathbf{K}\,e^{-\mathbf{A}_s}$. Consequently, $\mathbf{U}_s(\mathrm{Dom}\,\mathbf{K}) = \mathrm{Dom}\,\mathbf{K}$ and

$$\tilde{\kappa}_s\tilde{T}_s(K_s) = \mathbf{U}_s\mathbf{K}\,\mathbf{U}_s^{-1}. \tag{29}$$

Set $\mathbf{C}_s = \mathbf{U}_s\mathbf{K}\mathbf{U}_s^{-1} - \mathbf{K}$. According to (28), $\mathbf{C}_s \in \mathcal{B}(\mathcal{K})$. Now we can compute, using relation (29), a limit in $\mathcal{B}(\mathcal{K})$,

$$\begin{aligned}
\mathbf{C} &= \lim_{s\to\infty}\mathbf{C}_s = \lim_{s\to\infty}\kappa_s(\tilde{T}_s(K_s) - K_s)\\
&= \kappa_\infty\left(\lim_{s\to\infty}\iota_{\infty s}(\tilde{T}_s(K_s) - K_s)\right)\\
&= \kappa_\infty(\tilde{T}_\infty(K_\infty) - K_\infty).
\end{aligned}$$

So $\mathbf{K} + \mathbf{C} = \tilde{\kappa}_\infty(\tilde{T}_\infty(K_\infty))$. From the closeness of $\mathbf{K}$, the equality $\mathbf{U}_s\mathbf{K}\mathbf{U}_s^{-1} = \mathbf{K}+\mathbf{C}_s$, and from the fact that the sequences $\{\mathbf{U}_s^{\pm 1}\}$, $\{\mathbf{C}_s\}$ converge one deduces that $\mathbf{U}^{\pm 1}(\mathrm{Dom}\,\mathbf{K}) \subset \mathrm{Dom}\,\mathbf{K}$ and hence, in fact, $\mathbf{U}^{\pm 1}(\mathrm{Dom}\,\mathbf{K}) = \mathrm{Dom}\,\mathbf{K}$. In addition,

$$\mathbf{U}\mathbf{K}\mathbf{U}^{-1} = \mathbf{K} + \mathbf{C} = \tilde{\kappa}_\infty\tilde{T}_\infty(K_\infty). \tag{30}$$

Combining (27) and (30) one finds that

$$\tilde{\kappa}_\infty\tilde{T}_\infty(X) = \mathbf{U}\tilde{\kappa}_\infty(X)\mathbf{U}^{-1},\ \forall X \in \tilde{\mathfrak{X}}_\infty.$$

To conclude the proof it suffices to apply the mapping $\tilde{\kappa}_\infty$ to equality (20). □

5. Choice of the directed sequence of Banach spaces

Suppose that there are given a decreasing sequence of subsets of the interval $]0,+\infty[$, $\Omega_0 \supset \Omega_1 \supset \Omega_2 \supset \ldots$, a decreasing sequence of positive real numbers $\{\varphi_s\}_{s=0}^{\infty}$ and a strictly increasing sequence of positive real numbers $\{E_s\}_{s=0}^{\infty}$, $1 \le E_1 < E_2 < \ldots$.

We construct a complex Banach space ${}^0\mathfrak{X}_s$, $s \ge 0$, as a subspace

$${}^0\mathfrak{X}_s \subset L^\infty\left(\Omega_s \times \mathbb{Z} \times \mathbb{N} \times \mathbb{N}, \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}^{\oplus}\mathcal{B}(\mathcal{H}_m,\mathcal{H}_n)\right)$$

formed by those elements $X = \{X_{knm}(\omega)\}$ which satisfy

$$X_{knm}(\omega) \in \mathcal{B}(\mathcal{H}_m,\mathcal{H}_n),\ \forall\omega\in\Omega_s,\ \forall(k,n,m)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N},$$

and have finite norm

$$\|X\|_s = \sup_{\substack{\omega,\omega'\in\Omega_s\\ \omega\neq\omega'}}\ \sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{N}}\left(\|X_{knm}(\omega)\| + \varphi_s\,\|\partial X_{knm}(\omega,\omega')\|\right)e^{|k|/E_s} \tag{31}$$

where the symbol ∂ designates the discrete derivative in ω,

$$\partial X(\omega,\omega') = \frac{X(\omega) - X(\omega')}{\omega - \omega'}\,.$$

In fact, this norm is considered in Appendix B (c.f. (87)), and it is shown there that ${}^0\mathfrak{X}_s$ is an operator algebra with respect to the multiplication rule (89).

Let $\mathfrak{X}_s \subset {}^0\mathfrak{X}_s$ be a closed real subspace formed by those elements $X \in {}^0\mathfrak{X}_s$ which satisfy,

$$\forall (k,n,m) \in \mathbb{Z}\times\mathbb{N}\times\mathbb{N},\ \forall \omega \in \Omega_s, \quad X_{knm}(\omega)^* = X_{-k,m,n}(\omega) \in \mathcal{B}(\mathcal{H}_n,\mathcal{H}_m). \quad (32)$$

Note, however, that $\mathfrak{X}_s$ is not an operator subalgebra of ${}^0\mathfrak{X}_s$.

The sequence of Banach spaces, $\{\mathfrak{X}_s\}_{s=0}^{\infty}$, becomes directed with respect to mappings of restriction in the variable ω: if $u \geq s$ then we set

$$\iota_{us} : \mathfrak{X}_s \to \mathfrak{X}_u,\ \ \iota_{us}(X) = X|_{\Omega_u}.$$

Because of the monotonicity of the sequences $\{\varphi_s\}$ and $\{E_s\}$ we clearly have $\|\iota_{us}\| \leq 1$.

Next we introduce a bounded operator $\mathcal{D}_s \in \mathcal{B}(\mathfrak{X}_s)$ as an operator which extracts the diagonal part of a matrix,

$$\mathcal{D}_s(X)_{knm}(\omega) = \delta_{k0}\delta_{nm}X_{0nn}(\omega). \quad (33)$$

Clearly, $\|\mathcal{D}_s\| \leq 1$ and $\|1 - \mathcal{D}_s\| \leq 1$.

Let

$$V \in L^\infty\left(\mathbb{Z}\times\mathbb{N}\times\mathbb{N}, \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}^{\oplus}\mathcal{B}(\mathcal{H}_m,\mathcal{H}_n)\right)$$

be the element with the components $V_{knm} \in \mathcal{B}(\mathcal{H}_m,\mathcal{H}_n)$ given in (2). Since, by assumption, $V(t)$ is Hermitian for almost all t it hold true that

$$(V_{knm})^* = V_{-k,m,n}.$$

We still assume, as in Theorem 1, that there exists $r > 0$ such that

$$\epsilon_V = \sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{N}}\|V_{knm}\|\,\max\{|k|^r, 1\} < \infty\,. \quad (34)$$

Let us define elements $V_s \in \mathfrak{X}_s$, $s \geq 0$, by

$$\begin{aligned} (V_s)_{knm}(\omega) &= V_{knm} &&\text{if } |k| < E_s \\ &= 0 &&\text{if } |k| \geq E_s \end{aligned} \quad (35)$$

For $s \geq 1$ we get an estimate,

$$\begin{aligned} \|V_s - \iota_{s-1}(V_{s-1})\|_s &= \sup_{n\in\mathbb{N}} \sum_{\substack{k\in\mathbb{Z} \\ E_{s-1}\leq|k|<E_s}} \sum_{m\in\mathbb{N}} \|V_{knm}\| \, e^{|k|/E_s} \\ &\leq e \sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{N}} \|V_{knm}\| \, \frac{\max\{|k|^r, 1\}}{(E_{s-1})^r} \\ &= \frac{e\,\epsilon_V}{(E_{s-1})^r}\,. \end{aligned} \tag{36}$$

Similarly, for $s = 0$, we get

$$\|V_0\| \leq e\,\epsilon_V.$$

It is convenient to set $E_{-1} = 1$, $V_{-1} = 0$.

The sequence $\{K_s\}_{s=0}^{\infty}$ has the same meaning as in Section 3, i.e., each K_s is a distinguished basis vector in a one-dimensional vector space $\mathbb{R}K_s$. Furthermore, a sequence $\Theta_u^s \in \mathcal{B}(\mathfrak{X}_u)$, $0 \leq s < u$, is supposed to satisfy rule (6). Similarly as in Proposition 2 we construct sequences $T_s \in \mathcal{B}(\mathfrak{X}_s)$, $s \geq 1$, and $W_s \in \mathfrak{X}_s$, $s \geq 0$, using relations (7) and (8), respectively.

Proposition 7. *Suppose that it holds*

$$\|\Theta_u^s\| \leq \frac{5}{\varphi_{s+1}} \, \|W_s - \iota_{s-1}(W_{s-1})\|_s, \ \forall s, u, \ 0 \leq s < u, \tag{37}$$

and set

$$A_\star = 5e \sup_{s\geq 0} \frac{(E_s)^r}{\varphi_{s+1}(E_{s-1})^{2r}}, \ B_\star = 5e \sum_{s=0}^{\infty} \frac{1}{\varphi_{s+1}(E_{s-1})^r}, \ C_\star = 5e \sup_{s\geq 0} \frac{1}{\varphi_{s+1}(E_{s-1})^r}\,. \tag{38}$$

If

$$\epsilon_V B_\star \leq \frac{1}{3}\ln 2 \quad \textit{and} \quad \epsilon_V A_\star \phi(3\epsilon_V C_\star) \leq \frac{1}{9} \tag{39}$$

then the conclusions of Corollary 3 hold true, particularly, the objects $V_\infty, W_\infty \in \mathfrak{X}_\infty$, $T_\infty \in \mathcal{B}(\mathfrak{X}_\infty)$ *and* $\tilde{T}_\infty \in \mathcal{B}(\tilde{\mathfrak{X}}_\infty)$ *exist and satisfy the equality*

$$\tilde{T}_\infty(K_\infty + V_\infty) = K_\infty + \mathcal{D}_\infty(W_\infty)\,.$$

Remark. Respecting estimates (36) and (37) we set in what follows

$$F_s = \frac{5}{\varphi_{s+1}} \text{ and } v_s = \frac{e\,\epsilon_V}{(E_{s-1})^r}, \ s \geq 0. \tag{40}$$

Proof. Taking into account the defining relations (40) one finds that the constants A, B and C introduced in Proposition 3 may be chosen as

$$A = \epsilon_V A_\star, \ B = \epsilon_V B_\star \text{ and } C = \epsilon_V C_\star. \tag{41}$$

The assumption (39) implies that

$$B \leq \frac{1}{3}\ln 2 \quad \text{and} \quad A\,\phi(3C) \leq \frac{1}{9} \tag{42}$$

and so, according to the remark following Proposition 3, inequality (15) holds true with $d = 3$. Since $F_{\inf} = 5/\varphi_1 > 0$ assumption (18) of Corollary 4 as well as all assumptions of Proposition 3 are satisfied and so the conclusions of Corollary 4 hold true. □

6. Relation of the Banach spaces $\mathfrak{X}_s$ to Hermitian operators in $\mathcal{K}$

The real Banach spaces $\mathfrak{X}_s$ have been chosen in the previous section. Set

$$\Omega_\infty = \bigcap_{s=0}^{\infty} \Omega_s.$$

Suppose that $\Omega_\infty \neq \emptyset$ and fix $\omega \in \Omega_\infty$ (so $\omega > 0$).

To an operator-valued function $[\,0,T\,] \ni t \mapsto X(t) \in \mathcal{B}(\mathcal{H})$ there is naturally related an operator $\mathbf{X}$ in $\mathcal{K} = L^2([\,0,T\,],\mathcal{H},\mathrm{dt})$ defined by $(\mathbf{X}\psi)(t) = X(t)\psi(t)$. As is well known,

$$\|\mathbf{X}\| \leq \|X\|_{SH}$$

where $\|\cdot\|_{SH}$ is the so called Schur-Holmgren norm,

$$\begin{aligned} \|X\|_{SH} &= \max\Bigg\{ \sup_{(\ell,n)\in\mathbb{Z}\times\mathbb{N}} \sum_{(k,m)\in\mathbb{Z}\times\mathbb{N}} \big\|P_\ell \otimes Q_n \mathbf{X} P_k \otimes Q_m\big\|\,, \\ &\qquad \sup_{(k,m)\in\mathbb{Z}\times\mathbb{N}} \sum_{(\ell,n)\in\mathbb{Z}\times\mathbb{N}} \big\|P_\ell \otimes Q_n \mathbf{X} P_k \otimes Q_m\big\| \Bigg\} \\ &= \max\Bigg\{ \sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{N}} \|X_{knm}\|,\ \sup_{m\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{N}} \|X_{knm}\| \Bigg\}. \end{aligned} \tag{43}$$

Here

$$X_{knm} = \frac{1}{T}\int_0^T e^{-i\omega kt}\, Q_n X(t) Q_m \,\mathrm{dt}\,.$$

It is also elementary to verify that the Schur-Holmgren norm is an operator norm, $\|XY\|_{SH} \leq \|X\|_{SH}\|Y\|_{SH}$, with respect to the multiplication rule (89).

If $X(t)$ is Hermitian for (almost) every $t \in [\,0,T\,]$ then it holds, $\forall (k,n,m)$, $(X_{knm})^* = X_{-k,m,n}$, and so

$$\|X\|_{SH} = \sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{N}} \|X_{knm}\|\,.$$

Note also that, $\forall s \in \mathbb{Z}_+$, $\forall X \in \mathfrak{X}_s$,

$$\|X(\omega)\|_{SH} \leq \|X\|_s$$

and, consequently, the same is also true for $s = \infty$.

To an element $X \in {}^0\mathfrak{X}_s \subset L^\infty\Big(\Omega_s \times \mathbb{Z} \times \mathbb{N} \times \mathbb{N},\ \sum_{n\in\mathbb{N}} \sum^{\oplus}_{m\in\mathbb{N}} \mathcal{B}(\mathcal{H}_m, \mathcal{H}_n)\Big)$ such that $\|X(\omega)\|_{SH} < \infty$ we can relate an operator-valued function defined on the interval $[\,0, T\,]$,

$$t \mapsto \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} e^{ik\omega t}\, X_{knm}(\omega)\,.$$

The corresponding operator in $\mathcal{K}$ is denoted by $\kappa_s(X)$, with a norm being bounded from above by $\|X(\omega)\|_{SH}$. In particular, $\forall X \in \mathfrak{X}_s$,

$$\|\kappa_s(X)\| \le \|X(\omega)\|_{SH} \le \|X\|_s.$$

In addition, if $X \in \mathfrak{X}_s$ then the operator $\kappa_s(X)$ is Hermitian due to the property (32) of X. This way we have introduced the mappings $\kappa_s : \mathfrak{X}_s \to \mathcal{B}(\mathcal{K})$ for $s \in \mathbb{Z}_+$.

Another property we shall need is that κ_s is an algebra morphism in the sense: if $X, Y \in {}^0\mathfrak{X}_s$ such that $\|X(\omega)\|_{SH} < \infty$ and $\|Y(\omega)\|_{SH} < \infty$ then $\|(XY)(\omega)\|_{SH} < \infty$ and

$$\kappa_s(XY) = \kappa_s(X)\kappa_s(Y)\,.$$

Particularly this is true for all $X, Y \in \mathfrak{X}_s$.

Let $\mathbf{D} \in \mathcal{B}(\mathcal{B}(\mathcal{K}))$ be the operator on $\mathcal{B}(\mathcal{K})$ taking the diagonal part of an operator $X \in \mathcal{B}(\mathcal{K})$,

$$\mathbf{D}(X) = \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{N}} P_k \otimes Q_m X\, P_k \otimes Q_m.$$

Clearly, $\mathbf{D}\kappa_s = \kappa_s \mathcal{D}_s$. Since

$$\|\mathbf{D}(X)\| = \sup_{(k,m)\in\mathbb{Z}\times\mathbb{N}} \|P_k \otimes Q_m X\, P_k \otimes Q_m\| \le \|X\|$$

we have $\|\mathbf{D}\| \le 1$.

A consequence of (34) is that $V = \{V_{knm}\}$ has a a finite Schur-Holmgren norm, $\|V\|_{SH} < \infty$. Let $V_s \in \mathfrak{X}_s$, $s \in \mathbb{Z}_+$, be the cut-offs of V defined in (35). Then

$$\begin{aligned}
\|V - V_s\|_{SH} &= \sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{Z},\,|k|\ge E_s} \sum_{m\in\mathbb{N}} \|V_{knm}\| \\
&\le \frac{1}{(E_s)^r} \sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{N}} \|V_{knm}\| \,\max\{|k|^r, 1\} \\
&= \frac{\epsilon_V}{(E_s)^r}\,.
\end{aligned}$$

We shall impose an additional condition on the increasing sequence $\{E_s\}$ of positive real numbers that occur in the definition of the norm $\|\cdot\|_s$ in $\mathfrak{X}_s$ (c.f. (31)), namely we shall require

$$\lim_{s\to\infty} E_s = +\infty\,. \tag{44}$$

In this case $\lim_{s\to\infty} \|V - V_s\|_{SH} = 0$ and so

$$\mathbf{V} = \lim_{s\to\infty} \kappa_s(V_s) \quad \text{in the operator norm.} \tag{45}$$

We also assume that there exist $A_s \in \mathfrak{X}_{s+1}$, $s \in \mathbb{Z}_+$, such that

$$(A_s)_{knm}(\omega)^* = -(A_s)_{-k,m,n}(\omega), \tag{46}$$

and, using these elements, we define mappings ${}^0\Theta_u^s \in \mathcal{B}({}^0\mathfrak{X}_u)$, $u > s$, by

$${}^0\Theta_u^s(X) = [\, \iota_{u,s+1}(A_s), X\,] \tag{47}$$

(where the commutator on the RHS makes sense since ${}^0\mathfrak{X}_u$ is an operator algebra). Clearly, $\|{}^0\Theta_u^s\| \leq 2\|A_s\|_{s+1}$. One finds readily that $\mathfrak{X}_u \subset {}^0\mathfrak{X}_u$ is an invariant subspace with respect to the mapping ${}^0\Theta_u^s$ and so one may define $\Theta_u^s = {}^0\Theta_u^s\big|_{\mathfrak{X}_u} \in \mathcal{B}(\mathfrak{X}_u)$. Since $iA_s \in \mathfrak{X}_{s+1}$ we can set

$$\mathbf{A}_s = -i\,\kappa_{s+1}(iA_{s+1}) \in \mathcal{B}(\mathcal{K}).$$

Clearly, $\mathbf{A}_s$ is anti-Hermitian and satisfies $\|\mathbf{A}_s\| \leq \|A_s\|_{s+1}$. Note that (47) implies that, $\forall s, u,\ 0 \leq s < u$, $\forall X \in \mathfrak{X}_u$,

$$\kappa_u\big(\Theta_u^s(X)\big) = [\,\mathbf{A}_s, \kappa_u(X)\,]\,.$$

Lemma 8. *Let $\{W_s\}_{s=0}^\infty$ be a sequence of elements $W_s \in \mathfrak{X}_s$ and let $\tilde\Theta_u^s : \tilde{\mathfrak{X}}_u \to \tilde{\mathfrak{X}}_u$ be the extension of Θ_u^s, $0 \leq s < u$, defined in (9). Assume that the elements $A_s \in {}^0\mathfrak{X}_{s+1}$, $s \in \mathbb{Z}_+$, satisfy*

$$\begin{aligned}&(k\omega - \Delta_{mn})\,(A_s)_{knm}(\omega) \\ &\quad = \Big(\Theta_u^s\big(\iota_{us}\mathcal{D}_s(W_s)\big) + \iota_{us}(1-\mathcal{D}_s)\big(W_s - \iota_{s-1}(W_{s-1})\big)\Big)_{knm}(\omega),\end{aligned} \tag{48}$$

$\forall (k,m,n) \in \mathbb{Z}\times\mathbb{N}\times\mathbb{N}$, $\forall s,u,\ 0 \leq s < u$.

Then it holds true that,

$$\forall s \in \mathbb{Z}_+, \quad \mathbf{A}_s(\operatorname{Dom}\mathbf{K}) \subset \operatorname{Dom}\mathbf{K},$$

and

$$\forall s,u,\ 0 \leq s < u, \quad [\,\mathbf{A}_s, \mathbf{K}\,] = \kappa_u(\tilde\Theta_u^s(K_u))\big|_{\operatorname{Dom}(\mathbf{K})}.$$

Proof. Set

$$\mathbf{B}_s = -\kappa_u\big(\tilde\Theta_u^s(K_u)\big).$$

Since the RHS of (48) is in fact a matrix entry of $-\tilde\Theta_u^s(K_u)$ (c.f. (9)) this assumption may be rewritten as the equality

$$\mathbf{K}\,P_\ell \otimes Q_n \mathbf{A}_s P_k \otimes Q_m = P_\ell \otimes Q_n \mathbf{A}_s P_k \otimes Q_m \mathbf{K} + P_\ell \otimes Q_n \mathbf{B}_s P_k \otimes Q_m,$$

valid for all $(\ell, n), (k, m) \in \mathbb{Z}\times\mathbb{N}$. Since $\mathbf{K}$ is closed one easily derives from the last property that it holds true, $\forall (k,m) \in \mathbb{Z}\times\mathbb{N}$,

$$\mathbf{K}\,\mathbf{A}_s P_k \otimes Q_m = \mathbf{A}_s P_k \otimes Q_m \mathbf{K} + \mathbf{B}_s P_k \otimes Q_m. \tag{49}$$

Particularly, $\mathbf{A}_s \operatorname{Ran}(P_k \otimes Q_m) \subset \operatorname{Dom}(\mathbf{K})$. But $\operatorname{Ran}(P_k \otimes Q_m)$ are mutually orthogonal eigenspaces of $\mathbf{K}$. Consequently, if $v \in \operatorname{Dom}(\mathbf{K})$, then the sequence $\{v_N\}_{N=1}^\infty$,

$$v_N = \sum_{k,\,|k|\leq N}\ \sum_{m,\,m\leq N} P_k \otimes Q_m v$$

has the property: $v_N \to v$ and $\mathbf{K}v_N \to \mathbf{K}v$, as $N \to \infty$. Equality (49) implies that

$$\mathbf{K}\mathbf{A}_s v_N = \mathbf{A}_s\mathbf{K}v_N + \mathbf{B}_s v_N,\ \forall N.$$

Again owing to the fact that $\mathbf{K}$ is closed one concludes that $\mathbf{A}_s v \in \mathrm{Dom}(\mathbf{K})$ and $\mathbf{K}\mathbf{A}_s v = \mathbf{A}_s\mathbf{K}v + \mathbf{B}_s v$. □

Proposition 9. *Assume that* $\omega \in \Omega_\infty$ *and the norms* $\|\cdot\|_s$ *in the Banach spaces* $\mathfrak{X}_s$ *satisfy (44). Let* $\Theta_u^s \in \mathcal{B}(\mathbf{X}_u)$, $0 \le s < u$, *be the operators defined in (47) with the aid of elements* $A_s \in {}^0\mathfrak{X}_{s+1}$ *satisfying (46), and let* $W_s \in \mathfrak{X}_s$, $s \in \mathbb{Z}_+$, *be a sequence defined recursively in accordance with (8). Assume that the elements* A_s, $s \in \mathbb{Z}_+$, *satisfy condition (48) and that*

$$\|A_s\| \le \frac{5}{2\varphi_{s+1}}\,\|W_s - \iota_{s-1}(W_{s-1})\|,\ \forall s \in \mathbb{Z}_+. \tag{50}$$

Moreover, assume that the numbers $A_\star$, $B_\star$, $C_\star$, *as defined in (38), satisfy condition (39).*

Then there exist, in $\mathcal{K}$, *a unitary operator* $\mathbf{U}$ *and a bounded Hermitian operator* $\mathbf{W}$ *such that*

$$\mathbf{U}(\mathrm{Dom}\,\mathbf{K}) = \mathrm{Dom}\,\mathbf{K}$$

and

$$\mathbf{U}(\mathbf{K}+\mathbf{V})\mathbf{U}^{-1} = \mathbf{K} + \mathbf{D}(\mathbf{W}).$$

Proof. The norm of Θ_u^s may be estimated as

$$\|\Theta_u^s\| \le 2\|A_s\| \le \frac{5}{\varphi_{s+1}}\,\|W_s - \iota_{s-1}(W_{s-1})\|.$$

This way the assumptions of Proposition 7 are satisfied and consequently, according to Proposition 7 (and its proof), the same is true for Proposition 3 and Corollary 4 (with F_s and v_s defined in (40) and the constants A, B, C defined in (41)). Since it holds $\|\mathbf{A}_s\| \le \|A_s\| \le \frac{1}{2}F_s w_s$ (where $F_s = 5/\varphi_{s+1}$) and, by assumption, condition (15) is satisfied with $d = 3$ we get

$$\sum_{s=0}^{\infty}\|\mathbf{A}_s\| \le \frac{1}{2}\sum_{s=0}^{\infty}F_s w_s \le \frac{3}{2}\sum_{s=0}^{\infty}F_s v_s = \frac{3B}{2} < \infty.$$

This verifies assumption (23) of Proposition 5; the other assumptions of this proposition are verified as well as follows from Lemma 8. Note that, in virtue of (45), $\kappa_\infty(V_\infty) = \lim_{s\to\infty}\kappa_s(V_s)$ coincides with the given operator $\mathbf{V}$. Furthermore, $\mathbf{W} = \kappa_\infty(W_\infty) = \lim_{s\to\infty}\kappa_s(W_s)$ is a limit of Hermitian operators and so is itself Hermitian, and $\mathbf{U} = \lim_{s\to\infty} e^{\mathbf{A}_{s-1}}\ldots e^{\mathbf{A}_0}$ is unitary. Equality (25) holds true and this concludes the proof. □

7. Set of non-resonant frequencies

Let $J>0$ be fixed and assume that, $\forall s\in\mathbb{Z}_+$,

$$\Omega_s\subset\Big[\frac{8}{9}J,\frac{9}{8}J\Big].$$

The following definition concerns indices (k,n,m) corresponding to non-diagonal entries, i.e., those indices for which either $k\neq 0$ or $m\neq n$. The diagonal indices, with $k=0$ and $m=n$, will always be treated separately and, in fact, in a quite trivial manner.

Definition. We shall say that a multi-index $(k,n,m)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}$ is *critical* if $m\neq n$ and

$$\frac{kJ}{\Delta_{mn}}\in\Big]\frac{1}{2},2\Big[\tag{51}$$

(hence $\operatorname{sgn}(k)=\operatorname{sgn}(h_m-h_n)\neq 0$). In the opposite case the multi-index will be called *non-critical.*

Definition. Let $\psi(k,n,m)$ be a positive function defined on non-diagonal indices and $W\in\mathfrak{X}_s$. A frequency $\omega\in\Omega_s$ will be called (W,ψ)*–non-resonant* if for all non-diagonal indices $(k,n,m)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}$ it holds

$$\operatorname{dist}\left(\operatorname{Spec}(k\omega-\Delta_{mn}+W_{0nn}(\omega)),\,\operatorname{Spec}(W_{0mm}(\omega))\right)\geq\psi(k,n,m). \tag{52}$$

In the opposite case ω will be called (W,ψ)*–resonant.*

Note that, in virtue of (32), $W_{0mm}(\omega)$ is a Hermitian operator in $\mathcal{H}_m$.

Lemma 10. *Assume that* $\Omega_s\subset[\frac{8}{9}J,\frac{9}{8}J]$, $W\in\mathfrak{X}_s$ *and* ψ *is a positive function defined on non-diagonal indices and obeying a symmetry condition,*

$$\psi(-k,m,n)=\psi(k,n,m)\ \textit{for all}\ (k,n,m)\ \textit{non-diagonal.} \tag{53}$$

If

$$\forall m\in\mathbb{N},\ \ \forall\omega,\omega'\in\Omega_s,\ \omega\neq\omega',\ \|\partial W_{0mm}(\omega,\omega')\|\leq\frac{1}{4}, \tag{54}$$

and if condition (52) is satisfied for all $\omega\in\Omega_s$ *and all non-critical indices* (k,n,m) *then the Lebesgue measure of the set* $\Omega_s^{\mathrm{bad}}\subset\Omega_s$ *formed by* (W,ψ)*–resonant frequencies may be estimated as*

$$|\Omega_s^{\mathrm{bad}}|\leq 8\sum_{\substack{m,n\in\mathbb{N},\\ \Delta_{mn}>\frac{1}{2}J}}\ \sum_{\substack{k\in\mathbb{N},\\ \frac{\Delta_{mn}}{2J}<k<\frac{2\Delta_{mn}}{J}}}\frac{M_mM_n}{k}\,\psi(k,n,m)\,. \tag{55}$$

Proof. Let $\lambda_1^m(\omega)\leq\lambda_2^m(\omega)\leq\cdots\leq\lambda_{M_m}^m(\omega)$ be the increasingly ordered set of eigenvalues of $W_{0mm}(\omega)$, $m\in\mathbb{N}$. Set

$$\Omega_s^{\mathrm{bad}}(k,n,m,i,j)=\{\omega\in\Omega_s;\ |\omega k-\Delta_{mn}+\lambda_i^n(\omega)-\lambda_j^m(\omega)|<\psi(k,n,m)\}.$$

Then

$$\Omega_s^{\rm bad} = \bigcup_{(k,n,m)} \bigcup_{\substack{i,j\\ 1\le i\le M_n\\ 1\le j\le M_m}} \Omega_s^{\rm bad}(k,n,m,i,j).$$

By assumption, if (k,n,m) is a non-critical index then $\Omega_s^{\rm bad}(k,n,m,i,j)=\emptyset$ (for any i,j). Further notice that, due to the symmetry condition (53), $\Omega_s^{\rm bad}(k,n,m,i,j) = \Omega_s^{\rm bad}(-k,m,n,j,i)$.

According to Lidskii Theorem ([21], Chap. II §6.5), for any j, $1\le j\le M_m$, $\lambda_j^m(\omega)-\lambda_j^m(\omega')$ may be written as a convex combination (with non-negative coefficients) of eigenvalues of the operator $W_{0mm}(\omega)-W_{0mm}(\omega')$. Consequently,

$$\forall j,\ 1\le j\le M_m,\ \forall \omega,\omega'\in\Omega_s,\ \omega\ne\omega',\ |\partial\lambda_j^m(\omega,\omega')| \le \|\partial W_{0mm}(\omega,\omega')\| \le \frac{1}{4}\,.$$

If $\omega,\omega'\in\Omega_s^{\rm bad}(k,n,m,i,j)$, $\omega\ne\omega'$, then (k,n,m) is necessarily a critical index and

$$\begin{aligned}\frac{2\psi(k,n,m)}{|\omega-\omega'|} &> \left|\frac{(\omega k-\Delta_{mn}+\lambda_i^n(\omega)-\lambda_j^m(\omega))-(\omega' k-\Delta_{mn}+\lambda_i^n(\omega')-\lambda_j^m(\omega'))}{\omega-\omega'}\right|\\ &\ge |k|-\frac{1}{2} \ge \frac{1}{2}\,|k|\,.\end{aligned}$$

This implies that $|\Omega_s^{\rm bad}(k,n,m,i,j)|\le 4\psi(k,n,m)/|k|$ and so

$$|\Omega_s^{\rm bad}| \ \le\ 2 \sum_{\substack{(k,n,m)\\ k>0\\ \frac{\Delta_{mn}}{2J}<k<\frac{2\Delta_{mn}}{J}}} \sum_{\substack{i,j\\ 1\le i\le M_n\\ 1\le j\le M_m}} \frac{4}{k}\,\psi(k,n,m)\,.$$

This immediately leads to the desired inequality (55). □

8. Construction of the sequences $\{\Omega_s\}$ and $\{A_s\}$

For a non-diagonal multi-index (k,n,m) and $s\in\mathbb{Z}_+$ set

$$\begin{aligned}\psi_s(k,n,m) &= \frac{1}{2}\,\Delta_0 && \text{if } (k,n,m) \text{ is non-critical and } k=0,\\ &= \frac{7}{18}\,J\left(|k|-\frac{1}{2}\right) && \text{if } (k,n,m) \text{ is non-critical and } k\ne 0,\\ &= \varphi_{s+1}\left(\min\{M_m,M_n\}\right)^{1/2}|k|^{1/2}e^{-\varrho_s|k|/2} && \text{if } (k,n,m) \text{ is critical,}\end{aligned} \tag{56}$$

where

$$\varrho_s = \frac{1}{E_s}-\frac{1}{E_{s+1}}.$$

Observe that ψ_s obeys the symmetry condition (53). The choice of $\psi_s(k,n,m)$ for a non-critical index (k,n,m) was guided by the following lemma.

Lemma 11. *If* $\omega \in \Omega_s \subset [\frac{8}{9}J, \frac{9}{8}J]$, $(k,n,m) \in \mathbb{Z}\times\mathbb{N}\times\mathbb{N}$ *is a non-critical index and* $W \in \mathfrak{X}_s$ *satisfies*

$$\|W_{0mm}(\omega)\|, \|W_{0nn}(\omega)\| \leq \min\left\{\frac{1}{4}\,\Delta_0, \frac{7}{72}\,J\right\} \tag{57}$$

then the spectra $\mathrm{Spec}(k\omega-\Delta_{mn}+W_{0nn}(\omega))$, $\mathrm{Spec}(W_{0mm}(\omega))$ *are not interlaced (i.e., they are separated by a real point* p *such that one of them lies below and the other above* p*) and it holds*

$$\mathrm{dist}\left(\mathrm{Spec}(k\omega-\Delta_{mn}+W_{0nn}(\omega)), \mathrm{Spec}(W_{0mm}(\omega))\right) \geq \psi(k,n,m).$$

Proof. We distinguish two cases. If $k \neq 0$ then

$$|k\omega-\Delta_{mn}| = |k|\left|\omega-\frac{\Delta_{mn}}{k}\right| \geq \frac{7}{18}\,J|k|$$

since, by assumption,

$$\frac{\Delta_{mn}}{k}-\omega \in \,]-\infty, \frac{1}{2}\,J-\frac{8}{9}\,J\,] \cup [\,2J-\frac{9}{8}\,J, +\infty[\,.$$

So the distance may be estimated from below by

$$\frac{7}{18}\,J|k|-\|W_{0nn}(\omega)\|-\|W_{0mm}(\omega)\| \geq \frac{7}{18}\,J\left(|k|-\frac{1}{2}\right).$$

If $k=0$ then a lower bound to the distance is simply given by

$$\Delta_0-\|W_{0nn}(\omega)\|-\|W_{0mm}(\omega)\| \geq \frac{1}{2}\,\Delta_0.$$

□

Next we specify the way we shall construct the decreasing sequence of sets $\{\Omega_s\}_{s=0}^{\infty}$. Let $\Omega_0 = [\frac{8}{9}J, \frac{9}{8}J]$. If $W_s \in \mathfrak{X}_s$ has been already defined then we introduce $\Omega_{s+1} \subset \Omega_s$ as the set of (W_s,ψ_s)–non-resonant frequencies. Recall that the real Banach space $\mathfrak{X}_s$ is determined by the choice of data φ_s, E_s and Ω_s, as explained in Section 5.

As a next step let us consider , for $s \in \mathbb{Z}_+$, $\omega \in \Omega_{s+1}$ and a non-diagonal index (k,n,m), a commutation equation,

$$(k\omega-\Delta_{mn}+(W_s)_{0nn}(\omega))X - X\,(W_s)_{0mm}(\omega) = Y, \tag{58}$$

with an unknown $X \in \mathcal{B}(\mathcal{H}_m,\mathcal{H}_n)$ and a right hand side $Y \in \mathcal{B}(\mathcal{H}_m,\mathcal{H}_n)$. Since ω is (W_s,ψ_s)–non-resonant the spectra $\mathrm{Spec}(k\omega-\Delta_{mn}+(W_s)_{0nn}(\omega))$ and $\mathrm{Spec}((W_s)_{0mm}(\omega))$ don't intersect and so a solution X exists and is unique. This way one can introduce a linear mapping

$$(\Gamma_s)_{knm}(\omega) : \mathcal{B}(\mathcal{H}_m,\mathcal{H}_n) \to \mathcal{B}(\mathcal{H}_m,\mathcal{H}_n)$$

such that $X = (\Gamma_s)_{knm}(\omega)Y$ solves (58). Moreover, according to Appendix A,

$$\|(\Gamma_s)_{knm}(\omega)\| \leq \frac{(\min\{M_m,M_n\})^{1/2}}{\psi(k,n,m)} \tag{59}$$

in the general case, and provided the spectra $\mathrm{Spec}(k\omega - \Delta_{mn} + (W_s)_{0nn}(\omega))$ and $\mathrm{Spec}((W_s)_{0mm}(\omega))$ are not interlaced it even holds that

$$\|(\Gamma_s)_{knm}(\omega)\| \leq \frac{1}{\psi(k,n,m)}\,. \tag{60}$$

From the uniqueness it is clear that $\mathrm{Ker}((\Gamma_s)_{knm}(\omega)) = 0$.

We extend the definition of $(\Gamma_s)_{knm}$ to diagonal indices by letting $(\Gamma_s)_{0nn}(\omega) = 0 \in \mathcal{B}(\mathcal{B}(\mathcal{H}_n, \mathcal{H}_n))$. This way we get an element

$$\Gamma_s \in \mathrm{Map}\left(\Omega_{s+1} \times \mathbb{Z} \times \mathbb{N} \times \mathbb{N}, \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}}^{\oplus} \mathcal{B}(\mathcal{B}(\mathcal{H}_m, \mathcal{H}_n))\right), \tag{61}$$

which naturally defines a linear mapping, denoted for simplicity by the same symbol, $\Gamma_s : {}^0\mathfrak{X}_s \to {}^0\mathfrak{X}_{s+1}$, according to the rule

$$\Gamma_s(Y)_{knm}(\omega) := (\Gamma_s)_{knm}(\omega)(Y_{knm}(\omega))\,.$$

Lemma 12. *Assume that for all non-diagonal indices (k,n,m) and $\omega, \omega' \in \Omega_{s+1}$, $\omega \neq \omega'$, it holds*

$$\|\partial(\Gamma_s)^{-1}_{knm}(\omega,\omega')\| \leq |k| + \frac{1}{2}, \tag{62}$$

if $\omega \in \Omega_{s+1}$ and (k,n,m) is a non-critical index then the spectra $\mathrm{Spec}(k\omega - \Delta_{mn} + (W_s)_{0nn}(\omega))$ *and* $\mathrm{Spec}((W_s)_{0mm}(\omega))$ *are not interlaced and*

$$\varphi_{s+1} \leq \min\left\{\frac{2}{3}\,\Delta_0, \frac{1}{6}\,J\right\}. \tag{63}$$

Then the following upper estimate on the norm of $\Gamma_s \in \mathcal{B}({}^0\mathfrak{X}_s, {}^0\mathfrak{X}_{s+1})$ holds true:

$$\|\Gamma_s\| \leq \frac{5}{2\,\varphi_{s+1}}\,.$$

Proof. To estimate $\|\Gamma_s\|$ we shall use relation (94) of Proposition 15 in Appendix B. Note that

$$\begin{aligned}\|\partial(\Gamma_s)_{knm}(\omega,\omega')\| &= \|(\Gamma_s)_{knm}(\omega)\,\partial(\Gamma_s)^{-1}_{knm}(\omega,\omega')\,(\Gamma_s)_{knm}(\omega')\| \\ &\leq \|(\Gamma_s)_{knm}(\omega)\|\|(\Gamma_s)_{knm}(\omega')\|\left(|k| + \frac{1}{2}\right).\end{aligned} \tag{64}$$

If (k,n,m) is critical then we have, according to (59) and (56),

$$\|(\Gamma_s)_{knm}(\omega)\| \leq \frac{1}{\varphi_{s+1}|k|^{1/2}}\,e^{\varrho_s|k|/2}$$

and consequently

$$\begin{aligned} & e^{-\varrho_s|k|}\left(\|(\Gamma_s)_{knm}(\omega)\| + \varphi_{s+1}\|\partial(\Gamma_s)_{knm}(\omega,\omega')\|\right) \\ & \quad \le\ e^{-\varrho_s|k|}\left(\frac{1}{\varphi_{s+1}|k|^{1/2}}\,e^{\varrho_s|k|/2} + \frac{|k|+\frac{1}{2}}{\varphi_{s+1}|k|}\,e^{\varrho_s|k|}\right) \\ & \quad \le\ \frac{1}{\varphi_{s+1}}\left(1+1+\frac{1}{2|k|}\right)\ \le\ \frac{5}{2\,\varphi_{s+1}}. \end{aligned}$$

If (k,n,m) is non-critical and $k\neq 0$ then we have, according to (60) and (56),

$$\|(\Gamma_s)_{knm}(\omega)\|\ \le\ \frac{18}{7J\left(|k|-\frac{1}{2}\right)}$$

and consequently

$$\begin{aligned} & e^{-\varrho_s|k|}\left(\|(\Gamma_s)_{knm}(\omega)\| + \varphi_{s+1}\|\partial(\Gamma_s)_{knm}(\omega,\omega')\|\right) \\ & \quad \le\ \frac{18}{7J\left(|k|-\frac{1}{2}\right)}\left(1+\varphi_{s+1}\,\frac{18\left(|k|+\frac{1}{2}\right)}{7J\left(|k|-\frac{1}{2}\right)}\right) \\ & \quad \le\ \frac{1}{\varphi_{s+1}}\frac{1}{6}\frac{36}{7}\left(1+\frac{1}{6}\frac{54}{7}\right)\ <\ \frac{2}{\varphi_{s+1}}\,. \end{aligned}$$

In the case when (k,n,m) is non-critical and $k=0$ one gets similarly $\|(\Gamma_s)_{knm}(\omega)\|\le 2/\Delta_0$ and

$$\begin{aligned} & e^{-\varrho_s|k|}\left(\|(\Gamma_s)_{knm}(\omega)\| + \varphi_{s+1}\|\partial(\Gamma_s)_{knm}(\omega,\omega')\|\right) \\ & \quad \le\ \frac{2}{\Delta_0}\left(1+\varphi_{s+1}\,\frac{1}{\Delta_0}\right)\ \le\ \frac{1}{\varphi_{s+1}}\frac{4}{3}\left(1+\frac{2}{3}\right)\ <\ \frac{5}{2\,\varphi_{s+1}}\,. \end{aligned}$$

□

Now we are able to specify the mappings Θ_u^s. Set

$$A_s = \Gamma_s\big((1-\mathcal{D}_s)(W_s - \iota_{s-1}(W_{s-1}))\big) \in {}^0\mathfrak{X}_{s+1}. \tag{65}$$

$W_s\in\mathfrak{X}_s$ satisfies (32) and thus one finds, when taking Hermitian adjoint of (58), that

$$((\Gamma_s)_{knm}(\omega)Y)^* = -(\Gamma_s)_{-k,m,n}(\omega)(Y^*)\,.$$

This implies that A_s obeys condition (46). The mappings Θ_u^s, $s<u$, are defined by equality (47) (see also the comment following the equality).

9. Proof of Theorem 1

We start from the specification of the sequences $\{\varphi_s\}$ and $\{E_s\}$,

$$\varphi_s = a\,s^\alpha q^{-rs}\ \text{for}\ s\ge 1,\ \ E_s = q^{s+1}\ \text{for}\ s\ge 0, \tag{66}$$

where $\alpha>1$ and $q>1$ are constants that are arbitrary except of the restrictions

$$q^r \ge e^\alpha \quad\text{and}\quad q^{-r}\zeta(\alpha)\le 3\,\ln 2 \tag{67}$$

(ζ stands for the Riemann zeta function), and

$$a = 45\, e\, q^{2r} \epsilon_V . \tag{68}$$

For example, $\alpha = 2$ and $q^r = e^2$ will do. The value of $\varphi_0 \geq \varphi_1 = a\, q^{-r}$ doesn't influence the estimates which follow, and we automatically have $E_{-1} = 1$ (this is a convenient convention). Condition $r \ln(q) \geq \alpha$ guarantees that the sequence $\{\varphi_s\}$ is decreasing. Note also that

$$\varrho_s = \frac{1}{E_s} - \frac{1}{E_{s+1}} = \left(1 - \frac{1}{q}\right) q^{-s-1}.$$

Another reason for the choice (66) and (68) is that the constants $A_\star$, $B_\star$ and $C_\star$, as defined in (38), obey assumption (39) of Proposition 7. Particularly, a constraint on the choice of $\{\varphi_s\}$ and $\{E_s\}$, namely $\sum_{s=0}^{\infty} 1/(\varphi_{s+1}(E_{s-1})^r) < \infty$, is imposed by requiring $B_\star$ to be finite. However this is straightforward to verify. Actually, the constants may now be expressed explicitly,

$$A_\star = \frac{5e\, q^{2r}}{a},\ B_\star = \frac{5e\, q^r}{a}\, \zeta(\alpha),\ C_\star = \frac{5e\, q^r}{a}\, ,$$

and thus conditions (39) mean that

$$\epsilon_V \frac{5e\, q^r}{a}\, \zeta(\alpha) \leq \frac{1}{3}\, \ln 2,\ \epsilon_V \frac{5e\, q^{2r}}{a}\, \phi\left(\epsilon_V \frac{15e\, q^r}{a}\right) \leq \frac{1}{9}\, . \tag{69}$$

The latter condition in (69) is satisfied since the LHS is bounded from above by (c.f. (5))

$$\frac{1}{9}\, \phi\left(\frac{1}{3}\, q^{-r}\right) \leq \frac{1}{9}\, \phi\left(\frac{1}{3}\right) = 1 - \frac{2}{3}\, e^{1/3} < \frac{1}{9}\, .$$

Concerning the former condition, the LHS equals $q^{-r}\zeta(\alpha)/9$ and so it suffices to chose α and q so that (67) is fulfilled. An additional reason for the choice (66) will be explained later.

Let us now summarise the construction of the sequences $\{\mathfrak{X}_s\}$, $\{W_s\}$ and $\{\Theta_u^s\}_{s>u}$ which will finally amount to a proof of Theorem 1. Some more details were already given in Section 8. We set $\Omega_0 = [\,\frac{8}{9}J, \frac{9}{8}J\,]$ and $W_0 = V_0$. Recall that the cut-offs V_s of V were introduced in (35). In every step, numbered by $s \in \mathbb{Z}_+$, we assume that Ω_t and W_t, with $0 \leq t \leq s$, and A_t, with $0 \leq t \leq s-1$, have already been defined. The mappings Θ_u^t, with $u > t$, are given by $\Theta_u^t(X) = [\,\iota_{u,t+1}(A_t), X\,]$ provided $A_t \in {}^0\mathfrak{X}_{t+1}$ satisfies condition (46). We define $\Omega_{s+1} \subset \Omega_s$ as the set of (W_s, ψ_s)–non-resonant frequencies, with ψ_s introduced in (56). Consequently, the real Banach space $\mathfrak{X}_{s+1}$ is defined as well as its definition depends on the data Ω_{s+1}, φ_{s+1} and E_{s+1}. Then we are able to introduce an element Γ_s (in the sense of (61)) whose definition is based on equation (58) and which in turn determines a bounded operator $\Gamma_s \in \mathcal{B}({}^0\mathfrak{X}_s, {}^0\mathfrak{X}_{s+1})$ (with some abuse of notation). The element $A_s \in {}^0\mathfrak{X}_{s+1}$ is given by equality (65) and actually satisfies condition (46). Knowing W_t, $t \leq s$, and Θ_{s+1}^t, $t \leq s$, (which is equivalent to knowing A_t, $t \leq s$) one is able to evaluate the RHS of (8) defining the element W_{s+1}. Hence one proceeds one step further.

We choose $\epsilon_\star(r,\Delta_0,J)$ maximal possible so that

$$\frac{3e}{1-q^{-r}}\,\epsilon_\star(r,\Delta_0,J)\le\min\left\{\frac{1}{4}\,\Delta_0,\frac{7}{72}\,J\right\} \tag{70}$$

and

$$45\,e\,q^r\epsilon_\star(r,\Delta_0,J)\le\min\left\{\frac{2}{3}\,\Delta_0,\frac{1}{6}\,J\right\}. \tag{71}$$

We claim that this choice guarantees that the construction goes through. Basically this means that $\epsilon_V<\epsilon_\star(r,\Delta_0,J)$ is sufficiently small so that all the assumptions occurring in the preceding auxiliary results are satisfied in every step, with $s\in\mathbb{Z}_+$. This concerns assumption (57) of Lemma 11,

$$\|(W_s)_{0mm}(\omega)\|\le\min\left\{\frac{1}{4}\,\Delta_0,\frac{7}{72}\,J\right\},\quad\forall\omega\in\Omega_s,\ \forall m\in\mathbb{N}, \tag{72}$$

assumption (54) of Lemma 10,

$$\|\partial(W_s)_{0mm}(\omega,\omega')\|\le\frac{1}{4},\quad\forall\omega,\omega'\in\Omega_s,\ \omega\neq\omega',\ \forall m\in\mathbb{N}, \tag{73}$$

assumptions (62) and (63) of Lemma 12,

$$\|\partial(\Gamma_s)^{-1}_{knm}(\omega,\omega')\|\le|k|+\frac{1}{2},\quad\forall(k,n,m),\ \forall\omega,\omega'\in\Omega_s,\ \omega\neq\omega', \tag{74}$$

and

$$\varphi_{s+1}\le\min\left\{\frac{2}{3}\,\Delta_0,\frac{1}{6}\,J\right\}, \tag{75}$$

and assumption (50) of Proposition 9,

$$\|A_{s-1}\|\le\frac{5}{2\varphi_s}\,\|W_{s-1}-\iota_{s-2}(W_{s-2})\|. \tag{76}$$

We can immediately do some simplifications. As the sequence $\{\varphi_s\}$ is non-increasing condition (75) reduces to the case $s=0$. Since $\varphi_1=45\,e\,q^r\epsilon_V$ the upper bound (71) implies (75).

Note also that (74) is a direct consequence of (73). Actually, one deduces from the definition of $(\Gamma_s)_{knm}(\omega)$ (based on equation (58)) that, $\forall Y\in\mathcal{B}(\mathcal{H}_m,\mathcal{H}_n)$,

$$(\Gamma_s)^{-1}_{knm}(\omega)Y=(k\omega-\Delta_{mn}+(W_s)_{0nn}(\omega))Y-Y(W_s)_{0mm}(\omega).$$

Hence

$$\partial(\Gamma_s)^{-1}_{knm}(\omega,\omega')Y=(k+\partial(W_s)_{0nn}(\omega,\omega'))\,Y-Y\,\partial(W_s)_{0mm}(\omega,\omega')$$

and, assuming (73),

$$\|\partial(\Gamma_s)^{-1}_{knm}(\omega,\omega')\|\le|k|+\|\partial(W_s)_{0nn}(\omega,\omega'))\|+\|\partial(W_s)_{0mm}(\omega,\omega')\|\le|k|+\frac{1}{2}.$$

Let us show that in every step, with $s\in\mathbb{Z}_+$, conditions (72), (73) and (76) are actually fulfilled. For $s=0$, condition (76) is empty and condition (73) is obvious since $W_0 = V_0$ doesn't depend on ω. Condition (72) is obvious as well due to assumption (70) and the fact that $\|(W_0)_{0mm}(\omega)\| = \|(V_0)_{0mm}\| \leq \epsilon_V$.

Assume now that $t \in \mathbb{Z}_+$ and conditions (72), (73) and (76) are satisfied in each step $s \leq t$. Recall that in (40) we have set $F_s = 5/\varphi_{s+1}$ and $v_s = e\,\epsilon_V/(E_{s-1})^r$. We also keep the notation $w_s = \|W_s - \iota_{s-1}(W_{s-1})\|_s$, with the convention $W_{-1} = 0$.

We start with condition (76). Using the induction hypothesis, Lemma 11 and Lemma 12 one finds that $\|\Gamma_t\| \leq F_t/2$ and so $\|A_t\| \leq \|\Gamma_t\|\|W_t - \iota_{t-1}(W_{t-1})\| \leq F_t w_t/2$ (c.f. (65) and (4)).

By the induction hypothesis and the just preceding step, $\|A_s\| \leq F_s w_s$ for all $s \leq t$. As we already know the constants $A_\star$, $B_\star$ and $C_\star$ fulfil (39) and so the quantities A, B and C given by $A = \epsilon_V A_\star$, $B = \epsilon_V B_\star$ and $C = \epsilon_V C_\star$ (c.f. (41)) obey (42) and consequently inequality (15) with $d = 3$. By the very choice of A, B and C (c.f. (38) and (40)) the quantities also obey relations (12), (13) and (14). This means that all assumptions of Proposition 3 are fulfilled for $s \leq t$ (recall that $\|\Theta_u^s\| \leq 2\|A_s)$. One easily finds that the conclusion of Proposition 3, namely $w_s \leq d\,v_s$, holds as well for all s, $s \leq t+1$. Clearly, $\|(W_s)_{0mm}(\omega)\| \leq \|W_s\|_s$ for all s, and

$$\|W_{t+1}\|_{t+1} \leq \sum_{s=0}^{t+1} w_s \leq 3\sum_{s=0}^{\infty} v_s = 3e\,\epsilon_V \sum_{s=0}^{\infty} q^{-rs} = \frac{3e}{1-q^{-r}}\,\epsilon_V .$$

By (70) we conclude that (72) is true for $s = t+1$.

Finally, using once more that $w_s \leq 3v_s$ for $s \leq t+1$,

$$\begin{aligned} \|\partial(W_{t+1})_{0mm}(\omega,\omega')\| &\leq \sum_{s=0}^{t+1} \|\partial\big(W_s - \iota_{s-1}(W_{s-1})\big)_{0mm}(\omega,\omega')\| \\ &\leq \sum_{s=0}^{t+1} \frac{1}{\varphi_s}\,\|W_s - \iota_{s-1}(W_{s-1})\|_s \\ &\leq \sum_{s=0}^{\infty} \frac{3v_s}{\varphi_{s+1}}\,. \end{aligned}$$

However, the last sum equals (c.f. (40) and (42))

$$\frac{3}{5}\sum_{s=0}^{\infty} F_s v_s = \frac{3}{5}\,B \leq \frac{1}{5}\,\ln 2 < \frac{1}{4}\,.$$

This verifies (73) for $s = t+1$ and hence the verification of conditions (72), (73) and (76) is complete.

Set, as before, $\Omega_\infty = \bigcap_{s=0}^{\infty} \Omega_s$. Next we are going to estimate the Lebesgue measure of Ω_∞,

$$|\Omega_\infty| = |\Omega_0| - |\Omega_0 \setminus \Omega_\infty| = \frac{17}{72}\,J - \sum_{s=0}^{\infty} |\Omega_s \setminus \Omega_{s+1}| = \frac{17}{72}\,J - \sum_{s=0}^{\infty} |\Omega_s^{\mathrm{bad}}|\,.$$

Recalling Lemma 10 jointly with Lemma 11 showing that the assumptions of Lemma 10 are satisfied, and the explicit form of ψ (56) we obtain

$$\begin{aligned}
|\Omega_s^{\mathrm{bad}}| &\leq 8\varphi_{s+1} \sum_{\substack{m,n\in\mathbb{N},\\ \Delta_{mn}>\frac{1}{2}J}} \mu_{mn} \sum_{\substack{k\in\mathbb{N},\\ \max\{1,\frac{\Delta_{mn}}{2J}\}<k<\frac{2\Delta_{mn}}{J}}} k^{-1/2}e^{-\varrho_s k/2} \\
&\leq 8\varphi_{s+1} \sum_{\substack{m,n\in\mathbb{N},\\ \Delta_{mn}>\frac{1}{2}J}} \mu_{mn}\frac{2\Delta_{mn}}{J}\left(\frac{\Delta_{mn}}{2J}\right)^{-1/2} e^{-\varrho_s\Delta_{mn}/4J} \\
&= 32\,(2J)^{\sigma}\varphi_{s+1} \sum_{\substack{m,n\in\mathbb{N},\\ \Delta_{mn}>\frac{1}{2}J}} \frac{\mu_{mn}}{(\Delta_{mn})^{\sigma}}\left(\frac{\Delta_{mn}}{2J}\right)^{\sigma+\frac{1}{2}} e^{-\varrho_s\Delta_{mn}/4J} \\
&\leq 32\,2^{\sigma}\varphi_{s+1}\left(\frac{2\sigma+1}{e\varrho_s}\right)^{\sigma+\frac{1}{2}}\Delta_{\sigma}(J)
\end{aligned}$$

where we have used that if $\alpha>0$ and $\beta>0$ then $\sup_{x>0}x^{\alpha}e^{-\beta x}=(\frac{\alpha}{e\beta})^{\alpha}$. To complete the estimate we need that the sum $\sum_{s=0}^{\infty}\varphi_{s+1}/(\varrho_s)^{\sigma+\frac{1}{2}}$ should be finite which imposes another restriction on the choice of $\{\varphi_s\}$ and $\{E_s\}$. With our choice (66) this is guaranteed by the condition $r>\sigma+\frac{1}{2}$ since in that case

$$\sum_{s=0}^{\infty}\frac{\varphi_{s+1}}{(\varrho_s)^{\sigma+\frac{1}{2}}} = \frac{a}{\left(1-\frac{1}{q}\right)^{\sigma+\frac{1}{2}}}\sum_{s=0}^{\infty}(s+1)^{\alpha}q^{-(r-\sigma-\frac{1}{2})(s+1)} < \infty\,.$$

Hence

$$|\Omega_{\infty}| \geq \frac{17}{72}\,J - \delta_1(\sigma,r)\,\Delta_{\sigma}(J)\,\epsilon_V \tag{77}$$

where

$$\delta_1(\sigma,r) = 1440\,e\,q^{2r}2^{\sigma}\left(\frac{2\sigma+1}{\left(1-\frac{1}{q}\right)e}\right)^{\sigma+\frac{1}{2}}\mathrm{Li}_{-\alpha}(q^{-r+\sigma+\frac{1}{2}}) \tag{78}$$

Here $\mathrm{Li}_n(z)=\sum_{k=1}^{\infty}z^k/k^n$ ($|z|<1$) is the polylogarithm function. This shows (3).

To finish the proof let us assume that $\omega\in\Omega_{\infty}$. We wish to apply Proposition 9. Going through its assumptions one finds that it only remains to make a note concerning equality (48). In fact, this equality is a direct consequence of the construction of $A_s\in{}^0\mathfrak{X}_{s+1}$. Actually, by the definition of A_s (c.f. (65)), $A_s=\Gamma_s\big((1-\mathcal{D}_s)(W_s-\iota_{s-1}(W_{s-1}))\big)$, which means that for any $\omega\in\Omega_{s+1}$ and all indices (k,n,m),

$$\begin{aligned}
\big(k\omega-\Delta_{mn}+(W_s)_{0nn}(\omega)\big)(A_s)_{knm}(\omega)-(A_s)_{knm}(\omega)(W_s)_{0mm}(\omega)& \\
=\big((1-\mathcal{D}_s)(W_s-\iota_{s-1}(W_{s-1}))\big)_{knm}(\omega).&
\end{aligned} \tag{79}$$

On the other hand, by the definition of Θ_u^s (c.f. (47)) and the definition of $\mathcal{D}_s$ (c.f (33)), and since $\omega \in \Omega_\infty$, it holds true that, $\forall u,\ u > s$,

$$\begin{aligned} \Theta_u^s(\iota_{us}\mathcal{D}_s(W_s))_{knm}(\omega) &= \big([\,\iota_{u,s+1}(A_s), \iota_{us}\mathcal{D}_s(W_s)\,]\big)_{knm}(\omega) \\ &= (A_s)_{knm}(\omega)(W_s)_{0mm} - (W_s)_{0nn}(A_s)_{knm}(\omega). \end{aligned} \tag{80}$$

A combination of (79) and (80) gives (48). We conclude that according to Proposition 9 the operator $\mathbf{K}+\mathbf{V}$ is unitarily equivalent to $\mathbf{K}+\mathbf{D}(\mathbf{W})$ and hence has a pure point spectrum. This concludes the proof of Theorem 1.

10. Concluding remarks

The backbone of the proof of Theorem 1 forms an iterative procedure loosely called here and elsewhere the quantum KAM method. One of the improvements attempted in the present paper was a sort of optimalisation of this method, particularly from the point of view of assumptions imposed on the regularity of the perturbation V. In this final section we would like to briefly discuss this feature by comparing our presentation to an earlier version of the method. We shall refer to paper [9] but the main points of the discussion apply as well to other papers including the original articles [5], [6] where the quantum KAM method was established. For the sake of illustration we use a simple but basic model: $H = \sum_{m\in\mathbb{N}} m^{1+\alpha} Q_m$, i.e., $h_m = m^{1+\alpha}$, with $0 < \alpha \le 1$, and $\dim Q_m = 1$; thus $\mu_{mn} = 1$ and any $\sigma > 1/\alpha$ makes $\Delta_\sigma(J)$ finite. The perturbation V is assumed to fulfill (34) for a given $r \ge 0$.

According to Theorem 1, r is required to satisfy $r > \sigma + 1/2$ which may be compared to reference [9, Theorem 4.1] where one requires

$$r > \mathbf{r}_1 = 4\sigma + 6 + \left[\frac{(4\sigma+6)\sigma}{1+\sigma}\right] + 1. \tag{81}$$

The reason is that the procedure is done in two steps in the earlier version; in the first step preceding the iterative procedure itself the so-called adiabatic regularisation is applied on V in order to achieve a regularity in time and "space" (by the spatial part one means the factor $\mathcal{H}$ in $\mathcal{K} = L^2([\,0,T\,], dt) \otimes \mathcal{H}$) of the type

$$\exists r_1, r_2 > \mathbf{r}_2 = 4\sigma + 6, \quad \sup_{knm} |k|^{r_1} |n-m|^{r_2} |V_{knm}| < \infty. \tag{82}$$

The adiabatic regularisation brings in the summand $\left[\frac{(4\sigma+6)\sigma}{1+\sigma}\right] + 1$. In the present version both the adiabatic regularisation and condition (82) are avoided. This is related to the choice of the norm in the auxiliary Banach spaces $\mathfrak{X}_s$,

$$\|X\|_s = \sup_{\omega\neq\omega'} \sup_n \sum_{k,m} F_s(k,n,m) \left(|X_{knm}(\omega)| + \varphi_s |\partial X_{knm}(\omega,\omega')|\right).$$

In the earlier version the weights were chosen as $F_s(k,n,m) := \exp((|k| + |n - m|)/E_s)$ in order to compensate small divisors occurring in each step of the iterative method. A more careful control of the small divisors in the present version allows less restrictive weights, namely $F_s(k,n,m) = \exp(|k|/E_s)$. In more detail, indices labelling the small divisors are located in a critical subset of the lattice $\mathbb{Z}\times\mathbb{N}\times\mathbb{N}$. Definition (51) of the critical indices implies a simple estimate,

$$|k| \leq |k| + |n-m| \leq |k| + |\Delta_{mn}| \leq |k|(1+2J),$$

which explains why we effectively have, in the present version, $r_2 = 0$.

The second remark concerns Diophantine-like estimates of the small divisors governed by the sequence $\{\psi_s\}$. A bit complicated definition (56) is caused by the classification of the indices into critical and non-critical ones. However only the critical indices are of importance in this context and thus we can simplify, for the purpose of this discussion, the definition of ψ_s to

$$\psi_s = \gamma_s |k|^{1/2} e^{-\rho_s |k|/2}, \quad \varphi_{s+1} \geq \gamma_s > 0.$$

Let us compare it to the choice made in [9], namely $\psi_s = \gamma_s |k|^{-\sigma}$. The factors γ_s then occur in some key estimates; let us summarise them. The norm of the operators $\Gamma_s : \mathfrak{X}_s \to \mathfrak{X}_{s+1}$ are estimated as

$$\|\Gamma_s\| \leq \text{const}\, \frac{\varphi_{s+1}}{\gamma_s^2}$$

(this is shown in Lemma 12 but note that in this lemma we have set $\gamma_s = \varphi_{s+1}$). Another important condition is the convergence of the series

$$B_\star = \text{const} \sum_{s=0}^{\infty} \frac{\varphi_{s+1}}{\gamma_s^2 (E_{s-1})^r} < \infty$$

(c.f. (38) but there again $\gamma_s = \varphi_{s+1}$). Finally, the measure of the set of resonant frequencies, $|\cup_s \Omega_s^{\text{bad}}|$, is estimated by

$$\sum_{s=0}^{\infty} |\Omega_s^{\text{bad}}| \leq \text{const} \sum_{s=0}^{\infty} \frac{\gamma_s}{\rho_s^{\sigma+\frac{1}{2}}} < \infty, \quad \rho_s = \frac{1}{E_s} - \frac{1}{E_{s+1}}$$

(shown in the part of the proof of Theorem 1 preceding relation (77)). We recall that E_s denotes the width of the truncation of the perturbation V at step s of the algorithm (c.f. (35)). These conditions restrict the choice of the sequences $\{E_s\}$ and $\{\gamma_s\}$ which may also be regarded as parameters of the procedure. Specification (66) of these parameters, with $\gamma_s = \varphi_{s+1}$, can be compared to a polynomial behaviour of E_s and γ_s in the variable s in [9] where one sets $\varphi_{s+1} \equiv 1$ and

$$E_s = \text{const}\,(s+1)^{\nu-1}, \quad \nu > 2, \qquad \gamma_s = \text{const}\,(s+1)^{-\mu}, \quad \mu > 1.$$

The latter definition finally leads to the bound on the order of regularity of V

$$r > \frac{(2\sigma+1)\nu + 3}{\nu - 1}.$$

Thus in that case the bound varies from $r > 4\sigma + 5$ (for $\nu \to 2+$; this contributes to $\mathbf{r}_1$ in (81)) to $r > 2\sigma + 1$ ($\nu \to +\infty$). This shows why we have chosen here to truncate with exponential E_s, see (66).

In the last remark let us mention a consequence of the equality $\gamma_s = \varphi_{s+1}$. The conditions for convergence of $B_\star$ and $\cup_s \Omega_s^{\mathrm{bad}}$ become (notice that $\rho_s = \mathrm{const}/E_s$)

$$\sum_s \frac{1}{\varphi_{s+1}(E_{s-1})^r} < \infty \quad \text{and} \quad \sum_s \varphi_{s+1} E_s^{\sigma+\frac{1}{2}} < \infty$$

and are fulfilled for $r > \sigma + \frac{1}{2}$. There is however a drawback with this choice. Notice the role the coefficients φ_s play in the definition (31) of the norm $\|\cdot\|_s$. Since $\varphi_s \to 0$ as $s \to \infty$ one looses the control of the Lipschitz regularity in ω in the limit of the iterative procedure. This means that we have no information about the regularity of the eigenvectors and the eigenvalues of $\mathbf{K}+\mathbf{V}$ with respect to ω. With $r > 2\sigma+1$ we could have taken $\varphi_{s+1} = 1$ and obtained that these eigenvalues and vectors are indeed Lipschitz in ω.

Appendix A. Commutation equation

Suppose that $\mathfrak{X}$ and $\mathfrak{Y}$ are Hilbert spaces, $\dim \mathfrak{X} < \infty$, $\dim \mathfrak{Y} < \infty$, $A \in \mathcal{B}(\mathfrak{Y})$, $B \in \mathcal{B}(\mathfrak{X})$, both A and B are self-adjoint, and $V \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. If γ is a simple closed and positively oriented curve in the complex plane such that $\mathrm{Spec}(A)$ lies in the domain encircled by γ while $\mathrm{Spec}(B)$ lies in its complement then the equation

$$AW - WB = V \tag{83}$$

has a unique solution $W \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ given by

$$W = \frac{1}{2\pi\imath} \oint_\gamma (A-z)^{-1} V (B-z)^{-1} dz\,. \tag{84}$$

The verification is straightforward.

Denote $M_1 = \dim \mathfrak{X}$, $M_2 = \dim \mathfrak{Y}$. We shall need the following two estimates on the norm of $X \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$:

$$\|X\|^2 \;\leq\; \sum_{i=1}^{M_2} \sum_{j=1}^{M_1} |X_{ij}|^2 = \mathrm{Tr}\, X^* X \quad (\text{Hilbert} - \text{Schmidt norm}), \tag{85}$$

$$\|X\|^2 \;\geq\; \max\left\{ \max_{1 \leq i \leq M_2} \sum_{j=1}^{M_1} |X_{ij}|^2,\, \max_{1 \leq j \leq M_1} \sum_{i=1}^{M_2} |X_{ij}|^2 \right\}, \tag{86}$$

where (X_{ij}) is a matrix of X expressed with respect to any orthonormal bases in $\mathfrak{X}$ and $\mathfrak{Y}$.

If $\sup \mathrm{Spec}(A) < \inf \mathrm{Spec}(B)$ or $\sup \mathrm{Spec}(B) < \inf \mathrm{Spec}(A)$ we shall say that $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ are not interlaced.

Proposition 13. *If* $\mathrm{Spec}(A)$ *and* $\mathrm{Spec}(B)$ *are not interlaced then*

$$\|W\| \leq \frac{\|V\|}{\mathrm{dist}(\mathrm{Spec}(A), \mathrm{Spec}(B))}\,,$$

otherwise, if Spec(A) *and* Spec(B) *don't intersect but are interlaced,*

$$\|W\| \le (\min\{\dim\mathfrak{X},\dim\mathfrak{Y}\})^{1/2}\frac{\|V\|}{\operatorname{dist}(\operatorname{Spec}(A),\operatorname{Spec}(B))}\,.$$

Proof. (1) If $d = \inf\operatorname{Spec}(B) - \sup\operatorname{Spec}(A) > 0$ then, after a usual limit procedure, we can choose for the integration path in (84) the line which is parallel to the imaginary axis and intersects the real axis in the point $x_0 = (\sup\operatorname{Spec}(A)+\inf\operatorname{Spec}(B))/2$. So

$$\begin{aligned}\|W\|^2 &\le \frac{1}{2\pi}\int_{-\infty}^{\infty}\|(A-x_0-\imath s)^{-1}\|\|V\|\|(B-x_0-\imath s)^{-1}\|\,ds\\ &= \frac{\|V\|}{2\pi}\int_{-\infty}^{\infty}\frac{ds}{\left(\frac{d}{2}\right)^2+s^2}\\ &= \frac{\|V\|}{d}\,.\end{aligned}$$

(2) In the interlaced case we choose orthonormal bases in $\mathfrak{X}$ and $\mathfrak{Y}$ so that A and B are diagonal, $A = \operatorname{diag}(a_1,\dots,a_{M_2})$ and $B = (b_1,\dots,b_{M_1})$. For brevity let us denote $\operatorname{dist}(\operatorname{Spec}(A),\operatorname{Spec}(B))$ by d. Then $W_{ij} = V_{ij}/(a_i - b_j)$, and we can use (85), (86) to estimate

$$\begin{aligned}\|W\|^2 &\le \sum_{i=1}^{M_2}\sum_{j=1}^{M_1}\left|\frac{V_{ij}}{a_i-b_j}\right|^2 \le \sum_{i=1}^{M_2}\sum_{j=1}^{M_1}\frac{|V_{ij}|^2}{d^2}\\ &\le \sum_{i=1}^{M_2}\frac{\|V\|^2}{d^2} = M_2\frac{\|V\|^2}{d^2}\,.\end{aligned}$$

Symmetrically, $\|W\| \le M_1^{1/2}\|V\|/d$, and the result follows. □

Appendix B. Choice of a norm in a Banach space

Let

$$\mathcal{H} = \sum_{n\in\mathbb{N}}^{\oplus}\mathcal{H}_n$$

be a decomposition of a Hilbert space into a direct sum of mutually orthogonal subspaces, and $\Omega \subset \mathbb{R}$. To any couple of positive real numbers, φ and E, we relate a subspace

$$\mathfrak{A} \subset L^\infty\left(\Omega\times\mathbb{Z}\times\mathbb{N}\times\mathbb{N}, \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}^{\oplus}\mathcal{B}(\mathcal{H}_m,\mathcal{H}_n)\right)$$

formed by those elements $\mathcal{V}$ which satisfy

$$\mathcal{V}_{knm}(\omega) \in \mathcal{B}(\mathcal{H}_m,\mathcal{H}_n)$$

and have finite norm

$$\|\mathcal{V}\| = \sup_{\substack{\omega,\omega'\in\Omega\\ \omega\neq\omega'}} \sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{N}} \left(\|\mathcal{V}_{knm}(\omega)\| + \varphi\,\|\partial\mathcal{V}_{knm}(\omega,\omega')\|\right) e^{|k|/E} \tag{87}$$

where ∂ stands for the difference operator

$$\partial\mathcal{V}(\omega,\omega') = \frac{\mathcal{V}(\omega)-\mathcal{V}(\omega')}{\omega-\omega'}\,.$$

Note that the difference operator obeys the rule

$$\partial(\mathcal{U}\mathcal{V})(\omega,\omega') = \partial\mathcal{U}(\omega,\omega')\,\mathcal{V}(\omega') + \mathcal{U}(\omega)\partial\mathcal{V}(\omega,\omega')\,. \tag{88}$$

Proposition 14. *The norm in $\mathfrak{A}$ is an algebra norm with respect to the multiplication*

$$(\mathcal{U}\mathcal{V})_{knm}(\omega) = \sum_{\ell\in\mathbb{Z}} \sum_{p\in\mathbb{N}} \mathcal{U}_{k-\ell,n,p}(\omega)\,\mathcal{V}_{\ell pm}(\omega)\,. \tag{89}$$

Proof. We have to show that

$$\|\mathcal{U}\mathcal{V}\| \leq \|\mathcal{U}\|\|\mathcal{V}\|\,. \tag{90}$$

For brevity let us denote (in this proof)

$$\begin{aligned}
\mathcal{X}_p(\omega) &= \sum_{\ell\in\mathbb{Z}} \sum_{m\in\mathbb{N}} \|\mathcal{V}_{\ell pm}(\omega)\|\, e^{|\ell|/E},\\
\partial\mathcal{X}_p(\omega,\omega') &= \sum_{\ell\in\mathbb{Z}} \sum_{m\in\mathbb{N}} \|\partial\mathcal{V}_{\ell pm}(\omega,\omega')\|\, e^{|\ell|/E}.
\end{aligned}$$

Here $\partial\mathcal{X}$ is an "inseparable" symbol (which this time doesn't have the meaning ∂ of $\mathcal{X}$). It holds

$$\begin{aligned}
&\sum_k \sum_m \|(\mathcal{U}\mathcal{V})_{knm}(\omega)\|\, e^{|k|/E}\\
&\quad\leq \sum_k \sum_m \sum_\ell \sum_p \|\mathcal{U}_{k-\ell,n,p}(\omega)\|\, e^{|k-\ell|/E} \|\mathcal{V}_{\ell pm}(\omega)\|\, e^{|\ell|/E}\\
&\quad= \sum_k \sum_m \sum_\ell \sum_p \|\mathcal{U}_{knp}(\omega)\|\, e^{|k|/E} \|\mathcal{V}_{\ell pm}(\omega)\|\, e^{|\ell|/E}\\
&\quad= \sum_k \sum_p \|\mathcal{U}_{knp}(\omega)\|\, e^{|k|/E} \mathcal{X}_p(\omega)\,.
\end{aligned}$$

Similarly, using (88),

$$\begin{aligned}\sum_k \sum_m \|\partial(\mathcal{U}\mathcal{V})_{knm}(\omega)\|\, e^{|k|/E} &\leq \sum_k \sum_m \sum_\ell \sum_p \big(\|\mathcal{U}_{knp}(\omega)\|\, e^{|k|/E} \|\partial\mathcal{V}_{\ell pm}(\omega,\omega')\|\, e^{|\ell|/E} \\ &\qquad + \|\partial\mathcal{U}_{knp}(\omega,\omega')\|\, e^{|k|/E} \|\mathcal{V}_{\ell pm}(\omega')\|\, e^{|\ell|/E}\big) \\ &= \sum_k \sum_p (\|\mathcal{U}_{knp}(\omega)\|\, \partial\mathcal{X}_p(\omega,\omega') \\ &\qquad + \|\partial\mathcal{U}_{knp}(\omega,\omega')\|\, \mathcal{X}_p(\omega'))\, e^{|k|/E}\,.\end{aligned}$$

A combination of these two inequalities gives

$$\begin{aligned}&\sum_k \sum_m (\|(\mathcal{U}\mathcal{V})_{knm}(\omega)\| + \varphi\, \|\partial(\mathcal{U}\mathcal{V})_{knm}(\omega,\omega')\|)\, e^{|k|/E} \\ &\leq \sum_k \sum_p (\|\mathcal{U}_{knp}(\omega)\|(\mathcal{X}_p(\omega) + \varphi\, \partial\mathcal{X}_p(\omega,\omega')) + \varphi\, \|\partial\mathcal{U}_{knp}(\omega,\omega')\|\, \mathcal{X}_p(\omega'))\, e^{|k|/E} \\ &\leq \sup_{\omega,\omega'} \sup_p (\mathcal{X}_p(\omega) + \varphi\, \partial\mathcal{X}_p(\omega,\omega')) \sum_k \sum_p (\|\mathcal{U}_{knp}(\omega)\| + \varphi\, \|\partial\mathcal{U}_{knp}(\omega,\omega')\|) e^{|k|/E} \\ &= \|\mathcal{V}\| \sum_k \sum_p (\|\mathcal{U}_{knp}(\omega)\| + \varphi\, \|\partial\mathcal{U}_{knp}(\omega,\omega')\|) e^{|k|/E}\,.\end{aligned}$$

To obtain (90) it suffices to apply $\sup_{\omega,\omega'} \sup_n$ to this inequality. ☐

Suppose now that two couples of positive real numbers, (φ_1, E_1) and (φ_2, E_2), are given and that it holds

$$\varrho = \frac{1}{E_1} - \frac{1}{E_2} \geq 0 \quad \text{and} \quad \varphi_2 \leq \varphi_1\,. \tag{91}$$

Consequently, we have two Banach spaces, $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Furthermore, we suppose that there is given an element

$$\Gamma \in \mathrm{Map}\left(\Omega \times \mathbb{Z} \times \mathbb{N} \times \mathbb{N}, \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}}^{\oplus} \mathcal{B}(\mathcal{B}(\mathcal{H}_m, \mathcal{H}_n))\right), \tag{92}$$

such that for each couple $(\omega, k) \in \Omega \times \mathbb{Z}$ and each double index $(n, m) \in \mathbb{N} \times \mathbb{N}$, $\Gamma_{knm}(\omega)$ belongs to $\mathcal{B}(\mathcal{B}(\mathcal{H}_m, \mathcal{H}_n))$. Γ naturally determines a linear mapping, called for the sake of simplicity also Γ, from $\mathfrak{A}_1$ to $\mathfrak{A}_2$, according to the prescription

$$\Gamma(\mathcal{V})_{knm}(\omega) = \Gamma_{knm}(\omega)(\mathcal{V}_{knm}(\omega))\,. \tag{93}$$

Concerning the difference operator, in this case one can apply the rule

$$\partial\,(\Gamma(\mathcal{V}))\,(\omega,\omega') = \partial\Gamma(\omega,\omega')\,(\mathcal{V}(\omega')) + \Gamma(\omega)(\partial\mathcal{V}(\omega,\omega'))\,.$$

Proposition 15. *The norm of* $\Gamma : \mathfrak{A}_1 \to \mathfrak{A}_2$ *can be estimated as follows,*

$$\|\Gamma\| \leq \sup_{\substack{\omega,\omega'\in\Omega\\ \omega\neq\omega'}} \sup_{k\in\mathbb{Z}} \sup_{(n,m)\in\mathbb{N}\times\mathbb{N}} e^{-\varrho|k|}\, (\|\Gamma_{knm}(\omega)\| + \varphi_2\, \|\partial\Gamma_{knm}(\omega,\omega')\|)\,. \tag{94}$$

Proof. Notice that, if convenient, one can interchange ω and ω' in $\|\partial\mathcal{U}(\omega,\omega')\|$. It holds

$$\begin{aligned}\sum_k\sum_m &\left(\|\Gamma_{knm}(\omega)(\mathcal{V}_{knm}(\omega))\| + \varphi_2\,\|\partial\left(\Gamma_{knm}(\mathcal{V}_{knm})\right)(\omega,\omega')\|\right)e^{|k|/E_2}\\ &\le \sum_k\sum_m \Big(\|\mathcal{V}_{knm}(\omega)\|(\|\Gamma_{knm}(\omega)\| + \varphi_2\,\|\partial\Gamma_{knm}(\omega,\omega')\|)e^{-\varrho|k|}\\ &\qquad +\varphi_2\,\|\partial\mathcal{V}_{knm}(\omega,\omega')\|\,\|\Gamma_{knm}(\omega')\|e^{-\varrho|k|}\Big)\,e^{|k|/E_1}\\ &\le \sup_{\omega,\omega'}\sup_k\sup_{(n,m)} e^{-\varrho|k|}\left(\|\Gamma_{knm}(\omega)\| + \varphi_2\,\|\partial\Gamma_{knm}(\omega,\omega')\|\right)\\ &\qquad \times\sum_k\sum_m\left(\|\mathcal{V}_{knm}(\omega)\| + \varphi_1\,\|\partial\mathcal{V}_{knm}(\omega,\omega')\|\right)e^{|k|/E_1}\,.\end{aligned}$$

To finish the proof it suffices to apply $\sup_{\omega,\omega'}\sup_n$ to this inequality. □

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PROGRESSIVE DIAGONALIZATION AND APPLICATIONS

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ABSTRACT. We give a partial review of what is known so far on stability of periodically driven quantum systems versus regularity of the bounded driven force. In particular we emphasize the fact that unbounded degeneracies of the unperturbed Hamiltonian are allowed. Then we give a detailed description of an extension to some unbounded driven forces. This is done by representing the Schrödinger equation in the instantaneous basis of the time dependent Hamiltonian with a method that we call progressive diagonalization.

1. THE MAIN THEOREM

This paper is concerned with the spectral analysis of Floquet Hamiltonians associated to quantum systems which are periodically driven. They are described by the Schrödinger equation:

$$(1)\qquad (-\mathrm{i}\partial_t + H_0 + V(\omega t))\,\psi = 0, \quad \begin{cases} H_0 \text{ self-adjoint on } \mathcal{H}, \\ t \to V(t),\ 2\pi \text{ periodic}, \\ \omega > 0,\ \text{a real frequency}, \\ \mathbb{R} \ni t \to \psi(t) \in \mathcal{H}, \end{cases}$$

where $\mathcal{H}$ is a separable Hilbert space, and H_0 has the following type of spectral decomposition (E_n, P_n denoting respectively the eigenvalues in ascending order and the eigenprojections):

$$H_0 = \sum_{n=1}^{\infty} E_n P_n, \quad M_n := \dim P_n < \infty$$

with a growing gap condition of the type

$$(2)\qquad \exists\, \sigma > 0, \quad \frac{1}{(\Delta E_\sigma)^\sigma} := \sum_{m \neq n} \frac{M_m M_n}{|E_m - E_n|^\sigma} < \infty.$$

The driven force is given by a time dependent real potential V which is, in the first part of this paper, bounded in the norm

$$(3)\qquad \|V\|_r := \sup_{m\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{N}} \|V(k,m,n)\| \max\{|k|^r, 1\},$$

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where $\|V(k,m,n)\|$ denotes the operator norm of

$$V(k,m,n) := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kt} P_m V(t) P_n \, \mathrm{d}t : \mathcal{H} \to \mathcal{H}. \tag{4}$$

The following main theorem is about the self-adjoint operator $K := K_0 + V$ with $K_0 := -\mathrm{i}\omega\partial_t \otimes 1 + 1 \otimes H_0$ acting on the Hilbert space $\mathcal{K} := L^2(S^1) \otimes \mathcal{H}$, i.e. functions which are 2π-periodic in time.

Theorem 1.1. *Let $\omega_0 > 0$, $\Omega_0 := \left[\frac{8}{9}\omega_0, \frac{9}{8}\omega_0\right]$, assume* (2) *for some $\sigma > 0$, and let*

$$\Delta_0 := \min_{m \neq n} |E_m - E_n|.$$

Then $\forall r > \sigma + \frac{1}{2}$, $\exists\, C_1 > 0$ and $C_2(\sigma, r) > 0$ such that

$$\|V\|_r < \min\left\{ \frac{4\Delta_0}{C_1}, \frac{\omega_0}{C_1}, \frac{\omega_0}{C_2}\left(\frac{\Delta E_\sigma}{\omega_0}\right)^{\sigma} \right\}$$

implies $\exists\, \Omega_\infty \subset \Omega_0$ with

$$\frac{|\Omega_\infty|}{|\Omega_0|} \geq 1 - \frac{\|V\|_r}{\frac{\omega_0}{C_2}\left(\frac{\Delta E_\sigma}{\omega_0}\right)^{\sigma}},$$

so that K is pure point for all $\omega \in \Omega_\infty$. Here $|\Omega_*|$ denotes the Lebesgue measure of Ω_*.

The proof of this theorem and its complement that we state at the end of this section can be found in [7]. This theorem is a result in singular perturbation theory since, as shown in [9], one has

$$(2) \stackrel{\text{obviously}}{\Longrightarrow} \limsup_{n\to\infty} E_n = +\infty \stackrel{[9]}{\Longrightarrow} \forall \text{ a.a. } \omega,\ \operatorname{spect} K_0 = \mathbb{R},$$

i.e. for almost all ω, K_0 has a dense pure point spectrum. To be able to overcome this small divisors difficulty we use a technique which consists in applying to $K_0 + V$ an infinite sequence of unitary transformations so that at the s^{th} step

$$K_0 + V \sim K_0 + G_s + V_s, \quad \text{with } V_s = \mathcal{O}(\|V\|_{r-\sigma-\frac{1}{2}}^{2^{s-1}}),$$

i.e. $K_0 + V$ is unitary equivalent to a diagonal part $K_0 + G_s$ in the eigenbasis of K_0, plus an off diagonal part V_s which is super exponentially small in the s variable provided $\|V\|_r$ is small enough. This is why we like to call this method *progressive diagonalization* although it is known usually under the name KAM-type method, since this is an adaptation of the famous Kolmogorov-Arnold-Moser method originally invented to treat perturbations of integrable Hamiltonians in classical mechanics. An extension of the previous theorem to certain classes of unbounded perturbations V is given in Section 3 (see Theorem 3.3). We shall do it by (block-) diagonalizing $H_0 + V(t)$ for each t, i.e. by constructing a time dependent unitary transformation $J(t)$ such that $H_0 + V(t) = J(t)(H_0 + G(t))J(t)^*$, where $H_0 + G(t)$ commutes with H_0, thus

$$K_0 + V \sim -\mathrm{i}\omega\partial_t + H_0 + G(t) - \mathrm{i}\omega J(t)^* \dot{J}(t)$$

($\dot{J}$ denotes the time derivative of J). V and H_0 are such that the new perturbation $G(t) - \mathrm{i}\omega J(t)^*\dot{J}(t)$ is bounded, so that we can apply Theorem 1.1. This diagonalization of $H_0 + V$ will be done in detail with a progressive diagonalization method (PDM) which is simpler than the one used for Theorem 1.1 since we do not have small divisors here. We think this is a good starting point for readers who are not familiar with this PDM. This idea of regularizing an unbounded V by going to the instantaneous basis of $H_0 + V(t)$ is not new, (see e.g. [13, 1]). Let us also mention the recent work [4] which also treats the Schrödinger equation with unbounded perturbations which are quasi-periodic and ***analytic*** in time; here we treat the ***differential*** periodic case. The use of KAM technique to diagonalize quantum Floquet Hamiltonians first appeared in [3], where pulsed rotors of the type

(5) $$-\mathrm{i}\omega\partial_t + H_0 + f(t)W(x) \quad \text{acting on } L^2(S^1) \otimes L^2(S^d),$$

where $d = 1$, $H_0 = -\Delta$, f and W are analytic, were considered. Later on, the adaptation of the Nash-Moser ideas to treat non-analytic perturbations was done in [6] for the special case of (one dimensional) driven harmonic oscillators. These ideas where extended to a large class of models in [8]. However, to our knowledge, the above Theorem 1.1 is the first result that allows degeneracies of eigenvalues of H_0 which are not uniformly bounded with respect to the quantum number n. Consequently we can exhibit frequencies such that the quantum top model in arbitrary dimension, i.e. the higher dimensional versions of the pulsed rotor (see (5) and Section 4.1), is pure point. One of the main goals of the spectral analysis of these Floquet hamiltonians is the study of the stability of periodically driven quantum systems, since it is known that

(6) $$K_0 + V \text{ is pure point} \quad \overset{[10]}{\Longleftrightarrow} \quad \lim_{n\to\infty} \sup_{t \gtrless 0} \Big\| \sum_{m=n}^{\infty} P_m \psi(t) \Big\| = 0, \quad \forall \psi(0) \in \mathcal{H}$$

because $\exp(-\mathrm{i}T(K_0 + V))$ is unitary equivalent to $1 \otimes U(T, 0)$ where $U(T, 0)$ denotes the propagator over the period T associated to the Schrödinger equation (1), (see [11, 16]). The condition in the right-hand side of (6) says that the probability that the quantum trajectory with an arbitrary initial condition $\psi(0)$ explores in the full history the eigenstates of H_0 of energy higher than E_n becomes smaller and smaller as n gets larger and larger. On the other hand, if $\psi(0)$ belongs to the continuous spectral subspace of $U(T, 0)$ then (see [10]):

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \|P_m \psi(t)\| \, \mathrm{d}t = 0, \quad \forall m \in \mathbb{N},$$

which means that in the time average the probability that the trajectory stays in the m^{th} spectral subspace of H_0 vanishes. The conclusion that can be drawn from the articles [3, 8, 7] is that for non-resonant (i.e. diophantine) frequencies the pulsed rotor is stable if the driving force is sufficiently regular in time (see Figure 1 below) and sufficiently small in amplitude. In addition it is known (see [10]) that if f is sufficiently regular in time and ω is resonant (i.e. rational) the pulsed rotor is stable. The situation is different for the kicked rotor (i.e. $f(t) := \delta(t)$, the Dirac distribution): it was proved in [5] that if the frequency is rational or even Liouville, then one can find W's such that $U(T, 0)$ has a continuous spectral component. However nothing is known for non resonant frequencies. Since the kicked rotor corresponds to $r < -1$ in the notation of (3) and the known values of r for which $U(T, 0)$ is pure point are $r > \frac{3}{2}$ the sequences of papers [3, 8, 7] can be considered as reports on the efforts devoted to the long march from the pulsed rotor to the kicked rotor (in the non resonant case). In Figure 1 below we give a diagram which tells the history of this march. Since the regularity in the space variable has also played a role, we present this diagram in the plane of points (r_1, r_2) which says that the following generalization of (3)

$$\|V\|_{r_1,r_2} := \sup_{m\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{N}} \|V(k,m,n)\| \langle k\rangle^{r_1} \langle m-n\rangle^{r_2}$$

is finite, with $\langle x\rangle^2 := 1 + x^2$.

$\sigma = 1 + 0,\ \sup_n M_n < \infty,\ \omega$ non resonant and $\|V\|_{r_1,r_2}$ small enough

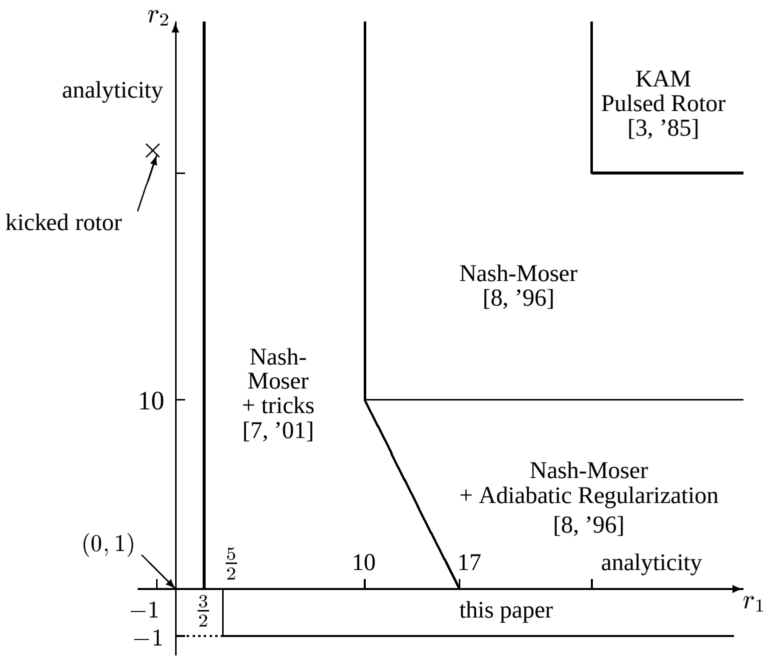

Figure 1. Historical diagram of progress toward the kicked rotor

The pure point property of K from which the stability (6) follows does not imply in general that

$$\sup_{t \gtrless 0} (H_0\psi(t), \psi(t)) < \infty \tag{7}$$

i.e. the energy is uniformly bounded. Notice that the converse is obviously true. It is believed that to get (7) one should require sufficient regularity of the eigenprojectors of K. That is why the following complement to Theorem 1.1 may be of interest. We have also added some explicit bound on the constants C_1 and C_2. It will be necessary in Section 3 to consider potentials V which depend on the frequency ω in a more elaborate way. Suppose that $V : \mathbb{R} \times \mathbb{R}_+ \to \mathcal{B}(\mathcal{H})$ is a bounded measurable function, which is 2π

periodic with respect to the first variable and such that for almost all $t \in \mathbb{R}$ and $\omega \in \mathbb{R}_+$, $V(t,\omega)^* = V(t,\omega)$. For such V we modify $\|V\|_r$ as follows:

$$\|V\|_r := \sup_{\omega,\omega'\in\Omega_0} \sup_{m\in\mathbb{N}} \sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}}} (\|V_{kmn}(\omega)\| + \omega_0\|\partial_\omega V_{kmn}(\omega,\omega')\|) \max\{|k|^r, 1\},$$

where

$$V_{kmn}(\omega) := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kt} P_m V(t,\omega) P_n \,\mathrm{d}t$$

and

$$\partial_\omega V_{kmn}(\omega,\omega') := \frac{V_{kmn}(\omega) - V_{kmn}(\omega')}{\omega - \omega'}.$$

Complement of Theorem 1.1. *In addition to the statements in Theorem* 1.1 *one also has:*

(a) *each eigenprojection P of K is bounded in the norm*

$$\|P\|_{r-\sigma-\frac{1}{2}} = \sup_{m\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{N}} \|P(k,m,n)\| \max\{|k|^{r-\sigma-\frac{1}{2}}, 1\};$$

(b) *the following values of the constants are allowed: $C_1 = 24305$, and*

$$C_2(\sigma,r) = \frac{C(\sigma)}{\min\{r-\sigma-\frac{1}{2}, \frac{7}{8}(2\sigma+1)\}^3},$$

with

$$C(\sigma) = 25223\,\pi(2\sigma+1)^3 \left(\frac{2(2\sigma+1)}{\mathrm{e}\Big(1-\exp\Big(\frac{-4}{2\sigma+1}\Big)\Big)}\right)^{\sigma+\frac{1}{2}};$$

(c) *Theorem* 1.1 *extends to $V : \mathbb{R}\times\mathbb{R}_+ \to \mathcal{B}(\mathcal{H})$ of the type described above.*

In the progressive diagonalization method one must solve at each step a commutator equation of the type

$$[K_0 + G_s, W_s] = V_s.$$

This is done block-componentwise, i.e. with the notation (4), solving for each $(k,m,n) \in \mathbb{Z}\times\mathbb{N}\times\mathbb{N}$ the following matrix equation in the unknown $W_s(k,m,n)$:

$$(\omega k + E_m + G_s(m))W_s(k,m,n) - W_s(k,m,n)(E_n + G_s(n)) = V_s(k,m,n).$$

We are interested in the best possible estimate of $\|W_s(k,m,n)\|$ in terms of $\|V_s(k,m,n)\|$. In Section 2 we report on a method to solve this equation which is, in our opinion, the best one known so far. Finally we present two applications in Section 4.

2. On the commutator equation

Let E and F be two Hilbert spaces and $\mathcal{B}(E), \mathcal{B}(F)$ the Banach spaces of bounded endomorphisms on E and F respectively, equipped with the usual operator norm. Let $A \in \mathcal{B}(E)$ and $B \in \mathcal{B}(F)$ be self-adjoint operators such that

$$d_{A,B} := \operatorname{dist}(\operatorname{spect} A, \operatorname{spect} B) > 0. \tag{8}$$

To each Y in the space $\mathcal{B}(F,E)$ of bounded operators from F into E, we want to associate $X \in \mathcal{B}(F,E)$ defined as follows:

$$\mathrm{ad}_{A,B}X = Y, \quad \text{where } \mathrm{ad}_{A,B}X := AX - XB$$

A review on answers about this question can be found in the beautiful paper [2]. In particular one can find there the following result.

Lemma 2.1. *Under the conditions described above* $\mathrm{ad}_{A,B}$ *is a bounded linear mapping which has a bounded inverse* $\Gamma_{A,B}$ *and*

$$\|\Gamma_{A,B}\| \leq \frac{\pi}{2}\,\frac{1}{d_{A,B}}\,.$$

Remark 2.2. (a) In fact, in some special cases the constant $\frac{\pi}{2}$ can be replaced by 1. We have not found useful to pay attention to these subtleties here.

(b) The solution X is given by

$$X := \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}tA} Y \mathrm{e}^{\mathrm{i}tB} f(t)\,\mathrm{d}t$$

for any $f \in L^1(\mathbb{R})$ such that its Fourier transform $\widehat{f}$ obeys $\sqrt{2\pi}\widehat{f}(s) = s^{-1}$ on the set $\mathrm{spect}\,A - \mathrm{spect}\,B$. Clearly this shows that $\|X\| \leq \|f\|_1\|Y\|$. Optimizing over such f leads to the constant $\frac{\pi}{2}$.

3. Unbounded perturbations

3.1. **The setting.** We start by the description of the class of unbounded perturbations we shall consider. Let H_0 be a *positive self-adjoint* operator on the Hilbert space $\mathcal{H}$ and $\{P_n\}_{n\in\mathbb{N}}$ be a complete set of mutually orthogonal projections which reduces H_0. We denote $E_n := P_nH_0P_n = H_0P_n$, $\mathcal{H}_n := \mathrm{Ran}\,P_n$, and let $\mathcal{H}^{(\mathrm{d})}$ be the algebraic direct sum $\bigoplus_{n\in\mathbb{N}} \mathrm{Ran}\,P_n$. We introduce the following Banach spaces: for all $1 \leq p \leq \infty$

$$L^p(\mathcal{H}^{(\mathrm{d})}) \ni u = \bigoplus_{n\in\mathbb{N}} u_n \quad \Leftrightarrow \quad \|u\|_p^p := \sum_{n\in\mathbb{N}} \|u_n\|^p < \infty,$$

where $\|\cdot\|$ is the norm of $\mathcal{H}$. Of course $L^2(\mathcal{H}^{(\mathrm{d})})$ is nothing but $\mathcal{H}$ and $\|u\|_\infty := \sup_n \|u_n\|$. Then $\mathcal{B}^{q,p}$, $1 \leq p,q \leq \infty$, will denote the Banach spaces of bounded operators defined on $L^p(\mathcal{H}^{(\mathrm{d})})$ with values in $L^q(\mathcal{H}^{(\mathrm{d})})$ and $\|\cdot\|_{q,p}$ its operator norm. We note that

$$\|X\|_{\infty,1} = \sup_{n,m\in\mathbb{N}} \|X(m,n)\|$$

and

$$\|X\|_{1,1} = \sup_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \|X(m,n)\|, \quad \|X\|_{\infty,\infty} = \sup_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}} \|X(m,n)\|,$$

where $X(m,n)$ is the block element of X which acts from $\mathcal{H}_n$ into $\mathcal{H}_m$ and $\|X(m,n)\|$ its norm as a bounded operator on $\mathcal{H}$. We shall say that $X \in \mathcal{B}^{q,p}$ is *symmetric*, respectively *antisymmetric*, if $X(n,m) = X(m,n)^*$, respectively $X(n,m) = -X(m,n)^*$, for all m,n. This definition coincides with the usual one in $\mathcal{B}^{2,2} \sim \mathcal{B}(\mathcal{H})$. We remark that if X is symmetric or antisymmetric then $X \in \mathcal{B}^{1,1}$ if and only if $X \in \mathcal{B}^{\infty,\infty}$, if and only if $X \in \mathcal{B}_{\rm SH} := \mathcal{B}^{1,1} \cap \mathcal{B}^{\infty,\infty}$; this last operator space is equipped with the norm $\|X\|_{\rm SH} := \max\{\|X\|_{1,1}, \|X\|_{\infty,\infty}\}$. It is known (see [14, Example III.2.3]) that $\mathcal{B}_{\rm SH}$ is contained in all $\mathcal{B}^{p,p}$, $1 \le p \le \infty$, and in particular in $\mathcal{B}(\mathcal{H})$. It is easy to check that $\mathcal{B}_{\rm SH}$ is a Banach algebra. We require the two following conditions on the spectra of H_0:

$$\text{(GGC}H_0\text{)} \qquad \frac{1}{\Delta E} := \sup_m \sum_{n \neq m} \frac{1}{\Delta_{m,n}} < \infty$$

with

$$\Delta_{m,n} := \operatorname{dist}\left(\operatorname{spect} E_m, \operatorname{spect} E_n\right),$$

which expresses that the distances between the spectrum of two blocks E_m and E_n grows sufficiently rapidly with $|m - n|$. The second condition says that each blocks E_n must be bounded:

$$\text{(BBC}H_0\text{)} \qquad E_n \in \mathcal{B}(\mathcal{H}), \quad \forall n.$$

3.2. **A class of unbounded perturbations.** We make the following assumptions on the perturbation of H_0 to be considered:

(UV) $\qquad V \in \mathcal{B}^{\infty,1}$ *and is symmetric*.

Strictly speaking such a V is not in general an operator acting on $\mathcal{H}$ but the following estimate shows that it can be seen as H_0-bounded in the quadratic form sense with zero relative bound: if $R_0(a) := (H_0 - a)^{-1}$ with $a < 0$, then

$$\|R_0(a)^{\frac{1}{2}} V R_0(a)^{\frac{1}{2}}\| \le \sum_{n \in \mathbb{N}} \frac{\|V\|_{\infty,1}}{\operatorname{dist}\left(a, \operatorname{spect} E_n\right)} \overset{a \to -\infty}{\longrightarrow} 0.$$

Indeed, since that $R_0(a)^{\frac{1}{2}}$ acts diagonally on $\mathcal{H}^{(\rm d)}$ one gets immediately from (GGCH_0) that

$$\max\left\{\|R_0(a)^{\frac{1}{2}}\|_{1,2}, \|R_0(a)^{\frac{1}{2}}\|_{2,\infty}\right\} \le \left(\sum_{m \in \mathbb{N}} \frac{1}{\operatorname{dist}\left(a, \operatorname{spect} E_m\right)}\right)^{\frac{1}{2}}.$$

This allows to consider $R_0(a)^{\frac{1}{2}} V R_0(a)^{\frac{1}{2}}$ as

$$L^2(\mathcal{H}^{(\rm d)}) \overset{R_0(a)^{\frac{1}{2}}}{\longrightarrow} L^1(\mathcal{H}^{(\rm d)}) \overset{V}{\longrightarrow} L^\infty(\mathcal{H}^{(\rm d)}) \overset{R_0(a)^{\frac{1}{2}}}{\longrightarrow} L^2(\mathcal{H}^{(\rm d)});$$

hence its above estimate and limiting behaviour as $a \to \infty$ follow easily.

3.3. **Progressive diagonalization of $H_0 + V$.** Here we show

Theorem 3.1. *Assume* $H_0 \ge 0$ *and* V *obey* (GGCH_0), (BBCH_0) *and* (UV). *If*

$$\|V\|_{\infty,1} \le \frac{\Delta E}{8},$$

then there exists $J \in \mathcal{B}_{\rm SH}$ *and* $G \in \mathcal{B}^{2,2}$ *such that*

$$H_0 + V = J(H_0 + G)J^*$$

with

(i) $[H_0, G] = 0$*;*

(ii) J *unitary in* $\mathcal{B}^{2,2}$*;*

(iii) $\|J\|_{\rm SH} \le \frac{3}{2}$ *and* $\|G\| \le 2\|V\|_{\infty,1}$*;*

(iv) $[H_0, J] \in \mathcal{B}^{\infty,1}$.

Remark 3.2. (a) Since ΔE is smaller than the smallest gap of H_0, the bound on $\|G\| \leq \frac{1}{4}\Delta E$ says in particular that each gap of H_0 remains open after perturbation by V. The bound on J will be used later on.

(b) The algorithm says that G belongs to $\mathcal{B}^{\infty,1}$, which combined with (i) gives $G \in \mathcal{B}^{2,2}$.

(c) The property (iv) is the key of the so-called “adiabatic regularization method” first proposed by Howland ([12]) for the case of bounded V. Its proof is immediate from the formula $H_0 + V = J(H_0 + G)J^*$ since it is equivalent to $[H_0, J] = JG - VJ$ and since $J \in \mathcal{B}^{1,1} \cap \mathcal{B}^{\infty,\infty}$, $G, V \in \mathcal{B}^{\infty,1}$. This trick was systematically used in [8, Section 3].

3.3.1. *The formal algorithm.* With $H_0 + V$ we form a first 4-tuple of operators

$$(U_0 := \mathrm{id}\,, G_1 := \mathrm{diag}\, V, V_1 := \mathrm{offdiag}\, V, H_1 := H_0 + G_1 + V_1)\,,$$

where

$$\mathrm{diag}\, X := \sum_{n\in\mathbb{N}} P_n X P_n, \quad \mathrm{offdiag}\, X := \sum_{m\neq n} P_m X P_n.$$

Clearly U_0 is unitary, G_1 diagonal (i.e. commutes with H_0), and V_1 is symmetric. Starting from this 4-tuple we generate recursively an infinite sequence of such 4-tuples as follows: let W_s be the solution of

$$[H_0 + G_s, W_s] = V_s \quad \text{and} \quad \mathrm{diag}\ W_s = 0;$$

we shall use the notations $\mathrm{ad}_A B := [A, B] := AB - BA$. Then we define

$$H_{s+1} := \mathrm{e}^{W_s} H_s \mathrm{e}^{-W_s} = H_0 + G_s + \sum_{k=1}^{\infty} \frac{k}{(k+1)!}\mathrm{ad}^k_{W_s} V_s \tag{9}$$

and set

$$U_s := \mathrm{e}^{W_s} U_{s-1}, \quad G_{s+1} := \mathrm{diag}\, H_{s+1} - H_0, \quad V_{s+1} = \mathrm{offdiag}\, H_{s+1}.$$

Since $H_0 + G_s$ and V_s are symmetric W_s is antisymmetric and therefore e^{W_s} and U_s are formally unitary. Consequently

$$H_0 + G_{s+1} + V_{s+1} = U_s(H_0 + V)U_s^{-1}, \tag{10}$$

and to achieve our goal we have to prove that $V_s \to 0$, $G_s \to G_\infty$ and $U_s \to U_\infty$ as $s \to \infty$.

3.3.2. *Convergence of the algorithm.* We solve the commutator equation $[H_0 + G_s, W_s] = V_s$ block-wise, i.e. for all $m \neq n$, we look for $W_s(m, n)$ such that

$$(E_m + G_s(m))W_s(m, n) - W_s(m, n)(E_n + G_s(n)) = V_s(m, n).$$

Notice the notation $G_s(m) := G_s(m, m)$. Assume for the moment that

$$4\|G_s(m)\| \leq \Delta_m := \inf_{m\neq n} \Delta_{m,n}, \quad \forall s \geq 1,\ \forall m \in \mathbb{N}. \tag{11}$$

This implies that $H_0 + G_s$ fulfills (BBCH_0) and

$$\mathrm{dist}\,(\mathrm{spect}\, E_m + G_s(m), \mathrm{spect}\, E_n + G_s(n)) \geq \frac{1}{2}\Delta_{m,n}, \quad \forall m \neq n.$$

Hence by the lemma of Section 2 we know that $W_s(m,n)$ is well defined and obeys

$$\|W_s(m,n)\| \le \pi \frac{\|V_s\|_{\infty,1}}{\Delta_{m,n}} \quad \Rightarrow \quad \|W_s\|_{\mathrm{SH}} \le \pi \frac{\|V_s\|_{\infty,1}}{\Delta E},$$

i.e. W_s belongs to $\mathcal{B}^{1,1} \cap \mathcal{B}^{\infty,\infty}$. This shows that $\mathrm{ad}_{W_s} : \mathcal{B}^{\infty,1} \to \mathcal{B}^{\infty,1}$ is bounded by $2\pi\|V_s\|_{\infty,1}\Delta E^{-1}$, and due to (9),

$$\|V_{s+1}\|_{\infty,1} \le \Phi\Big(\frac{2\pi\|V_s\|_{\infty,1}}{\Delta E}\Big)\|V_s\|_{\infty,1},$$

where $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is the strictly increasing analytic function defined by $\Phi(x) := \mathrm{e}^x - \frac{1}{x}(\mathrm{e}^x - 1)$ whose Taylor expansion is $\sum_{k\ge 1} \frac{k}{(k+1)!} x^k$.

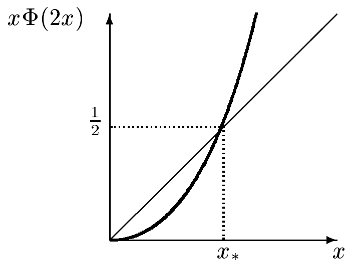

Figure 2. Graph of $x \mapsto x\Phi(2x)$ and its fixed point $x_* = \frac{1}{2}$

With $x_s := \pi\|V_s\|_{\infty,1}\Delta E^{-1}$, the above inequality becomes $x_{s+1} \le \Phi(2x_s)x_s$. This is an elementary exercise to check that the series $\{x_s\}_s$ is summable if $x_1 < x_* := \frac{1}{2}$. Thus we get

$$\|V\|_{\infty,1} \le \frac{\Delta E}{8} \quad \Rightarrow \quad \sum_{s=1}^{\infty} \|W_s\|_{\mathrm{SH}} \le \sum_{s=1}^{\infty} x_s \le \frac{x_1}{1-\Phi(2x_1)}.$$

The summability of $\{x_s\}_s$ implies that $\|V_s\|_{\infty,1} \to 0$ as $s \to \infty$, and that $\sum_{s\ge 1} \|W_s\|_{\mathrm{SH}} < \infty$. This last property shows that U_s is convergent in $\mathcal{B}_{\mathrm{SH}}$ to some U_∞ as $s \to \infty$. We must check now whether the required property on G_s, i.e. (11), is verified. Since $G_{s+1} - G_s = \mathrm{diag}\,\Phi(\mathrm{ad}_{W_s})V_s$ and $\Delta_m > \Delta E$, we have successively

$$\begin{aligned}
(11) \quad &\Leftarrow \quad \sum_{s=1}^{\infty} \|G_{s+1} - G_s\| + \|G_1\| \le \frac{1}{4}\Delta_m \\
&\Leftarrow \quad \sum_{s=1}^{\infty} x_s \Phi(2x_s)\frac{1}{\pi}\Delta E + \|G_1\| \le \frac{1}{4}\Delta E \\
&\Leftarrow \quad \|G_1\| \le 0.13\,\Delta E \quad \Leftarrow \quad \|V\|_{\infty,1} \le \frac{\Delta E}{8},
\end{aligned}$$

since one can check numerically that $\frac{1}{4} - \frac{1}{\pi}\sum_{s=1}^{\infty} x_s\Phi(2x_s) \ge 0.13$ if $x_1 \le \frac{\pi}{8}$ (see below for this bound on x_1). Thus (11) is true and we have also shown that G_s converges to some diagonal and bounded G_∞ as $s \to \infty$. To pass from (10) to $H_0 + G_\infty = U_\infty(H_0+V)U_\infty^{-1}$ using the three ingredients $\|V_s\|_{\infty,1} \to 0$, $G_s \to G_\infty$ and $U_s \to U_\infty$ is not as obvious as it seems; we have to adapt the technique of [8, Section 2.4]. We have renamed G_∞ by G and U_∞ by J for later convenience. Finally we derive the bound on $\|U_\infty\|_{\mathrm{SH}}$ and $\|G_\infty\|$.

$$\|U_\infty\|_{\mathrm{SH}} \le \exp\left(\sum_{k=1}^{\infty}\|W_k\|_{\mathrm{SH}}\right) \le \frac{3}{2},$$

since one can check numerically that $\exp\left(\sum_{s=1}^{\infty} x_s\Phi(2x_s)\right) \le \frac{3}{2}$, with

$$x_1 := \frac{\pi\|V_1\|_{\infty,1}}{\Delta E} \le \frac{\pi}{8}.$$

Concerning G_s, notice that $\|X\|_{\mathrm{SH}} = \|X\|$ if X is diagonal. Then we get

$$\begin{aligned}
\|G_\infty\| &\le \|G_1\| + \sum_{s=1}^{\infty}\|G_{s+1}-G_s\| \le \|V\|_{\infty,1} + \sum_{s=1}^{\infty} x_s\Phi(2x_s)\frac{1}{\pi}\Delta E\\
&\le \|V\|_{\infty,1} + \left(\Phi(2x_1) + \frac{\Phi(2x_1)\Phi(2x_1\Phi(2x_1))}{1-\Phi(2x_1\Phi(2x_1)\Phi(2x_1\Phi(2x_1)))}\right)x_1\frac{\Delta E}{\pi}\\
&= \|V\|_{\infty,1}\left(1+\Phi(2x_1) + \frac{\Phi(2x_1)\Phi(2x_1\Phi(2x_1))}{1-\Phi(2x_1\Phi(2x_1)\Phi(2x_1\Phi(2x_1)))}\right)\\
&\le 2\|V\|_{\infty,1}.
\end{aligned}$$

The above analytic bound on $\sum_{s=1}^{\infty} x_s\Phi(2x_s)$ is obtained with elementary manipulation and we end up with a numerical computation with $x_1 = \frac{\pi}{8}$ (notice that Φ is increasing).

3.4. **Pure pointness of K_0+V.** Let $V:\mathbb{R}\to\mathcal{B}^{\infty,1}$ be a 2π-periodic symmetric function, with the notation (4) we define the new norm

$$\|\!|V|\!\|_r := \sup_{m,n\in\mathbb{N}}\sum_{k\in\mathbb{Z}}\|V(k,m,n)\|\max\{|k|^r,1\}. \tag{12}$$

We shall prove that $K := K_0+V$ is self-adjoint on a suitable domain and

Theorem 3.3. *Let* $\omega_0 > 0$, $\Omega_0 := \left[\frac{8}{9}\omega_0, \frac{9}{8}\omega_0\right]$, *assume* (2) *for some* $\sigma > 0$, *and let* $\Delta_0 := \min_{m\neq n}|E_m - E_n|$. *Then* $\forall r > \sigma + \frac{3}{2}$, $\exists\, C_1 > 0$ *and* $\widetilde{C}_2(\sigma,r) > 0$, *such that*

$$\|\!|V|\!\|_r < \frac{1}{2(1+8\frac{\omega_0}{\Delta E})}\min\left\{\frac{4\Delta_0}{C_1}, \frac{\omega_0}{C_1}, \frac{\omega_0}{\widetilde{C}_2}\left(\frac{\Delta E_\sigma}{\omega_0}\right)^{\sigma}, 2^{-r}\frac{\Delta E + 8\omega_0}{4}\right\}$$

implies

$$\exists\,\Omega_\infty \subset \Omega_0, \quad \textit{with } \frac{|\Omega_\infty|}{|\Omega_0|} \ge 1 - \frac{2(1+8\frac{\omega_0}{\Delta E})\|\!|V|\!\|_r}{\frac{\omega_0}{\widetilde{C}_2}\left(\frac{\Delta E_\sigma}{\omega_0}\right)^{\sigma}},$$

so that K is pure point for all $\omega\in\Omega_\infty$.

In addition each eigenprojection P of K is bounded in the norm

$$\|P\|_{r-\sigma-\frac{3}{2}} = \sup_{m\in\mathbb{N}}\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{N}}\|P(k,m,n)\|\max\{|k|^{r-\sigma-\frac{3}{2}},1\}$$

and $\widetilde{C}_2(\sigma,r) = C_2(\sigma,r-1)$, *where C_1 and C_2 are the constants from Theorem* 1.1.

Proof. (a) As mentioned in the introduction the strategy consists in proving that $H_0 + V(t) = J(t)(H_0 + G(t))J^*(t)$ using Theorem 3.1 for each t. Then

$$K_0 + V = -\mathrm{i}\omega\partial_t + H_0 + V = J(-\mathrm{i}\omega\partial_t + H_0 + \widetilde{V})J^*$$

where $\widetilde{V}(t) := G(t) - \mathrm{i}\omega J^*(t)\dot{J}(t)$ will be seen to fulfill Theorem 1.1.

(b) The self-adjointness of $K_0 + V$ is not an easy matter since the quadratic form technique cannot be used here because K_0 is not bounded below. We shall establish it indirectly. First with the PDM we shall get the existence of the strongly C^1 map $J : S^1 \to \mathcal{B}_{\mathrm{SH}}$ such that $1 \otimes H_0 + V = J(1 \otimes H_0 + G)J^*$. Then it is easily verified that $K_0 + V$ is self-adjoint on $J\mathrm{dom}\, K_0$ since $\widetilde{V}$ is bounded.

(c) Let $w_r(k) := 2^r \max\{|k|^r, 1\}$ for some $r \geq 0$. We shall use the notation

$$w_r V := \{w_r(k)V(k,m,n) \mid k \in \mathbb{Z}, m, n \in \mathbb{N}\}\ .$$

It is straightforward to check that

$$\mathcal{E}_r := \{V : S^1 \to \mathcal{B}^{\infty,1} \mid \|\!|w_r V|\!\|_0 < \infty\} \quad \text{and}$$
$$\mathcal{A}_r := \{V : S^1 \to \mathcal{B}^{\infty,1} \mid \|w_r V\|_0 < \infty\}$$

are respectively a Banach space and a Banach algebra, with $\mathcal{A}_r \subset \mathcal{E}_r$ and with

$$\mathcal{A}_r\mathcal{E}_r \subset \mathcal{E}_r \quad \text{and} \quad \mathcal{E}_r\mathcal{A}_r \subset \mathcal{E}_r.$$

We simply follow Section 3.3 with $\mathcal{H}, H_0, V$, $\mathcal{B}^{\infty,1}$ and $\mathcal{B}_{\mathrm{SH}}$ replaced respectively by $\mathcal{K}$, $1 \otimes H_0$, $S^1 \ni t \to V(t)$, $\mathcal{E}_r$ and $\mathcal{A}_r$ so that we get as for Theorem 3.1:

If $\|\!|w_r V|\!\|_0 \leq \frac{\Delta E}{8}$*, there exists* $J \in \mathcal{A}_r$ *and* $G \in \mathcal{E}_r$ *such that* $(1 \otimes H_0) + V = J((1 \otimes H_0) + G)J^*$ *together with*

$$\|w_r J\|_0 \leq \frac{3}{2} \quad \textit{and} \quad \|w_r G\|_0 \leq 2\|\!|w_r V|\!\|_0.$$

Therefore $\|w_{r-1}J\|_0 \leq \frac{3}{4}$ and $\|w_{r-1}G\|_0 \leq \|\!|w_r V|\!\|_0$ since $w_{r-1} \leq \frac{1}{2w_r}$. Of course it follows that $\|w_{r-1}J^*\|_0 \leq \frac{3}{4}$. It remains to estimate $\|w_r\dot{J}\|_0$. One has, with $J = \prod_{s=1}^{\infty} \mathrm{e}^{W_s}$ and $x_1 := \pi\|\!|w_r V|\!\|_0 \Delta E^{-1} \leq \frac{\pi}{8}$,

$$\begin{aligned}
\|w_{r-1}\dot{J}\|_0 &\leq \sum_{s=1}^{\infty} \|w_{r-1}\dot{W}_s\|_0 \exp\left(\sum_{s=1}^{\infty} \|w_{r-1}W_s\|_0\right) \\
&= \frac{1}{2}\sum_{s=1}^{\infty} \|w_r W_s\|_0 \exp\left(\frac{1}{2}\sum_{s=1}^{\infty} \|w_r W_s\|_0\right) \\
&\leq \frac{1}{2}\frac{x_1}{1 - \Phi(2x_1)} \exp\left(\frac{1}{2}\frac{x_1}{1 - \Phi(2x_1)}\right) \\
&\leq \pi\frac{\|\!|w_r V|\!\|_0}{\Delta E}3 = 3\pi\frac{\|\!|w_r V|\!\|_0}{\Delta E}
\end{aligned}$$

since one can check numerically that $(2(1-\Phi(2x_1))^{-1}\exp\left(\frac{x_1}{2(1-\Phi(2x_1))}\right)$ is less than 3 if $x_1 \leq \frac{\pi}{8}$. Thus we have obtained for all $\omega \in \Omega_0$

$$2^{r-1}\|\widetilde{V}\|_{r-1} = \|w_{r-1}\widetilde{V}\|_0 \leq \|w_{r-1}G\|_0 + \frac{9}{8}\omega_0\|w_{r-1}J^*\|\,\|w_{r-1}\dot{J}\|$$
$$\leq \left(1+\frac{9}{8}\cdot\frac{3}{4}3\pi\frac{\omega_0}{\Delta E}\right)\|w_rV\|_0 \leq \left(1+8\frac{\omega_0}{\Delta E}\right)2^r\|V\|_r.$$

Finally we apply Theorem 1.1 to $K_0+\widetilde{V}$ with r replaced by $r-1$ and $\|V\|_r$ by $2\left(1+8\frac{\omega_0}{\Delta E}\right)\cdot\|V\|_r$. We also have to impose the additional condition $\|w_rV\|_0 \leq \frac{\Delta E}{8}$. □

4. Applications

4.1. **The d dimensional quantum top.** Here we give an example of Theorem 1.1 with unbounded multiplicities of the spectrum of H_0. We consider the model (5). H_0 is the Laplace-Beltrami operator on the d-dimensional sphere S^d. Then the n^{th} eigenvalue obeys

$$E_n = n(n+d-1) \quad \text{with } M_n = \binom{n+d}{d} - \binom{n+d-2}{d} \underset{n\to\infty}{\sim} \frac{2n^{d-1}}{(d-1)!}$$

so that the growing gap condition (2) is fulfilled if and only if

$$\sum_{m>n}\frac{(mn)^{d-1}}{(m^2-n^2)^\sigma} < \infty \quad \Leftrightarrow \quad \sigma > 2d-1.$$

If $f \in C^s(\mathbb{R})$ and $W \in C^u(S^d)$ with

$$s > r+1 > \sigma + \frac{1}{2} + 1 > 2d + \frac{1}{2} \quad \text{and} \quad u \geq 4,$$

then Theorem 1.1 applies (see [7] for details). This model has already been studied by Nenciu in [15] who found a sufficient condition to rule out the absolutely continuous spectrum. We have gathered in Figure 3 below what is known so far about this model.

4.2. **The pulsed rotor with a δ point interaction.** As an application of Theorem 3.3 we shall consider the pulsed rotor (5) with $f \in C^s(S^1)$ and W the delta point interaction located at 0. We recall that this is the interaction associated to the quadratic form on $L^2(S^1)$ defined by $u \to |u(0)|^2$. One has for the n^{th} eigenprojection of H_0

$$P_n = (\,\cdot\,,\varphi_{-n})\varphi_{-n} + (\,\cdot\,,\varphi_n)\varphi_n, \quad \text{with } \varphi_n(x) := \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}nx},$$

except for $P_0 = (\,\cdot\,,\varphi_0)\varphi_0$. (UV) is true since $\|\delta\|_{\infty,1} = \pi^{-1}$ because

$$\|P_m\delta P_n\| = \left\|\frac{1}{2\pi}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\right\| = \frac{1}{\pi} \quad m,n \neq 0$$
$$\|P_m\delta P_0\| = \left\|\frac{1}{2\pi}\begin{pmatrix}1 & 1\end{pmatrix}\right\| = \frac{1}{\sqrt{2}\pi} \quad m \neq 0$$
$$\|P_0\delta P_0\| = \frac{1}{2\pi}\,.$$

Moreover

$$\frac{1}{\Delta E} = \sup_{m\in\mathbb{N}} \sum_{\mathbb{N}\ni n\neq m} \frac{1}{|m^2-n^2|} = \frac{7}{4} \quad \Rightarrow \quad (\mathrm{GGC}H_0)$$

$$\|E_n\| = \left\| n^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\| = n^2 < \infty \quad \Rightarrow \quad (\mathrm{BBC}H_0).$$

Let

$$f(t) = \sum_{k\in\mathbb{Z}} \widehat{f}_k \mathrm{e}^{\mathrm{i}kt}$$

be the Fourier expansion of f. Then

$$|\!|\!| fW |\!|\!|_r \leq \frac{1}{\pi} \sum_{k\in\mathbb{Z}} |\widehat{f}_k| \max\{|k|^r, 1\}.$$

Since the eigenvalues of H_0 are $\{n^2\}_{n\in\mathbb{N}}$, every $\sigma > 1$ will insure that $\Delta E_\sigma < \infty$. Thus in order to apply Theorem 3.3 one needs $r > \sigma + \frac{3}{2}$ i.e. $r > \frac{5}{2}$ and finally $s > \frac{7}{2}$ to insure that $|\!|\!| fW |\!|\!|_r$ is finite. We have proved:

Let $f \in C^s(\mathbb{R}, R)$ be a 2π-periodic function with $s > \frac{7}{2}$ and g a real constant. The Floquet operator associated to the time dependent Schrödinger operator $-\Delta + gf(\omega t)\delta(x)$ on $L^2(S^1)$ is pure point provided g is small enough and for appropriate frequencies ω. In such conditions this quantum system is stable in the sense of equation (6).

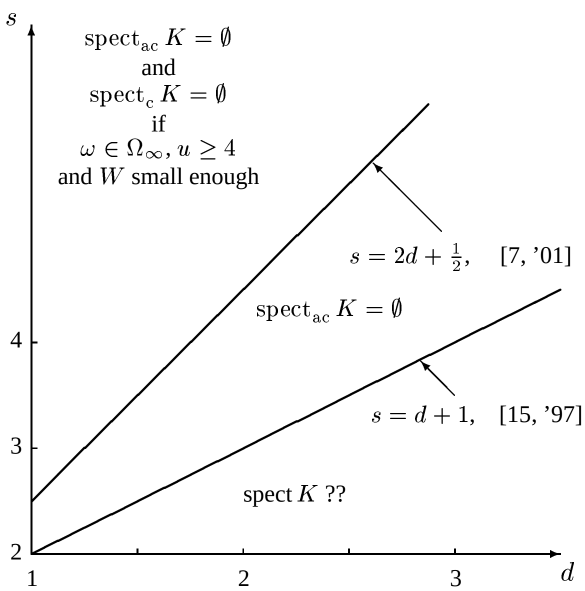

Figure 3. About the quantum top

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On a semiclassical formula for non-diagonal matrix elements

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Abstract

Let $H(\hbar) = -\hbar^2 d^2/dx^2 + V(x)$ be a Schrödinger operator on the real line, $W(x)$ be a bounded observable depending only on the coordinate and k be a fixed integer. Suppose that an energy level E intersects the potential $V(x)$ in exactly two turning points and lies below $V_\infty = \liminf_{|x|\to\infty} V(x)$. We consider the semiclassical limit $n \to \infty$, $\hbar = \hbar_n \to 0$ and $E_n = E$ where E_n is the nth eigen-energy of $H(\hbar)$. An asymptotic formula for $\langle n|W(x)|n+k\rangle$, the non-diagonal matrix elements of $W(x)$ in the eigenbasis of $H(\hbar)$, has been known in the theoretical physics for a long time. Here it is proved in a mathematically rigorous manner.

Keywords: semiclassical limit, non-diagonal matrix elements, WKB method

1 Introduction

In the quantum mechanics the matrix elements of an observable occur in various situations. Let us mention few of them. They measure transition probabilities between two states and the coefficients in the stationary perturbation theory are expressed in terms of the matrix elements of the perturbation. The distribution of matrix elements is of interest for quantum systems stemming from classically chaotic systems, see for example [9, 6] and references in the latter paper. Our immediate motivation to study the matrix elements was the quantum version of the Kolmogorov-Arnold-Moser method [1], [8]. One of the assumptions under which this method is applicable is that a time-dependent perturbation of a quantum system must be sufficiently small with respect to certain norm which is also expressed in terms of matrix elements.

One may hope to obtain at least a qualitative information about the behavior of matrix elements when considering the semiclassical limit. In fact this idea goes back to the very origins of the quantum mechanics. A semiclassical formula for non-diagonal matrix elements in the one-dimensional case has been suggested already a long time ago [12]. In [9] one can find another derivation, also on the level of rigor usual in the theoretical physics, for absolute values of the non-diagonal matrix elements.

Despite of the ancient history rigorous mathematical results have been published essentially more recently. Moreover, they cover only some particular cases even though the technical tools necessary for the derivation may be at hand nowadays. One usually assumes that the corresponding classical system is either ergodic [5], [6] or completely integrable [19], [2], [15], [7]. The semiclassical limit of diagonal matrix elements is now treated in detail [5]. In the case of multi-dimensional completely integrable systems a formula for non-diagonal matrix elements was proved in [19], [15], [7], see also [16] for some generalizations. The one-dimensional case seems to be rather particular. In [14] one can find a derivation of the semiclassical formula for pseudo-differential operators in one variable such that the Weyl symbol of the Hamiltonian is a real polynomial on the phase space while imposing an additional assumption on the discreteness of the operator spectrum.

The present paper aims to provide a mathematically rigorous verification of the semiclassical limit of non-diagonal matrix elements for Schrödinger operators on the real line. We prove the formula under mild assumptions on the potential. In addition, we take care about identifying the quantum number coming from the Bohr-Sommerfeld quantization condition with the index determined by the natural enumeration of eigenvalues in ascending order. Our approach relies on a transparent application of some well established tools in the spectral and semiclassical analysis. So we briefly recall the corresponding results while adjusting their formulation to our purposes. On the other hand, the chosen method restrict us to considering observables which depend on the coordinate only. This particular case was sufficient for the applications we originally had in mind, as mentioned above.

Let us now formulate precisely in what sense the semiclassical limit is understood. Set

$$H(\hbar) = -\hbar^2 \frac{d^2}{dx^2} + V(x) \quad \text{in } L^2(\mathbb{R}, dx). \tag{1}$$

We consider a fixed energy E and an observable $W = W(x)$ depending only on the coordinate x. The assumptions are as follows.

We suppose that $V(x)$ is bounded from below and three times continuously differentiable, $W(x)$ is bounded and continuously differentiable,

$$E < V_\infty := \liminf_{|x|\to\infty} V(x). \tag{2}$$

We assume that at the energy E there are exactly two regular turning points, i.e., $V^{-1}(E) = \{x_-, x_+\}$, $x_- < x_+$, and $V'(x_\pm) \neq 0$. Set

$$f(x) = V(x) - E. \tag{3}$$

In addition we introduce an assumption making it possible to apply the WKB approximation, namely we assume that

$$\int_{\mathbb{R}\setminus[-a,a]} \left| \frac{1}{f^{1/4}} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left(\frac{1}{f^{1/4}} \right) \right| \mathrm{d}x < \infty \tag{4}$$

where a is a positive number chosen so that $f(x) \geq \delta > 0$ for $|x| \geq a$. Notice that

$$\frac{1}{f^{1/4}} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left(\frac{1}{f^{1/4}} \right) = \frac{5(V')^2 - 4(V-E)V''}{16(V-E)^{5/2}} \,.$$

It may be convenient to replace condition (4) by two simpler conditions,

$$\int_{\mathbb{R}\setminus[-a,a]} \frac{|V'|^2}{(V-E)^{5/2}} \,\mathrm{d}x < \infty, \quad \int_{\mathbb{R}\setminus[-a,a]} \frac{|V''|}{(V-E)^{3/2}} \,\mathrm{d}x < \infty. \tag{5}$$

The part of the spectrum of $H(\hbar)$ lying below V_∞ is known to be formed exclusively of simple isolated eigenvalues. We fix the phase of an eigenfunction ψ_n corresponding to an eigenvalue $E_n < V_\infty$ by requiring ψ_n to be positive on a neighborhood of $+\infty$. Moreover, there exists a strictly decreasing sequence of positive numbers tending to 0, $\{\hbar_n\}_{n=n_0}^{\infty}$, and a constant $\hbar_0 > 0$ such that for $\hbar \in]0, \hbar_0]$, E belongs to the spectrum of $H(\hbar)$ if and only if $\hbar = \hbar_n$ and in that case $E = E_n$ is the nth eigenvalue of $H(\hbar)$ provided the enumeration of eigenvalues starts from the index $n = 0$.

Under these assumptions we claim that if $k \in \mathbb{Z}$ is fixed, $n \to \infty$, $\hbar = \hbar_n \to 0$, with $E = E_n$, then

$$\langle n|W(x)|n+k\rangle \to \frac{1}{T} \int_0^T W(q(t))\, e^{ik\omega t} \,\mathrm{d}t \tag{6}$$

where $(q(t), p(t))$, $t \in [0, T]$, is the classical trajectory in the phase space at the energy E and with the initial point chosen so that the kinetic energy vanishes, i.e., $p(0) = 0$, and $q(0)$ coincides the right turning point x_+. Furthermore, $T > 0$ is the period of the classical motion and $\omega = 2\pi/T$ is the frequency.

Remark. If the phase of the wave function ψ_n was chosen so that ψ_n was positive on a neighborhood of $-\infty$ then formula (6) would be again true with $(q(0), p(0)) = (x_-, 0)$.

As already said, we have confined ourselves to observables depending only on the coordinate because our method of proof is based on the WKB approximation. One naturally expects, however, that for any smooth bounded classical observable $A(q, p)$,

$$\langle n|\hat{A}|n+k\rangle \to \frac{1}{T} \int_0^T A(q(t), p(t))\, e^{ik\omega t} \mathrm{d}t$$

where $\hat{A}$ is a suitable quantization of A. We have already mentioned that this result is actually proved in [14] in the case when the potential $V(x)$ is a polynomial.

Let us rewrite the RHS in formula (6). The equation of the classical trajectory in the phase space reads $p^2 + V(x) = E$ and its period equals

$$T = \int_{x_-}^{x_+} \frac{\mathrm{d}x}{\sqrt{E - V(x)}} \,. \tag{7}$$

For $x \in [x_-, x_+]$ set

$$\tau(x) = \frac{1}{2} \int_x^{x_+} \frac{\mathrm{d}y}{\sqrt{E - V(y)}} \,. \tag{8}$$

Then $\tau(x_+) = 0$, $\tau(x_-) = T/2$, $q(\tau(x)) = x$, and

$$\int_0^T W(q(t))\, e^{ik\omega t}\, \mathrm{d}t = \int_{x_-}^{x_+} \frac{W(x)}{\sqrt{E - V(x)}} \cos\!\left(\frac{2\pi k}{T}\, \tau(x)\right) \mathrm{d}x.$$

The paper is organized as follows. In Sections 2 through 4 we recall some preliminaries that we need for the proof of the formula. Section 2 is devoted to the basic spectral properties of the Schrödinger operator, Section 3 is concerned with the Weyl asymptotic formula and some basic facts about the WKB approximation are summarized in Section 4. By counting the zeroes of wave functions we show in Section 5 that the quantum number coming from the Bohr-Sommerfeld quantization condition equals the index of the corresponding eigenvalue. The semiclassical formula is then proved in Section 6.

2 Properties of the spectrum lying below V_∞

Here we briefly recall two well known properties of Schrödinger operators. In the monographs they are usually formulated and derived for potentials diverging at infinity. We just wish to point up that the same assertions apply also for more general potentials provided one takes care only about the part of the spectrum lying below V_∞. The corresponding proofs can be taken almost literally from the cited monographs.

In this section (and only in it) the Planck constant is not relevant and so we set it equal to 1 and consider the Hamiltonian

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) \quad \text{in } L^2(\mathbb{R}, \mathrm{d}x).$$

The following theorem is in fact widely used. We recall it in a form which is a direct modification of Theorem XIII.16 in [17]. Its proof is based on the min-max principle and is applicable in any dimension of the underlying Euclidean space. Moreover, the differentiability of $V(x)$ is not required.

Theorem 1. *Let V be a measurable function in $\mathbb{R}^n$ which is bounded from below. Define $H = -\Delta + V$ as the sum of quadratic forms in $L^2(\mathbb{R}^n, \mathrm{d}^n x)$. Then the lower edge of the essential spectrum of H, if any, is greater than or equal to $V_\infty = \liminf_{|x|\to\infty} V(x)$.*

Let us note that in the one-dimensional case and provided the potential is continuous Theorem 1 also follows from a well known estimate on the number of negative eigenvalues.

Here and everywhere in what follows, if A is a self-adjoint operator then $P(A;\cdot)$ designates the associated projector-valued measure, and for $K \in \mathbb{R}$ we denote

$$N(A, K) = \operatorname{rank} P(A;]-\infty, K[).$$

Further, for a real-valued function $W(x)$ we set

$$W_-(x) = \max\{0, -W(x)\}.$$

It holds (see, for example, Theorem 5.3 in [3])

$$N(H,0) \le 1 + \int_{\mathbb{R}} |x|\, V_-(x)\,\mathrm{d}x.$$

In particular, if $V(x)$ is continuous and bounded from below then for any $c < V_\infty$ the function $(V-c)_-(x)$ has a compact support and, by this estimate, $N(H,c) < \infty$. This again implies that the lower edge of the essential spectrum of H is greater than or equal to V_∞.

The next property is specific for the one-dimensional case. The potential $V(x)$ is supposed to be continuous and bounded from below.

As is well known from the theory of ordinary differential equations, for $E < V_\infty$, any nontrivial solution of the Schrödinger equation either grows at least exponentially or decays at least exponentially at $+\infty$ (see, for example, Corollary 1 in [3, Section II]). The latter solution is called recessive at $+\infty$ and is unique up to a multiplicative constant. Of course, an analogous assertion is also true for $-\infty$. It immediately follows that all eigenvalues of the Hamiltonian H lying below V_∞ are simple. Moreover, in virtue of Theorem 1, they have no accumulation points below V_∞. Consequently, the eigenvalues of H below V_∞ can be arranged into a strictly increasing sequence, empty or finite or infinite,

$$E_0 < E_1 < E_2 < \ldots < V_\infty.$$

The following theorem is a straightforward modification of Theorem 3.5 in [3, Chapter II].

Theorem 2. *The number of zeroes of the mth eigenfunction of H corresponding to the eigenvalue $E_m < V_\infty$ is exactly equal to m.*

3 The Weyl asymptotic formula

In this section we aim to recall the Weyl asymptotic formula generalized to Schrödinger operators. It can be derived from the Gutzwiller trace formula [10] which was rigorously proved in [4] under the assumption that the potential is positive and infinitely differentiable. In [18] there is given a short review of the history and the Weyl asymptotic formula is recalled even under stricter assumptions which among others mean that the potential does not grow faster than polynomially. A weaker version of the formula is also stated in [17, Theorem XIII.79] but only for compactly supported potentials.

Here we wish to point out that the proof of Theorem XIII.79 in [17] can be extended in a straightforward manner and thus the Weyl asymptotic formula can be derived just under the assumption that the potential is semi-bounded and continuous. We restrict ourselves, however, to the one-dimensional case only. In addition, this approach is quite simple as it is based merely on an application of the min-max principle and the Dirichlet-Neumann bracketing. On the other hand, if compared to the result based on the trace formula, as presented in [18], the control of the error term is essentially worse;

it is known to be of order $O(1)$ while the present method only yields the asymptotic behavior of the type $o(\hbar^{-1})$.

From now on, the Planck constant is again relevant. This means that the discussion concerns the Hamiltonian $H(\hbar)$ introduced in (1). Since what follows is nothing but a slight modification of known results we just indicate the basic steps.

First let us recall a definition from [17, XIII.15] making it possible to compare self-adjoint operators defined in different Hilbert spaces. The symbol $Q(A)$ stands for the form domain of A. If $\psi \in Q(A)$ then the scalar product $\langle \psi, A\psi \rangle$ is automatically understood in the form sense.

Definition. Let $\mathcal{H}_1 \subset \mathcal{H}$ be a closed subspace, let A be a semi-bounded self-adjoint operator in $\mathcal{H}$ and let B be a semi-bounded self-adjoint operator in $\mathcal{H}_1$. We shall write $A \le B$ if and only if it holds

(i) $Q(A) \supset Q(B)$,

(ii) $\forall \psi \in Q(B)$, $\langle \psi, A\psi \rangle \le \langle \psi, B\psi \rangle$.

With the aid of the min-max principle one can show [17, XIII.15] that if $A \le B$ then

(i) $\forall K \in \mathbb{R}$, $\operatorname{rank} P(A;]-\infty, K[) \ge \operatorname{rank} P(B;]-\infty, K[)$,

(ii) $\forall K \in \mathbb{R}$, $\operatorname{rank} P(A;]-\infty, K]) \ge \operatorname{rank} P(B;]-\infty, K])$.

The following lemma is analogous to Proposition 2 in [17, XIII.15] in the one-dimensional case and its proof is based on rather elementary explicit computations of the eigenvalues for the involved operators.

Lemma 1. *Let $I = [a, b]$ be a compact interval. Let us introduce H_D, H_N and H_M as self-adjoint operators in $L^2(I, \mathrm{d}x)$ such that all of them act as the differential operator $-\hbar^2\, \mathrm{d}^2/\mathrm{d}x^2$ and whose domain is respectively determined by the Dirichlet, Neumann and mixed boundary conditions. Then for all $K > 0$ it holds*

$$-1 \le \operatorname{rank} P(H;]-\infty, K[) - \frac{\ell}{\pi\hbar}\sqrt{K} \le \operatorname{rank} P(H;]-\infty, K]) - \frac{\ell}{\pi\hbar}\sqrt{K} \le 1,$$

where H is any of the operators H_D, H_N, H_M, and $\ell = b - a$ is the length of the interval.

The following lemma coincides with Proposition 4 in [17, XIII.15] in the one-dimensional case.

Lemma 2. *Let $-\infty < a < b < c < +\infty$ and let H be a self-adjoint operator in $L^2([a, c], \mathrm{d}x)$ which acts as the differential operator $-\mathrm{d}^2/\mathrm{d}x^2$ with either the Dirichlet or the Neumann boundary condition imposed at each of the points a and c (mixed boundary conditions are admitted). Let $H_D^{(1)}$ and $H_N^{(1)}$ be the self-adjoint operators in $L^2([a, b], \mathrm{d}x)$ also acting as $-\mathrm{d}^2/\mathrm{d}x^2$ and with the domain being determined by the same boundary condition at the point a as imposed in the case of the operator H and by the*

Dirichlet or Neumann boundary condition at the point b, respectively. Analogously one introduces the self-adjoint operators $H_D^{(2)}$ *and* $H_N^{(2)}$ *in* $L^2([b,c],\mathrm{d}x)$*. Then it holds*

$$H_N^{(1)} \oplus H_N^{(2)} \leq H \leq H_D^{(1)} \oplus H_D^{(2)}.$$

First let us state the Weyl asymptotic formula for a finite interval. It can be prover in a way very close to the proof of Theorem XIII.79 in [17]. So we do not reproduce the proof but let us note that it is based on a limit procedure when the interval is split into N subintervals of equal length with N tending to ∞. In the course of the proof one uses Lemma 1 and 2, the additivity of the numbers $N(A,K)$, i.e.,

$$N(A_1 \oplus A_2 \oplus \ldots \oplus A_N, K) = N(A_1,K) + N(A_2,K) + \ldots + N(A_N,K),$$

and the fact that the integral on the RHS of (9) exists in the Riemann sense.

Theorem 3. *Let* $-\infty < a < b < +\infty$, $V \in C([a,b])$, *and let*

$$H_f(\hbar) = -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$$

be a self-adjoint operator in $L^2([a,b],\mathrm{d}x)$ *with either the Dirichlet or Neumann boundary condition imposed at each of the boundary points a and b (mixed boundary conditions are admitted). Then for all* $K \in \mathbb{R}$,

$$\lim_{\hbar\to 0+} \hbar\, N(H_f(\hbar),K) = \frac{1}{\pi} \int_a^b \sqrt{(V-K)_-(x)}\,\mathrm{d}x. \tag{9}$$

Finally let us proceed to the case of the Hamiltonian $H(\hbar)$.

Theorem 4. *Let* $V \in C(\mathbb{R})$ *be a real-valued function which is bounded from below. Then for all* $K < V_\infty$ *it holds true that*

$$\lim_{\hbar\to 0+} \hbar\, N(H(\hbar),K) = \frac{1}{2\pi} \mathrm{Vol}_Z\left(\mathscr{H}^{-1}(]-\infty,K[)\right) = \frac{1}{\pi}\int_{\mathbb{R}} \sqrt{(V-K)_-(x)}\,\mathrm{d}x \tag{10}$$

where $\mathscr{H}(x,p) = p^2 + V(x)$ *and* $\mathrm{Vol}_Z(X)$ *designates the Lebesgue measure of a measurable set* X *in the phase space.*

Proof. If $K < V_\infty$ then the support of $(V-K)_-$ is compact. Suppose that $\mathrm{supp}(V-K)_- \subset [a,b]$, $-\infty < a < b < +\infty$. Set

$$H_1(\hbar) = -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} - (V-K)_-(x) \quad \text{in } L^2(\mathbb{R},\mathrm{d}x)$$

and

$$H_2(\hbar) = -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) - K \quad \text{in } L^2([a,b],\mathrm{d}x)$$

with the Dirichlet boundary condition imposed at the points a and b. Observe that $-(V-K)_-(x) \leq V(x) - K$ on $\mathbb{R}$ and so $Q(H(\hbar)-K) \subset Q(H_1(\hbar))$. Furthermore,

$L^2([a,b],\mathrm{d}x)$ can be naturally regarded as a subspace in $L^2(\mathbb{R},\mathrm{d}x)$. If $\psi \in Q(H_2(\hbar))$ then $\tilde{\psi}$ defined by $\tilde{\psi}(x)=\psi(x)$ for $x\in[a,b]$, $\tilde{\psi}(x)=0$ for $x\in\mathbb{R}\setminus[a,b]$, belongs to $Q(H(\hbar)-K)$ ($\tilde{\psi}$ is an absolutely continuous function). This implies that $Q(H_2(\hbar))\subset Q(H(\hbar)-K)$. We have find that $H_1(\hbar)\le H(\hbar)-K\le H_2(\hbar)$. Hence

$$N(H_2(\hbar),0)\le N(H(\hbar),K)\le N(H_1(\hbar),0).$$

Formula (10) for compactly supported potentials is stated in [17, Theorem XIII.79]. Hence it holds

$$\lim_{\hbar\to0+}\hbar\,N(H_1(\hbar),0)=\frac{1}{\pi}\int_{\mathbb{R}}\sqrt{(V-K)_-(x)}\,\mathrm{d}x,$$

and from Theorem 3 we know that

$$\lim_{\hbar\to0+}\hbar\,N(H_2(\hbar),0)=\frac{1}{\pi}\int_a^b\sqrt{(V-K)_-(x)}\,\mathrm{d}x=\frac{1}{\pi}\int_{\mathbb{R}}\sqrt{(V-K)_-(x)}\,\mathrm{d}x.$$

Formula (10) for a general potential then follows by bracketing. □

For our purposes the following immediate corollary of Theorem 4 will be sufficient. Suppose that $V(x)$ is continuously differentiable and an interval $]a,b[$, $a<b\le V_\infty$, contains at least one regular value of the classical Hamiltonian $\mathscr{H}(x,p)$, i.e., there exists $\lambda\in\,]a,b[$ satisfying $\mathscr{H}^{-1}(\{\lambda\})\neq\emptyset$ and $V(x)=\lambda$ implies $V'(x)\neq0$. Then the number of eigenvalues of $H(\hbar)$ in the interval $]a,b[$ tends to infinity as $\hbar\to0+$.

4 The WKB method for one and two turning points

Here we summarize some basic facts about the WKB approximation, also called Liouville-Green approximation, that we need for the proof of the formula in Section 6. At the same time we introduce the necessary notation. We stick to the presentation given in the monograph [13] whose distinguished feature is that it provides explicit bounds on the error terms.

Let us first consider the situation with one turning point. Let $]a,b[\subset\mathbb{R}$ be an interval, finite or infinite, $x_0\in\,]a,b[$, and $f(x)$ be a real-valued function defined on $]a,b[$ such that $f(x)/(x-x_0)$ is positive and twice continuously differentiable (hence $f(x_0)=0$, $f'(x_0)>0$). For $x\in\,]a,b[$ set

$$\frac{2}{3}\,\zeta^{3/2}=\int_{x_0}^{x}\sqrt{f(t)}\,\mathrm{d}t\qquad\text{if } x\ge x_0,\tag{11a}$$

$$\frac{2}{3}\,(-\zeta)^{3/2}=\int_{x}^{x_0}\sqrt{-f(t)}\,\mathrm{d}t\quad\text{if } x<x_0.\tag{11b}$$

Then $\zeta(x)$ is strictly monotone, $\zeta(x)/(x-x_0)$ is positive and twice continuously differentiable in $]a,b[$, see Lemma 3.1 in [13, Chapter 11].

Assume further that

$$\int_{x_0}^{b}\sqrt{f(t)}\,\mathrm{d}t=\infty\tag{12}$$

and

$$\int_{]a,b[\setminus U_0} \frac{|f''|}{|f|^{3/2}}\,\mathrm{d}t < \infty, \quad \int_{]a,b[\setminus U_0} \frac{(f')^2}{|f|^{5/2}}\,\mathrm{d}t < \infty, \tag{13}$$

where $U_0 = [x_0 - \varepsilon, x_0 + \varepsilon]$ and ε is any positive number such that $a < x_0 - \varepsilon$ and $x_0 + \varepsilon < b$.

Notice also that

$$\zeta' = \left(\frac{f}{\zeta}\right)^{1/2} \text{ and } \zeta'(x_0) = f'(x_0)^{1/3}. \tag{14}$$

Denote by ξ the inverse function to ζ. Theorem 3.1 in [13, Chapter 11, §3.3] can be rephrased as follows.

Theorem 5. *Under the above assumptions, the solution of the differential equation*

$$\hbar^2\,\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = f(x)w \tag{15}$$

which is recessive as x tends to b exists on $]a,b[$, is unique up to a multiplicative constant and equals

$$\psi(x) = \left(\frac{\zeta}{f}\right)^{1/4} \left(\mathrm{Ai}(\hbar^{-2/3}\zeta) + \varepsilon(\hbar, x)\right) \tag{16}$$

with the error term satisfying the estimates

$$|\varepsilon(\hbar, x)| \leq \Phi_0(\hbar^{-2/3}\zeta)\,\hbar, \quad \left|\frac{\partial\varepsilon(\hbar, x)}{\partial x}\right| \leq \left(\frac{f}{\zeta}\right)^{1/2} \Phi_1(\hbar^{-2/3}\zeta)\,\hbar^{1/3},$$

where $\Phi_0(x)$, $\Phi_1(x)$ are certain continuous positive functions on $\mathbb{R}$ such that

$$\Phi_0(x) \sim \begin{cases} \text{const}\,\dfrac{\exp\left(-\frac{2}{3}x^{3/2}\right)}{x^{1/4}} & \text{as } x \to +\infty, \\ \text{const}\,\dfrac{1}{|x|^{1/4}} & \text{as } x \to -\infty, \end{cases}$$

$$\Phi_1(x) \sim \begin{cases} \text{const}\,\exp\left(-\dfrac{2}{3}x^{3/2}\right) & \text{as } x \to +\infty, \\ \text{const} & \text{as } x \to -\infty. \end{cases}$$

Let us now turn to the case when $f(x)$ is given by (3) and so is defined on the entire real line. From now on the potential V satisfies all assumptions as formulated in the Introduction. In particular, it follows that the function

$$\frac{V(x) - E}{(x - x_-)(x - x_+)} \text{ is positive on } \mathbb{R} \text{ and belongs to } C^2(\mathbb{R}). \tag{17}$$

Moreover, there exists an open neighborhood of E, $U_E =]E_-, E_+[$, $E_- < E < E_+$, such that these assumptions apply for any $\lambda \in \overline{U_E}$ as well.

For $\lambda \in U_E$ set

$$\gamma_\lambda = \mathscr{H}^{-1}(\{\lambda\})$$

where $\mathscr{H}(x,p) = p^2 + V(x)$. Thus γ_λ is a closed curve in the phase space and the energy takes on it the value λ. Let us further introduce the action integral,

$$J(\lambda) = \int_{\mathscr{H}(x,p)\leq\lambda} \mathrm{d}x\mathrm{d}p = \int_{\gamma_\lambda} p\,\mathrm{d}x = 2\int_{x_-(\lambda)}^{x_+(\lambda)} \sqrt{\lambda - V(x)}\,\mathrm{d}x \tag{18}$$

where $x_-(\lambda) < x_+(\lambda)$ are the turning points at the energy λ. Then

$$T(\lambda) = J'(\lambda) = \int_{x_-(\lambda)}^{x_+(\lambda)} \frac{\mathrm{d}x}{\sqrt{\lambda - V(x)}} \tag{19}$$

is the period of the classical trajectory in the phase space.

In the following theorem we summarize the result derived in [13, Chapter 13, §8.2].

Theorem 6. *Under the assumptions on V formulated in the Introduction (in particular, we assume that condition (17) is fulfilled as well as the convergence of the integrals in (5)) there exist a neighborhood U_E of E, $\hbar_0 > 0$, $n_0 \in \mathbb{N}$ and for every $\lambda \in U_E$ a sequence $\{\hbar_n(\lambda)\}_{n=n_0}^{\infty}$, $\hbar_0 > \hbar_{n_0}(\lambda) > \hbar_{n_0+1}(\lambda) > \hbar_{n_0+2}(\lambda) > \ldots > 0$, such that for $\hbar \in]0,\hbar_0[$ the energy λ is an eigenvalue of $H(\hbar)$ if and only if $\hbar = \hbar_n(\lambda)$ for some $n \geq n_0$. Moreover, the sequence $\{\hbar_n(\lambda)\}$ asymptotically behaves like*

$$\hbar_n(\lambda)^{-1} = (2n+1)\pi J(\lambda)^{-1} + O(n^{-1}) \tag{20}$$

where the error term $O(n^{-1})$ decays in n uniformly with respect to $\lambda \in U_E$.

Remark. It is known that if $V \in C^r(\mathbb{R})$, with $r \geq 1$, and $E < V_\infty$ is a regular value of $V(x)$ then the action integral $J(\lambda)$ defined in (18) is r times continuously differentiable on some neighborhood of E (see, for example, [18]).

The verification of this assertion is quite elementary in the one-dimensional case and with two turning points at the energy E. For a sufficiently small neighborhood $U_E =]E_-, E_+[$ the function $V(x)$ is strictly decreasing on the interval $[x_-(E_+), x_-(E_-)]$ and strictly increasing on $[x_+(E_-), x_+(E_+)]$, with nowhere vanishing derivative. Let us write

$$\begin{aligned} T(\lambda) &= \left(\int_{x_-(\lambda)}^{x_-(E_-)} + \int_{x_-(E_-)}^{x_+(E_-)} + \int_{x_+(E_-)}^{x_+(\lambda)}\right) \frac{\mathrm{d}x}{\sqrt{\lambda - V(x)}} \\ &= T_-(\lambda) + T_0(\lambda) + T_+(\lambda). \end{aligned}$$

Clearly, $T_0(\lambda) \in C^\infty(U_E)$. Thus it is sufficient to verify that $T_-(\lambda), T_+(\lambda) \in C^{r-1}(U_E)$. Let us focus only on the latter function. Set $W_+ = \left(V\big|_{[x_+(E_-),x_+(E_+)]}\right)^{-1}$. Hence W_+

is r times continuously differentiable. After some elementary manipulations one can show that

$$T_+(\lambda) = \int_{x_+(E_-)}^{x_+(\lambda)} \frac{\mathrm{d}x}{\sqrt{\lambda - V(x)}} = 2\sqrt{\lambda - E_-} \int_0^1 \frac{\mathrm{d}t}{V'\big(W_+\big(\lambda(1-t^2) + E_- t^2\big)\big)}\,.$$

From the last expression it is obvious that $T_+(\lambda)$ is $r-1$ times continuously differentiable.

5 Number of zeroes derived from the WKB method

We need to show that if $\hbar = \hbar_m(\lambda)$ and hence λ is an eigenvalue of $H(\hbar)$, as claimed in Theorem 6, then λ is exactly the mth eigenvalue of $H(\hbar)$. According to Theorem 2, the index of an eigenvalue lying below V_∞ equals the number of zeroes of the corresponding eigenfunction. Fortunately, the WKB approximation, as explained in [13], is precise enough to control the number of zeroes.

Let us recall some facts concerning the Airy functions. Let us denote by a_n and b_n the zeroes of the Airy functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$, respectively, arranged in ascending order of the absolute value, i.e., $\ldots < b_3 < a_2 < b_2 < a_1 < b_1 < 0$. It is known that

$$a_n = -\left(\frac{3}{2}\,\pi\left(n - \frac{1}{4}\right) + \mathfrak{z}\left(n - \frac{1}{4}\right)\right)^{2/3}, \quad b_n = -\left(\frac{3}{2}\,\pi\left(n - \frac{3}{4}\right) + \mathfrak{z}\left(n - \frac{3}{4}\right)\right)^{2/3}, \tag{21}$$

where $\mathfrak{z}(x) = O(x^{-1})$.

First we again consider the situation with one turning point. Recall defining relations (11a), (11b) for ζ. In the following theorem we summarize the results from §§ 6.1, 6.2 and 6.3 in [13, Chapter 11].

Theorem 7. *Under the same assumptions as in Theorem 5, let $w(x)$ be a nonzero solution of the differential equation (15) on $]a,b[$ which is recessive as x tends to b (hence $w(x)$ is unique up to a multiplicative constant). Then the set of zeroes of $w(x)$ in $]a,b[$, denoted $\{z_n\}_{n\geq 1}$ and arranged in descending order, is at most countable. Any such a zero z fulfills $\zeta(z) < \hbar^{2/3} b_1$. Furthermore, for all sufficiently small $\hbar$ it is true that if $\zeta(a) < \hbar^{2/3} b_{n+1}$ then the nth zero, z_n, does exist and obeys the estimate*

$$\hbar^{2/3} b_{n+1} < \zeta(z_n) < \hbar^{2/3} b_n.$$

Moreover, it holds

$$|\zeta(z_n) - \hbar^{2/3} a_n| = O(n^{-1/3})\hbar$$

where the symbol $O(n^{-1/3})$ is uniform with respect to $\hbar$.

Remarks. From Theorem 7 it immediately follows that there are no zeroes in the interval $[x_0, b[$. Furthermore, the number of zeroes of $w(x)$ in any fixed nonempty subinterval $]c,d[\, \subset\,]a,x_0[$ tends to infinity as $\hbar \to 0+$.

Now we come back to the case when $f(x)$ is given by (3), with $V(x)$ satisfying the assumptions from the Introduction. In particular, there are two turning points at the energy E, x_- and x_+, and $V(x)$ satisfies (17) and (5). Then for any a, $x_- < a < x_+$, the function $f(x)$ satisfies the assumptions of Theorem 7 with $b = +\infty$ and x_0 being replaced by x_+. Actually, condition (5) implies (13) and condition (12) is fulfilled automatically for $E < V_\infty$. Analogous arguments apply also for the other turning point x_-.

According to Theorem 6 there exist $\hbar_0 > 0$ and a sequence $\{\hbar_n\}_{n=n_0}^\infty$, $\hbar_0 > \hbar_{n_0} > \hbar_{n_0+1} > \hbar_{n_0+2} > \ldots > 0$, such that for $\hbar \in]0, \hbar_0[$, E is an eigenvalue of $H(\hbar)$ if and only $\hbar = \hbar_n$ for some $n \geq n_0$. Let $\psi_n(x)$ be an eigenfunction of $H(\hbar_n)$ corresponding to the eigenvalue E. Thus $\psi_n(x)$ is recessive both at $+\infty$ and $-\infty$ and is unique up to a multiplicative constant. We can suppose that $\hbar_0$ is sufficiently small so that $\psi_n(x)$ has at least one zero in the interval $]x_-, x_+[$. By Theorem 7, $\psi_n(x)$ has no zeroes in the set $\mathbb{R}\setminus]x_-, x_+[$.

Let us choose a point $x_1 \in]x_-, x_+[$ independently of n. Let x_1' be the zero of ψ_n which is nearest to x_1. This means that x_1' depends on n but the distance between x_1 and x_1' tends to zero as n tends to infinity. Denote by m_+ and m_- the number of zeroes of ψ_n in the interval $[x_1', x_+[$ and $]x_-, x_1']$, respectively (hence the zero x_1' is counted both in m_+ and m_-). Denote by $\zeta_+(x)$ the function defined by relations (11a) and (11b), with x_0 being replaced by x_+. In virtue of Theorem 7, there exists a constant $c_+ \geq 0$ (independent of n) such that

$$|\zeta_+(x_1') - \hbar_n^{2/3} a_{m_+}| \leq \frac{c_+ \hbar_n}{m_+^{1/3}}$$

for all $n \geq n_0$. An application of the mean value theorem,

$$|u^{3/2} - v^{3/2}| \leq \frac{3}{2} \left(\max\{u, v\}\right)^{1/2} |u - v| \quad \text{for } u > 0, v > 0,$$

yields the inequality

$$\left| |\zeta_+(x_1')|^{3/2} - \hbar_n |a_{m_+}|^{3/2} \right| \leq \frac{3}{2} \left(\frac{3}{2} \int_{x_-}^{x_+} \sqrt{E - V(x)}\, \mathrm{d}x \right)^{1/3} \frac{c_+ \hbar_n}{m_+^{1/3}} \tag{22}$$

which is valid for all sufficiently large n. Analogously, for the other turning point we get the estimate

$$\left| |\zeta_-(x_1')|^{3/2} - \hbar_n |a_{m_-}|^{3/2} \right| \leq \frac{3}{2} \left(\frac{3}{2} \int_{x_-}^{x_+} \sqrt{E - V(x)}\, \mathrm{d}x \right)^{1/3} \frac{c_- \hbar_n}{m_-^{1/3}} \tag{23}$$

where again $c_- \geq 0$ is a constant independent of n. Set

$$c = \left(\frac{3}{2} \int_{x_-}^{x_+} \sqrt{E - V(x)}\, \mathrm{d}x \right)^{1/3} \max\{c_-, c_+\}.$$

Combining (22) and (23) we arrive at the inequality

$$\left|\frac{1}{\hbar_n}\int_{x_-}^{x_+}\sqrt{E-V(x)}\,\mathrm{d}x-\frac{2}{3}\left(|a_{m_-}|^{3/2}+|a_{m_+}|^{3/2}\right)\right|\leq c\left(\frac{1}{m_-^{1/3}}+\frac{1}{m_+^{1/3}}\right).$$

Let $m=m(n)$ be the number of zeroes of $\psi_n(x)$. Obviously, $m=m_-+m_+-1$. Recalling the asymptotic behavior of $\hbar_n$, as stated in (20) (see also (18)), as well as the asymptotic formulas (21) for the roots of the Airy functions we finally find that

$$\left|n-m+O(n^{-1})-3\left(m_--\frac{1}{4}\right)-3\left(m_+-\frac{1}{4}\right)\right|\leq\frac{c}{\pi}\left(\frac{1}{m_-^{1/3}}+\frac{1}{m_+^{1/3}}\right).$$

By Theorem 7, both m_- and m_+ tend to infinity as n tends to infinity. This implies that $m(n)=n$ for all sufficiently large n and therefore, in virtue of Theorem 2, E is the nth eigenvalue of the Hamiltonian $H(\hbar_n)$ (with the numbering starting from $n=0$).

All estimates can be carried out in a uniform manner for E being replaced by λ running over some neighborhood of E. We conclude that

with the assumptions on $V(x)$ formulated in the Introduction, there exist $n_0\in\mathbb{N}$ and a neighborhood U_E of E such that for all $n\geq n_0$ and $\lambda\in U_E$, λ equals exactly the nth eigenvalue of $H\big(\hbar_n(\lambda)\big)$ (with $\hbar_n(\lambda)$ introduced in Theorem 6).

6 Proof of the formula

Here we prove the limit (6). We know that there exists a sequence of positive numbers, $\{\hbar_n\}_{n=n_0}^{\infty}$, such that E is the nth eigenvalue of $H(\hbar_n)$ (Theorem 6). This sequence is strictly decreasing and tends to 0. We even known that $\hbar_n\sim n^{-1}$ as $n\to\infty$ (see (20)). Therefore everywhere in what follows the symbol $O(\hbar)$ should be understood as a substitute for $O(n^{-1})$.

Let us fix $x_1,x_1',x_1''\in\,]x_-,x_+[$, $x_1'<x_1<x_1''$. For a given $\hbar=\hbar_n$ we shall denote by ψ a conveniently normalized eigenfunction corresponding to the eigenvalue $E=E_n$. Hence ψ is recessive both at $+\infty$ and $-\infty$. The normalization is fixed by requiring the eigenfunction ψ to coincide on the interval $]x_1',+\infty[$ with the solution described in Theorem 5 (with $f(x)=V(x)-E$ and $x_0=x_+$ being the single turning point in this interval). Theorem 5 is also applicable to the interval $]-\infty,x_1''[$ containing the turning point x_-. On this interval, ψ equals κ times the solution described in Theorem 5 for some $\kappa\in\mathbb{C}\setminus\{0\}$.

There exists a neighborhood of E, $U_E=\,]E_-,E_+[$, such that any $\lambda\in U_E$ satisfies the same assumptions as those imposed on E. Recall that we have fixed $k\in\mathbb{Z}$. For all sufficiently large n, the $(n+k)$th eigenvalue of $H(\hbar_n)$, called E_{n+k}, exists and lies in U_E. For brevity we shall denote E_{n+k} sometimes by $\tilde{E}$. We show below that $\tilde{E}-E=O(\hbar)$, see (24). The eigenfunction of $H(\hbar_n)$ corresponding to the eigenvalue $\tilde{E}=E_{n+k}$ and coinciding on $]x_1',+\infty[$ with the solution from Theorem 5 will be denoted by $\tilde{\psi}$. In this case, too, there exists $\tilde{\kappa}\in\mathbb{C}\setminus\{0\}$ such that on the interval $]-\infty,x_1''[$, $\tilde{\psi}$ equals

$\tilde{\kappa}$ times the solution from Theorem 5. Furthermore, denote by $\tilde{x}_\pm$ the turning points corresponding to $\tilde{E}$, i.e., $V(\tilde{x}_\pm) = \tilde{E}$. Since $V(\tilde{x}_\pm) - V(x_\pm) = \tilde{E} - E$ and $V'(x_\pm) \neq 0$ it is clear that $\tilde{x}_\pm - x_\pm = O(\hbar)$ as well.

The verification of (6) is based on a series of estimates relying on Theorem 5. This will be done in several steps.

(1) Relation between $\tilde{E}$ and E. Let $E_m(\hbar)$ be the mth eigenvalue of $H(\hbar)$. From the perturbation theory [11] one deduces that if it exists and lies below V_∞ then $E_m(\hbar)$ is strictly increasing and real analytic as a function of $\hbar$. According to the conclusion of Section 5, $E_m(\hbar)$ and $\hbar_m(\lambda)$ are mutually inverse functions. Therefore if $\hbar = \hbar_n(E)$ then $\hbar = \hbar_{n+k}(\tilde{E})$. Thus we have

$$\hbar_n(E) = \hbar_{n+k}(\tilde{E})$$

and from the asymptotic formula (20) we get

$$(2n + 2k + 1)J(E) - (2n + 1)J(\tilde{E}) = O(n^{-1}).$$

Since

$$J(\tilde{E}) = J(E) + \frac{\partial J(E)}{\partial \lambda}(\tilde{E} - E) + O((E' - E)^2)$$

we finally arrive at the equation

$$\frac{2k}{2n+1}\,\frac{J(E)}{T(E)} - \tilde{E} + E = O(n^{-2}) + O((\tilde{E} - E)^2)$$

whose solution satisfies

$$\tilde{E} = E + \frac{J(E)}{T(E)}\,\frac{k}{n} + O(n^{-2}). \tag{24}$$

(2) Asymptotic behavior of κ and $\tilde{\kappa}$. On the interval $]x_1', x_1''[$ one can compare the asymptotics of the solutions which are respectively recessive at $+\infty$ and $-\infty$ and infer this way the asymptotic behavior of κ as $\hbar \to 0$. For a moment we shall distinguish by a subscript the functions $\zeta_\pm$ related to the turning points $x_\pm$ and defined respectively on the intervals $[x_1', +\infty[$ and $]-\infty, x_1'']$. Thus

$$\frac{2}{3}\,|\zeta_+|^{2/3} = \left|\int_{x_+}^{x} |f(t)|\,\mathrm{d}t\right|, \quad \frac{2}{3}\,|\zeta_-|^{2/3} = \left|\int_{x}^{x_-} |f(t)|\,\mathrm{d}t\right|,$$

and both ζ_+/f and ζ_-/f are positive functions on their domains. We have

$$\psi(x) = \left(\frac{\zeta_+}{f}\right)^{1/4}\big(\mathrm{Ai}(\hbar^{-2/3}\zeta_+) + \varepsilon_+(\hbar, x)\big)$$

for $x \geq x_1'$, and

$$\psi(x) = \kappa\left(\frac{\zeta_-}{f}\right)^{1/4}\big(\mathrm{Ai}(\hbar^{-2/3}\zeta_-) + \varepsilon_-(\hbar, x)\big)$$

for $x \le x_1''$. Suppose that $x \in [x_1', x_1'']$. Recalling that

$$\mathrm{Ai}(-z) = \frac{1}{\pi^{1/2} z^{1/4}} \left(\cos\!\left(\frac{2}{3}\, z^{3/2} - \frac{\pi}{4} \right) + O(z^{-3/2}) \right) \quad \text{as } z \to +\infty \tag{25}$$

and the error term estimates from Theorem 5 we arrive at the equality

$$\cos\!\left(\frac{2}{3}\, \hbar^{-1} |\zeta_+|^{3/2} - \frac{\pi}{4} \right) + O(\hbar) = \kappa \left(\cos\!\left(\frac{2}{3}\, \hbar^{-1} |\zeta_-|^{3/2} - \frac{\pi}{4} \right) + O(\hbar) \right).$$

Furthermore, in virtue of (20) it holds

$$\frac{2}{3}\, \hbar^{-1} \left(|\zeta_+|^{2/3} + |\zeta_-|^{2/3} \right) = \hbar^{-1} \int_{x_-}^{x_+} |f(t)| \, \mathrm{d}t = \left(n + \frac{1}{2} \right) \pi + O(\hbar).$$

Combining the last two equalities we find that

$$\cos\!\left(\frac{2}{3}\, \hbar^{-1} |\zeta_+|^{3/2} - \frac{\pi}{4} \right) + O(\hbar) = \kappa \left((-1)^n \cos\!\left(\frac{2}{3}\, \hbar^{-1} |\zeta_+|^{3/2} - \frac{\pi}{4} \right) + O(\hbar) \right).$$

For $\hbar$ sufficiently small it clearly exists $x \in [x_1', x_1'']$ such that

$$\cos\!\left(\frac{2}{3}\, \hbar^{-1} |\zeta_+|^{3/2} - \frac{\pi}{4} \right) = 1.$$

It follows immediately that

$$\kappa = (-1)^n + O(\hbar). \tag{26}$$

Similarly,

$$\tilde{\kappa} = (-1)^{n+k} + O(\hbar). \tag{27}$$

(3) The leading asymptotic term on the interval $]x_+ - \delta, \infty[$. Fix $\delta > 0$ sufficiently small (at least $x_1 < x_+ - \delta$). Let us show that

$$\int_{x_+ - \delta}^{\infty} \psi^2 \, \mathrm{d}x = \delta^{1/2} O(\hbar^{1/3}), \quad \int_{-\infty}^{x_- + \delta} \psi^2 \, \mathrm{d}x = \delta^{1/2} O(\hbar^{1/3}). \tag{28}$$

We shall verify only the first equality in (28). In view of (26) and (27), the verification of the second one is analogous.

Here and everywhere in what follows the symbol $O(\hbar^{\varepsilon})$ should be interpreted properly. It means that there exists a constant $c \ge 0$ (independent of δ) and $\hbar_0(\delta) > 0$ such that for all $\hbar$, $0 < \hbar < \hbar_0(\delta)$, it holds $|O(\hbar^{\varepsilon})| \le c\hbar^{\varepsilon}$.

First let us estimate the contribution from the leading asymptotic term of ψ. Applying the substitution $x = \xi(\hbar^{2/3} z)$ we get the expression

$$\int_{x_+ - \delta}^{\infty} \left(\frac{\zeta}{f} \right)^{1/2} \mathrm{Ai}(\hbar^{-2/3} \zeta)^2 \, \mathrm{d}x = \hbar^{4/3} \int_{\hbar^{-2/3} \zeta(x_+ - \delta)}^{\infty} \frac{z}{f\big(\xi(\hbar^{2/3} z)\big)} \, \mathrm{Ai}(z)^2 \, \mathrm{d}z. \tag{29}$$

By the assumptions, there exist $x_2 > x_+$ and $c_1 > 0$ such that $f(x) \geq c_1$ for $x \geq x_2$. The function $\zeta(x)/f(x)$ is continuous on the interval $[x_1, x_2]$ and therefore it is majorized on this interval by a constant $c_2 \geq 0$. This also means that

$$0 < \frac{y}{f\big(\xi(y)\big)} \leq c_2 \quad \text{for } \zeta(x_1) \leq y \leq \zeta(x_2).$$

This way we get the following upper bound on (29), namely

$$\begin{aligned}
&\hbar^{2/3} \int_{\hbar^{-2/3}\zeta(x_+-\delta)}^{\hbar^{-2/3}\zeta(x_2)} c_2 \operatorname{Ai}(z)^2 \, \mathrm{d}z + \hbar^{4/3} \int_{\hbar^{-2/3}\zeta(x_2)}^{\infty} \frac{z}{c_1} \operatorname{Ai}(z)^2 \, \mathrm{d}z \\
&\leq c_2 \hbar^{2/3} \big(\operatorname{Ai}'(x)^2 - x \operatorname{Ai}(x)^2\big)\Big|_{x=\hbar^{-2/3}\zeta(x_+-\delta)} + o(\hbar^{4/3}).
\end{aligned}$$

Here we have used the knowledge of the primitive function

$$\int \operatorname{Ai}(x)^2 \, \mathrm{d}x = x \operatorname{Ai}(x)^2 - \operatorname{Ai}'(x)^2.$$

In addition to formula (25) let us recall also the asymptotic behavior of the derivative of the Airy function,

$$\operatorname{Ai}'(-z) = \frac{z^{1/4}}{\pi^{1/2}} \left(\sin\!\left(\frac{2}{3}\, z^{3/2} - \frac{\pi}{4} \right) + O(z^{-3/2}) \right) \quad \text{as } z \to +\infty. \tag{30}$$

Since $\zeta(x_+-\delta) = -\zeta'(y)\delta$ for some $y \in [x_+-\delta, x_+]$ we find that for $x = \hbar^{-2/3}\zeta(x_+-\delta)$ it holds

$$|\hbar^{2/3} \operatorname{Ai}'(x)^2| \leq \text{const}\, \hbar^{2/3} \left(\hbar^{-2/3}\delta\right)^{1/2} = \text{const}\, \hbar^{1/3}\delta^{1/2}$$

and

$$|\hbar^{2/3} x \operatorname{Ai}(x)^2| \leq \text{const}\, \hbar^{2/3}\hbar^{-2/3}\delta \left(\hbar^{-2/3}\delta\right)^{-1/2} = \text{const}\, \hbar^{1/3}\delta^{1/2}.$$

We have shown that

$$\int_{x_+-\delta}^{\infty} \left(\frac{\zeta}{f}\right)^{1/2} \operatorname{Ai}(\hbar^{-2/3}\zeta)^2 \, \mathrm{d}x = \delta^{1/2} O(\hbar^{1/3}).$$

(4) The error term on the interval $]x_+ - \delta, \infty[$. Further let us write

$$\psi^2 = \left(\frac{\zeta}{f}\right)^{1/2} \operatorname{Ai}(\hbar^{-2/3}\zeta)^2 + \varepsilon_2(\hbar, x).$$

It is known that

$$\operatorname{Ai}(x) \leq \frac{1}{2\sqrt{\pi}}\, x^{-1/4} \exp\!\left(-\frac{2}{3}\, \hbar^{-1} x^{3/2} \right) \quad \text{for } x > 0,$$

see [13, Chapter 11]. Using also the estimates of error terms from Theorem 5 one can check that

$$|\varepsilon_2(\hbar, x)| \leq \text{const}\, f^{-1/2} \exp\!\left(-\frac{4}{3}\, \hbar^{-1} \zeta^{3/2} \right) \hbar^{4/3} \quad \textit{for } x > x_+.$$

It follows that

$$\begin{aligned}\left|\int_{x_+}^{\infty}\varepsilon_2(\hbar,x)\,\mathrm{d}x\right| &\leq \text{const}\,\hbar^{4/3}\int_{x_+}^{\infty}f^{-1/2}\exp\!\left(-\frac{4}{3}\,\hbar^{-1}\zeta^{3/2}\right)\mathrm{d}x\\ &= \text{const}\,\hbar^{4/3}\int_0^{\infty}\frac{y^{1/2}}{f\big(\xi(y)\big)}\exp\!\left(-\frac{4}{3}\,\hbar^{-1}y^{3/2}\right)\mathrm{d}y. \qquad (31)\end{aligned}$$

There exists $c \geq 0$ such that for $y > 0$, $f\big(\xi(y)\big)^{-1} \leq c(1+y^{-1})$. Hence (31) is majorized by

$$\text{const}\,\hbar^{4/3}\int_0^{\infty}(y^{1/2}+y^{-1/2})\exp\!\left(-\frac{4}{3}\,\hbar^{-1}y^{3/2}\right)\mathrm{d}y = O(\hbar^{5/3}).$$

The asymptotic formula (25) implies that $|\operatorname{Ai}(x)| \leq \text{const}\,|x|^{-1/4}$ for $x < 0$. Recalling once more Theorem 5 we have

$$\left|\int_{x_1}^{x_+}\varepsilon_2(\hbar,x)\,\mathrm{d}x\right| \leq \text{const}\,\hbar^{4/3}\int_{x_1}^{x_+}|f|^{-1/2}\,\mathrm{d}x = O(\hbar^{4/3}). \qquad (32)$$

This concludes the verification of (28).
(5) Oscillating integral on the interval $]x_1, x_+ - \delta[$. By the usual integration by parts one can verify the following claim.
Let $[a,b]$ be a compact interval, $F \in C^1([a,b])$, $\mu \in C^2([a,b])$ and $\nu(\hbar,z)$ be twice continuously differentiable in z on $[a,b]$. Assume that $\mu'(z)$ nowhere vanishes on $[a,b]$ and

$$\sup_{z\in[a,b]}|\partial_z\nu(\hbar,z)| = O(1), \quad \sup_{z\in[a,b]}|\partial_z^2\nu(\hbar,z)| = O(1).$$

Then for all sufficiently small $\hbar$ it holds true that

$$\left|\int_a^b F(z)\sin\big(\hbar^{-1}\mu(z)+\nu(\hbar,z)\big)\,\mathrm{d}z\right| \leq \text{const}\,\hbar$$

where the constant depends only on the length of the interval $[a,b]$ and on the quantities

$$\mu_0^{-1}\|F\|_C,\ \mu_0^{-2}\|F\|_C\|\mu''\|_C,\ \mu_0^{-1}\|F'\|_C,$$

with

$$\mu_0 = \min_{z\in[a,b]}|\mu'(z)|$$

and $\|\cdot\|_C$ standing for the norm in the Banach space $C([a,b])$.

As a consequence we find that if $W \in C^1(\mathbb{R})$ then

$$\int_{x_1}^{x_+-\delta}\frac{W}{\sqrt{E-V}}\sin\!\left(\frac{2}{3}\,\hbar^{-1}\big(|\zeta|^{3/2}+|\tilde{\zeta}|^{3/2}\big)\right)\mathrm{d}x = \delta^{-1}O(\hbar). \qquad (33)$$

To show this asymptotics it suffices to set in the above claim $F = W/\sqrt{E-V}$, $\mu = (4/3)|\zeta|^{3/2}$ and

$$\begin{aligned}\nu(\hbar,z) &= \frac{2}{3}\,\hbar^{-1}\big(|\tilde{\zeta}(z)|^{3/2}-|\zeta(z)|^{3/2}\big)\\ &= \hbar^{-1}\left(\int_z^{\tilde{x}_+}\sqrt{\tilde{E}-V(t)}\,\mathrm{d}t-\int_z^{x_+}\sqrt{E-V(t)}\,\mathrm{d}t\right).\end{aligned}$$

Hence $\mu'(z) = -2\sqrt{E - V(z)}$ and

$$\begin{aligned}
\partial_z \nu(\hbar, z) &= \frac{E - \tilde{E}}{\hbar} \left(\sqrt{E - V(z)} + \sqrt{\tilde{E} - V(z)} \right)^{-1}, \\
\partial_z^2 \nu(\hbar, z) &= \frac{E - \tilde{E}}{2\hbar} V'(z) \big(E - V(z)\big)^{-1/2} \big(\tilde{E} - V(z)\big)^{-1/2} \\
&\quad \times \left(\sqrt{E - V(z)} + \sqrt{\tilde{E} - V(z)} \right)^{-1}.
\end{aligned}$$

(6) The leading asymptotic term on the interval $]x_1, x_+ - \delta[$. Let us check the contribution to the matrix element coming from the interval $[x_1, x_+ - \delta]$. The leading asymptotic term in the expansion of ψ is given in (16). We also need the asymptotic behavior of the Airy function (25) and the fact that the function f/ζ is continuous and hence bounded on the interval $[x_1, x_+]$. We conclude that

$$\psi \sim \left(\frac{\zeta}{f} \right)^{1/4} \mathrm{Ai}(\hbar^{-2/3}\zeta) = \frac{\hbar^{1/6}}{\sqrt{\pi}\, |f|^{1/4}} \cos\!\left(\frac{2}{3}\, \hbar^{-1} |\zeta|^{3/2} - \frac{\pi}{4} \right) + \frac{1}{|f|^{7/4}}\, O(\hbar^{7/6}).$$

Observe that

$$\hbar^{4/3} \int_{x_1}^{x_+ - \delta} \frac{\mathrm{d}x}{|f|^2} = \delta^{-1} O(\hbar^{4/3}),$$

and on the interval $[x_1, x_+ - \delta]$,

$$(\tilde{E} - V)^{-1/4} = (E - V)^{-1/4} \big(1 + \delta^{-1} O(\hbar)\big).$$

From the boundedness of W and from an estimate similar to (32) it follows that

$$\begin{aligned}
\int_{x_1}^{x_+ - \delta} W \psi \tilde{\psi}\, \mathrm{d}x &= \int_{x_1}^{x_+ - \delta} W \left(\frac{\zeta}{f} \right)^{1/4} \left(\frac{\tilde{\zeta}}{f} \right)^{1/4} \mathrm{Ai}(\hbar^{-2/3}\zeta)\, \mathrm{Ai}(\hbar^{-2/3}\tilde{\zeta})\, \mathrm{d}x + O(\hbar^{4/3}) \\
&= \frac{\hbar^{1/3}}{\pi} \int_{x_1}^{x_+ - \delta} \frac{W}{|f|^{1/2}} \big(1 + \delta^{-1} O(\hbar)\big) \\
&\qquad \times \cos\!\left(\frac{2}{3}\, \hbar^{-1} |\zeta|^{3/2} - \frac{\pi}{4} \right) \cos\!\left(\frac{2}{3}\, \hbar^{-1} |\tilde{\zeta}|^{3/2} - \frac{\pi}{4} \right) \mathrm{d}x \\
&\quad + \delta^{-1} O(\hbar^{4/3}).
\end{aligned}$$

Using the asymptotic behavior (33) we have

$$\int_{x_1}^{x_+ - \delta} W \psi \tilde{\psi}\, \mathrm{d}x = \frac{\hbar^{1/3}}{2\pi} \int_{x_1}^{x_+ - \delta} \frac{W}{\sqrt{E - V}} \cos\!\left(\frac{2}{3}\, \hbar^{-1} (|\zeta|^{3/2} - |\tilde{\zeta}|^{3/2}) \right) \mathrm{d}x + \delta^{-1} O(\hbar^{4/3}). \tag{34}$$

(7) The argument of the cosine on the interval $]x_1, x_+ - \delta[$. Let us show that for $x \in [x_1, x_+ - \delta]$,

$$\frac{2}{3}\, \hbar^{-1} (|\zeta|^{3/2} - |\tilde{\zeta}|^{3/2}) = -\frac{2\pi k}{T}\, \tau(x) + \delta^{1/2} O(1) \tag{35}$$

where $\tau(x)$ was defined in (8). We have

$$\begin{aligned}
\frac{2}{3}\,\hbar^{-1}(|\zeta|^{3/2}-|\tilde\zeta|^{3/2}) &= \hbar^{-1}\left(\int_x^{x_+}\sqrt{E-V}\,\mathrm{d}t-\int_x^{\tilde x_+}\sqrt{\tilde E-V}\,\mathrm{d}t\right)\\
&= \hbar^{-1}\Bigg(\int_x^{x_+-\delta}\left(\sqrt{E-V}-\sqrt{\tilde E-V}\right)\mathrm{d}t\\
&\qquad +\int_{x_+-\delta}^{x_+}\sqrt{E-V}\,\mathrm{d}t-\int_{x_+-\delta}^{\tilde x_+}\sqrt{\tilde E-V}\,\mathrm{d}t\Bigg).
\end{aligned}$$

Set temporarily

$$g(y)=\int_{x_+-\delta}^{y}\sqrt{V(y)-V(t)}\,\mathrm{d}t.$$

Then for y lying between x_+ and $\tilde x_+$ it holds

$$\begin{aligned}
|g'(y)| &= \left|\frac{1}{2}\int_{x_+-\delta}^{y}\frac{V'(y)}{\sqrt{V(y)-V(t)}}\,\mathrm{d}t\right|\le\frac{1}{2}\,\mathrm{const}\int_{x_+-\delta}^{y}\frac{\mathrm{d}t}{\sqrt{y-t}}\\
&\le \mathrm{const}\,\sqrt{|x_+-\tilde x_+|+\delta}.
\end{aligned}$$

Hence

$$\begin{aligned}
\left|\int_{x_+-\delta}^{x_+}\sqrt{E-V}\,\mathrm{d}t-\int_{x_+-\delta}^{\tilde x_+}\sqrt{\tilde E-V}\,\mathrm{d}t\right| &= |g(x_+)-g(\tilde x_+)|\\
&\le \mathrm{const}\,\sqrt{|x_+-\tilde x_+|+\delta}\,|x_+-\tilde x_+|\\
&= \delta^{1/2}O(\hbar). \qquad (36)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\sqrt{E-V}-\sqrt{\tilde E-V}-\frac{E-\tilde E}{2\sqrt{E-V}} &= \frac{(E-\tilde E)^2}{2\left(\sqrt{E-V}+\sqrt{\tilde E-V}\right)^2\sqrt{E-V}}\\
&\le \frac{(E-\tilde E)^2}{2(E-V)^{3/2}}
\end{aligned}$$

and

$$\int_x^{x_+-\delta}\frac{(E-\tilde E)^2}{(E-V)^{3/2}}\,\mathrm{d}t=\delta^{-1/2}O(\hbar^2).$$

From (24) it follows that

$$\hbar^{-1}(\tilde E-E)=\frac{2\pi k}{T}+O(\hbar)$$

where T is the period of the classical motion, see (7). Altogether this means that

$$\begin{aligned}
\hbar^{-1}\int_{x}^{x_+-\delta}\left(\sqrt{E-V}-\sqrt{\tilde{E}-V}\right)\mathrm{d}t &= -\left(\frac{2\pi k}{T}+O(\hbar)\right)\int_{x}^{x_+-\delta}\frac{\mathrm{d}t}{2\sqrt{E-V(t)}}\\
&\quad +\delta^{-1/2}O(\hbar)\\
&= -\frac{2\pi k}{T}\int_{x}^{x_+}\frac{\mathrm{d}t}{2\sqrt{E-V(t)}}\\
&\quad +\delta^{1/2}O(1)+\delta^{-1/2}O(\hbar).
\end{aligned} \tag{37}$$

Relations (36) and (37) jointly imply (35).

(8) The final step. From (34) and (35) we derive that

$$\begin{aligned}
\int_{x_1}^{x_+-\delta}W\psi\tilde{\psi}\,\mathrm{d}x &= \frac{\hbar^{1/3}}{2\pi}\Bigg(\int_{x_1}^{x_+-\delta}\frac{W(x)}{\sqrt{E-V(x)}}\\
&\qquad\times\cos\left(\frac{2\pi k}{T}\,\tau(x)+\delta^{1/2}O(1)+\delta^{-1/2}O(\hbar)\right)\mathrm{d}x+\delta^{-1}O(\hbar)\Bigg)\\
&= \frac{\hbar^{1/3}}{2\pi}\left(\int_{x_1}^{x_+}\frac{W(x)}{\sqrt{E-V(x)}}\cos\left(\frac{2\pi k}{T}\,\tau(x)\right)\mathrm{d}x+\delta^{1/2}O(1)\right).
\end{aligned} \tag{38}$$

The interval $[x_-+\delta,x_1]$ can be treated similarly. We have

$$\int_{x_-+\delta}^{x_1}W\psi\tilde{\psi}\,\mathrm{d}x=\kappa\tilde{\kappa}\,\frac{\hbar^{1/3}}{2\pi}\left(\int_{x_-}^{x_1}\frac{W(x)}{\sqrt{E-V(x)}}\cos\left(\frac{2\pi k}{T}\,\tau_-(x)\right)\mathrm{d}x+\delta^{1/2}O(1)\right)$$

where

$$\tau_-(x)=\frac{1}{2}\int_{x_-}^{x}\frac{\mathrm{d}y}{\sqrt{E-V(y)}}=\frac{1}{2}\,T-\tau(x).$$

Taking into account also (26) and (27) we finally find that

$$\int_{x_-+\delta}^{x_1}W\psi\tilde{\psi}\,\mathrm{d}x=\frac{\hbar^{1/3}}{2\pi}\left(\int_{x_-}^{x_1}\frac{W(x)}{\sqrt{E-V(x)}}\cos\left(\frac{2\pi k}{T}\,\tau(x)\right)\mathrm{d}x+\delta^{1/2}O(1)\right). \tag{39}$$

From the boundedness of W and relations (28), (38) and (39) it follows that

$$\int_{\mathbb{R}}W\psi\tilde{\psi}\,\mathrm{d}x=\frac{\hbar^{1/3}}{2\pi}\left(\int_0^T W\big(q(t)\big)e^{ik\omega t}\,\mathrm{d}t+\delta^{1/2}O(1)\right). \tag{40}$$

As a particular case, with $W(x)=1$ and $k=0$, we have

$$\int_{\mathbb{R}}\psi^2\,\mathrm{d}x=\frac{\hbar^{1/3}}{2\pi}\big(T+\delta^{1/2}O(1)\big). \tag{41}$$

The same relation holds also for the squared norm of $\tilde{\psi}$.

Relations (40) and (41) imply that there exists $c\geq 0$ such that for all sufficiently small positive δ and all n, $n\geq n_0(\delta)$, it holds

$$\left|\langle n|W(x)|n+k\rangle-\frac{1}{T}\int_0^T W(q(t))\,e^{ik\omega t}\mathrm{d}t\right|\leq c\delta^{1/2}.$$

Since δ is arbitrary this concludes the verification of the limit (6).

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On the energy growth of some periodically driven quantum systems with shrinking gaps in the spectrum

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Abstract

We consider quantum Hamiltonians of the form $H(t) = H + V(t)$ where the spectrum of H is semibounded and discrete, and the eigenvalues behave as $E_n \sim n^\alpha$, with $0 < \alpha < 1$. In particular, the gaps between successive eigenvalues decay as $n^{\alpha-1}$. $V(t)$ is supposed to be bounded, continuously differentiable in the strong sense and such that the matrix entries with respect to the spectral decomposition of H obey the estimate $\|V(t)_{m,n}\| \leq \varepsilon\,|m-n|^{-p}\max\{m,n\}^{-2\gamma}$ for $m \neq n$ where $\varepsilon > 0$, $p \geq 1$ and $\gamma = (1-\alpha)/2$. We show that the energy diffusion exponent can be arbitrarily small provided p is sufficiently large and ε is small enough. More precisely, for any initial condition $\Psi \in \mathrm{Dom}(H^{1/2})$, the diffusion of energy is bounded from above as $\langle H\rangle_\Psi(t) = O(t^\sigma)$ where $\sigma = \alpha/(2[p]\gamma - \frac{1}{2})$. As an application we consider the Hamiltonian $H(t) = |p|^\alpha + \varepsilon v(\theta, t)$ on $L^2(S^1, \mathrm{d}\theta)$ which was discussed earlier in the literature by Howland.

1 Introduction

One of the basic questions one can ask about time-dependent quantum systems is the growth of energy on a long time scale for a given initial condition. Unfortunately the quantum dynamics in the time-dependent case proved itself to be rather difficult to analyze in its full generality and complexity. The systems which allow for at least partially analytical treatment and whose dynamics has been perhaps best studied from various points of view are either driven harmonic oscillators or periodically kicked quantum Hamiltonians [2, 14, 7, 8, 9, 3, 5]. On a more general level, it is widely believed that there exist close links between long time behavior of a quantum system and its spectral properties. For time-independent quantum systems such a relation is manifested by the famous RAGE theorem, see [25] for a summary. In a modified form this theorem has been extended to periodic and quasi-periodic quantum systems [12, 18, 24]. In this case the relevant operator whose spectral properties are of interest is the Floquet (monodromy) operator. Let us mention that a refined analysis of how the spectral properties determine the quantum dynamics is now available, see for example [13, 6, 11] and other papers, but here we are not directly concerned with this question.

Thus for periodically time-dependent systems one can distinguish as a related problem the spectral analysis of the Floquet operator under certain assumptions on the quantum Hamiltonian. Frequently one writes the time-dependent Hamiltonian in the form $H(t) = H + V(t)$ while imposing assumptions on the spectral properties of the unperturbed part H and requiring some sort of regularity from the perturbation $V(t)$. For our purposes an approach is rather important which is based on the adiabatic methods and which was initiated by Howland [15, 16] and further extended in [22, 19]. An essential property imposed on the unperturbed Hamiltonian in this case is the discreteness of the spectrum with increasing gaps between successive eigenvalues.

Under this hypothesis Nenciu in [23] was not only able to strengthen the results due to Howland but he derived in addition an upper bound on the diffusive growth of the energy having the form $\mathrm{const}\, t^{a/n}$ where $a > 0$ is given by the spectral properties of H and n is the order of differentiability of $V(t)$. Inspired by this result on the energy growth, Joye in [20] considered another class of time-dependent quantum Hamiltonians with rather mild assumptions on the spectral properties of H but on the other hand assuming that the strength of the perturbation $V(t)$ is in some sense small with respect to H. Moreover, as far as the energy diffusion is discussed, the periodicity of $V(t)$ is required neither in [23] nor in [20].

It is worthwhile to mention that Howland in [17] succeeded to treat also the case when the spectrum of H is discrete but the gaps between successive eigenvalues are decreasing. To achieve this goal he restricted himself to certain classes of perturbations $V(t)$ characterized by the behavior of matrix entries with respect to the eigen-basis of H. In particular, he discussed as an example the following model: $H(t) = |p|^\alpha + v(\theta, t)$ in $L^2(S^1, \mathrm{d}\theta)$ where $0 < \alpha < 1$ and $v(\theta, t)$ is in $C^\infty(S^1 \times S^1)$. It seems to be natural to look in this case, too, for a result parallel to that due to Nenciu [23] and to attempt a derivation of a nontrivial bound on the diffusive growth of energy. But we are aware of only one contribution in this direction made by Barbaroux and Joye [1]; it is based on the general scheme proposed in [20].

In this paper we wish to complete or to strengthen the results from [1] while making use of some ideas from [20]. Thus we aim to consider other classes of time-dependent Hamiltonians whose unperturbed part H has a discrete spectrum with decreasing gaps. In particular, the derived results are applicable to the Howland's model introduced in [17]. In more detail, we deal with a quantum system described by the Hamiltonian $H(t) := H + V(t)$ acting on a separable Hilbert space $\mathscr{H}$ and such that H is semibounded and has a pure point spectrum with the spectral decomposition

$$H = \sum_{n \in \mathbb{N}} E_n P_n.$$

Assume that the eigen-values $E_1 < E_2 < \dots$ obey the shrinking gap condition

$$c_H \frac{|m-n|}{\max\{m,n\}^{2\gamma}} \leq |E_m - E_n| \leq C_H \frac{|m-n|}{\max\{m,n\}^{2\gamma}} \tag{1}$$

for $\gamma \in]0, \frac{1}{2}[$ and strictly positive constants c_H, C_H. Notice that condition (1) implies $E_n \sim n^\alpha$ where $\alpha = 1 - 2\gamma \in]0,1[$ (more precisely, (1) implies that the sequence $E_n n^{-\alpha}$ is bounded both from below and from above by strictly positive constants for all sufficiently large n). To simplify the discussion let us assume, without loss of generality, that H is strictly positive, i.e., $E_1 > 0$.

The time-dependent perturbation $V(t) \in \mathscr{B}(\mathscr{H})$ is supposed to be T-periodic and C^1 in the strong sense. From the strong differentiability it follows that the propagator $U(t,s)$ associated to the Hamiltonian $H + V(t)$ exists and preserves the domain $\mathrm{Dom}(H)$ (see, e.g., [21]).

Let us suppose that V is small with respect to the norm

$$\|V\|_{p,\gamma} := \sup_{t \in [0,T]} \sup_{m,n \in \mathbb{N}} \langle m-n \rangle^p \max\{m,n\}^{2\gamma} \|V(t)_{m,n}\|, \tag{2}$$

where p is such that $[p] > 1/(2(1-\alpha))$,

$$\langle m-n \rangle := \max\{1, |m-n|\},$$

and $\|V(t)_{m,n}\|$ denotes the norm of the operator

$$V(t)_{m,n} := P_m V(t) P_n : \mathrm{Ran}\, P_n \to \mathrm{Ran}\, P_m.$$

We claim that the propagator $U(t,s)$ preserves the form domain $Q_H = \mathrm{Dom}(H^{1/2})$ and for any Ψ from Q_H one can estimate the long-time behavior of the energy expectation value by

$$\langle U(t,0)\Psi, HU(t,0)\Psi \rangle = O(t^\sigma), \text{ with } \sigma = \frac{2\alpha}{2[p](1-\alpha) - 1}$$

(more details are given in Theorem 5 below). Here $[p]$ is standing for the integer part of p.

Provided that $[V(t), V(s)] = 0$ for every t, s and $\int_0^T V(t)\, \mathrm{d}t = 0$, the assumption $\|V\|_{p,\gamma} \leq \varepsilon$ can be replaced by $\|V\|_{p+1,0} \leq \varepsilon$, i.e.,

$$\|P_m V(t) P_n\| \leq \frac{\varepsilon}{\langle m-n \rangle^{p+1}}.$$

The condition $[V(t), V(s)] = 0$ is satisfied for example when $V(t)$ is a potential (i.e., a multiplication operator by a function on a certain L^2 space) or when the time dependence of $V(t)$ is factorized, i.e., $V(t) = f(t)v$ where $f(t)$ is a real-valued (T-periodic and C^1) function and v is a time-independent operator on $\mathscr{H}$.

Let us compare the result of the current paper, as briefly described above, to the results derived in [20] and [1]. Paper [20] focuses on the general scheme and is not so much concerned with particular cases as that one we are going to deal with here. Nevertheless a possible application to the Howland's classes of perturbations is shortly discussed in Proposition 5.1 and Lemma 5.1. The Howland's classes are determined by a norm which somewhat differs from (2), as explained in more detail in Subsection 2.1. But the difference is not so essential to prevent a comparison. To simplify the discussion let us assume that the eigenvalues of H are simple and behave asymptotically as $E_n \sim \mathrm{const}\, n^\alpha$, with $0 < \alpha < 1$. In the particular case when $\|V\|_{p,\gamma} < \infty$ for some $p > 1$ and $\gamma = (1-\alpha)/2$ the bound on the energy diffusion exponent derived in [20] equals $\alpha/(2\gamma - \frac{1}{2})$ provided $\gamma > (1+\alpha)/4$, i.e., $\alpha < 1/3$. Our bound $\alpha/(2[p]\gamma - \frac{1}{2})$, valid for $0 < \alpha < 1$ and provided $[p] > 1/(4\gamma)$, is achieved by making use of the rapid decay of matrix entries of V in the direction perpendicular to the diagonal. It follows that we can make the growth of the energy $\langle H \rangle_\Psi$ arbitrarily small by imposing more restrictive assumptions on the perturbation V, i.e., by letting the parameter p be sufficiently large.

In paper [1] one treats in fact a larger class of perturbations than we do since one requires only the finiteness of the norm $\|V\|_{p,0} < \infty$ for p sufficiently large. In other words, no decay of matrix entries of V along the diagonal is supposed. On the other hand, one assumes that the initial quantum state belongs to the domain $\mathrm{Dom}(H^\beta)$ for β sufficiently large; β is never assumed therein to be smaller than $3/2$. Furthermore, there is no assumption on the periodicity of $H(t)$ both in [1] and [20]. On the other hand, our assertion concerns all initial states from the domain $\mathrm{Dom}(H^{1/2})$ but we need a decay of matrix entries of V along the diagonal at least of order $2\gamma = 1 - \alpha$. For the sake of comparison let us also recall the bound on the energy diffusion exponent which has been derived in [1]. It is roughly of the form $\alpha/(1 - f(p))^2$ where α has the same meaning as above, $f(p)$ is positive and $f(p) = O(p^{-1})$ as $p \to \infty$. Hence this bound is never smaller than α and approaches this value as the parameter p tends to infinity.

2 Upper bound on the energy growth

2.1 The gap condition and the modified Howland's classes

On the contrary to Howland who introduced in [17] the classes $\mathcal{X}(p,\delta)$ equipped with the norm

$$\|A\|^H_{p,\delta} = \sup_{m,n} \{(mn)^\delta \langle m - n \rangle^p \, \|A_{m,n}\|;\ m, n \geq 1\},$$

we prefer to work with somewhat modified classes, called $\mathcal{Y}(p,\delta)$, whose definition is adjusted to the gap condition (1). Our choice is dictated by an expected asymptotic behavior of eigenvalues of H in a typical situation. Let us briefly explain where condition (1) comes from.

We expect the eigenvalues to behave asymptotically as $E_n = \mathrm{const}\, n^\alpha(1 + o(1))$ where the error term $o(1)$ is supposed to tend to zero sufficiently fast. The spectral gaps $E_{n+1} - E_n$ tend to zero as $n \to \infty$ if $\alpha \in]0,1[$. Keeping the notation $\gamma := (1-\alpha)/2$ we wish to estimate the difference $|E_m - E_n|$. To this end we replace E_n simply by the power sequence

n^α. Then one gets

$$\frac{m^\alpha - n^\alpha}{m-n}(mn)^\gamma = \frac{\sinh(\alpha y)}{\sinh(y)} = e^{-(1-\alpha)|y|}\frac{1-e^{-2\alpha|y|}}{1-e^{-2|y|}}$$

where $e^{2y} := m/n$. Since the fraction $(1-e^{-2\alpha|y|})/(1-e^{-2|y|})$ can be estimated by positive constants both from above and from below we finally find that

$$C_1\,\frac{|m-n|}{\max\{m,n\}^{2\gamma}} \le |m^\alpha - n^\alpha| \le C_2\,\frac{|m-n|}{\max\{m,n\}^{2\gamma}}$$

for some $C_1, C_2 > 0$ and all $m, n \in \mathbb{N}$.

Definition 1. Let $p \ge 1$, $\delta \ge 0$ and $p + 2\delta > 1$. We say that an operator $A \in \mathscr{B}(\mathscr{H})$ belongs to the class $\mathcal{Y}(p,\delta)$ if and only if

$$\|A\|_{p,\delta} := \sup_{m,n\in\mathbb{N}} \langle m-n\rangle^p \max\{m,n\}^{2\delta}\,\|A_{m,n}\| < \infty. \tag{3}$$

Let $A(t)$ be a T-periodic function with values in the space $\mathcal{Y}(p,\delta)$. With some abuse of notation we shall also write

$$\|A\|_{p,\delta} := \sup_{t\in[0,T]}\sup_{m,n\in\mathbb{N}} \langle m-n\rangle^p \max\{m,n\}^{2\delta}\,\|A(t)_{m,n}\| < \infty.$$

Remarks. (i) It is straightforward to check that $\|\cdot\|_{p,\delta}$ is indeed a norm. Let us note that an equivalent norm is obtained if one replaces $\max\{m,n\}$ by $(m+n)$ in (3).

(ii) Obviously, $\mathcal{Y}(p,\delta) \subset \mathcal{X}(p,\delta)$. Notice that $\mathcal{Y}(p,\delta)$ is a Banach space equipped with the norm $\|\cdot\|_{p,\delta}$.

(iii) For the sake of convenience we have chosen the norm (3) with the restrictions $p \ge 1$, $\delta \ge 0$ and $p+2\delta > 1$ so that if it is finite for a matrix $\{A_{mn}\}$, $A_{mn} \in \mathscr{B}(\operatorname{Ran} P_n, \operatorname{Ran} P_m)$, then the matrix corresponds to a bounded operator $A \in \mathscr{B}(\mathscr{H})$. Indeed, it is so since one can estimate the operator norm $\|A\|$ by the Shur-Holmgren norm

$$\|A\|_{SH} := \max\left\{\sup_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}}\|A_{m,n}\|, \sup_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}\|A_{m,n}\|\right\}.$$

It clearly holds

$$\|A\|_{SH} \le \|A\|_{p,\delta}\sup_{m\in\mathbb{N}}\sum_{n=1}^{\infty}\frac{1}{\langle m-n\rangle^p \max\{m,n\}^{2\delta}}\,.$$

The sum on the RHS equals

$$\begin{aligned}\frac{1}{m^{2\delta}} + \sum_{n=1}^{m-1}\frac{1}{(m-n)^p m^{2\delta}} + \sum_{n=m+1}^{\infty}\frac{1}{(n-m)^p n^{2\delta}} \quad &\le\quad 2 + \frac{1}{m^{2\delta}}\int_1^m \frac{\mathrm{d}x}{x^p} + \sum_{k=1}^{\infty}\frac{1}{k^{p+2\delta}}\\ &=\quad 2 + \frac{1-m^{-p+1}}{(p-1)m^{2\delta}} + \zeta(p+2\delta).\end{aligned}$$

Setting temporarily $x = \ln(m)$ and $\epsilon = p - 1$ one can make use of the inequality

$$\frac{1}{\epsilon}\left(e^{-2\delta x} - e^{-(\epsilon+2\delta)x}\right) \leq \frac{1}{\epsilon + 2\delta}$$

which is true for all $x \geq 0$ provided $\epsilon \geq 0$, $\delta \geq 0$ and $\epsilon + 2\delta > 0$. Thus one arrives at the estimate

$$\|A\|_{SH} \leq \left(2 + \frac{1}{p + 2\delta - 1} + \zeta(p + 2\delta)\right) \|A\|_{p,\delta}.$$

Here $\zeta(u) := \sum_{k=1}^{\infty} k^{-u}$ denotes the Riemann's zeta function.

(iv) Finally let us note that the value $p = \infty$ is admissible. We shall use the norm $\|\cdot\|_{\infty,\delta}$ exclusively in the case of diagonal matrices when it simply reduces to

$$\|A\|_{\infty,\delta} := \sup_{n\in\mathbb{N}} n^{2\delta}\, \|A_{n,n}\|.$$

From Definition 1 one immediately deduces the following lemma.

Lemma 2. *Suppose that H is an operator on $\mathscr{H}$ with pure point spectrum whose eigenvalues $E_1 < E_2 < \ldots$ obey the upper bound in (1). Let $p \geq 2$. If $A \in \mathcal{Y}(p, \delta)$ then the commutator $[A, H]$ lies in $\mathcal{Y}(p - 1, \delta + \gamma)$ and*

$$\|[A, H]\|_{p-1,\delta+\gamma} \leq C_H \|A\|_{p,\delta}.$$

A basic technical tool we need is the following lemma concerned with products of two classes $\mathcal{Y}$. For its proof as well as for the remainder of the paper the following two elementary inequalities will be useful. According to the first one, for every $m, k \geq 1$ it holds

$$\frac{m}{k} \leq 2\langle m - k\rangle. \tag{4}$$

In fact, this is a direct consequence of the implication $a, b \geq 1 \implies a + b \leq 2ab$.

The second inequality claims that if $a, b \geq 0$ then

$$\frac{\langle a + b\rangle}{\langle a\rangle\langle b\rangle} \leq \frac{2}{\langle \min\{a, b\}\rangle}\,.$$

This can be reduced to the inequality $\langle 2a\rangle \leq 2\langle a\rangle$ which is quite obvious.

Lemma 3. *Consider two classes $\mathcal{Y}(p_1, \delta_1)$, $\mathcal{Y}(p_2, \delta_2)$, with $p_1, p_2 \geq 1$, $\delta_1, \delta_2 \geq 0$, and $p_1 + 2\delta_1, p_2 + 2\delta_2 > 1$. Suppose that the numbers p, δ satisfy the inequalities*

$$1 \leq p \leq \min\{p_1, p_2\},\ \ \max\{\delta_1, \delta_2\} \leq \delta \leq \delta_1 + \delta_2,$$
$$1 < p + 2\delta \leq \min\{p_1 + 2\delta_1, p_2 + 2\delta_2\},\ p + 2\delta < \max\{p_1 + 2\delta_1, p_2 + 2\delta_2\}.$$

If $A \in \mathcal{Y}(p_1, \delta_1)$ and $B \in \mathcal{Y}(p_2, \delta_2)$ then

$$\|AB\|_{p,\delta} \leq C(p, p_1, p_2, \delta, \delta_1, \delta_2)\, \|A\|_{p_1,\delta_1} \|B\|_{p_2,\delta_2} \tag{5}$$

where

$$C(p, p_1, p_2, \delta, \delta_1, \delta_2) = 2^{\max\{p_1,p_2\}+2(\delta-\delta_0)}$$
$$\times \left(3 + 3\zeta(\max\{p_1 + 2\delta_1, p_2 + 2\delta_2\} - 2\delta) + \frac{1}{e(\max\{p_1 + 2\delta_1, p_2 + 2\delta_2\} - p - 2\delta)}\right)$$

and $\delta_0 = \min\{\delta_1, \delta_2\}$. Consequently, $\mathcal{Y}(p_1, \delta_1)\mathcal{Y}(p_2, \delta_2) \subset \mathcal{Y}(p, \delta)$.

Proof. Under the assumptions we have

$$\langle m-n\rangle^p \max\{m,n\}^{2\delta}\|(AB)_{mn}\| \le \langle m-n\rangle^p \max\{m,n\}^{2\delta}\sum_{\ell=1}^{\infty}\|A_{m\ell}\|\|B_{\ell n}\|$$

which is less than or equal to

$$\|A\|_{p_1,\delta_1}\|B\|_{p_2,\delta_2}\sum_{\ell=1}^{\infty}\frac{\langle m-n\rangle^p\max\{m,n\}^{2\delta}}{\langle m-\ell\rangle^{p_1}\max\{m,\ell\}^{2\delta_1}\langle n-\ell\rangle^{p_2}\max\{n,\ell\}^{2\delta_2}}\,. \tag{6}$$

Observe that

$$\frac{\max\{m,n\}^{2\delta}}{\max\{m,\ell\}^{2\delta_1}\max\{n,\ell\}^{2\delta_2}} \le 2^{2(\delta-\delta_0)}\langle m-\ell\rangle^{2(\delta-\delta_1)}\langle n-\ell\rangle^{2(\delta-\delta_2)}$$

Without loss assume that $\max\{p_1+2\delta_1, p_2+\delta_2\} = p_1+2\delta_1 > p+2\delta$. The summand in (6) can be estimated from above by

$$\frac{2^{2(\delta-\delta_0)}\langle|m-\ell|+|n-\ell|\rangle^p}{\langle m-\ell\rangle^{p_1+2(\delta_1-\delta)}\langle n-\ell\rangle^{p_2+2(\delta_2-\delta)}} \le \frac{2^{p+2(\delta-\delta_0)}}{\langle\min\{|m-\ell|,|n-\ell|\}\rangle^p\langle m-\ell\rangle^{p_1-p+2(\delta_1-\delta)}}$$

For definiteness let us suppose that $m>n$. With the help of inequality $\sum_{k=1}^{i}k^{-1}\le 1+\log i$ the sum in (6) is estimated from above by

$$\begin{aligned}
&2^{p+2(\delta-\delta_0)}\Bigg(\sum_{\ell=1}^{n}\langle n-\ell\rangle^{-p}\langle m-\ell\rangle^{p-p_1+2(\delta-\delta_1)}\\
&\quad+\sum_{\ell=n+1}^{[\frac{m+n}{2}]}(\ell-n)^{-1}(m-\ell)^{p-p_1+2(\delta-\delta_1)}+\sum_{\ell=[\frac{m+n}{2}]+1}^{\infty}\langle m-\ell\rangle^{-p_1+2(\delta-\delta_1)}\Bigg)\\
&\le 2^{p+2(\delta-\delta_0)}\left(\left(\frac{2}{m-n}\right)^{p_1-p+2(\delta_1-\delta)}(1+\log(\frac{m-n}{2}))+2+3\zeta(p_1+2(\delta_1-\delta))\right)
\end{aligned}$$

Using the fact that $x^{-\alpha}\log x\le 1/(e\alpha)$ holds true for $x,\alpha>0$ one gets easily (5). The cases $m<n, m=n$ may be investigated similarly. □

Corollary 4. *Let $p\ge 2$, $i\ge 1$ and $\gamma\in]0,\frac{1}{2}[$. Then the following product formulas hold true:*

$$\begin{aligned}
&\mathcal{Y}(p,i\gamma)\,\mathcal{Y}(p,i\gamma)\subset\mathcal{Y}(p-1,(i+1)\gamma)\\
&\mathcal{Y}(p,(i-1)\gamma)\,\mathcal{Y}(p-1,i\gamma)\subset\mathcal{Y}(p-1,i\gamma)\\
&\mathcal{Y}(p+1,(i-1)\gamma)\,\mathcal{Y}(p-1,(i+1)\gamma)\subset\mathcal{Y}(p-1,(i+1)\gamma)
\end{aligned}$$

The formulas are also true for the opposite order of factors on the LHS. Moreover, if operators A and B belong to the corresponding classes on the LHS then

$$\begin{aligned}
&\|AB\|_{p-1,(i+1)\gamma}\le C_{p,\gamma}\,\|A\|_{p,i\gamma}\|B\|_{p,i\gamma}\\
&\|AB\|_{p-1,i\gamma}\le C_{p,\gamma}\,\|A\|_{p,(i-1)\gamma}\|B\|_{p-1,i\gamma}\\
&\|AB\|_{p-1,(i+1)\gamma}\le 2C_{p,\gamma}\,\|A\|_{p+1,(i-1)\gamma}\|B\|_{p-1,(i+1)\gamma},
\end{aligned}$$

where

$$C_{p,\gamma} := 2^{p+2}(3 + 3\,\zeta(p-2\gamma) + \frac{1}{e(1-2\gamma)}). \tag{7}$$

The norm estimates hold true also for the opposite order of factors A and B in the product.

2.2 The main theorem

Theorem 5. *Let a quantum system be described by a Hamiltonian of the form*

$$H(t) = H + V(t) \ \text{ on } \mathscr{H}$$

where H is a self-adjoint operator with a pure point spectrum and the spectral decomposition

$$H = \sum_{n\in\mathbb{N}} E_n P_n.$$

Suppose that the eigen-values of H are ordered increasingly and obey the gap condition (1) with $\gamma \in \left]0, \frac{1}{2}\right[$. Set $\alpha = 1 - 2\gamma$. Assume that

$$[p] > \frac{1}{2(1-\alpha)}. \tag{8}$$

Then there exists $\varepsilon > 0$ such that if $V(t)$ is T-periodic, symmetric, continuously differentiable in the strong sense and obeys $\|V\|_{p,\gamma} \leq \varepsilon$ then the propagator $U(t,s)$ associated to the Hamiltonian $H + V(t)$ maps Q_H, the form domain of H, onto itself and for every $\Psi \in Q_H$ it holds

$$\langle H\rangle_\Psi(t) := \langle U(t,0)\Psi, HU(t,0)\Psi\rangle = O(t^\sigma) \tag{9}$$

where

$$\sigma = \frac{2\alpha}{2[p](1-\alpha)-1}\,.$$

Remark 6. *(i) There is no assumption on the dimension of $\operatorname{Ran} P_n$. The multiplicities of eigenvalues may grow arbitrarily, they can even be infinite.*

(ii) Suppose that $V(t) \in \mathcal{Y}(p+1, 0)$, with $p > 2$, is T-periodic, symmetric, continuously differentiable in the strong sense and such that $[V(t), V(s)] = 0$ for every t, s, and $\bar{V} := T^{-1}\int_0^T V(t)\,\mathrm{d}t = 0$. Then one arrives at the same estimate (9). Let us outline the proof for this case. According to Remark 12 one can transform anti-adiabatically $H+V(t)$ to $H+V_1(t)$, with $V_1(t) \in \mathcal{Y}(p,\gamma)$, and afterwards one can apply Theorem 5. The first diagonalization procedure is not necessary since $\bar{V} = 0$.

(iii) Provided that $H(t) = H + V(t)$, with V in $C^1(\mathbb{R}, \mathscr{B}(\mathscr{H}))$ in the strong sense there exists a trivial bound (see [23]) which does not depend on the spectral properties of H, namely

$$|\langle U(t,0)\Psi, H(t)U(t,0)\Psi\rangle| \leq |\langle \Psi, H(0)\Psi\rangle| + |t| \sup_{s\in\mathbb{R}} \|\dot{V}(s)\|\|\Psi\|^2. \tag{10}$$

For the derivation it suffices to notice that

$$\partial_t \langle U(t,0)\Psi, H(t)U(t,0)\Psi\rangle = \langle U(t,0)\Psi, \dot{V}(t)U(t,0)\Psi\rangle$$

where $\dot{V}(s)$ denotes the time derivative in the strong sense. The estimate given by Theorem 5 is better than this trivial bound if

$$[p] > p_{min} := \frac{2\alpha+1}{2(1-\alpha)}.$$

For example, in the case of $\alpha = 2/3$ (the quantum ball) we get $p_{min} = 7/2$. The condition $[p] > p_{min}$ is fulfilled if $p \geq 4$ and then Theorem 5 tells us that $\langle H\rangle_\Psi(t) = O(t^{4/5})$.

2.3 An application to the Howland's model

Let us apply the results of Theorem 5 to the model introduced by Howland in [17] and described by the Hamiltonian $|p|^\alpha + \varepsilon v(\theta,t)$, with $\alpha \in]0,1[$, which is supposed to act on $L^2(S^1, \mathrm{d}\theta)$ and to be 2π-periodic in time. Set $H := |p|^\alpha$. The spectral decomposition of H reads

$$H = \sum_{n\geq 0} n^\alpha P_n \;\text{ where }\; P_n\Psi(\theta) = \frac{1}{\pi}\int_0^{2\pi} \cos\left(n(\theta-s)\right)\Psi(s)\,\mathrm{d}s.$$

Except of the first one the multiplicities of the eigen-values are equal 2. Using integration by parts one derives that any multiplication operator a by a function $a(\theta) \in C^k$ obeys the estimate

$$\|P_m\, a\, P_n\| \leq \frac{2\sqrt{2\pi}\,\|a^{(k)}\|}{\langle m-n\rangle^k}\,.$$

Hence $a \in \mathcal{Y}(k,0)$. Applying Theorem 5 and Remark 6 ad (ii) we get

Proposition 7. *Let $\alpha \in]0,1[$ and $v(\theta,t)$ be a real-valued function which is 2π-periodic both in the space and in the time variable. Suppose that $v(\theta,t)$ is C^k in θ and C^1 in t and such that $\int_0^{2\pi} v(\theta,t)\,\mathrm{d}t = 0$. If $k > 3$ and $k > (3-2\alpha)/(2(1-\alpha))$ then there exists $\varepsilon_0 > 0$ such that for every real ε, $|\varepsilon| < \varepsilon_0$, the propagator $U(t,s)$ associated to*

$$H(t) := |p|^\alpha + \varepsilon v(\theta,t) \;\text{ on } L^2(S^1,\mathrm{d}\theta)$$

preserves the domain $\mathrm{Dom}(|p|^{\alpha/2})$ and for every Ψ from this domain it holds true that

$$\langle U(t,0)\Psi, H(t)U(t,0)\Psi\rangle = O(t^\sigma)$$

where

$$\sigma = \frac{2\alpha}{2(k-1)(1-\alpha)-1}\,.$$

Let us summarize that the energy diffusion exponent in the Howland's model can be made arbitrarily small provided the potential on the circle is sufficiently smooth and the coupling constant is sufficiently small.

3 Derivation of the main result

3.1 Two additional theorems

The proof of Theorem 5 is based on the following two theorems, Theorem 8 and Theorem 9. In what follows we use the notation $D := -\mathrm{i}\partial_t$ on the interval $[0, T]$ with the periodic boundary condition.

Theorem 8. *Let $K = D + H + V(t)$ be a Floquet Hamiltonian on $L^2([0,T], \mathscr{H})$, with H and $V(t)$ satisfying the assumptions of Theorem 5. Let $p \geq 1$ and $q \leq p-1$ be a natural number or zero. Then there exists $\varepsilon > 0$ such that $\|V\|_{p,\gamma} \leq \varepsilon$ implies the existence of a T-periodic family of unitary operators $J(t)$ on $\mathscr{H}$ which is continuously differentiable in the strong sense and such that*

$$K = J(t)(D + H + A + B(t))J(t)^*$$

where $B(t) \in \mathcal{Y}(p-q, (q+1)\gamma)$ is T-periodic, Hermitian and strongly continuously differentiable, and A is bounded, symmetric and commutes with H.

The remainder of the current paper is concerned with the proof of Theorem 8. Theorem 9 to follow is a mere modification of Proposition 5.1 in [20] in combination with some ideas from [1, Section 2]. This is why we present its proof in a rather sketchy form. Let us also note that the basic idea standing behind the estimates goes back to Nenciu [23].

Theorem 9. *Let H be a positive operator with a pure point spectrum and the spectral decomposition $H = \sum_n E_n P_n$. Assume that the eigen-values $0 < E_1 < E_2 < \ldots$ satisfy $E_n = O(n^\alpha)$, with $\alpha > 0$. Set $Q_n = 1 - P_n$. Let an operator-valued function $W(t) \in \mathscr{B}(\mathscr{H})$ be Hermitian, C^1 in the strong sense and such that*

$$\forall n \in \mathbb{N}, \ \ \|P_n W(t) Q_n H^{-1/2}\| \leq \frac{\text{const}}{n^{\mu + \frac{\alpha}{2}}}$$

uniformly in time for some $\mu > 1/2$. Then the propagator $U(t,s)$ associated with $H + W(t)$ preserves Q_H, the form domain of H, and for every Ψ from Q_H,

$$\langle U(t,0)\Psi, HU(t,0)\Psi \rangle = O(t^{2\alpha/(2\mu-1)}).$$

Remark. The bound on the energy expectation value is nontrivial if $\mu > \frac{1}{2} + \alpha$.

Proof. Let

$$W_d(t) := \sum_{n=1}^{\infty} P_n W(t) P_n$$

be the diagonal part of $W(t)$. It is straightforward to see that $W_d(t)$ is again C^1 in the strong sense. Let $U_d(t,s)$ be the propagator associated to $H + W_d(t)$. Since $W_d(t)$ commutes with H the same if true for $U_d(t,s)$. Equivalently this means that $U_d(t,s)$ commutes with all projectors P_n. From the Duhamel's formula we have

$$R(t) := U(t,0) - U_d(t,0) = -\mathrm{i} \int_0^t U_d(t,s)\big(W(s) - W_d(s)\big)U(s,0)\, \mathrm{d}s.$$

Fix $t > 0$ and choose $\Psi \in \mathrm{Dom}(H) \subset \mathrm{Dom}(H^{1/2})$. Notice that $P_n\big(W(s) - W_d(s)\big) = P_n W(s) Q_n$. For any t', $0 \le t' \le t$, it holds

$$\|H^{1/2}U(t',0)\Psi\|^2 \le \sum_{n=1}^{\infty} E_n \|P_n U(t',0)\Psi\|^2 \le E_N \|\Psi\|^2 + \sum_{n=N+1}^{\infty} E_n \|P_n U(t',0)\Psi\|^2.$$

Furthermore,

$$\|P_n U(t',0)\Psi\|^2 \le 2(\|P_n\Psi\|^2 + \|P_n R(t')\Psi\|^2)$$

and

$$\begin{aligned} \|P_n R(t')\Psi\| &\le \int_0^t \|P_n W(s) Q_n H^{-1/2}\| \,\mathrm{d}s \sup_{0\le s\le t} \|H^{1/2}U(s,0)\Psi\| \\ &\le \frac{c\,t}{n^{\mu+\frac{\alpha}{2}}} \sup_{0\le s\le t} \|H^{1/2}U(s,0)\Psi\|. \end{aligned}$$

From these estimates one concludes that for any $t > 0$, all $\Psi \in \mathrm{Dom}(H)$, $N \in \mathbb{N}$ and some positive constants c_1, c_2 independent of t, Ψ and N it holds

$$\left(1 - \frac{c_1 t^2}{N^{2\mu-1}}\right) \sup_{0\le s\le t} \|H^{1/2}U(s,0)\Psi\|^2 \le c_2 N^{\alpha} \|\Psi\|^2 + 2\|H^{1/2}\Psi\|^2.$$

Setting $N = [C t^{2/(2\mu-1)}]$ where $C > 0$ is a sufficiently large constant one deduces that there exists $c_3 > 0$ such that it holds

$$\|H^{1/2}U(t,0)\Psi\|^2 \le c_3 \left(t^{2\alpha/(2\mu-1)} \|\Psi\|^2 + \|H^{1/2}\Psi\|^2\right) \tag{11}$$

for all $t \ge 1$ and $\Psi \in \mathrm{Dom}(H)$.

One can extend the validity of (11) to $\Psi \in \mathrm{Dom}(H^{1/2})$. To this end it suffices to use the fact that $\mathrm{Dom}(H^{1/2})$ is a Banach space with respect to the norm $\|\Psi\|_* = (\|\Psi\|^2 + \|H^{1/2}\Psi\|^2)^{1/2}$, and $\mathrm{Dom}(H) \subset \mathrm{Dom}(H^{1/2})$ is a dense subspace. Choosing $\Psi \in \mathrm{Dom}(H^{1/2})$ one can find a sequence $\{\Psi_k\}$ in $\mathrm{Dom}(H)$ such that $\Psi_k \to \Psi$ in $\mathrm{Dom}(H^{1/2})$. Then (11) implies that $\{U(t,0)\Psi_k\}$ is a Cauchy sequence in $\mathrm{Dom}(H^{1/2})$ whose limit necessarily equals $U(t,0)\Psi$. Hence $\mathrm{Dom}(H^{1/2})$ is $U(t,0)$–invariant and (11) is valid also for all $\Psi \in \mathrm{Dom}(H^{1/2})$. This concludes the proof. □

3.2 Proof of Theorem 5

Here we show how Theorem 5 follows from Theorem 8 and Theorem 9.

Lemma 10. *Assume that H is a positive operator with a pure point spectrum and the spectral decomposition $H = \sum_{n=1}^{\infty} E_n P_n$, and such that the eigen-values satisfy* $\inf E_n n^{-\alpha} > 0$, *with $\alpha > 0$. Set $Q_n = 1 - P_n$. Then for any $p \ge 1$ there exist a constant $c(p,\alpha) > 0$ such that for all $\delta > 0$,*

$$\forall W \in \mathcal{Y}(p,\delta), \forall n \in \mathbb{N},\ \ \|P_n W Q_n H^{-1/2}\| \le c(p,\alpha)\, \frac{\|W\|_{p,\delta}}{n^{2\delta+\frac{\alpha}{2}}}\,.$$

Proof. Suppose that $W \in \mathcal{Y}(p,\delta)$. By the assumptions, $E_n \geq c\, n^{\alpha}$ for all n and some $c > 0$. We have

$$\|P_n W Q_n H^{-1/2}\|^2 \leq \sum_{m, m\neq n} \frac{\|W_{n,m}\|^2}{E_m} \leq \frac{1}{c} \sum_{m,m\neq n} \frac{\|W\|_{p,\delta}^2}{|m-n|^{2p} \max\{m,n\}^{4\delta} m^{\alpha}}\,.$$

Now one splits the range of summation in m into three segments: $1 \leq m < n/2$, $n/2 \leq m < n$ and $n < m$. For each case one can apply elementary and rather obvious estimates to show that the expression decays in n at least as $n^{-4\delta-\alpha}$. In the first case one has to use the fact that $\alpha < 1$. We omit the details. ☐

Proof of Theorem 5. Theorem 8, with $q := [p-1]$, implies the existence of a transformation

$$K = J(t)(D + H + A + B(t))J(t)^* \tag{12}$$

where A is bounded and diagonal and $B(t) \in \mathcal{Y}(p-q,(q+1)\gamma)$. Since $q = [p-1]$ we have $p - q \geq 1$. Set $W(t) := A + B(t)$. Then $P_n W(t) Q_n = P_n B(t) Q_n$. The gap condition (1) guarantees that the assumptions of Lemma 10 are satisfied and thus one finds that $\|P_n W(t) Q_n H^{-1/2}\| \leq \text{const} \cdot n^{-\mu-\frac{\alpha}{2}}$, with $\mu = 2(q+1)\gamma = [p](1-\alpha)$. Notice that assumption (8) means that $\mu > 1/2$. In virtue of Theorem 9, the propagator $\tilde{U}(t,s)$ associated to $H + W(t)$ maps the form domain Q_H onto itself and fulfills

$$\langle \tilde{U}(t,0)\tilde{\Psi}, H\tilde{U}(t,0)\tilde{\Psi}\rangle = O(t^{\sigma}),\ \text{with}\ \sigma = \frac{2\alpha}{2[p](1-\alpha)-1}\,,$$

for every $\tilde{\Psi} \in Q_H$.

Equality (12) implies that

$$H + V(t) = J(t)HJ(t)^* + \mathrm{i}\dot{J}(t)J(t)^* + J(t)W(t)J(t)^*. \tag{13}$$

Since the family $J(t)$ is known to be continuously differentiable in the strong sense it follows from the uniform boundedness principle that the derivative $\dot{J}(t)$ is a bounded operator. Moreover, using the periodicity and applying the uniform boundedness principle once more one finds that $\|\dot{J}(t)\|$ is bounded uniformly in t. Hence all operators occurring in equality (13), except of H, are bounded. One deduces from (13) that $J(t)$ maps $\mathrm{Dom}\, H$ onto itself for every t and that the same is also true for the form domain. Set $U(t,s) := J(t)\tilde{U}(t,s)J(s)^*$. Then $U(t,s)$ is the propagator corresponding to $H + V(t)$. For any $\Psi \in Q_H$ we have

$$\begin{aligned}\langle H\rangle_{\Psi}(t) &= \langle U(t,0)\Psi, H\,U(t,0)\Psi\rangle = \langle U(t,0)\Psi, J(t)HJ(t)^*U(t,0)\Psi\rangle + O(1)\\ &= \langle \tilde{U}(t,0)\tilde{\Psi}, H\tilde{U}(t,0)\tilde{\Psi}\rangle + O(1) = O(t^{\sigma})\end{aligned}$$

where $\tilde{\Psi} := J(0)^*\Psi$. This proves the theorem. ☐

3.3 The idea of the proof of Theorem 8

It remains to prove Theorem 8. The proof is somewhat lengthy and the remainder of the paper is devoted to it. Let us explain the main idea. The proof combines the anti-adiabatic transformation due to Howland (see Section 4) with a (properly modified) diagonalization method, as presented in [10] (see Section 5). This procedure is applied repeatedly until achieving the required properties of the perturbation. Let us describe one step in this approach when starting from the Floquet Hamiltonian

$$K_\triangle := D + H + Y + Z(t)$$

where $Y \in \mathcal{Y}(\infty, \gamma)$ is Hermitian and diagonal (i.e., commuting with H) and $Z(t) \in \mathcal{Y}(r, i\gamma)$ is symmetric, T-periodic and strongly C^1. The parameters are supposed to satisfy $i \geq 1$, $r \geq 2$.

Firstly, using the anti-adiabatic transform we try to improve the decay of entries of $Z(t)$ along the main diagonal when paying for it by a worse decay of elements in the direction perpendicular to the diagonal. In more detail, we would like to transform $Z(t) \in \mathcal{Y}(r, i\gamma)$ into $Z_\diamondsuit(t) \in \mathcal{Y}(r-1, (i+1)\gamma)$. Unfortunately, we are not able to get rid of the extra term $\bar{Z} \in \mathcal{Y}(r, i\gamma)$, the time average of $Z(t)$. The anti-adiabatic transform can be schematically described as

$$K_\triangle = D + H + Y + Z(t) \to K_\diamondsuit = D + H + Y + \bar{Z} + Z_\diamondsuit(t).$$

To cope with the unwanted extra term we apply afterwards a diagonalization procedure which in fact means the transform

$$K_\diamondsuit = D + H + Y + \bar{Z} + Z_\diamondsuit(t) \to K_\heartsuit := D + H + A + B(t)$$

where A and $B(t)$ already have the desired properties, i.e., $B(t) \in \mathcal{Y}(r-1, (i+1)\gamma)$ is symmetric, T-periodic and strongly C^1, and $A \in \mathcal{Y}(\infty, \gamma)$ is Hermitian and commuting with H.

4 The anti-adiabatic transform

In this section we adapt the strategy of Howland [17] and make precise the mapping $K_\triangle \to K_\diamondsuit$, as announced in Subsection 3.3. Using the anti-adiabatic transform, i.e., roughly speaking, applying the commutator with H, one can improve the decay of matrix entries of the perturbation along the main diagonal at the expense of a slower decay in the direction perpendicular to the diagonal. Using the language of classes $\mathcal{Y}(p, \delta)$, the anti-adiabatic transform may be viewed as passing from a perturbation $Z(t) \in \mathcal{Y}(p, \delta)$ to a new perturbation $Z_1(t) \in \mathcal{Y}(p-1, \delta+\gamma)$ where γ comes from the gap condition (1) (see Lemma 2).

Let us introduce the transform in detail. Let $K_\triangle$ be a Floquet Hamiltonian of the form

$$K_\triangle = D + H + Y + Z(t),$$

with H satisfying the assumptions of Theorem 5, $Y \in \mathcal{Y}(\infty, \gamma)$ being Hermitian and commuting with H, and $Z(t) \in \mathcal{Y}(r, i\gamma)$ being Hermitian, T-periodic and continuous in the strong sense. By the uniform boundedness principle, $\|Z(t)\|$ is bounded uniformly in t. The parameters are supposed to satisfy $r \geq 2$, $i \geq 1$. Set

$$\bar{Z} := \frac{1}{T} \int_0^T Z(t)\, \mathrm{d}t, \quad \tilde{Z}(t) = Z(t) - \bar{Z}.$$

Define

$$F(t) := \int_0^t \tilde{Z}(s)\,\mathrm{d}s,$$

so that $F(t)$ is Hermitian, T-periodic, strongly C^1 and lying in $\mathcal{Y}(r, i\gamma)$. Let us define $K_\diamondsuit$ by the gauge-type transformation of $K_\triangle$,

$$K_\diamondsuit := e^{\mathrm{i}F(t)} K_\triangle e^{-\mathrm{i}F(t)} = D + H + Y + \bar{Z} + Z_\diamondsuit(t),$$

with

$$Z_\diamondsuit(t) = e^{\mathrm{i}F(t)} \left(D + H + Y + Z(t)\right) e^{-\mathrm{i}F(t)} - \left(D + H + Y + \bar{Z}\right). \tag{14}$$

The main result related to the anti-adiabatic transform is as follows.

Proposition 11. *Let $r \geq 2$, $i \geq 1$, $\gamma \in]0, \frac{1}{2}[$, and H be a self-adjoint operator with a pure point spectrum and the spectral decomposition $H = \sum_n E_n P_n$. Assume that the eigen-values $\{E_n\}_{n=1}^\infty$ are ordered increasingly and satisfy the inequality*

$$|E_m - E_n| \leq C_H \frac{|m-n|}{\max\{m,n\}^{2\gamma}}.$$

Furthermore, Y and $Z(t)$ obey the assumptions formulated above.

Then $Z_\diamondsuit(t)$ defined in (14) is T-periodic, continuous in the strong sense, Hermitian, and lies in $\mathcal{Y}(r-1, (i+1)\gamma)$. The norm of $Z_\diamondsuit$ obeys the bound

$$\|Z_\diamondsuit\|_{r-1,(i+1)\gamma} \leq \frac{\exp(4C_{r,\gamma}T\,\|Z\|_{r,i\gamma}) - 1}{2C_{r,\gamma}} \left(C_H + 4\|Y\|_{\infty,\gamma} + 2C_{r,\gamma}\|Z\|_{r,i\gamma}\right), \tag{15}$$

with the constant $C_{r,\gamma}$ defined in (7). The operator-valued function $e^{\mathrm{i}F(t)}$ is C^1 in the strong sense. Moreover, if $Z(t)$ is C^1 in the strong sense then the same is true for $Z_\diamondsuit(t)$.

Proof. The periodicity and the differentiability are clear from the above discussion. The RHS of (14) can be expanded according to the formula

$$e^A B e^{-A} = B + \sum_{j=1}^\infty \frac{1}{j!}\,\mathrm{ad}_A^j(B).$$

Here we use the notation $\mathrm{ad}_A(B) := [A,B] = AB - BA$. Since $\mathrm{ad}_{F(t)}\,D = \mathrm{i}\dot{F}(t) = \mathrm{i}\tilde{Z}(t)$ we get

$$\begin{aligned} Z_\diamondsuit(t) &= \sum_{j=1}^\infty \frac{\mathrm{i}^j}{j!}\;\mathrm{ad}_{F(t)}^{j-1}\left(\mathrm{i}\tilde{Z}(t) + [F(t), H + Y + Z(t)]\right) + \tilde{Z}(t) \\ &= \sum_{j=1}^\infty \frac{\mathrm{i}^j}{j!}\;\mathrm{ad}_{F(t)}^{j-1} X(t) \end{aligned} \tag{16}$$

where

$$X(t) := \mathrm{ad}_{F(t)}\left(H + Y + Z(t) - \frac{1}{j+1}\tilde{Z}(t)\right) = \mathrm{ad}_{F(t)}\left(H + Y + \frac{j}{j+1}Z(t) + \frac{1}{j+1}\bar{Z}\right).$$

By Lemma 2, $\operatorname{ad}_{F(t)} H \in \mathcal{Y}(r-1,(i+1)\gamma)$, and according to Corollary 4, the same holds true for $\operatorname{ad}_{F(t)} Z(t)$ and $\operatorname{ad}_{F(t)} \bar{Z}$. Notice also that $\|\bar{Z}\|_{p,\delta} \leq \|Z\|_{p,\delta}$. Furthermore, since $Y \in \mathcal{Y}(\infty,\gamma)$ is diagonal we have

$$\begin{aligned}&\langle m-n\rangle^{r-1}\max\{m,n\}^{2(i+1)\gamma}\|(F(t)Y)_{m,n}\|\\ &\leq \frac{1}{\langle m-n\rangle}\left(\frac{\max\{m,n\}}{n}\right)^{2\gamma} n^{2\gamma}\|F\|_{r,i\gamma}\|Y_{n,n}\| \leq 2^{2\gamma}\|F\|_{r,i\gamma}\|Y\|_{\infty,\gamma}.\end{aligned}$$

Hence $\|F(t)Y\|_{r-1,(i+1)\gamma} \leq 2\|F\|_{r,i\gamma}\|Y\|_{\infty,\gamma}$. The same estimate is true for $\|YF(t)\|_{r-1,(i+1)\gamma}$ and therefore $\|\operatorname{ad}_F Y\|_{r-1,(i+1)\gamma} \leq 4\|F\|_{r,i\gamma}\|Y\|_{\infty,\gamma}$. We conclude that $X(t)$ belongs to $\mathcal{Y}(r-1,(i+1)\gamma)$ and

$$\|X\|_{r-1,(i+1)\gamma} \leq \|F\|_{r,i\gamma}\left(C_H + 4\|Y\|_{\infty,\gamma} + 2C_{r,\gamma}\|Z\|_{r,i\gamma}\right). \tag{17}$$

Recalling Corollary 4 once more we have

$$\mathcal{Y}(r-1,(i+1)\gamma)\mathcal{Y}(r,i\gamma),\ \mathcal{Y}(r,i\gamma)\mathcal{Y}(r-1,(i+1)\gamma) \subset \mathcal{Y}(r-1,(i+1)\gamma)$$

and so $\operatorname{ad}_{F(t)}^{j-1} X(t)$ lies in $\mathcal{Y}(r-1,(i+1)\gamma)$ as well and

$$\|\operatorname{ad}_F^{j-1} X\|_{r-1,(i+1)\gamma} \leq \left(2C_{r,\gamma}\|F\|_{r,i\gamma}\right)^{j-1}\|X\|_{r-1,(i+1)\gamma}. \tag{18}$$

Consequently, the series (16) converges in the Banach space $\mathcal{Y}(r-1,(i+1)\gamma)$. To derive inequality (15) from (17) and (18) one applies the estimate $\|F\|_{r,i\gamma} \leq 2T\|Z\|_{r,i\gamma}$ which immediately follows from the definition of $F(t)$ and $\tilde{Z}(t)$. This completes the proof. ☐

Remark 12. *The proposition holds also true for* $i=0$ *provided* $[Z(t),Z(s)]=0$ *for every* t,s. *In this case* $F(t)$ *commutes with* $Z(t)$ *and* $\bar{Z}$, *and the formula (16) holds true with* $X(t)=\operatorname{ad}_{F(t)}(H+Y)$. *Repeating the steps from the proof of the proposition one arrives at the inequality*

$$\|Z_\diamondsuit\|_{r-1,(i+1)\gamma} \leq \frac{\exp(4C_{r,\gamma}T\,\|Z\|_{r,i\gamma})-1}{2C_{r,\gamma}}\left(C_H + 2\|Y\|_{\infty,\gamma}\right).$$

5 The diagonalization procedure

5.1 Formulation of the result

The main result of this section is formulated in the following proposition.

Proposition 13. *Let* $i \geq 1$ *be a natural number,* $\gamma \in]0,\frac{1}{2}[$, *and* H *be a self-adjoint operator with a pure point spectrum and the spectral decomposition* $H=\sum_n E_n P_n$. *Assume that the eigen-values* $\{E_n\}_{n=1}^\infty$ *are ordered increasingly and satisfy the inequality*

$$|E_m - E_n| \geq c_H \frac{|m-n|}{\max\{m,n\}^{2\gamma}}. \tag{19}$$

Let $Y \in \mathcal{Y}(\infty, \gamma)$ be Hermitian and commuting with H. Suppose that $\bar{Z}$ is Hermitian and belongs to the class $\mathcal{Y}(r, i\gamma)$ for some $r \geq 2$. Finally, assume that

$$\|Y\|_{\infty,\gamma} + \|\bar{Z}\|_{r,i\gamma} \leq \frac{c_H}{4\pi\, C_{r+1,\gamma}}, \tag{20}$$

with the constant $C_{r+1,\gamma}$ given by (7).

Then there exists U, a unitary operator on $\mathscr{H}$, such that it holds

$$U(H + Y + \bar{Z})U^* = H + A \tag{21}$$

where $A \in \mathcal{Y}(\infty, \gamma)$ commutes with H and obeys

$$\|A\|_{\infty,\gamma} \leq 2\left(\|Y\|_{\infty,\gamma} + \|\bar{Z}\|_{r,i\gamma}\right). \tag{22}$$

Moreover, for every operator $X \in \mathcal{Y}(r-1, (i+1)\gamma)$ we have the estimate

$$\|UXU^*\|_{r-1,(i+1)\gamma} \leq \exp\!\left(2\,\frac{C_{r,\gamma}}{C_{r+1,\gamma}}\right) \|X\|_{r-1,(i+1)\gamma}. \tag{23}$$

Since U does not depend on time this result can be interpreted in the following way.

Corollary 14. *Let us consider a Floquet Hamiltonian of the form*

$$K_\diamondsuit = D + H + Y + \bar{Z} + Z_\diamondsuit(t)$$

where H, Y and $\bar{Z}$ obey the same assumptions as in Proposition 13, with $r \geq 2$ and $i \geq 1$, and $Z_\diamondsuit(t) \in \mathcal{Y}(r-1, (i+1)\gamma)$ is T-periodic, continuously differentiable in the strong sense and Hermitian.

Then there exists a unitary operator U on $\mathscr{H}$ such that for the transformed Floquet Hamiltonian

$$K_\heartsuit := UK_\diamondsuit U^* = D + H + A + B(t)$$

it holds: $A \in \mathcal{Y}(\infty, \gamma)$ commutes with H and fulfills (22),

$$B(t) := UZ_\diamondsuit(t)U^* \in \mathcal{Y}(r-1, (i+1)\gamma)$$

is T-periodic, continuously differentiable in the strong sense, Hermitian and satisfies

$$\|B\|_{r-1,(i+1)\gamma} \leq \exp\!\left(2\,\frac{C_{r,\gamma}}{C_{r+1,\gamma}}\right) \|Z_\diamondsuit\|_{r-1,(i+1)\gamma}.$$

The proof of Proposition 13 is a modification (to the case of shrinking gaps) of a diagonalization procedure introduced in [10] and conventionally called the progressive diagonalization method.

5.2 The algorithm

The diagonalization procedure is constructed iteratively, let us first describe the algorithm. Starting from $H + Y + \bar{Z}$ we construct the first 4-tuple of operators

$$U_0 := 1,\ G_1 := Y + \operatorname{diag} \bar{Z},\ V_1 := \operatorname{offdiag} \bar{Z},\ H_1 := H + G_1 + V_1,$$

where

$$\operatorname{diag} X := \sum_{n\in\mathbb{N}} P_n X P_n,\ \ \operatorname{offdiag} X := \sum_{m\neq n} P_m X P_n$$

denote the diagonal and the off diagonal part of the matrix of an operator X with respect to the eigen-basis of H. We define recursively a sequence of operators H_s, G_s, V_s, W_s and U_s by the following rules: provided G_s and V_s have been already defined let W_s be the solution of

$$[H + G_s, W_s] = V_s \ \text{ and } \ \operatorname{diag} W_s = 0. \tag{24}$$

We define

$$H_{s+1} := e^{W_s} H_s e^{-W_s}. \tag{25}$$

Finally, we set

$$U_s := e^{W_s} U_{s-1},\ G_{s+1} := \operatorname{diag} H_{s+1} - H,\ V_{s+1} := \operatorname{offdiag} H_{s+1}. \tag{26}$$

Since $H_s = H + G_s + V_s$ for all s and with the aid of (24) one derives from (25) that

$$\begin{aligned} H_{s+1} &= H_s + \sum_{k=1}^{\infty} \frac{1}{k!}\ \operatorname{ad}_{W_s}^{k-1}[W_s, H_s] \ = \ H + G_s + V_s + \sum_{k=1}^{\infty} \frac{1}{k!}\ \operatorname{ad}_{W_s}^{k-1}(-V_s + [W_s, V_s]) \\ &= H + G_s + \Phi(\operatorname{ad}_{W_s})V_s \end{aligned} \tag{27}$$

where

$$\Phi(x) := \sum_{k=1}^{\infty} \frac{k}{(k+1)!} x^k = e^x - \frac{1}{x}(e^x - 1) \tag{28}$$

Observe also that in the course of the algorithm, G_s is always diagonal (commuting with H) and symmetric, V_s is symmetric and off diagonal, W_s is antisymmetric and off diagonal. Therefore e^{W_s} and U_s are unitary. It is straightforward to prove by induction that for every $s = 1, 2, \ldots$,

$$H + G_{s+1} + V_{s+1} = U_s(H + Y + \bar{Z})U_s^*. \tag{29}$$

5.3 Auxiliary facts

To solve the commutator equation (24) we need the following result taken from a paper by Bhatia and Rosenthal.

Lemma 15 ([4]). *Let E and F be two Hilbert spaces. Let A and B be Hermitian operators (i.e., bounded and self-adjoint) on E and F, respectively, such that* $\operatorname{dist}(\sigma(A), \sigma(B)) > 0$.

Then for every bounded operator $Y : F \to E$ there exists a unique bounded operator $X : F \to E$ such that

$$AX - XB = Y.$$

Moreover, the inequality

$$\|X\| \leq \frac{\pi}{2\,\mathrm{dist}(\sigma(A), \sigma(B))}\,\|Y\|,$$

holds true.

Remark. The solution X is given by

$$X = \int_{\mathbb{R}} e^{-\mathrm{i}tA} Y e^{\mathrm{i}tB} f(t)\,\mathrm{d}t$$

for any $f \in L^1(\mathbb{R})$ such that its Fourier image obeys $\hat{f}(s) = 1/\sqrt{2\pi}s$ on the set $\sigma(A) - \sigma(B)$. This implies $\|X\| \leq \|f\|_1 \|Y\|$, and optimizing over such f one gets the constant $\pi/2$.

In the algorithm plays a certain role the function $\Phi(x)$ introduced in (28). It is supposed to be defined on the interval $[0, \infty[$. Let us point out here some of its elementary properties. This is a strictly increasing function mapping the interval $[0, \infty[$ onto itself. It holds $\Phi(0) = 0$, $\Phi(1) = 1$, and so the function maps also the interval $]0, 1[$ onto itself. Moreover, $\Phi(x)$ is a convex function and so

$$\forall x \in]0, 1[,\ \Phi(x) < x. \tag{30}$$

Further, let us consider a sequence $\{x_s\}_{s=1}^{\infty}$ formed by nonnegative numbers obeying the inequalities

$$\forall s \in \mathbb{N},\ x_{s+1} \leq \Phi(x_s) x_s. \tag{31}$$

If $x_1 < 1$ then the sequence is non-increasing and (30), (31) imply that $x_{s+1} \leq x_s^2$. It follows that

$$\forall s \in \mathbb{N},\ x_s \leq x_1^{2^{s-1}},$$

and

$$\sum_{s=1}^{\infty} x_s \leq \frac{x_1}{1 - x_1} < \infty. \tag{32}$$

5.4 Convergence of the algorithm

Proof of Proposition 13. We have to prove that $V_s \to 0$, $G_s \to A$ and $U_s \to U$. The key ingredient of the algorithm is the control of the size of W_s given as the off diagonal solution to the commutator equation (24). For every $m \neq n$ we seek $W_s(m, n)$ such that

$$(E_m + (G_s)_{m,m})\,(W_s)_{m,n} - (W_s)_{m,n}\,(E_n + (G_s)_{n,n}) = (V_s)_{m,n}.$$

Suppose for the moment that G_s lies in $\mathcal{Y}(\infty, \gamma)$ for every $s \in \mathbb{N}$ with

$$\|G_s\|_{\infty,\gamma} \leq \frac{c_H}{6}. \tag{33}$$

The norm $\|\cdot\|_{\infty,\gamma}$ makes sense in this case since G_s is diagonal for every $s \in \mathbb{N}$. The spectrum of $E_n + (G_s)_{n,n}$ is a subset of the interval

$$\Big[\, E_n - \frac{\|G_s\|_{\infty,\gamma}}{n^{2\gamma}}, E_n + \frac{\|G_s\|_{\infty,\gamma}}{n^{2\gamma}} \,\Big].$$

Owing to (19) the distance between the spectrum of $E_m + (G_s)_{m,m}$ and $E_n + (G_s)_{n,n}$ can be estimated from below by

$$\begin{aligned} |E_m - E_n| - \|G_s\|_{\infty,\gamma}\left(m^{-2\gamma} + n^{-2\gamma}\right) &\geq c_H \frac{|m-n|}{\max\{m,n\}^{2\gamma}} - \frac{c_H}{6}(m^{-2\gamma} + n^{-2\gamma}) \\ &\geq \frac{c_H |m-n|}{2\max\{m,n\}^{2\gamma}} \,. \end{aligned} \tag{34}$$

The last inequality in (34) is a consequence of the following estimate where we assume for definiteness that $m > n$ (recall that $2\gamma < 1$):

$$\frac{3(m-n)}{m^{2\gamma}} \geq m^{-2\gamma} + \frac{m}{n}\, m^{-2\gamma} \geq m^{-2\gamma} + n^{-2\gamma}.$$

Applying Lemma 15 we conclude that

$$\|(W_s)_{m,n}\| \leq \frac{\pi \max\{m,n\}^{2\gamma}}{c_H |m-n|} \|(V_s)_{m,n}\|. \tag{35}$$

Set

$$M := \frac{c_H}{2\pi C_{r+1,\gamma}}, \;\; x_s := \frac{\|V_s\|_{r,i\gamma}}{M}, \tag{36}$$

If V_s lies in the class $\mathcal{Y}(r, i\gamma)$ then one derives from (35) that $W_s \in \mathcal{Y}(r+1, (i-1)\gamma)$ and

$$\|W_s\|_{r+1,(i-1)\gamma} \leq \frac{\pi}{c_H} \|V_s\|_{r,i\gamma} = \frac{x_s}{2C_{r+1,\gamma}}. \tag{37}$$

From Corollary 4 it follows that $\operatorname{ad}^k_{W_s} V_s \in \mathcal{Y}(r, i\gamma)$ and

$$\|\operatorname{ad}^k_{W_s} V_s\|_{r,i\gamma} \leq \left(2C_{r+1,\gamma}\|W_s\|_{r+1,(i-1)\gamma}\right)^k \|V_s\|_{r,i\gamma} \leq {x_s}^k \|V_s\|_{r,i\gamma}, \tag{38}$$

Since V_{s+1} is defined as the off diagonal part of H_{s+1} we get from (27) and (38) that

$$V_{s+1} = \operatorname{offdiag}(\Phi(\operatorname{ad}_{W_s})V_s)\,.$$

and so

$$\|V_{s+1}\|_{r,i\gamma} \leq \Phi(x_s)\|V_s\|_{r,i\gamma}.$$

Hence the sequence $\{x_s\}$ defined in (36) fulfills inequalities (31).

Since $\|V_1\|_{r,i\gamma} \leq \|\bar{Z}\|_{r,i\gamma}$ assumption (20) implies $x_1 \leq 1/2$. We know from the discussion at the end of Subsection 5.3 that in that case the series $\sum x_s$ is convergent. It follows that $\|V_s\|_{r,i\gamma} \to 0$ and, using the estimate

$$\|W_s\| \leq \|W_s\|_{SH} \leq (1 + 2\zeta(r+1))\, \|W_s\|_{r+1,(i-1)\gamma}$$

and (37), also that U_s converges to a unitary operator U in $\mathscr{B}(\mathscr{H})$. Furthermore, from (27) and (26) one deduces that

$$G_{s+1} - G_s = \mathrm{diag}(\Phi(\mathrm{ad}_{W_s})V_s)\,.$$

Since G_s is diagonal and $i \geq 1$ we have

$$\|G_{s+1} - G_s\|_{\infty,\gamma} = \|G_{s+1} - G_s\|_{r,\gamma} \leq \|G_{s+1} - G_s\|_{r,i\gamma} \leq \|\Phi(\mathrm{ad}_{W_s})V_s\|_{r,i\gamma}.$$

Using once more (37) and (38) one finds that

$$\|G_{s+1} - G_s\| = \|G_{s+1} - G_s\|_{\infty,0} \leq \|G_{s+1} - G_s\|_{\infty,\gamma} \leq M\Phi(x_s)x_s. \tag{39}$$

From here one concludes that $\{G_s\}$ is a Cauchy sequence both in $\mathcal{Y}(\infty,\gamma)$ and $\mathscr{B}(\mathscr{H})$. Hence G_s converges to a diagonal operator A which lies in $\mathcal{Y}(\infty,\gamma)$.

We must verify that condition (33) is actually fulfilled. Observe from (7) that $C_{r+1,\gamma} \geq 2^6 \cdot 3$ if $r \geq 2$. By the assumptions,

$$\|G_1\|_{\infty,\gamma} \leq \|Y\|_{\infty,\gamma} + \|\bar{Z}\|_{r,i\gamma} < \frac{c_H}{12}.$$

Furthermore, from (39) it follows that

$$\|G_{s+1}\|_{\infty,\gamma} \leq \|G_1\|_{\infty,\gamma} + \sum_{j=1}^{s} \|G_{s+1} - G_s\|_{\infty,\gamma} \leq \frac{c_H}{12} + M\sum_{j=1}^{\infty} x_j\Phi(x_j). \tag{40}$$

Recalling that $x_1 \leq 1/2$ one gets

$$M\sum_{j=1}^{\infty} x_j\Phi(x_j) \leq \frac{Mx_1^2}{1-x_1} \leq Mx_1 \leq \|\bar{Z}\|_{r,i\gamma} < \frac{c_H}{12}. \tag{41}$$

The last inequality is again a consequence of assumption (20). One concludes that condition (33) is fulfilled for all s.

Since all operators occurring in (29) except of H are bounded one deduces from this equality that U_s preserves the domain of H for all s. Since H is a closed operator the limit in equality (29), as $s \to \infty$, can be carried out and results in equality (23).

From the computations in (40), (41) it also follows that

$$\|G_{s+1}\|_{\infty,\gamma} \leq \|G_1\|_{\infty,\gamma} + Mx_1 = \|G_1\|_{\infty,\gamma} + \|V_1\|_{r,i\gamma} \leq \|Y\|_{\infty,\gamma} + 2\|\bar{Z}\|_{r,i\gamma}.$$

Sending s to infinity one verifies the estimate (22). Further, estimate (37) implies

$$\sum_{s=1}^{\infty} \|W_s\|_{r+1,(i-1)\gamma} \leq \frac{1}{2C_{r+1,\gamma}} \sum_{s=1}^{\infty} x_s \leq \frac{x_1}{2C_{r+1,\gamma}(1-x_1)} \leq \frac{1}{2C_{r+1,\gamma}}.$$

From Corollary 4 we deduce that the operator ad_{W_s} is well defined on the Banach space $\mathcal{Y}(r-1,(i+1)\gamma)$, with a norm bounded from above by $4C_{r,\gamma}\|W_s\|_{r+1,(i-1)\gamma}$. Thus for

$X \in \mathcal{Y}(r-1,(i+1)\gamma)$ one can estimate

$$\begin{aligned}\|UXU^*\|_{r-1,(i+1)\gamma} &= \lim_{s\to\infty} \|e^{W_s}e^{W_{s-1}}\cdots e^{W_1}Xe^{-W_1}\cdots e^{-W_{s-1}}e^{-W_s}\|_{r-1,(i+1)\gamma}\\ &\leq \exp\left(4C_{r,\gamma}\sum_{s=1}^{\infty}\|W_s\|_{r+1,(i-1)\gamma}\right)\|X\|_{r-1,(i+1)\gamma}\\ &\leq \exp\left(2\,\frac{C_{r,\gamma}}{C_{r+1,\gamma}}\right)\|X\|_{r-1,(i+1)\gamma}.\end{aligned}$$

This shows (23). The proof is complete. □

6 Proof of Theorem 8

As already announced, the proof of Theorem 8 is based on a combination of the anti-adiabatic transform (Proposition 11) and the progressive diagonalization method (Corollary 14). Let us formulate it as a corollary.

Corollary 16. *Let $r \geq 2$, $i \geq 1$, $\gamma \in]0, \frac{1}{2}[$, and H be a self-adjoint operator with a pure point spectrum and the spectral decomposition $H = \sum_n E_n P_n$. Assume that the eigenvalues $\{E_n\}_{n=1}^{\infty}$ are ordered increasingly and satisfy (1). Further assume that $Y \in \mathcal{Y}(\infty,\gamma)$ is Hermitian and commutes with H, and $Z(t) \in \mathcal{Y}(r, i\gamma)$ is Hermitian, T-periodic and C^1 in the strong sense. If*

$$\|Y\|_{\infty,\gamma} + \|Z\|_{r,i\gamma} \leq \frac{c_H}{4\pi\, C_{r+1,\gamma}}$$

then there exists a family $\mathcal{U}(t)$ of unitary operators on $\mathscr{H}$ which is T-periodic and C^1 in the strong sense and such that

$$\mathcal{U}(t)\,(D + H + Y + Z(t))\,\mathcal{U}(t)^* = D + H + A + B(t)$$

where $A \in \mathcal{Y}(\infty,\gamma)$ is Hermitian, commutes with H and fulfills

$$\|A\|_{\infty,\gamma} \leq 2\left(\|Y\|_{\infty,\gamma} + \|Z\|_{r,i\gamma}\right),$$

and $B(t) \in \mathcal{Y}(r-1,(i+1)\gamma)$ is T-periodic, Hermitian, continuously differentiable in the strong sense and satisfies

$$\begin{aligned}\|B\|_{r-1,(i+1)\gamma} \leq\ & \frac{1}{2C_{r,\gamma}}\exp\left(2\,\frac{C_{r,\gamma}}{C_{r+1,\gamma}}\right)\\ & \times \big(\exp(4C_{r,\gamma}T\,\|Z\|_{r,i\gamma}) - 1\big)\left(C_H + 4\|Y\|_{\infty,\gamma} + 2C_{r,\gamma}\|Z\|_{r,i\gamma}\right).\end{aligned}$$

To prove Corollary 16 it suffices to set $\mathcal{U}(t) = U\exp(\mathrm{i}F(t))$ where $F(t)$ comes from Proposition 11 and U comes from Corollary 14. Apart of this one applies the following elementary estimate: if the norm $\|X\|_{p,\delta}$ of a T-periodic family $X(t)$ formed by bounded operators is finite for some $p > 1$ and $\delta \geq 0$ then the time average $\bar{X}$ of $X(t)$ over the period T fulfills $\|\bar{X}\|_{p,\delta} \leq \|X\|_{p,\delta}$.

Equipped with Corollary 16 we are ready to approach the proof of Theorem 8.

Proof of Theorem 8. One starts from the Floquet Hamiltonian $K = D + H + V(t)$ and applies to it q times Corollary 16, with each step being enumerated by $i = 1, 2, \ldots, q$. In the ith step one assumes that a strongly continuous function $J_{i-1}(t)$ with values in unitary operators on $\mathscr{H}$ has been already constructed so that

$$K = J_{i-1}(t)\,(D + H + A_{i-1} + B_{i-1}(t))\,J_{i-1}(t)^*,$$

with $A_{i-1} \in \mathcal{Y}(\infty, \gamma)$ being Hermitian and commuting with H, and $B_{i-1}(t) \in \mathcal{Y}(p - i + 1, i\gamma)$ being symmetric, T-periodic and C^1 in the strong sense. In the first step one sets $A_0 := 0$, $B_0(t) := V(t)$ and $J_0(t) := 1$.

Corollary 16 can be applied to the Floquet Hamiltonian $K_{i-1} := D + H + A_{i-1} + B_{i-1}(t)$, with $r = p - i + 1$, provided there is satisfied the assumption

$$\|A_{i-1}\|_{\infty,\gamma} + \|B_{i-1}\|_{p-i+1,i\gamma} \leq \frac{c_H}{4\pi C_{p-i+2,\gamma}}. \tag{42}$$

Recall that the constant C_p is given by (7). Under this assumption, there exists a strongly differentiable family of unitary operators $\mathcal{U}_i(t)$ such that

$$K_i := D + H + A_i + B_i(t) = \mathcal{U}_i(t) K_{i-1} \mathcal{U}_i(t)^*$$

where $A_i \in \mathcal{Y}(\infty, \gamma)$ is symmetric and diagonal, and $B_i(t) \in \mathcal{Y}(p - i, (i + 1)\gamma)$ is T-periodic, symmetric and strongly C^1. Moreover,

$$\|A_i\|_{\infty,\gamma} \leq 2\left(\|A_{i-1}\|_{\infty,\gamma} + \|B_{i-1}\|_{p-i+1,i\gamma}\right) \tag{43}$$

and

$$\begin{aligned}\|B_i\|_{p-i,(i+1)\gamma} \quad &\leq \quad \frac{1}{2C_{p-i+1,\gamma}} \exp\!\left(2\,\frac{C_{p-i+1,\gamma}}{C_{p-i+2,\gamma}}\right)\left(\exp(4C_{p-i+1,\gamma}T\,\|B_{i-1}\|_{p-i+1,i\gamma}) - 1\right) \\ &\qquad \times\ \left(C_H + 4\|A_{i-1}\|_{\infty,\gamma} + 2C_{p-i+1,\gamma}\|B_{i-1}\|_{p-i+1,i\gamma}\right). \end{aligned} \tag{44}$$

Finally, $J_i(t) := J_{i-1}(t)\mathcal{U}_i(t)^*$ is a family of unitary operators which is continuously differentiable in the strong sense and such that

$$K = J_i(t)\,(D + H + A_i + B_i(t))\,J_i(t)^*.$$

To finish the proof we have to choose $\varepsilon > 0$ sufficiently small so that if $\|V\|_{p,\gamma} < \varepsilon$ then condition (42) is satisfied in each step $i = 1, 2, \ldots, q$.

From (43) one derives by induction

$$\|A_i\|_{\infty,\gamma} \leq \sum_{j=0}^{i-1} 2^{i-j}\|B_j\|_{p-j,(j+1)\gamma}.$$

From here we deduce that inequalities (42) are satisfied for $i = 1, 2, \ldots, k$, provided the inequalities

$$\sum_{j=0}^{i-1} 2^{i-1-j}\|B_j\|_{p-j,(j+1)\gamma} \leq \frac{c_H}{4\pi C_{p-i+2,\gamma}} \tag{45}$$

are satisfied for the same range of indices. Furthermore, relations (42) and (44) imply that

$$\|B_i\|_{p-i,(i+1)\gamma} \leq \phi_i(\|B_{i-1}\|_{p-i+1,i\gamma}) \tag{46}$$

where

$$\phi_i(y) := \frac{\exp\left(2\,\frac{C_{p-i+1,\gamma}}{C_{p-i+2,\gamma}}\right)}{2C_{p-i+1,\gamma}}\big(\exp(4C_{p-i+1,\gamma}T\,y)-1\big)\left(C_H + \frac{c_H}{\pi C_{p-i+2,\gamma}} + (2C_{p-i+1,\gamma}-4)y\right).$$

Set

$$F_i(y) := 2^{i-1}y + \sum_{j=1}^{i-1} 2^{i-1-j}\phi_j \circ \phi_{j-1} \circ \cdots \circ \phi_1(y), \ \ i = 1, 2, \ldots, q.$$

It follows from (46) that inequalities (45) are satisfied for $i = 1, 2, \ldots, k$, if it holds

$$F_i(\|B_0\|_{p,\gamma}) \leq \frac{c_H}{4\pi C_{p-i+2,\gamma}}$$

for the same range of indices.

Recall that $B_0(t) = V(t)$. From this discussion it is clear that condition (42) is satisfied in all steps $i = 1, 2, \ldots, q$, provided $\|V\|_{p,\gamma} \leq \varepsilon$ and $\varepsilon > 0$ is chosen so that

$$\forall i \in \{1, 2, \ldots, q\}, \forall y \in \ [0, \varepsilon\,],\ F_i(y) \leq \frac{c_H}{4\pi C_{p-i+2,\gamma}}\,.$$

But all functions $\phi_i(y)$ are continuous, strictly increasing and satisfy $\phi_i(0) = 0$. Consequently, the same is true for all functions $F_i(y)$. Hence the following choice of ε will do:

$$\varepsilon = \min\left\{F_i^{-1}\left(\frac{c_H}{4\pi C_{p-i+2,\gamma}}\right);\ 1 \leq i \leq q\right\}.$$

This completes the proof of Theorem 8. □

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