České vysoké učení technické v Praze
Fakulta jaderná a fyzikálně inženýrská

Katedra matematiky
Obor: Matematické inženýrství
Zaměření: Matematická fyzika

Hamiltoniány s konstantními spektrálními mezerami a časově závislou poruchou
Hamiltonians with constant spectral intervals and time-dependent perturbation

DIPLOMOVÁ PRÁCE
DIPLOMA THESIS

Vypracoval: Bc. Václav Košař
Vedoucí práce: prof. Ing. Pavel Šťovíček, CSc.
Rok: 2012

Před svázáním místo téhle stránky vložíte zadání práce s podpisem děkana (bude to jediný oboustranný list ve Vaší práci) !!!!

Prohlášení

Prohlašuji, že jsem svou diplomovou práci vypracoval samostatně a použil jsem pouze podklady (literaturu, projekty, SW atd.) uvedené v přiloženém seznamu.

Nemám závažný důvod proti užití tohoto školního díla ve smyslu §60 Zákona č.121/2000 Sb. , o právu autorském, o právech souvisejících s právem autorským a o změně některých zákonů (autorský zákon).

I declare that I have written this diploma thesis independently using the listed references.

V Praze dne ....

Bc. Václav Košař

Acknowledgement

I would like to thank prof. Ing. Pavel Šťovíček, CSc. for his perfect guidance, infinite patience and many corrections of my work. I would also like to thank my constant for many usefull advices, tips and corrections.

Bc. Václav Košař

Název práce:
Hamiltoniány s konstantními spektrálními mezerami a časově závislou poruchou

Autor: Bc. Václav Košař

Obor: Matematické inženýrství
Druh práce: Diplomová práce
Vedoucí práce: prof. Ing. Pavel Šťovíček, CSc.
Katedra matematiky, FJFI, ČVUT
Konzultant: Ing. Tomáš Kalvoda
FIT ČVUT

Abstrakt: Předmětem studia této práce jsou kvantové systémy učené časově závislými Hamiltonovy operatory. Důraz je kladen na návrh nové metody pro studium dříve neřešených otázek stability kvantových systémů určených Hamiltonovými operátory tvaru $H(t) = H_0 + V(t)$, kde $V(t)$ je porucha a H_0 je samosdružený operator s čistě bodovým spektrem a konstantními mezerami mezi vlastními hodnotami ve spektru $\sigma(H_0)$. Existujicí teorie týkajíí se stability kvatových systémů s Hamiltonovými operatory uvedeného tvaru, kde H_0 je samosdružený operator s čistě bodovým spektrem a rostoucími nebo zmenšujícími se mezerami mezi vlastními hodnotami ve spektru $\sigma(H_0)$ je uvedena v příslušné kapitole. Kvůli neaplikovatelnosti předchozích výsledků se autor se pokouší nalézt nový přístup ke studiu výše zmíněného problému pomocí pojmu "střední hodnota Hamiltova operatoru přes nekonečný časový interval". V poslední kapitole autor studuje jednoduchý příklad a aplikuje navržené postupy.

Klíčová slova: časově závislé Hamiltoniány, stabilita kvatového systému

Title:
Hamiltonians with constant spectral intervals and time-dependent perturbation

Author: Bc. Václav Košař

Abstract: This work deals with quantum systems determined by time-dependent Hamilton operators. Family of quantum systems, whose Hamilton operators take form $H(t) = H_0 + V(t)$, where $V(t)$ is perturbation and H_0 is self-adjoint with pure-point spectrum and constant gaps between eigenvalues in spectrum $\sigma(H_0)$. Existing theory dealing with stability of quantum systems with Hamilton operators of above form, where H_0 is self-adjoint with pure-point spectrum and growing or shrinking gaps between eigenvalues in spectrum $\sigma(H_0)$ is given in corresponding chapter. Because of non-applicability of existing theory to the studied cases the author attempts to device a new approach based on term "mean of Hamilton operator over infinite time interval". In the last chapter is devoted to the study of simple example and to the application of devised theory.

Key words: time-dependent Hamiltonians, stability of a quantum system

Contents

Preface **7**

List of symbols **8**

1 Time-dependent Hamiltonians in classical mechanics **9**

2 Existence of evolution **10**

3 The Floquet operator **14**

4 Positive operator valued measure **17**

5 The RAGE theorem and the time-mean of Hamiltonian **22**

5.1 The RAGE theorem 22

5.2 The time-mean of Hamiltonian 23

6 Methods of studying stability **28**

7 Example **31**

7.1 Notation 31

7.2 The problem 31

7.3 Floquet Hamiltonian and its first order approximation 32

7.4 Discrete symmetry of the problem 33

7.5 Approximated propagator 36

7.6 The simplest case 37

7.7 Remarks and suggestions 37

7.8 Numerical analysis 39

Conclusion **46**

Bibliography **48**

Preface

The subject of this thesis is the study of time-dependent quantum systems, i.e. systems whose Hamiltonian $H(t)$ depends on time. The special interest is paid to the cases where Hamiltonian of the quantum system takes form $H(t) = H_0 + \varepsilon V(t)$, where H_0 is semi-bounded operator with pure point spectra with constant gaps between distinct eigenvalues and $\varepsilon V(t)$ is periodically time-dependent perturbation.

The subject is indeed very complicated and even the existence of time evolution is nontrivial as seen in the chapter 2. That is why the most attempts for analytical study of this subject address only the most elementary problems like stability of the evolution, i.e. boundedness of energy in time for some sets of initial states. To author's knowledge all existing results regarding stability problems are based on the assumptions on gaps between distinct eigenvalues in the pure point spectra. So far there are only results in cases where the gaps between distinct eigenvalues are either shrinking or growing fast enough. The main results of the previous research are included in this work in the chapter 6.

Since essence of the existing theory is assumption that the gaps between distinct eigenvalues are either shrinking or growing fast enough, the author did not expect to obtain any results for the case were the gaps between distinct eigenvalues are constant by any simple modification of the previous results and thus tried to take a new approach to the problem.

The chapter 7 is devoted to the study of the simple case of the studied problem, which is well defined for arbitrary dimension of the separable Hilbert space (even for the infinite one). There author presents his results: the first order perturbation of the time evolution for arbitrary dimension, applicable discrete symmetries and numerical analysis.

When studying the finite dimensional problem mentioned in the previous paragraph the operator "mean of Hamiltonian over infinite time interval" is naturally introduced. However introducing similar result for unbounded operator was complicated and nonintuitive. This led author to the definition of positive operator valued measure in chapter 4 and to the study of existing results in this area as 6. Attempt to correctly define all terms regarding integral with respect to positive operator valued measure took the author great effort since he was unaware of any complete and suitable source on positive operator measures and source on this subject, although it probably exists. Note that the paper [16] is unavailable to the author and the paper [17] covers only integration over bounded function in section 7.1. That is why some terms and relations regarding positive operator measures and integration with respect to positive operator measure may be largely reinvention of existing terms. The author's main theoretical results of this paper are in chapter 5. The theorem 14 has special importance.

However due to the lack of time, it is unknown whether the theorem 14 leads to some results regarding stability of some family of quantum systems. There are proposals for possible theorems regarding stability of the infinite dimensional case of the simple problem studied in the last section 7.7 of chapter 7.

List of symbols

$\mathbb{C}$	complex numbers.
$\mathbb{R}$	real numbers.
$\mathbb{R}^n := \mathbb{R} \times \mathbb{R} \cdots \times \mathbb{R}$	Cartesian product of real numbers.
$\mathbb{C}^n := \mathbb{C} \times \mathbb{C} \cdots \times \mathbb{C}$	Cartesian product of complex numbers.
$\mathbb{N}$	natural numbers.
$\mathbb{Z}$	integral numbers.
$\cup, \cap, \subset$	union, intersection, inclusion .
i	imaginary unit (should be evident from context).
1	identity operator (should be evident from context).
$\langle . \vert . \rangle$	inner product in corresponding $\mathcal{H}$.
$\vert\psi\rangle\langle\phi\vert$	operator in corresponding Hilbert space $\mathcal{H}$ acting as $\forall \eta \in \mathcal{H}, \vert\psi\rangle\langle\phi\vert\eta = \langle\phi\vert\eta\rangle\psi$.
$[a, b]$	closed interval.
(a, b)	open interval.
:	"such that" (e.g. $\forall \varepsilon > 0 : \varepsilon < 1$ is equivalent to $\forall \varepsilon \in (0, 1)$)
$\upharpoonright$	restriction.
$\overline{a}$	complex conjugation of a or closure of a set (should be evident from context).
∂_x	partial derivative with respect to x.
$L^2([a, b], \mathcal{X})$	set of X-valued functions so that $\int_a^b \Vert f \Vert_{\mathcal{X}}^2 \, \mathrm{d}t < \infty$, where X is Banach space.
$L^\infty([a, b], \mathcal{X})$	set of X-valued functions so that $\mathrm{supess}_{t \in [a,b]} \Vert f(t) \Vert_{\mathcal{X}} < \infty$, where X is Banach space.
μ-a.a.	should be read as "for μ almost all", where μ is measure.
$\mu_{leb.}$	Lebesgue measure on $\mathbb{R}$.
$\mathcal{H}$	complex separable Hilbert space.
$\mathfrak{B}(\mathcal{H})$	set of all bounded operators on the Hilbert space $\mathcal{H}$.
$\mathfrak{B}_s(\mathcal{H})$	strong-topology space of bounded operators on the Hilbert space $\mathcal{H}$.
$\mathcal{C}^n(\mathbb{R}^m)$	set of functions $\mathbb{R}^m \to \mathbb{C}$ with continuous n-th derivative.
$\mathcal{C}^n(\mathbb{R}^m, \mathfrak{B}_s(\mathcal{H}))$	set of functions $\mathbb{R}^m \to \mathfrak{B}(\mathcal{H})$ with strongly continuous n-th derivative.
a-periodical function	function with period $a > 0$.
$\mathcal{C}^\infty(\mathbb{R})$	set of infinitely continuously differentiable functions $\mathbb{R}^m \to \mathbb{C}$.
$\mathrm{st-}\lim$	limit in strong resolvent sense.
$\mathrm{u-}\lim$	limit in norm resolvent sense.
$\sigma(A)$	see the chapter 4.
$\otimes$	tensor product of Hilbert spaces resp. vectors resp. operators.
$\oplus$ resp. $\oplus_\perp$	direct sum of Hilbert spaces resp mutually orthogonal Hilbert spaces.
Dom	domain of mapping.
Ran	range of mapping.
χ_M	characteristic function of the set M.
A^*	adjoint operator of linear operator A on some $\mathcal{H}$.
Tr	trace of of linear operator A on some $\mathcal{H}$.

Chapter 1

Time-dependent Hamiltonians in classical mechanics

Theorem 1 (Hamilton's equations for flat phase-space)**.** *Let*

$$(1) H \in \mathcal{C}^1(\mathbb{R}^{2n+1})$$
$$(2) \forall q_0, p_0 \in \mathbb{R}^n, \forall i \in \{1, 2, \dots n\},\ \partial_{q_i} H(q_0, p_0, t), \partial_{p_i} H(q_0, p_0, t) \in \mathcal{C}^1(\mathbb{R})$$

Then $\forall i \in \{1, \dots, n\}, \forall q_0, p_0 \in \mathbb{R}^n, \exists_1 q_i, p_i \in \mathcal{C}^1(\mathbb{R}) :$

$$\bullet q_i(0) = q_0 \text{ and } p_i(0) = p_0,$$
$$\bullet \forall t \in \mathbb{R}, \partial_t q_i = \partial_{p_i} H(q, p, t),$$
$$\partial_t p_i = -\partial_{q_i} H(q, p, t).$$

Proof. Basic result of theory of ordinary differential equations. (cf. [3]). □

Some theorems for Hamiltonians, which do not dependent on time, can be generalized to the time dependent case. This is done by defining special function called Floquet Hamiltonian and interpreting it as Hamiltonian on extended phase-space, where two new coordinates t, E are introduced.

Definition 1 (Floquet Hamiltonian for flat phase-space)**.**

Let $H(p, q, t) \in \mathcal{C}^2(\mathbb{R}^{2n+1})$ *be Hamiltonian on phase-space* $\mathbb{R}^{2n}$.
Then the function $K(t, q, E, p) := E + H(q, p, t)$ *is called Floquet Hamiltonian.*

Hamilton's equations for Floquet Hamiltonian with new time variable σ taking t resp. E as spacial resp. momentum coordinate are equivalent to the former equations. Nevertheless K is time-independent Hamiltonian and methods developed for time-independent Hamiltonians can be now applied to study properties of the former problem.

Chapter 2

Existence of evolution

In the following text $\hbar = 1$ and $H(t)$ is mapping from $\mathbb{R}$ to self-adjoint operators on $\mathcal{H}$.

Question of existence of evolution operator as a solution of Schrödinger equation is treated first.

Definition 2 (strongly continuous unitary propagator)**.** *Two parametric jointly strongly continuous unitary operator valued function $U(t,s)$ which fulfills:*

$$\forall t, r, s \in \mathbb{R} \text{ (1) } U(t,r)U(r,s) = U(t,s)$$
$$\text{(2) } U(t,t) = 1$$

is called **strongly continuous unitary propagator.**

Following theorem is mainly due to Krein [20].

Theorem 2 (Existence of evolution operator)**.** *Assume that*

$$\text{(1) } \forall t \in \mathbb{R}, \mathrm{Dom} H(t) = D \subset \mathcal{H}$$
$$\text{(2) } B(t,s) := \frac{H(t) - H(s)}{t - s}(H(s) + i)^{-1} \in C(\mathbb{R}^2, \mathfrak{B}_s(\mathcal{H}))$$

i.e. can be extended by limit as a strongly continuous function for $t = s$.

Then

$$\bullet \forall t, s \in \mathbb{R}, \exists K > 0, \quad \sup_{r \in [t,s]} \left\| (H(r) + i)U(r,s)(H(s) + i)^{-1} \right\| \leq K$$

$\bullet \exists_1$ *strongly continuous unitary propagator $U(t,s)$ so that*

$$\forall t, s \in \mathbb{R} \quad U(t,s)D = D \text{ and } \forall \psi \in D, i\frac{d}{dt}U(t,s)\psi = H(t)U(t,s)\psi.$$

Proof. Proof of more general theorem regarding one-parametric semi-groups can be found in [2] (theorem X.70) and is also based on solving easier problem with $H(s)$ by function constant on small intervals. For $H_n(t) := H(\frac{[nt]}{n})$ above Schrödinger equation on D is due to the Stone's theorem uniquely solved by strongly continuous unitary propagator U_n:

$$U_n(t,s) = \mathrm{e}^{-i(t - \frac{[nt]}{n})H(\frac{[nt]}{n})} \left(\prod_{j=[ns]+1}^{[nt]-1} \mathrm{e}^{\frac{-i}{n}H(\frac{j}{n})} \right) \mathrm{e}^{-i(\frac{[ns]}{n} - s)H(\frac{[ns]}{n})}.$$

Stone's theorem also implies that $U_n(t,s)D = D$.

It follows that $\forall j,k \in \mathbb{Z}, U_n(t,s) \in \mathcal{C}^1((\frac{j}{n},\frac{j+1}{n}) \times (\frac{k}{n},\frac{k+1}{n}), \mathfrak{B}(\mathcal{H}))$ as $H_n(s)$ is constant on $(\frac{j}{n},\frac{j+1}{n})$ for all $j \in \mathbb{Z}$.

Thus to prove convergence of U_n as $n \to \infty$ Bochner integral can be used in following way.

$$\|(U_{n+p}(t,s) - U_n(t,s))\psi\| = \left\|[U_{n+p}(t,r)U_n(r,s)]_{r=s}^{r=t}\psi\right\| \leq \tag{2.1}$$

$$\leq \int_s^t \left\|\frac{d}{dr}U_{n+p}(t,r)U_n(r,s)\psi\right\| dr = \int_s^t \|U_{n+p}(t,r)(H_{n+p}(r) - H_n(r))U_n(r,s)\psi dr\| \tag{2.2}$$

$$\leq \frac{a|t-s|}{n} \sup_{r\in[t,s]} \|(H_n(r)+i)U_n(r,s)\psi\|\,, \tag{2.3}$$

where $\sup_{t',s'\in[t,s]} \|B(t',s')\| \leq a$, which holds true due to uniform continuity of $B(t,s)$ on every compact subinterval of $\mathbb{R}^2$. It is now enough to estimate supremum on RHS of (2.3) independently of n.

One can reduce previous problem using following

$$\|(H_n(r)+i)U_n(r,s)\psi\| \leq \left\|(H_n(r)+i)U_n(r,s)(H_n(s)+i)^{-1}\right\| \|H_n(s)+i)\psi\|\,,$$

since $\psi \in \mathrm{Dom}H_n(s)$ and by the closed-graph theorem operator $(H_n(r)+i)U_n(r,s)(H_n(s)+i)^{-1}$ is bounded.

$$\begin{aligned}&\left(H(\frac{[nr]-1}{n})+i\right)U_n(\frac{[nr]-1}{n},\frac{[ns]+1}{n})\left(H(\frac{[ns]+1}{n})+i\right)^{-1} = \\ &= \prod_{j=[ns]+1}^{[nr]-1}\left(H(\frac{j}{n})+i\right)\mathrm{e}^{\frac{-i}{n}H(\frac{j}{n})}\left(H(\frac{j-1}{n})+i\right)^{-1}\end{aligned}$$

Now RHS of the above equation can be easily iteratively estimated.

$$\begin{aligned}&\left\|\left(H(\frac{[nr]-1}{n})+i\right)U_n(\frac{[nr]-1}{n},\frac{[ns]+1}{n})\left(H(\frac{[ns]+1}{n})+i\right)^{-1}\right\| \leq \\ &\leq (\frac{a}{n}+1)\left\|\left(H(\frac{[nr]-2}{n})+i\right)U_n(\frac{[nr]-2}{n},\frac{[ns]+1}{n})\left(H(\frac{[ns]+1}{n})+i\right)^{-1}\right\| \\ &\leq (1+\frac{a}{n})^{[nr]-[ns]-2} \leq \mathrm{e}^{a|r-s|}\,.\end{aligned}$$

Where commutation properties of propagators and their generators on D from Stone theorem were used. This can be used to estimate above supremum in following way. Let $\phi := (H_n(s)+i)\psi$.

$$\sup_{r\in[t,s]}\left\|(H_n(r)+i)U_n(r,s)(H_n(s)+i)^{-1}\right\| \leq (1+a)^2\,\mathrm{e}^{a|t-s|}$$

Thus $U_n(t,s)$ strongly converges as $n \to \infty$ to a unitary operator $U(t,s)$ as $\|U_n(t,s)\| = 1$, $\mathfrak{B}_s(\mathcal{H})$ is complete, scalar product is continuous and all $U_n(t,s)$ are unitary.

In an analogous way it can proven, considering uniform boundedness of operator $(H_n(r)+i)U_n(r,s)(H_n(s)+i)^{-1}$ and closedness of space $\mathfrak{B}_s(\mathcal{H})$, that

$$\mathrm{st-}\lim_{n\to\infty}(H_n(r)+i)U_n(r,s)(H_n(s)+i)^{-1} = (H(r)+i)U(r,s)(H(s)+i)^{-1}.$$

Now using fact that both $U(t,s)$ and $(H(r)+i)U(r,s)(H(s)+i)^{-1}$ are on every compact subset of $\mathbb{R}^2$ uniform limits of jointly continuous functions one can also get joint continuity of $U(t,s)$ and $(H(r)+i)U(r,s)(H(s)+i)^{-1}$.

To prove that $U(t,s)$ is differentiable in both arguments one can consider:

$$\forall\psi\in D, U(t,s)\psi=\int_t^s H(r)U(r,s)\psi\mathrm{d}r=\int_t^s H(r)U(r,s)(H(s)+i)^{-1}(H(s)+i)\psi\mathrm{d}r$$

Thus using continuity of bounded operator function $r\to H(r)U(r,s)(H(s)+i)^{-1}$ one obtains differentiability of $U(t,s)$.

To prove uniqueness of solution of Schrödinger equations one has to consider following. Let there be vector-valued function $\psi(t)$ such that

$$i\frac{\mathrm{d}}{\mathrm{d}t}\psi(t)=H(t)\psi(t)\text{ and }\psi(s)=\psi_0\in\mathcal{H}$$

Then one easily see that

$$i\frac{\mathrm{d}}{\mathrm{d}t}U(t,s)\psi(t)=U(t,s)(H(t)-H(t))\psi(t)=0.$$

Thus using initial conditions $U(s,s)\psi(s)=\psi_0$ one has

$$\psi(t)=U(t,s)\psi_0.$$

□

Remark 1. *It can be seen from 14 that even if $\forall t\in\mathbb{R}, H(t+T)=H(t)$ there still may be states so that $\lim_{t\to\infty}\left\|(H_n(t)+i)U_n(t,0)(H_n(0)+i)^{-1}\psi\right\|=+\infty$. It is also true that the limit above is in quite general case infinite if vector ψ has nonzero orthogonal projection to the complement of all eigensubspaces of so called Floquet operator $U(T,0)$.*

From proof of theorem 2 one can easily get following corollary.

Corollary 1. *Let*

(1) $\forall t\in\mathbb{R}, \mathrm{Dom}H(t)=D\subset\mathcal{H}$

(2) $B(t,s):=\dfrac{H(t)-H(s)}{t-s}(H(s)+i)^{-1}\in C(\mathbb{R}^2,\mathfrak{B}_s(\mathcal{H})))$

(3) $\forall t\in\mathbb{R}, H(t+T)=H(t).$

Then $\exists K>0, \sup_{t,s\in\mathbb{R}}\|B(t,s)\|<K$ and of course the main implication in the previous theorem holds true.

Lemma 1 (Strong-continuity of evolution operator with respect to some parameter)**.** *Let $\forall\varepsilon\in[0,1], H^{(\varepsilon)}(t)$ is self-adjoint-operator-valued function.*

Let $\forall\varepsilon\in[0,1], \mathrm{Dom}H^{\varepsilon}(t)=D\subset\mathcal{H}$, and that there exist corresponding unique solutions of Schrödinger equation i.e

$\forall\varepsilon\in[0,1], \exists_1$ *strongly continuous unitary propagator* $U^{(\varepsilon)}(t,s)$ *so that*

$\forall t,s\in\mathbb{R}$ • $U(t,s)^{(\varepsilon)}D=D$

• $\forall\psi\in D, i\dfrac{d}{dt}U^{(\varepsilon)}(t,s)\psi=H^{(\varepsilon)}(t)U^{(\varepsilon)}(t,s)\psi.$

Let following additional assumptions holds true

$(1)\left\|(H^{(\varepsilon)}(t)-H^{(0)}(t))(H^{(0)}(0)+i)^{-1}\right\|$ *is locally bounded function*,

$(2)\forall t\in\mathbb{R}, \lim\limits_{\varepsilon\to0}\left\|(H^{(\varepsilon)}(t)-H^{(0)}(t))(H^{(0)}(0)+i)^{-1}\right\|=0,$

$(3)\forall\psi\in D, \left\|H^{(0)}(0)U^{(0)}(t,s)\psi\right\|$ *is locally bounded.*

Then

$$\forall t,s\in\mathbb{R}, \mathrm{st}-\lim_{\varepsilon\to0}U^{(\varepsilon)}(t,s)=U^{(0)}(t,s).$$

Proof. The proof is based on one the ideas of the previous proof. Let $\psi \in D$ and $t, s \in \mathbb{R}$: $t - s > 0$ then one has following estimate.

$$\left\| U^{(0)}(t,s)\psi - U^{(\varepsilon)}(t,s)\psi \right\| = \left\| [U^{(\varepsilon)}(t,r)U^{(0)}(r,s)\psi]_s^t \right\| \leq \int_s^t \left\| \frac{\mathrm{d}}{\mathrm{d}r} U^{(\varepsilon)}(t,r)U^{(0)}(r,s)\psi]_s^t \right\| \mathrm{d}r$$
$$\leq \sup_{u\in[t,s]} \left\| (H^{(0)}(0)+i)U^{(0)}(t,s)\psi \right\| \int_s^t \left\| (H^{(\varepsilon)}(r) - H^{(0)}(r))(H^{(0)}(0)+i)^{-1} \right\| \mathrm{d}r.$$

Thus using Lebesgue theorem one has

$$\forall \psi \in D, \lim_{\varepsilon\to 0} \left\| U^{(0)}(t,s)\psi - U^{(\varepsilon)}(t,s)\psi \right\| = 0.$$

Since $\left\| U^{(\varepsilon)}(t,s) \right\| = 1$ and D is dense this completes the proof of the lemma. □

Chapter 3

The Floquet operator

In quantum mechanics there is, in analogy to classical mechanics 1 , also a way to generalize theorems for time-independent Hamilton operator to time-dependent case. It is done by considering larger Hilbert space, defining on it special operator and solving generalized Schrödinger equation. Exact definition however depends on application. This paper focuses on Hamilton operators that are periodic in time. **Note that for simplicity period will be considered to be 1 and functions from $\mathcal{K} := L^2([0,1],\mathcal{H})$ will be sometimes treated as 1-periodic functions on $\mathbb{R}$.** Last note should become clear after reading the following definition.

Definition 3.

$$\forall f \in L^2([0,1],\mathcal{H}), \forall t \in [0,1], \quad (T_\sigma f)(t) := f(t-\sigma)).$$
$$\forall f \in L^2(\mathbb{R},\mathcal{H}), \forall t \in \mathbb{R} \quad (\widetilde{T_\sigma} f)(t) := f(t-\sigma)).$$

Note that $T_\sigma \in \mathfrak{B}(L^2([0,1],\mathcal{H}))$, $\widetilde{T_\sigma} \in \mathfrak{B}(L^2(\mathbb{R},\mathcal{H}))$.

Definition 4 (Bounded multiplication operator)**.** *Let*

(1) $a, b \in [-\infty, +\infty], \quad a < b.$
(2) $\Phi \in L^\infty([a,b],\mathfrak{B}(\mathcal{H})).$
(3) $\forall f \in L^2([a,b],\mathcal{H}), \quad (Af)(t) = \Phi(t)f(t).$

*Then the operator $A \in \mathfrak{B}(L^2([a,b],\mathcal{H}))$ will be called **bounded multiplication operator in** $L^2([a,b],\mathcal{H})$ **generated by** Φ.*

*Bounded multiplication operator generated by scalar function Φ will be called **bounded scalar multiplication operator in** $L^2([a,b],\mathcal{H})$ **generated by** Φ.*

Following theorem is proposition 1 in [9].

Theorem 3 (Bounded multiplication operator)**.** *Let $A \in \mathfrak{B}(L^2([a,b],\mathcal{H})$. Then*

A is bounded multiplication operator $\Leftrightarrow$ A commutes with all scalar multiplication operators $\Leftrightarrow$ A commutes with all scaler multiplication operators generated by characteristic functions of finite open subintervals of $[a,b]$.

Definition 5 (Unitary evolution group)**.** *Let*

(1.) $A(\sigma) \in \mathcal{C}(\mathbb{R},\mathfrak{B}_s(\mathcal{K}))$ be strongly continuous unitary group in $\mathcal{K}$
(2.) $A(\sigma)T^*_\sigma$ be bounded multiplication operator in $\mathcal{K}$.

*Then $A(\sigma)$ will be called **unitary evolution group** and its self-adjoint generator K (Stone's theorem) will be called **Floquet Hamiltonian.***

Definition 6. *Let*

(1) $U(t,s)$ *be strongly continuous unitary propagator*
(2) is 1-periodic propagator i.e.$\forall t,s \in \mathbb{R} \quad U(t+1,s+1) = U(t,s)$

Then following notation will be used:

$V(\sigma) \in \mathcal{C}(\mathbb{R}, \mathfrak{B}_s(\mathcal{K}))$ *is called* $U(t,s)$***-associated evolution group*** *iff*
$\forall f \in \mathcal{K}, \mu_{leb.} - a.a.t \in [0,1], \quad (V(\sigma)f)(t) = U(t,t-\sigma)f(t-\sigma),$

Note that $L^2([0,1],\mathcal{H}) \cong L^2([0,1],dt) \otimes \mathcal{H}$.

Lemma 2 (Correctness of the previous definition)**.** *Let* $V(\sigma)$ *be* $U(t,s)$*-associated evolution group. Then* $V(\sigma)$ *is strongly continuous unitary group on* $\mathcal{K}$ *and thus corresponding Floquet Hamiltonian exists and is self-adjoint operator.*

Proof. One needs only to take into consideration that continuous function on compact interval is uniformly continuous and that span of the set $\{\eta \otimes \psi \in \mathcal{K} : \eta \in L^2([0,1],dt), \psi \in \mathcal{H}\}$ is dense in $L^2([0,1],\mathcal{H})$ i.e. $L^2([0,1],\mathcal{H}) \cong L^2([0,1],dt) \otimes \mathcal{H}$. □

Theorem 4 (Propagator and evolution group)**.** *Let* $A(\sigma)$ *be strongly continuous unitary group in* $\mathcal{K}$*. Then* $A(\sigma)$ *is unitary evolution group if and only if* $\exists U \in L^\infty(\mathbb{R},\mathcal{H})$ *so that*

- $U(0) = 1,$
- $\forall t \in \mathbb{R}, \quad U(t) = U(t+1),$
- $A(\sigma) = \mathfrak{U} T_\sigma \mathfrak{U}^{-1},$

where $\forall f \in \mathcal{K}, a.a.t \in [0,T], \quad (\mathfrak{U}f)(t) := U(t)f(t)$*. Further more* $U(t)$ *is uniquely determined by stated properties and A(1)=* $\mathfrak{U}(1 \otimes U(1))\mathfrak{U}^{-1}$.

Proof. Original proof is due to Howland [7]. Only sketch of the proof will be given here. (Some technical details will be omitted). Proof of "⇒" is based on Stone-von Neumann theorem e.g. theorem VIII.14 in [1]. One can define mappings

$$\forall k \in \mathbb{Z}, g_k \in \Big(L^2(\mathbb{R},\mathcal{H}) \to L^2([0,1],\mathcal{H})\Big) : (\forall t \in [0,1], \quad g_k(f)(t) := f(t+k)),$$

which can be used to define following embedding

$$h \in \Big(\mathfrak{B}(L^2([0,1],\mathcal{H})) \hookrightarrow \mathfrak{B}(L^2(\mathbb{R},\mathcal{H}))\Big),$$

such that

$$\forall B \in \mathfrak{B}(L^2([0,1],\mathcal{H})), \forall f \in L^2(\mathbb{R},\mathcal{H}), \forall s \in \mathbb{R}, (h[B]f)(s) := \sum_{k \in \mathbb{Z}} \chi_{[k,k+1]}(s)(Bg_k(f))(s-k).$$

One can now see that $h(A(\sigma)T^*_\sigma)$ is a multiplication by periodic unitary-operator valued function on $L^2(\mathbb{R},\mathcal{H})$. Let us now define

$$\widetilde{A}(\sigma) := h(A(\sigma)T^*_\sigma)\widetilde{T_\sigma}.$$

From $h(A(\sigma_1)T^*_{\sigma_1})h(A(\sigma_2)T^*_{\sigma_2}) = h(A(\sigma_1{+}\sigma_2)T^*_{\sigma_1+\sigma_2})$ and $h(A(\sigma_1)T^*_{\sigma_1})\widetilde{T_{\sigma_2}} = \widetilde{T_{\sigma_2}}h(A(\sigma_1{+}\sigma_2)T^*_{\sigma_1+\sigma_2})$ and strong continuity of $A(\sigma)$ one gets that $\widetilde{A}(\sigma)$ is strongly continuous unitary group.

One can see that spectral form of Weyl's form of commutation relations holds true:

$$\widetilde{A}(\sigma)E(S)\widetilde{A}^{-1} = E(S+\sigma),$$

where $E(S)$ is bounded multiplication operator generated by χ_S, S is finite open interval in $\mathbb{R}$.

Thus one can use Stone-von Neumann theorem to prove that there is unitary operator $\mathfrak{U}$ on $L^2(\mathbb{R}, \mathcal{H})$ such that

$$\mathfrak{U}^* \widetilde{A}(\sigma) \mathfrak{U} = \widetilde{T_\sigma}, \tag{3.1}$$

$$\mathfrak{U}^* E(S) \mathfrak{U} = E(S). \tag{3.2}$$

Due to (3.2) and 3 one can see that $\mathfrak{U}$ is bounded multiplication operator on $L^2(\mathbb{R}, \mathcal{H})$ generated by unitary operator valued function. Equation (3.1) can be rewritten as:

$$\widetilde{A(\sigma)} \widetilde{T^*_\sigma} = \mathfrak{U} \widetilde{T_\sigma} \mathfrak{U}^* \widetilde{T^*_\sigma}.$$

This implies that operator on the RHS is bounded multiplication generated by unitary-operator valued function with period 1 on $L^2(\mathbb{R}, \mathcal{H})$. Due to $\mathfrak{U}$ is bounded multiplication operator on $L^2(\mathbb{R}, \mathcal{H})$ generated by unitary operator valued function one has

$$\forall f \in \mathcal{K}, \forall \sigma \in \mathbb{R}, a.a.t \in \mathbb{R} \quad (A(\sigma)f)(t) = U(t)U^{-1}(t-\sigma)f(t-\sigma),$$

$$a.a.t, s \in \mathbb{R}, \quad U(t)U^{-1}(s) = U(t+1)U^{-1}(s+1).$$

In addition one have uniqueness of $U(t)$ up to multiplication by constant unitary operator, thus one can get uniqueness requiring $U(0) = 1$. Due to $T_1 = 1$ and $U(t)U^{-1}(s) = U(t+1)U^{-1}(s+1)$ one gets

$$A(1) = \mathfrak{U} U(1) \mathfrak{U}^{-1}.$$

Proof of "$\Leftarrow$" is easy. □

Note that $U(t)$ is not necessarily a group thus not all unitary evolution groups are $U(t,s)$-associated evolution groups. However we may now define following term.

Definition 7. *Let $A(\sigma)$ be unitary evolution group and $U(t)$ be unitary operator valued function. Then unitary operator $U(1)$ will be called **Floquet operator**.*

Following theorem is due to P. Duclos E. Soccorsi P. Šťovicek [4] and it specifies the form of Floquet Hamiltonian for physically-common cases.

Theorem 5 (Form of Floquet Hamiltonian)**.** *Let*

(1) $\forall t \in \mathbb{R}, \mathrm{Dom} H(t) = D \subset \mathcal{H}$.
(2) $\forall t \in \mathbb{R} \quad H(t+1) = H(t)$.
(3) $\exists$ *strongly continuous unitary propagator* $U(t,s)$ *so that*
$\forall \psi \in D, \quad i\partial_t U(t,s)\psi = H(t)U(t,s)\psi$.
(4) $\mathbb{R} \ni t \to \left\| H(t)(H(0)+i)^{-1} \right\|$ *is locally bounded.*
(5) $\forall \psi \in D, \quad \mathbb{R} \ni t \to \|H(t)U(t,0)\psi\|$ *is locally square integrable.*

Then

$\bullet \mathrm{Dom} K = \{f \in \mathcal{K} : \quad \forall \psi \in D, \quad \langle \psi | f(t) \rangle_{\mathcal{H}}$ *is absolutely continuous and*
$\exists g_f \in \mathcal{K}, \forall \psi \in D, \quad -i\partial_t \langle \psi | f(t) \rangle_{\mathcal{H}} + \langle H(t)\psi | f(t) \rangle_{\mathcal{H}} = \langle \psi | g_f(t) \rangle_{\mathcal{H}} \}$.
$\bullet \forall f \in \mathrm{Dom} K, \quad Kf = g_f$.
$\bullet K = \overline{K^0}$, *where* $\mathrm{Dom} K^0 := \{f \in \mathcal{C}^\infty(\mathbb{R}) \otimes D : \forall t \in \mathbb{R}, f(t) = f(t+1)\}$
and $\forall \eta \otimes \psi \in \mathrm{Dom} K^0, \quad (K^0(\eta \otimes \psi))(t) := -i\eta'(t) \otimes \psi + \eta(t) \otimes H(t)\psi$.

Corollary 2. *Above theorem holds true under assumptions of corollary 1.*

Chapter 4

Positive operator valued measure

Some additional information on the subject of positive operator valued measures can be found in [17] or in the original paper on Naimarks's dilation theorem [16]. Note that the author was unaware on any suitable source on positive operator measures and integration with respect to positive operator measure, although it probably exists. That is why this section may be largely reinvention of the existing terms. Note that [16] was unavailable to the author and paper [17] covers only integration over bounded function in the section 7.1.

Definition 8 (POVM := Positive operator valued measure)**.** *Let $\mathcal{B} \subset 2^{\mathbb{R}}$ be σ-algebra of Borel sets, $\mu(.) : \mathcal{B} \to \mathcal{B}(\mathcal{H}_1)$, where $\mathcal{H}_1$ is complex separable Hilbert space*

(1) $\forall M \in \mathcal{B}, \mu(M)$ is bounded positive operator on $\mathcal{H}_1$,
(2) $\mu(\mathbb{R}) = 1$,
(3) If $\{M_k\}_{k=0}^{\infty} \subset \mathcal{B}$ are mutually disjoint sets then

$$\forall\psi \in \mathcal{H}_1, \quad \langle\psi|\mu(\bigcup_{k=0}^{\infty} M_s)\psi\rangle = \lim_{N\to\infty} \langle\psi|\sum_{k=0}^{N} \mu(M_s)\psi\rangle.$$

Then in this paper mapping μ will be called **Positive operator valued measure** *or* **POVM**. *Mapping $\mu(.) : \mathcal{B} \to \mathfrak{B}(\mathcal{H}_1)$ for which axioms (1),(3) holds true together with relaxed version of the second axiom "$\mu(\mathbb{R})$ is bounded operator" will be called* **POVM*** .

Any set $M \subset \mathbb{R}$ such that $\mu(M) = 0$ will called ***set of μ-measure zero*** *i.e. $\forall\psi \in \mathcal{H}_1, \langle\psi|\mu(M)\psi\rangle = 0$. Using previous notions $\mu - a.a.$ and $\mu - a.e.$ are defined in a usual way using previous as well as sets of concentration of measure and absolute continuity of two POVM measures.*

Following notation will be used: $\mu_\lambda := \mu((-\infty, \lambda])$. In fact in the paper [17] POVM is defined in terms of μ_λ.

Lemma 3. *The third axiom of previous definition together with other axioms imply*

If $\{M_s\}_{s=0}^{\infty} \subset \mathcal{B}$ are disjoint sets then

$$\mu(\bigcup_{s=0}^{\infty} M_s) = \mathrm{st-}\lim_{N\to\infty} \sum_{s=0}^{N} \mu(M_s)$$

Proof. Due to the fact that bounded positive operators are uniquely defined by their quadratic forms, $0 \leq \mu(\bigcup_{s=0}^{N} M_s) \leq \mu(\bigcup_{s=0}^{N} M_s) \leq 1$ and well-known theorem (c.f. theorem 5.3.4 in [21]). $\square$

There is a unique way of decomposing of Borel probability measure $\mu : \mathcal{B} \to [0, 1]$ into sum of three Borel positive bounded measures $\mu = \mu_{pp} + \mu_{ac} + \mu_{sc}$ called "extended"-Lebesgue-decomposition. These three measures are all absolutely continuous with respect to μ i.e. $\forall M \in \mathcal{B}, \mu(M) = 0 \Rightarrow \mu_{pp}(M)$, $\mu_{ac}(M)$,$\mu_{sc}(M) = 0$ and are mutually singular i.e. they are concentrated on some mutually disjoint sets, which have particular relation to the original measure. In detail μ_{pp} is concentrated on the set $M_{pp} = \{m \in \mathbb{R} : \mu(m) \neq 0\}$ and μ_{sc},μ_{ac} are obtained from Lebesgue's decomposition theorem of $\mu - \mu_{pp}$ so that μ_{sc} is concentrated on the set $\tilde{M}_{sc}$ and μ_{ac} is concentrated on $\tilde{M}_{ac}$, where $\tilde{M}_{ac} \cap \tilde{M}_{ac} = \emptyset$. Because $\mu_{ac}(M_{pp}) = \mu_{sc}(M_{pp}) = 0$ one can define three mutually disjoint sets on which corresponding measures of decomposition are concentrated: M_{pp}, $M_{sc} := \tilde{M}_{sc} \setminus M_{pp}$, $M_{ac} := \tilde{M}_{ac} \setminus M_{pp}$.

Previous decomposition regarding Borel probability measures can be generalized to the case of POVM resulting into not only decomposition of the set $\mathbb{R}$ but in case of PVM also in direct decomposition of carrier Hilbert space defined using the support of certain quadratic forms.

Let μ be POVM. Since bounded operator is uniquely defined by it's quadratic form, one can apply previous decomposition of probability measures to the POVM and define POVM* μ_{pp}, μ_{ac}, μ_{sc} using decomposition of $\langle\psi|\mu\psi\rangle$ for some $\psi \in \mathcal{H}$ as their evaluation on vector ψ i.e. $\forall\psi$, $\langle\psi|\mu_{pp}\psi\rangle := (\langle\psi|\mu\psi\rangle)_{pp}$ and so on. It is easy to see that obtained operator valued functions fulfill axioms of POVM*. This is summarized in the following definition.

Definition 9 (extended Lebesgue decomposition of POVM)**.** *Let μ be POVM. Three members of unique decomposition of μ into POVM* studied in above comment will be denoted in the same way μ_{pp},μ_{ac},μ_{sc} i.e.*

$$\mu = \mu_{pp} + \mu_{sc} + \mu_{ac}$$

and regarded as **extended Lebesgue decomposition of POVM** .

Since POVM has greater theoretical importance in physics, because it has probability interpretation, it is natural to ask on whether there are some subspaces of $\mathcal{H}$ for POVM* μ_{pp},μ_{ac},μ_{sc} from extended Lebesgue decomposition of some POVM μ on which μ_{pp},μ_{ac},μ_{sc} behave like POVM. In case of μ_{pp} one can look for a such set $\mathcal{H}^{\mu}_{pp}$ that $\psi \in \mathcal{H}^{\mu}_{pp} \Leftrightarrow \langle\psi|\mu_{pp}(\mathbb{R})\psi\rangle = \langle\psi|\psi\rangle \Leftrightarrow \mu_{pp}(\mathbb{R})|\psi\rangle = |\psi\rangle$, where last equivalence can be obtained form estimate

$$\|(1-\mu_{pp}(\mathbb{R}))\psi\|^2 = \langle(1-\mu_{pp}(\mathbb{R}))\psi|(1-\mu_{pp}(\mathbb{R}))\psi\rangle \leq \|(1-\mu_{pp}(\mathbb{R})\| \, \langle\psi|(1-\mu_{pp}(\mathbb{R}))\psi\rangle,$$

which holds true since $1-\mu_{pp}(\mathbb{R})$ is positive operator. Previous statement holds true also for subspaces $\mathcal{H}^{\mu}_{ac}, \mathcal{H}^{\mu}_{sc}$ defined in analogous way. Previous two statements can now be harvested showing that subspaces $\mathcal{H}^{\mu}_{pp}, \mathcal{H}^{\mu}_{ac}, \mathcal{H}^{\mu}_{sc}$ are mutually orthogonal. This can be done by following estimate

$$\forall\psi \in \mathcal{H}^{\mu}_{pp}, \forall\phi \in \mathcal{H}^{\mu}_{sc}, |\langle\psi|\phi\rangle|^2 = |\langle\psi|\mu_{sc}(\mathbb{R})\phi\rangle|^2 = |\langle\mu_{sc}(\mathbb{R})\psi|\phi\rangle|^2 \leq \langle\psi|\mu_{sc}(\mathbb{R})\psi\rangle \, \|\phi\|^2 = 0$$

Indeed $\|\psi\|^2 = \langle\psi|\mu(\mathbb{R})\psi\rangle = \|\psi\|^2 + \langle\psi|\mu_{sc}(\mathbb{R})\psi\rangle + \langle\psi|\mu_{ac}(\mathbb{R})\psi\rangle$.

Previous justifies following definition.

Definition 10. *Let μ be POVM. Closure of largest subspaces on which three members of unique decomposition of μ into POVM* μ_{pp},μ_{ac},μ_{sc} act like POVM will be denoted $\mathcal{H}^{\mu}_{pp}, \mathcal{H}^{\mu}_{ac}, \mathcal{H}^{\mu}_{sc}$.*

$$\mathcal{H}^{\mu}_{pp} \oplus_{\perp} \mathcal{H}^{\mu}_{ac} \oplus_{\perp} \mathcal{H}^{\mu}_{sc} \subset \mathcal{H}.$$
$$\mathcal{H}_{cont} := \overline{\mathrm{span}\{\mathcal{H}^{\mu}_{ac}, \mathcal{H}^{\mu}_{ac}\}}$$

The simplest case when " $\subset''$ in previous is not equality is any one dimensional POVM with at least two non-zero members of decomposition $\mu = \mu_{pp} + \mu_{sc} + \mu_{ac}$.

Term of following definition has particular importance in theory of self-adjoint operators.

Definition 11 (spectrum (support) of POVM*)**.** *Let ν be POVM*.*

$$\sigma(\nu) := \{x \in \mathbb{R} : (\forall U = U^{\circ} \in \mathcal{B} : x \in U, \nu(U) \neq 0)\}.$$

Lemma 4. *Let μ be PVM.*

- $\forall M, N, L \in \mathcal{B}, \mu_{pp}(M), \mu_{ac}(N), \mu_{sc}(L)$ *are mutually orthogonal projectors,*
- $\mu_{pp}, \mu_{ac}, \mu_{sc}$ *act like PVM on* $\mathcal{H}^{\mu}_{pp}, \mathcal{H}^{\mu}_{ac}, \mathcal{H}^{\mu}_{sc}$
- $\mathcal{H}^{\mu}_{pp} \oplus_{\perp} \mathcal{H}^{\mu}_{ac} \oplus_{\perp} \mathcal{H}^{\mu}_{sc} = \mathcal{H}$.

Proof. Since $\mu_{pp}(M) \leq \mu_{pp}(\mathbb{R})$ one has $\mathrm{Ran}\mu_{pp}(M) \subset \mathrm{Ran}\mu_{pp}(\mathbb{R}) = \mathcal{H}^{\mu}_{pp}$ thus one has the first part of the lemma and in analogous way also the second part.

Considering that $\mu_{pp}(\mathbb{R}) + \mu_{sc}(\mathbb{R}) + \mu_{ac}(\mathbb{R}) = 1$ and together with mutual orthogonality of summed projectors and well-known theorem (c.f. theorem 5.4.4 in [21]) one has

$$\mathrm{Ran}\mu_{pp}(\mathbb{R}) \oplus_{\perp} \mathrm{Ran}\mu_{sc}(\mathbb{R}) \oplus_{\perp} \mathrm{Ran}\mu_{ac}(\mathbb{R}) = \mathrm{Ran}1 = \mathcal{H}.$$

□

Having measure one can also define integral. Simple functions and the integral will be defined in the usual sense:

Definition 12. *The set*

$$\mathcal{S}(\mathcal{B}) = \{\sum_{j\in I} f_j \chi_{M_j} : I \subset \mathbb{Z}, |I| < \infty, \text{ mut. disjoint } \{M_j\}_{j\in I} \subset \mathcal{B}, \{f_j\}_{j\in I} \in \mathbb{C}\}$$

will be called **set of simple functions.** *The operator*

$$\int s_n(\lambda)\mathrm{d}\mu_\lambda := \sum_{j\in I} f_j \mu(M_j) \in \mathcal{B}(\mathcal{H}_1)$$

will be called **integral of a simple function** .

For every f Borel function, such that $\mu - a.a.x \in \mathbb{R}, |f(x)| < \infty$ there exists a sequence $s_n \in \mathcal{S}(\mathcal{B})$, $\mu - a.a.x \in \mathbb{R}, |s_n(x)| \leq |f(x)|, s_n(x) \to f(x)$ and thus one can introduce generally unbounded operator defined by the following limit on dense domain

$$\int f(\lambda)\mathrm{d}\mu_\lambda\psi := \lim_{n\to\infty} \int s_n(\lambda)\mathrm{d}\mu_\lambda\psi$$
$$\mathrm{Dom}\Big(\int f(\lambda)\mathrm{d}\mu_\lambda\Big) := \{\psi \in \mathcal{H}_1 : \quad \int |f(\lambda)|^2 \langle\psi|\mathrm{d}\mu_\lambda\psi\rangle < \infty\},$$

This operator is called **integral of Borel function f**. *Correctness of definition of operator in previous will be proven in following.*

If $\forall M \in \mathcal{B}, \mu(M)$ is projector then then we call mapping μ **projection valued measure** *or* **PVM**. *This is also equivalent to condition $\forall M, N \in \mathcal{B}, \mu(M \cap N) = \mu(M)\mu(N) = \mu(N)\mu(M)$. In following will be show that the general case can be in a sense reduced to this special one.*

The following important theorem is due to Naimark (1940) [16] an alternative proof can be found in [17].

Theorem 6 (Naimark's dilation theorem)**.** *Let μ be POVM in a complex separable Hilbert space $\mathcal{H}_1$.*

Then there is Hilbert space $\mathcal{H}_2$, bounded linear map $V \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and PVM P so that

$$\mu = V^*PV.$$

Remark 2. *The linear map V in the previous theorem is injective and it's left inverse is V^* due to second axiom of POVM. However even though $V^*V = 1$ holds true, $VV^* = 1$ holds true only when μ is PVM. Note that $\|V^*\| = \|V\| = 1$, $\|V\psi\| = \|\psi\|$, V^*V is orthogonal projector on $\mathcal{H}_2$. Thus one can interpret $\mathcal{H}_1$ as closed subspace of $\mathcal{H}_2$, V as "identity" embedding $\mathcal{H}_1 \hookrightarrow \mathcal{H}_2$ V^* as orthogonal projector on the subspace $\mathcal{H}_1$.*

Following theorem is Theorem 9.6.4. in [21], where detailed information can be found regarding this problematics. There is also a short note regarding this problem preceding theorem VIII.6 in [1].

Theorem 7 (Correctness of definition for PVM)**.** *Let P be PVM, f be Borel and $P - a.a.x, |f(x)| < \infty$.*

Then operator $\int f(\lambda)\mathrm{d}\mu_\lambda$ exists, is unique, densely defined on the domain stated in the definition. Note that following equation holds true

$$\forall \psi \in \operatorname{Dom} \int f \mathrm{d}P_\lambda, \left\| \int f(\lambda) \mathrm{d}P_\lambda \psi \right\|^2 = \int |f(\lambda)|^2 \mathrm{d}\langle \psi | P_\lambda \psi \rangle$$

The limit by which the integral is defined does not depend on the choice of the sequence of convergent simple function.

Here the author suggest an analogous theorem for POVM.

Theorem 8 (Correctness of definition for POVM)**.** *Let μ be POVM, f is Borel measurable and $\mu - a.a.x, |f(x)| < \infty$. Then operator $\int f(\lambda)\mathrm{d}\mu_\lambda$ exists, is unique, densely defined operator on the domain stated in the definition. The limit by which integral is defined does not depend on the choice of the sequence of convergent simple functions.*

$$\left\| \int f(\lambda) \mathrm{d}P_\lambda \right\|^2 \leq \int |f(\lambda)|^2 \mathrm{d}\langle \psi | \mu_\lambda \psi \rangle$$

Proof. Using previous Naimark's dilation theorem 6 one sees that there is $\mathcal{H}_2, V, P$ so that $\forall s \in \mathcal{S}(\mathcal{B}), \int s(\lambda)\mathrm{d}\mu_\lambda = V^* \int s(\lambda)\mathrm{d}P_\lambda V$. Since one has only $\forall M \in \mathbb{R}, P(M) = 0 \Rightarrow \mu(M) = 0$ then needs to redefine f to by zero on sets of measure zero. This does not change the original integral, but is important in the following construction. Since now conditions of the previous theorem are fulfiled for the measure P and since the f has been suitably redefined one gets

$$\begin{aligned} & s_n \in \mathcal{S}(\mathcal{B}), \mu - a.a.x \in \mathbb{R}, |s_n(x)| \leq |f(x)|, s_n(x) \to f(x) \Rightarrow \\ & P - a.a.x \in \mathbb{R}, |s_n(x)| \leq |f(x)|, s_n(x) \to f(x). \end{aligned}$$

Considering

$$\left\| \int f_\lambda \mathrm{d}\mu_\lambda \psi \right\|^2 \leq \left\| \int f_\lambda \mathrm{d}P_\lambda V \psi \right\|^2 = \int |f(\lambda)|^2 \mathrm{d}\langle \psi | \mu_\lambda \psi \rangle$$

one has $\operatorname{Dom}\left(\int f(\lambda)\mathrm{d}\mu_\lambda \right) = V^* \operatorname{Dom}\left(\int f(\lambda)\mathrm{d}P_\lambda \right)$. From previous estimate and previous theorem 7 one has also that integral does not depend on the choice of sequence pf simple functions. One is now left with proving that domain of the operator is dense. Since one obviously has

$$\forall \psi \in \mathcal{H}_1, \forall \varepsilon > 0, \exists \phi_\varepsilon \in \mathcal{H}_2, \phi_\varepsilon \in \operatorname{Dom}\left(\int f(\lambda)\mathrm{d}P_\lambda \right) \text{ et } \|V\psi - \phi_\varepsilon\|_{\mathcal{H}_2} < \varepsilon.$$

Using $\|V\psi - \phi_\varepsilon\|_{\mathcal{H}_2} < \varepsilon \Rightarrow \|\psi - V^*\phi_\varepsilon\|_{\mathcal{H}_1} < \varepsilon \|V^*\|$ one gets denseness of the set

$$\operatorname{Dom}\left(\int f(\lambda)\mathrm{d}\mu_\lambda \right) = V^* \operatorname{Dom}\left(\int f(\lambda)\mathrm{d}P_\lambda \right).$$

Thus the operator is densely defined.

□

Remark 3. *The domain of integral with respect to POVM μ is chosen so that there are no problems with the definition, because problem then can be reduced to the problem of PVM. The author cannot exclude cases where the integral can be defined more naturally on some larger set.*

Lemma 5. *Let μ be POVM, f be Borel function and $\mu - a.a.x \in \mathbb{R}, |f(x)| < \infty$. Then subspaces $\mathcal{H}^\mu_{pp}, \mathcal{H}^\mu_{ac}, \mathcal{H}^\mu_{sc}$ are invariant under integral of f with respect to μ. The restriction of the integral to subspace $\mathcal{H}^\mu_{pp} \oplus_\perp \mathcal{H}^\mu_{ac} \oplus_\perp \mathcal{H}^\mu_{sc}$ can be written as a direct sum of integrals of f on subspaces $\mathcal{H}^\mu_{pp}, \mathcal{H}^\mu_{ac}, \mathcal{H}^\mu_{sc}$ with respect to μ_{pp},μ_{ac},μ_{sc}.*

Following theorem is due to von Neumann (1929), Stone (1932) and Riesz (1930).

Theorem 9 (Spectral decomposition theorem)**.** *There is one-to-one map between the set of all PVM and self-adjoint operators B on Hilbert space $\mathcal{H}$ given by*

$$B = \int \lambda \mathrm{d}P_\lambda^{(B)}$$

The PVM $P^{(B)}$ will in following denote PVM corresponding to the self-adjoint operator B.

The author expects that first part of the theorem can be made even stronger in following sense. For a fixed real Borel strictly monotone function f and fixed real Borel g, there is one-to-one map between set of all PVM and the operators of form $\int f(\lambda) + ig(\lambda)\mathrm{d}P_\lambda$, where P is some PVM.

Definition 13 (Decomposition of spectrum)**.** *Let B be self-adjoint operator. Then following of notation will be used*

$$\begin{aligned}
&\bullet \sigma(B) := \sigma(P^{(B)}),\\
&\bullet \sigma_{pp}(B) := \sigma(P_{pp}^{(B)}), \sigma_{sc}(B) := \sigma(P_{sc}^{(B)}), \sigma_{ac}(B) := \sigma(P_{ac}^{(B)}),\\
&\bullet \mathcal{H}_{pp}^B := \mathcal{H}_{pp}^{P^{(B)}}, \mathcal{H}_{ac}^B := \mathcal{H}_{ac}^{P^{(B)}}, \mathcal{H}_{sc}^B := \mathcal{H}_{sc}^{P^{(B)}}.\\
&\bullet \mathcal{H}_{cont}^B := \mathcal{H}_{cont}^{P^{(B)}} = \overline{\mathrm{span}\{\mathcal{H}_{sc}^B, \mathcal{H}_{ac}^B\}}.
\end{aligned}$$

*If $\sigma_{sc}(B) = 0, \sigma_{ac}(B) = 0$, then it will be said that B **has pure-point spectrum**.*

There is way of defining similar terms also for unitary operators and as author expects even for a larger subset of normal operators: for the integral of $f + ig$ with respect to some PVM, where f,g are Borel functions with continuous derivative on $\mathbb{R}$ and non-zero derivative up to measure zero. However only the case of unitary operators, where one has $f = \cos$ and $g = \sin$, will be studied in this paper. Variant of following theorem is theorem 10.2.6 [21].

Theorem 10 (Spectral decomposition theorem)**.** *There is one-to-one map between the set of all PVM concentrated on $[0, 2\pi)$ and unitary operators U on Hilbert space $\mathcal{H}$ given by*

$$U = \int \mathrm{e}^{i\lambda}\, \mathrm{d}P_\lambda^{(U)}$$

The PVM $P^{(U)}$ will in following denote PVM corresponding to the unitary operator U. Following notation will be used

$$\begin{aligned}
&\bullet \sigma(U) := \sigma(P^{(U)}),\\
&\bullet \sigma_{pp}(U) := \sigma(P_{pp}^{(U)}), \sigma_{sc}(U) := \sigma(P_{sc}^{(U)}), \sigma_{ac}(U) := \sigma(P_{ac}^{(U)}),\\
&\bullet \mathcal{H}_{pp}^U := \mathcal{H}_{pp}^{P^{(U)}}, \mathcal{H}_{ac}^U := \mathcal{H}_{ac}^{P^{(U)}}, \mathcal{H}_{sc}^U := \mathcal{H}_{sc}^{P^{(U)}}.\\
&\bullet \mathcal{H}_{cont}^U := \mathcal{H}_{cont}^{P^{(U)}} = \overline{\mathrm{span}\{\mathcal{H}_{sc}^U, \mathcal{H}_{ac}^U\}}
\end{aligned}$$

Chapter 5

The RAGE theorem and the time-mean of Hamiltonian

5.1 The RAGE theorem

This section is devoted to the theoretical results of Ruelle, Amrein, Georgescu and Enss (1973-1982) in [18].

Theorem 11 (RAGE)**.** *Let H be self-adjoint operator in $\mathcal{H}$, $C \in \mathfrak{B}(\mathcal{H})$ relatively compact with respect to H i.e. $C(H+i)^{-1}$ is compact.*

Then $\forall \phi \in \mathcal{H}_{cont}^{(H)}$

$$\bullet \lim_{\tau\to\pm\infty} \frac{1}{\tau}\int_0^\tau \mathrm{d}t \left\|C\,\mathrm{e}^{-itH}\,\phi\right\| = 0, \tag{5.1}$$

$$\bullet \lim_{\tau\to\pm\infty} \frac{1}{\tau}\int_0^\tau \mathrm{d}t \left\|C\,\mathrm{e}^{-itH}\,\phi\right\|^2 = \lim_{\tau\to\pm\infty} \frac{1}{\tau}\int_0^\tau \mathrm{d}t\langle\phi|\mathrm{e}^{itH}\,C^*\,C\,\mathrm{e}^{-itH}\,\phi\rangle = 0, \tag{5.2}$$

where (5.2) *implies* (5.1) *due to Schwarz inequality.*

One can geometrically interpret previous theorem in spaces $L^2(\mathbb{R}^n, \mathrm{d}^n x)$ taking typical Hamiltonian $H = -\triangle + V$ and $C_R = \chi_{\{x\in\mathbb{R}^n:\|x\|\leq R\}}$ i.e. bounded multiplication operator by characteristic function of set $\{x \in \mathbb{R}^n : \|x\| \leq R\}$. Thus C_R is orthogonal projector and is relatively compact with respect to H for large family of potentials V. The RAGE theorem then for C_R, which is relatively compact with respect to H, gives geometrical unboundedness of evolution of arbitrary state $\phi \in \mathcal{H}$.

Some previous results can be generalized to periodically time-dependent Hamiltonians $H(t+T) = H(t)$ for which corresponding jointly continuous T-periodic unitary propagator $U(t,s)$ exists.

Definition 14 (Bounded and free states)**.** *Let $U(t,s)$ be jointly continuous unitary propagator,*

$$\mathcal{H}_\pm^{bound.} := \{\psi \in \mathcal{H} : \{U(t,0)\psi : \pm t > 0\} \text{ is precompact.}$$

$$\mathcal{H}_\pm^{free} := \{\psi \in \mathcal{H} : \forall K \text{ is compact op. }, \lim_{\tau\to\pm\infty} \frac{1}{\tau}\int_0^\tau \mathrm{d}t\,\|KU(t,0)\psi\|^2 = 0\},$$

Theorem 12 (equivalence of definitions)**.** *Let $U(t,s)$ be jointly continuous unitary T-periodic propagator. Then*

$$\mathcal{H}_\pm^{free} = \mathcal{H}_{cont}(U(T,0)) \text{ and } \mathcal{H}_\pm^{bound.} = \mathcal{H}_{pp}(U(T,0))$$

Definition 15 (geometrically bounded and free states)**.** *Let $U(t,s)$ be jointly continuous unitary propagator; $P := \{P_R\}_{R>0}$ family of Hermitean operators; $\|P_R\| \leq 1$ and $\mathrm{st-}\lim_{R\to\infty} P_R = 1$. Then*

$$\mathcal{M}_{\pm}^{free}(P) := \{\psi \in \mathcal{H} : \forall R > 0, \lim_{\tau\to\pm\infty} \frac{1}{\tau}\int_0^\tau \mathrm{d}t\, \|P_R U(t,0)\psi\| = 0\},$$
$$\mathcal{M}_{\pm}^{bound.}(P) := \{\psi \in \mathcal{H} : \lim_{R\to\infty} \sup_{\pm t>0} \|(1-P_R)U(t,0)\psi\| = 0\}.$$

Theorem 13 (T-periodic version of RAGE)**.** *Let $P := \{P_R\}_{R>0}$ be family of bounded operators; $\|P_R\| \leq 1$; $\mathrm{st-}\lim_{R\to\infty} P_R = 1$; $\forall R > 0, \forall\psi \in \mathcal{H}, P_R\{U(t,0)\psi : \pm t > 0\}$ is precompact. Then*

$$\mathcal{H}_{\pm}^{free} = M_{\pm}^{free}(P) = \mathcal{H}_{cont}(U(T,0)) \text{ and } \mathcal{H}_{\pm}^{bound.} = \mathcal{M}_{\pm}^{bound.}(P) = \mathcal{H}_{pp}(U(T,0))$$

Previous theorem can be seen as generalization of the RAGE theorem since in time-independent case relative compactness of P_R with respect to H for all $R > 0$ implies precompactness of $\forall R > 0, \forall\psi \in \mathcal{H}, P_R\{U(t,0)\psi : \pm t > 0\}$. This can be seen rewriting $P_R U(t,0)\psi = P_R(H(t)+i)^{-1}(H(t)+i)U(t,0)(H(0)+i)^{-1}(H(0)+i)\psi$. Note that under some regularity conditions (e.g. theorem 2) one can get local boundedness of operator $(H(t)+i)U(t,0)(H(0)+i)^{-1}$, however one does not have uniform boundedness in general (e.g. theorem 14).

5.2 The time-mean of Hamiltonian

Inspired by the RAGE theorem the author proposes following theorems.

Lemma 6. *Let*

- *B be self-adjoint operator in complex separable Hilbert space $\mathcal{H}_1$,*
- *$A(t)$ unitary-operator valued function,*
- *$A(t)\mathrm{Dom}B = \mathrm{Dom}B$*
- *$\left\|BA(t)(B+i)^{-1}\right\|$ is locally bounded,*
- *$B = \int \lambda \mathrm{d}P_\lambda^{(B)}$*
- *$\tau > 0$; $\mu_\lambda := \frac{1}{\tau}\int_0^\tau A(t)^* P_\lambda^{(B)} A(t)\mathrm{d}t$ is operator valued function,*
- *$B_\tau\psi := \int \lambda \mathrm{d}\mu_\lambda$.*

Then

$$\bullet \mu \text{ is POVM},$$
$$\bullet \forall\psi \in \mathrm{Dom}B, B_\tau\psi = \frac{1}{\tau}\int_0^\tau A(t)^* BA(t)\psi \mathrm{d}t,$$
$$\bullet \mathrm{Dom}B_\tau = \mathrm{Dom}B.$$

Proof. RHS is well defined using theory of Bochner integral as given in [21] since

$$\|A(t)^* BA(t)\psi\| \leq \left\|BA(t)(B+i)^{-1}\right\| \|(B+i)\psi\|.$$

Proving axioms 1,2 of POVM for μ is straight forward. Third axiom also holds true due to $\forall M \in \mathcal{B}, \forall\psi \in \mathcal{H}_1, \langle\psi|\mathcal{A}^*(t)\mu(M)A(t)\psi\rangle \leq 1$ and Lebesgue theorem.

One can define family of mutually absolutely continuous POVM measures $\forall t \in \mathbb{R}, \widetilde{\mu}_\lambda[t] := A(t)^* P_\lambda A(t)$. The equality of the operators is not hard to prove using mutually absolute continuity of POVM measures i.e. $\forall t \in \mathbb{R}, \forall M \in \mathcal{B}, \widetilde{\mu}[t](M) = 0 \Leftrightarrow \widetilde{\mu}[0](M) = 0$, definition of the integral using a $a.e.\widetilde{\mu}[0]$-pointwise convergence of simple functions for which $|s_n(\lambda)| \leq |\lambda|$ and Lebesgue theorem, which uses estimate

$$\forall \psi \in \mathrm{Dom} B, \left\| \int s_n(\lambda) \mathrm{d}\widetilde{\mu}[t]_\lambda \psi \right\|^2 = \left\| \int s_n(\lambda) \mathrm{d}P_\lambda A(t)\psi \right\|^2 =$$

$$= \int |s_n(\lambda)|^2 \mathrm{d}\langle A(t)\psi | P_\lambda A(t)\psi \rangle \leq \int |\lambda|^2 \mathrm{d}\langle A(t)\psi | P_\lambda A(t)\psi \rangle =$$

$$= \|BA(t)\psi\|^2,$$

where last term is locally bounded. From the previous one easily gets also equality of domains:

$$\int \lambda^2 \mathrm{d}\langle \psi | \mu_\lambda \psi \rangle = \frac{1}{\tau} \int_0^\tau \langle A(t)\psi | BA(t)\psi \rangle \mathrm{d}t < \infty \Leftrightarrow \psi \in \mathrm{Dom} B.$$

□

Following lemma is generalization of lemma 2.4 in paper [18] due to Enss and Veselic .

Lemma 7. *Let*

- C *be compact operator,*
- $A(t)$ *unitary operator valued function,*
- $A := A(1)$*,*
- $\forall t \in \mathbb{R}, j \in \mathbb{Z}, A(t+j) = A(t)A^j$*,*
- $A \restriction_{\mathcal{H}_{pp}(A)} = \sum_{\mathrm{e}^{i\lambda_j} \in \sigma_p(A)} \mathrm{e}^{i\lambda_j} P_j^{(A)}$*, where* $P_j^{(A)}$ *are orthogonal projectors on eigensubspaces corresponding to distinct eigenvalues* $\mathrm{e}^{i\lambda_j}$*.*

Then

$$u - \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau A^*(t) C A(t) \mathrm{d}t = \sum_{\mathrm{e}^{i\lambda_j} \in \sigma_p(A)} P_j^{(A)} \int_0^1 A^*(t) C A(t) \mathrm{d}t P_j^{(A)}$$

Proof. Lemma 2.4 from paper [18] states that

$$u - \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau A^*(t) C A(t) P_{cont}^{(A)}(\mathbb{R}) = 0.$$

Using fact that $\mathbb{C}^*$ is also compact and $\forall B \in \mathcal{B}(\mathcal{H}), \|B\| = \|B^*\|$ one can see that

$$u - \lim_{\tau \to \infty} P_{cont}^{(A)}(\mathbb{R}) \frac{1}{\tau} \int_0^\tau A^*(t) C A(t) = 0.$$

Thus one can reduce problem to the case where $A = A_p := AP_{pp}^{(A)}(\mathbb{R}) = \sum_{\mathrm{e}^{i\lambda_j} \in \sigma_p(A)} \mathrm{e}^{i\lambda_j} P_j^{(A)}$.

Integral can be reduced into sum as can be seen in the following

$$\frac{[\tau]}{\tau} \sum_{n=0}^{[\tau]-1} \frac{1}{[\tau]} A^{-n} \int_0^1 A(t)^* C A(t) \mathrm{d}t A^n - \frac{1}{\tau} \int_{\tau - [\tau]}^1 A^{-[\tau]} A(t)^* C A(t) \mathrm{d}t A^{[\tau]}$$

Since $\left\| \int_{\tau-[\tau]}^1 A^{-[\tau]} A(t)^* C A(t) \mathrm{d}t A^{[\tau]} \right\| \leq \|C\|$ last term can be omitted thus

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau A^*(t) C A(t) = \lim_{N \to \infty} \sum_{n=0}^{N-1} \frac{1}{N} A^{-n} \int_0^1 A(t)^* C A(t) \mathrm{d}t A^n.$$

One can see that $C' := \int_0^1 A(t)^* C A(t) \mathrm{d}t$ is compact operator since it is integral of uniformly continuous compact operator valued function. Since compact operator is operator limit of finite dimensional operator one can reduce problem to case where $C' = |\psi\rangle\langle\phi|$. Since $P_j^{(A)}$ are mutually orthogonal projectors from spectral decomposition of A one has for $\forall x \in \mathcal{H}$

$$\left\| \Big(\sum_{n=0}^{N-1} \frac{1}{N} A^{-n} |\psi\rangle\langle\phi| A^n - \sum_{\mathrm{e}^{i\lambda_j} \in \sigma_p(A)} P_j^{(A)} |\psi\rangle\langle\phi| P_j^{(A)} \Big) |x\rangle \right\|^2 \leq \tag{5.3}$$

$$\leq \sum_{\mathrm{e}^{i\lambda_j} \in \sigma_p(A)} \left\| P_j^{(A)} |\psi\rangle \right\|^2 |\langle\phi| \Big(\sum_{n=0}^{N-1} \frac{1}{N} A^n \, \mathrm{e}^{-in\lambda_j} - P_j^{(A)} \Big) |x\rangle|^2 = \tag{5.4}$$

$$= \sum_{\mathrm{e}^{i\lambda_j} \in \sigma_p(A)} \left\| P_j^{(A)} |\psi\rangle \right\|^2 \sum_{\mathrm{e}^{i\lambda_k} \in \sigma_p(A): \lambda_k \neq \lambda_j} \left\| \sum_{n=0}^{N-1} \frac{1}{N} \mathrm{e}^{in(\lambda_k - \lambda_j)} P_k^{(A)} |\phi\rangle \right\|^2 \|x\|^2 \tag{5.5}$$

Because sum in (5.5) can be estimated by $\|\psi\|^2 \|\phi\|^2 \|x\|^2$ one can exchange limit $\lim_{N\to\infty}$ and summations in (5.5) and using the following limit

$$\lim_{N\to\infty} \sum_{n=0}^{N-1} \frac{1}{N} \mathrm{e}^{ix} = \begin{cases} 1 & x = 2\pi k : k \in \mathbb{Z} \\ 0 & \text{elsewhere.} \end{cases}$$

One can complete the proof. Note that the author expects that the proof of lemma 2.4 [18] can be done in analogous way. ☐

Theorem 14 (T-mean theorem)**.** *Let*

- *B be semi-bounded operator with pure point on spectrum complex separable Hilbert space $\mathcal{H}$,*
- $B = \int \lambda \mathrm{d}P_\lambda^{(B)}$,
- $\forall \lambda \in \mathbb{R}, \dim \mathrm{Ran} P_\lambda^{(B)} < \infty$,
- *$A(t)$ unitary-operator valued function,*
- $A(t)\mathrm{Dom}B = \mathrm{Dom}B$,
- *$\left\|BA(t)(B+i)^{-1}\right\|$ is locally bounded,*
- *$\forall \tau \in \mathbb{R}, \mu[\tau]_\lambda := \frac{1}{\tau}\int_0^\tau A(t)^* P_\lambda^{(B)} A(t)\mathrm{d}t$ is POVM,*
- *$\forall M \in \mathcal{B}, \mu[\infty](M) := \lim_{\tau\to\infty} \mu[\tau](M)$,*
- *$A := A(1)$,*
- *$\forall t \in \mathbb{R}, \forall j \in \mathbb{Z}, A(t+j) = A(t)A^j$,*
- *$A \restriction_{\mathcal{H}_{pp}(A)} = \sum_{\mathrm{e}^{i\lambda_j} \in \sigma_p(A)} \mathrm{e}^{i\lambda_j} P_j^{(A)}$, where $P_j^{(A)}$ are orthogonal projectors on eigensubspaces corresponding to distinct eigenvalues $\mathrm{e}^{i\lambda_j}$.*

Then

1. *$\forall \lambda \in \mathbb{R}, \mu[\infty]_\lambda = \lim_{\tau\to\infty} \mu[\tau]_\lambda = \sum_{\mathrm{e}^{i\lambda_j} \in \sigma_p(A)} P_j^{(A)} \mu[1](M) P_j^{(A)}$,*
2. *$\mu[\infty](\mathbb{R}) = 1$,*
3. *$\mu[\infty]$ is POVM $\Leftrightarrow \mathcal{H}_{cont}(A) = \{0\}$,*

4. $\mu[\infty]\restriction_{\mathcal{H}_{pp}(A)}$ *is POVM on Hilbert space* $\mathcal{H}_{pp}(A)$*,*

5. $B_\infty := \int \lambda \mathrm{d}\mu[\infty]\restriction_{\mathcal{H}_{pp}(A)}$ *is symmetric operator on* $\mathcal{H}_{pp}(A)$*,*

6. $\mathrm{Dom}B_\infty = \{\psi \in \mathcal{H}_{pp}(A) : \sum_{\mathrm{e}^{i\lambda_j}\in\sigma_p(A)} \int_0^1 \left\|BA(t)P_j^{(A)}\psi\right\|^2 \mathrm{d}t < \infty\}$*,*

7. $\mathrm{Dom}B_\infty \subset \mathrm{Dom}B \cap \mathcal{H}_{pp}(A)$*,*

8. $\forall\psi \in \mathrm{Dom}B_\infty, B_\infty\psi = \sum_{\mathrm{e}^{i\lambda_j}\in\sigma_p(A)} P_j^{(A)} B_1 P_j^{(A)}\psi$*,*

9. $\psi \notin \mathcal{H}_{pp}(A)$*,* $\lim_{\tau\to\infty}\langle\psi|B_\tau\psi\rangle = \infty$*,*

10. $\forall\psi \in \mathrm{Dom}B_\infty, \langle\psi|B_\infty\psi\rangle \leq \liminf_{\tau\to\infty}\langle\psi|B_\tau\psi\rangle$*,*

11. $\forall\psi \in \mathrm{Dom}B_\infty, \|B_\infty\psi\| \leq \liminf_{\tau\to\infty}\|B_\tau\psi\|$*.*

Proof. The first two implications (1.,2.) of the theorem can be proven using previous lemma and fact that $\forall\tau \in \mathbb{R}, \mu[\tau](\mathbb{R}) = 1$. Since

$$(\forall\psi, \phi \in \mathcal{H} : \psi \notin \mathcal{H}_{pp}(A)),\ \lim_{\lambda\to\infty}\langle\phi|\mu[\infty]_\lambda\psi\rangle = 0 \neq \langle\phi|\psi\rangle$$

one can see that while axioms of POVM holds true for $\mu[\infty]$, axiom (3) does not. However axiom (3) of POVM holds true for $\mu[\infty]\restriction_{\mathcal{H}_{pp}(A)}$ on Hilbert space $\mathcal{H}_{pp}(A)$ thus it is POVM.

Domain defined for integral with respect to POVM $\mu[\infty]$ is

$$\mathrm{Dom}B_\infty = \{\psi \in \mathcal{H}_{pp}(A) : \int \lambda^2 \mathrm{d}\langle\psi|\mu[\infty]\psi\rangle < \infty\}.$$

Using $\forall M \in \mathcal{B}, \mu[1] = 0 \Rightarrow \mu[\infty] = 0$ one can choose proper set of simple functions to prove that the sum $\sum_{\mathrm{e}^{i\lambda_j}\in\sigma_p(A)}$ and the POVM integral can be exchanged.

$$\int \lambda^2 \mathrm{d}\langle\psi|\sum_{\mathrm{e}^{i\lambda_j}\in\sigma_p(A)} P_j^{(A)}\mu[1]_\lambda P_j^{(A)}\psi\rangle = \sum_{\mathrm{e}^{i\lambda_j}\in\sigma_p(A)}\int \lambda^2 \mathrm{d}\langle\psi|P_j^{(A)}\mu[1]_\lambda P_j^{(A)}\psi\rangle$$

Now using lemma 6 one prove next part of the theorem

$$\begin{aligned}
&\sum_{\mathrm{e}^{i\lambda_j}\in\sigma_p(A)}\int \lambda^2 \mathrm{d}\langle\psi|P_j^{(A)}\mu[1]_\lambda P_j^{(A)}\psi\rangle = \\
&= \sum_{\mathrm{e}^{i\lambda_j}\in\sigma_p(A)}\int_0^1\int \lambda^2 \mathrm{d}\langle\psi|P_j^{(A)}A(t)^* P_\lambda^{(B)} A(t) P_j^{(A)}\psi\rangle \mathrm{d}t = \\
&= \sum_{\mathrm{e}^{i\lambda_j}\in\sigma_p(A)}\int_0^1 \left\|BA(t)P_j^{(A)}\psi\right\|^2 \mathrm{d}t.
\end{aligned}$$

To prove $\mathrm{Dom}B \cap \mathcal{H}_{pp}(A) \subset \mathrm{Dom}B_\infty$ one needs only to for $\psi \in \mathcal{H}$ find appropriate estimate for $\|B_1\psi\|$. This can be done as follows

$$\|B_1\psi\| \leq \sum_{\mathrm{e}^{i\lambda_j}\in\sigma_p(A)}\int_0^1 \left\|BA_tP_j^{(A)}\psi\right\|\mathrm{d}t \leq \sqrt{\sum_{\mathrm{e}^{i\lambda_j}\in\sigma_p(A)}\int_0^1 \left\|BA_tP_j^{(A)}\psi\right\|^2\mathrm{d}t}.$$

To prove $B_\infty \subset \sum_{\mathrm{e}^{i\lambda_j}\in\sigma_p(A)} P_j^{(A)} B_1 P_j^{(A)}$ i.e.

$$\forall\psi \in \mathrm{Dom}B_\infty, B_\infty\psi = \sum_{\mathrm{e}^{i\lambda_j}\in\sigma_p(A)} P_j^{(A)} B_1 P_j^{(A)}\psi$$

one can use following estimate together with Lebesgue theorem. In the following estimate $s_n = \sum_k f_{n,k}\chi_{M_k}$ is set of simple functions used to define $\int \lambda \mathrm{d} \sum_{\mathrm{e}^{i\lambda_j} \in \sigma_p(A)} P_j^{(A)} \mu[1]_\lambda P_j^{(A)}$. Also in the following estimate Naimark's theorem 6 is used to imply $\|\mu[1]\phi\|^2 \leq \langle \phi | \mu[\mathit{1}]\phi \rangle$.

$$\begin{aligned}
&\left\| \sum_{j\in I, |I|<\infty} \int s_n(\lambda) \mathrm{d} P_j^{(A)} \mu[1]_\lambda P_j^{(A)} \psi \right\| = \left\| \sum_{j\in I, |I|<\infty} \sum_k f_{n,k} P_j^{(A)} \mu[1](M_k) P_j^{(A)} \psi \right\| \leq \\
&\leq \sum_{j\in I, |I|<\infty} \sum_k |f_{n,k}| \left\| P_j^{(A)} \mu[1](M_k) P_j^{(A)} \psi \right\| \leq \sum_{j\in I, |I|<\infty} \sum_k |f_{n,k}| \left\| \mu[1](M_k) P_j^{(A)} \psi \right\| \leq \\
&\leq \sqrt{\sum_{j\in I, |I|<\infty} \sum_k |f_{n,k}|^2 \left\| \mu[1](M_k) P_j^{(A)} \psi \right\|^2} \leq \\
&\leq \sqrt{\sum_{j\in I, |I|<\infty} \sum_k |f_{n,k}|^2 \langle \psi | P_j^{(A)} \mu[\mathit{1}](M_k) P_j^{(A)} \psi \rangle} \leq \\
&\leq \sum_{\mathrm{e}^{i\lambda_j} \in \sigma_p(A)} \int \lambda^2 \mathrm{d} \langle \psi | P_j^{(A)} \mu[\mathit{1}]_\lambda P_j^{(A)} \psi \rangle < \infty.
\end{aligned}$$

To prove

$$\psi \notin \mathcal{H}_{pp}(A), \lim_{\tau\to\infty} \langle \psi | B_\tau \psi \rangle = \infty,$$

one can use following estimate and without loss of generality consider that $B \geq 0$ i.e. is positive.

$\forall \psi \in \mathrm{Dom} B, \forall n \in \mathbb{N}$,

$$\begin{aligned}
&\langle \psi | B_\tau \psi \rangle = \int \lambda \mathrm{d} \langle \psi | \mu[\tau]_\lambda \psi \rangle \geq n \langle \psi | \mu[\tau]((n,\infty)) \psi \rangle \geq \\
&\geq n \langle \psi_{cont} | \mu[\tau]((n,\infty)) \psi_{cont} \rangle + 2n \mathrm{Re}(\langle \psi_{pp} | \mu[\tau]((n,\infty)) \psi_{cont} \rangle),
\end{aligned}$$

where $\psi_{cont} = P_{cont}^{(B)}(\mathbb{R})\psi$ and $\psi_{pp} = P_{pp}^{(B)}(\mathbb{R})\psi$. Since one has

$$\lim_{\tau\to\infty} \langle \psi_{pp} | \mu[\tau]((n,\infty)) \psi_{cont} \rangle = 0,$$

and

$$\lim_{\tau\to\infty} \langle \psi_{cont} | \mu[\tau]((n,\infty)) \psi_{cont} \rangle = 1,$$

it is straightforward that addressed part of the theorem holds.

Proving the remaining part of the theorem is now straightforward using analogous approach as used in the well-known proof of Fatou's lemma. Note that it is important to take into consideration that $\forall M \in \mathcal{B}, \mu[\tau](M) = 0 \Leftrightarrow \mu[1](M) = 0 \Rightarrow \mu[\infty](M) = 0$. $\square$

Chapter 6

Methods of studying stability

This section is devoted to the existing methods of studying time evolution generated by time-dependent Hamiltonians. Three well known methods that are mentioned below are all based on iterative conjugation of Floquet Hamiltonian K by unitary operators so that partial resp. complete diagonalization is after finitely resp. infinitely many steps achieved.

First two methods which are summarized in following two theorems 16,17 were initiated by Howland. They are both based on the result from scattering theory, which states that the absolutely continuous spectrum is invariant under trace class perturbation. This result is summarized in the following theorem, which is in fact consequence of Theorem 5 in [7].

Theorem 15. *Let H_0 be self-adjoint operator in $\mathcal{H}$, $H(t) = H(t+T) = H_0 + V(t)$ for which associated propagator $U(t,s)$ exists,*

$$\begin{aligned} &\bullet V(t) \textit{ is measurable,} \\ &\bullet V(t) \textit{ is self-adjoint}, \\ &\bullet V(t) \textit{ is trace class } \Leftrightarrow \operatorname{Tr}\sqrt{V(t)^* V(t)} < \infty, \\ &\bullet \int_0^T \operatorname{Tr}\sqrt{V(t)^* V(t)} \mathrm{d}t < \infty. \end{aligned}$$

Then

$$\sigma_{ac}(H_0) = \emptyset \Rightarrow \sigma_{ac}(U(T,0)) = \emptyset$$

Since there are no restrictions on singular spectrum previous theorem does not resolve problem of stability completely, because there still may be propagating states $P_{cont}^{(U(T,0))}\psi \neq 0$.

The previous theorem was used by Howland in papers [5, 6] and further developed in [10, 11] to develop method called the adiabatic method. Basic result from [5] is given in the following theorem.

Theorem 16 (The adiabatic method)**.** *Let H_0 is self-adjoint operator in $\mathcal{H}$ with non-degenerate pure point spectrum $\sigma(H_0) = \{E_m\}_{m=1}^{\infty}$, such that growing gap condition holds true:*

$$\exists c > 0, \alpha > 0, \quad \forall n \in \mathbb{N} \quad E_{n+1} - E_n \geq cn^{\alpha},$$

and $V(t) \in \mathcal{C}^2(\mathbb{R}, \mathfrak{B}(\mathcal{H})_s))$, $\forall t \in \mathbb{R}, V(t+T) = V(t)$. Then Floquet Hamiltonian spectrum $\sigma_{ac}(K) = \sigma_{ac}(\mathfrak{U}(1 \otimes U(T,0))\mathfrak{U}^*) = \emptyset$.

Proof. Only short summary of the main idea of the proof is given in this work.

Unitary operators $\mathrm{e}^{W(t)}$ used to achieve partial diagonalization are chosen to be generated by symmetric operator map $iW(t)$, which fulfills $[W(t), H_0] = -V(t)$. Previous equation is non-uniquely solved by operator with following matrix entries in the eigenvector basis of H_0:

$$W_{m,n}(t) = \begin{cases} \frac{-V_{m,n}(t)}{E_m - E_n} & \text{for } m \neq n \\ 0 & else. \end{cases}.$$

In expansion of conjugated Floquet Hamiltonian $\mathrm{e}^{W(t)}\, K\, \mathrm{e}^{-W(t)}$ former potential $V(t)$ is effectively replaced by $i\partial_t W(t)$, which due to adiabatic gap condition has better decay properties than former potential $V(t)$.

By iterating previous procedure one, after finite steps, arrives to equivalent Floquet Hamiltonian with uniformly-trace-class potential. Thus one can then use 15 to prove the theorem.

The main difficulty of proving this theorem is showing that other members of expansion of unitary conjugations are "less important". □

Another application of 15 given in the following theorem was introduced by Howland in [9].

Theorem 17 (The anti-adiabatic method)**.** *Let H_0 be self-adjoint operator in $\mathcal{H}$ with non-degenerate pure point spectrum $\sigma(H_0) = \{E_m\}_{m=1}^{\infty}$. Assume that anti-adiabatic gap condition holds true:*

$$\forall m, n \in \mathbb{N}, \exists \gamma \in (0, 1/2), \exists a_1 > 0, \quad |E_m - E_n| < \frac{a_1 |m-n|}{(mn)^{\gamma}}$$

and that $H(t) := H(t+T) = H_0 + V(t)$ for which associated propagator $U(t,s)$ exists. Further more assume that

$$\forall m, n \in \mathbb{N}, \exists r > 1 + \frac{2}{\gamma}, \exists a_2 > 0, \quad |V(t)_{m,n}| \leq \frac{a_2}{(mn)^{\gamma} (\max\{1, |m-n|\})^r},$$

where $V(t)_{m,n}$ denotes matrix entries in eigenvector basis of H_0.

Then Floquet Hamiltonian spectrum $\sigma_{ac}(K) = \sigma_{ac}(\mathfrak{U}(1 \otimes U(T,0))\mathfrak{U}^*) = \emptyset$

Proof. Only the main idea of the proof will be given in this paper. The proof is based on the same iteration only $W(t)$ is chosen in a different way. Here the generator $iW(t)$ of the unitary operator $\mathrm{e}^{W(t)}$ used for conjugation is

$$W(t) = i \int_0^t \mathrm{d}u \bigg(V(u) - \frac{1}{T} \int_0^T \mathrm{d}s V(s) \bigg).$$

So that $V(t)$ is effectively replaced by $[W(t), i\partial_t]$. Using anti-adiabatic condition one can prove that

$$|[W(t), H_0]_{m,n}| \leq \frac{|W(t)_{m,n}||m-n|}{(mn)^{\gamma}}.$$

Thus decay of on-diagonal elements is improved. After finite iterative steps one arrives to unitary equivalent Floquet Hamiltonian with $V(t)$ effectively replaced by uniformly-trace-class operator and thus one can use theorem 15.

The main difficulty of proving this theorem is showing that other members of expansion of unitary conjugations are "less important". □

Quantum KAM method, which is inspired by result of Kolmogorov, Arnold and Moser in the classical mechanics, was proposed by Bellissard [12] and further developed in [13, 14, 15]. Following theorem is taken from [22].

Theorem 18. *Let H_0 be self-adjoint operator in $\mathcal{H}$ with non-degenerate pure point spectrum $\sigma(H_0) = \{E_m\}_{m=1}^{\infty}$, such that growing gap condition holds true:*

$$\exists c > 0, \alpha > 0, \quad \forall n \in \mathbb{N} \quad E_{n+1} - E_n \geq c n^{\alpha}.$$

Assume that $H(t) := H(t+2\pi) = H_0 + V(t)$ for which associated propagator $U(t,s)$ exists. Furthermore assume that

$$\bullet V(t) \in \mathcal{C}(\mathbb{R}, \mathfrak{B}_s(\mathcal{H})),$$
$$\bullet \forall t \in [0, 2\pi], \quad V(t) = V(t)^* \textit{ i.e. is Hermitean}$$
$$\bullet \omega_1, \omega_2 \in \mathbb{R} : \omega_2 - \omega_1 > 0,$$

Then $\exists p(\alpha) > 0, \forall r > p(\alpha), \exists q(\alpha, r) > 0, \exists u(\alpha, r) > 0$ such that the following implication holds true:

$$\sup_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \left| \int_0^{2\pi} \mathrm{e}^{-i2\pi kt}\, V(t)_{m,n} \mathrm{d}t \right| (\max\{1, |k|\})^r < q(\alpha, r)$$

implies

$\exists \Omega_{res} \subset [\omega_1, \omega_2] \subset \mathbb{R}$ *so that*

$$\bullet \int_{\Omega_{res}} 1 \mathrm{d}t \le u(\alpha, r)(\omega_2 - \omega_1) \sup_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \left| \int_0^{2\pi} \mathrm{e}^{-i2\pi kt}\, V(t)_{m,n} \mathrm{d}t \right| (\max\{1, |k|\})^r,$$

$\bullet \forall \omega \in [\omega_1, \omega_2] \setminus \Omega_{res}, \quad K_\omega = -i\partial_t + H_0 + V(\omega t)$ *has pure point spectrum.*

Where $V(t)_{m,n}$ are matrix entries in eigenvector basis of H_0.

Proof. Only the main idea of the proof will be given in this paper. The proof is based on iterative conjugation of Floquet Hamiltonian by unitary operators such that in the limit complete diagonalization is achieved. This then implies that perturbed operator has also pure point spectrum under assumptions of the theorem. □

Chapter 7

Example

7.1 Notation

Vectors of standard orthonormal basis in the finite dimensional Hilbert space $\mathbb{C}^n$ will be denoted $\mathcal{E}_1 := \{e_j\}_{j=1}^n \subset \mathbb{C}^n$ or $\{|e_j\rangle\}_{j=1}^n \subset \mathbb{C}^n$.

Definition 16. *Let $\mathcal{H}$ be separable Hilbert space, $\mathcal{E} = \{f_k\}_{k\in I}$ is its orthonormal basis, where $I = \mathbb{Z}$ resp. $I = \mathbb{N} \cup \{0\}$. Assume that $a, b \in \mathbb{C}$ and $\forall k \in I, a_k \in \mathbb{C}$. Then following notation will be used:*

$$\mathrm{jac}_{\mathcal{E}}(a,b) := \sum_{j,j+1\in I} a|f_j\rangle\langle f_{j+1}| + b|f_{j+1}\rangle\langle f_j|,$$

$$\mathrm{diag}_{\mathcal{E}}(\{a_k\}_{k\in I}) := \sum_{j\in I} a_j|f_j\rangle\langle f_j|.$$

Note that $\mathrm{jac}_{\mathcal{E}}(a,b) \in \mathfrak{B}(\mathcal{H})$.

7.2 The problem

Let us define for some $\omega, \Omega > 0$ periodically time-dependent Hamiltonian $H(t)$ with period $T = 2\pi/\Omega$ that acts in $\mathcal{H}$,

$$H(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & 0 & \dots & 0 \\ 0 & 0 & 2\omega & 0 & \dots & 0 \\ 0 & 0 & 0 & 3\omega & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & (n-1)\omega \end{pmatrix} + \epsilon\sin(\Omega t)\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ \vdots & & & & & \ddots & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

That is

$$H(t) = H_0 + \epsilon H_1(t),$$
where $H_0 = \omega\mathrm{diag}(0, 1, \dots, n-1)$, $H_1(t) = \sin(\Omega t)\mathrm{jac}_{\mathcal{E}_1}(1,1)$.

Main goal is to study the unitary evolution propagator fulfilling the Schrödinger equation. It can be easily seen that this problem is equivalent to the problem of finding Floquet operator generated by the following Hamiltonian.

$$\widetilde{H}(t) := \mathrm{e}^{-itH_0}\, H_1(t)\, \mathrm{e}^{itH_0} = \epsilon \sin(\Omega t) \mathrm{jac}_{\mathcal{E}_1}(\mathrm{e}^{-i\omega t}, \mathrm{e}^{i\omega t}). \tag{7.1}$$

Note that $\widetilde{H}(t)$ need not to be periodical for general choice of ω, Ω. The first order time-averaging approximation of this Hamiltonian, taking into account

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau \mathrm{e}^{ixt}\,\mathrm{d}t = \begin{cases} 1 & \text{for } \forall k \in \mathbb{Z} \quad x = 2\pi k \\ 0 & \text{elsewhere.} \end{cases},$$

can be easily calculated:

$$\widetilde{H_{(1)}} := \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau \widetilde{H}(t)\mathrm{d}t = \begin{cases} \frac{i\epsilon}{2}\mathrm{jac}_{\mathcal{E}_1}(-1,1) & \text{for } \omega = \Omega \\ 0 & \text{elsewhere.} \end{cases}. \tag{7.2}$$

Alternative proof will be given in following section using well known fact that time-averaging in case of periodic Hamiltonians is in some sense equivalent to the classical first-order perturbation of Floquet Hamiltonian. One can define approximated Hamiltonian of studied problem:

$$H_{(1)}(t) = H_0 + \mathrm{e}^{itH_0}\, \widetilde{H_{(1)}}\, \mathrm{e}^{-itH_0} = \begin{cases} H_0 + \frac{i\epsilon}{2}\mathrm{jac}_{\mathcal{E}_1}(-\,\mathrm{e}^{i\omega t}, \mathrm{e}^{-i\omega t}) & \omega = \Omega, \\ H_0 & \omega \neq \Omega, \end{cases} \tag{7.3}$$

whose unitary propagator is simple to derive as can be seen from the section 7.5.

7.3 Floquet Hamiltonian and its first order approximation

Since corollary after (5) obviously holds true for the problem defined above one can directly consider Floquet Hamiltonian in the following form.

$$K := -\,i\partial_t + 1 \otimes H_0 + \varepsilon H_1(t) = K_0 + \epsilon K_1 = \tag{7.4}$$

$$= \quad -i\partial_t + 1 \otimes \omega \mathrm{diag}_{\mathcal{E}_1}(\{0,1,\ldots,n-1) + \frac{i\varepsilon}{2}\mathrm{jac}_{\mathcal{E}_2}(-1,1) \otimes \mathrm{jac}_{\mathcal{E}_1}(1,1) \tag{7.5}$$

where $\sin(\Omega t)$ from $H_1(t)$ acts as multiplication by function on $\mathcal{K} = L^2([0,T],\mathbb{C}^n) = L^2([0,T],\mathrm{d}t) \otimes \mathbb{C}^n$, $K_0 = \partial_t + 1 \otimes H_0$ and $K_1 = \epsilon H_1(t)$. Let us define orthonormal basis on $L^2([0,T],\mathrm{d}t)$ as $\mathcal{E}_2 := \{f_k\}_{k\in\mathbb{Z}}$, where $f_k(t) := \frac{\mathrm{e}^{itk\Omega}}{\sqrt{T}}$ and unitary operator on $\mathcal{K}$:

$$Qf_k \otimes e_j := \mathrm{e}^{-itj\omega}\, f_k \otimes e_j = \begin{cases} f_{k-m'} \otimes e_j & \text{for } j\omega = m'\Omega, \\ \sum_{m\in\mathbb{Z}} \frac{\mathrm{e}^{-ijT\omega}-1}{iT((k-m)\Omega - j\omega)} f_m \otimes e_j & \text{elsewhere: } \forall m' \in \mathbb{Z}, j\omega \neq m'\Omega. \end{cases}.$$

Using previous definitions one can rewrite Floquet Hamiltonian into the following form.

If $\exists k \in \mathbb{N}, \omega = k\Omega$ then

$$K = \quad Q^* - i\partial_t Q + \frac{i\varepsilon}{2}\mathrm{jac}_{\mathcal{E}_2}(-1,1) \otimes \mathrm{jac}_{\mathcal{E}_1}(1,1) = \tag{7.6}$$

$$= \quad Q^*\left(-i\partial_t + \frac{i\varepsilon}{2}\,\mathrm{e}^{-i\omega t}\,\mathrm{jac}_{\mathcal{E}_2}(-1,1) \otimes \mathrm{jac}_{\mathcal{E}_1}(1,0) + \frac{i\varepsilon}{2}\,\mathrm{e}^{i\omega t}\,\mathrm{jac}_{\mathcal{E}_2}(-1,1) \otimes \mathrm{jac}_{\mathcal{E}_1}(0,1)\right)Q \tag{7.7}$$

Since K_0 has only isolated eigenvalues in the spectrum, the first order perturbation $K_{(1)}$ can be derived using well-known equation from [19].

$$K_{(1)} = K_0 + \varepsilon \sum_{\lambda\in\sigma(K_0)} P_\lambda K_1 P_\lambda,$$

where P_λ is orthogonal projector on eigensubspace corresponding to $\lambda \in \sigma(K_0)$. Since K_1 acts on $\mathcal{H}$ as $\mathrm{jac}_{\mathcal{E}_2} \otimes \mathrm{jac}_{\mathcal{E}_1}$ the perturbation is non-trivial only if $\exists k, j \in \mathbb{Z}, \exists \lambda \in \sigma(K_0)$ so that $f_k \otimes e_j, f_{k+1} \otimes e_{j+1} \in \mathrm{Ran} P_\lambda$ or $f_{k+1} \otimes e_j, f_k \otimes e_{j+1} \in \mathrm{Ran} P_\lambda$. Using definition of K_0 one can see that $f_k \otimes e_j \in \mathrm{Ran} P_\lambda \Rightarrow \lambda = k\Omega + j\omega$. Since $\omega, \Omega > 0$ previous gives that perturbation is non-trivial only if $(k+1)\Omega + j\omega = k\Omega + (j+1)\omega$, which is equivalent to condition $\omega = \Omega$. For $\omega = \Omega$ the projectors on eigensubspaces get simple form:

$$\omega = \Omega \quad \Rightarrow \quad K_0 = \sum_{k \in \mathbb{Z}} k\omega P_k, \quad P_k = \sum_{j=1}^{n} |f_{k-j+1}\rangle\langle f_{k-j+1}| \otimes |e_j\rangle\langle e_j|.$$

$$\omega = \Omega \quad \Rightarrow \quad P_k K_1 P_k = Q\Big(|f_k\rangle\langle f_k| \otimes \frac{i}{2}\mathrm{jac}_{\mathcal{E}_1}(-1,1)\Big)Q^*,$$

Thus first order perturbation $K_{(1)}$ can be written in following form:

$$K_{(1)} = \begin{cases} Q\Big(-i\partial_t + 1 \otimes \frac{i\varepsilon}{2}\mathrm{jac}_{\mathcal{E}_1}(-1,1)\Big)Q^* & \text{for } \omega = \Omega, \\ K_0 & \text{elsewhere.} \end{cases} \tag{7.8}$$

Thus for approximation of $H(t)$ one has indeed the same result as when using time-averaging method (7.3). It is also interesting that

$$\mathrm{e}^{-iTK_{(1)}} = \begin{cases} Q\Big(1 \otimes \mathrm{e}^{\frac{T\varepsilon}{2}\mathrm{jac}_{\mathcal{E}_1}(-1,1)}\Big)Q^* & \text{for } \omega = \Omega, \\ 1 \otimes 1 & \text{elsewhere.} \end{cases} \tag{7.9}$$

7.4 Discrete symmetry of the problem

Definition 17.

$$R := \sum_{j=1}^{n} |e_{n+1-j}\rangle\langle e_j|$$

Note that $R = R^{-1} = R^*$.

Known symmetries of the studied Hamiltonians 7.2 follows:

$$\begin{aligned} H(t) =& R\overline{H(t)}R = (n-1)\omega - RH(-t))R & (7.10)\\ \widetilde{H}(t) =& R\overline{\widetilde{H}(t)}R = -R\widetilde{H}(-t))R & (7.11)\\ \widetilde{H}_{(1)} =& R\overline{\widetilde{H}_{(1)}}R = -R\widetilde{H}_{(1)}R, & (7.12)\\ \widetilde{H}_{(1)} =& \mathrm{e}^{-i\frac{T}{2}H_0}\, R\widetilde{H}_{(1)}R\, \mathrm{e}^{i\frac{T}{2}H_0}\,. & (7.13) \end{aligned}$$

The first symmetry from (7.10) can be applied directly to Schrödinger equation for the propagator $U(t,0)$ generated by $H(t)$ with initial conditions $\forall s \in \mathbb{R}, \quad U(s,s) = 1$.

$$\begin{aligned} i\frac{\mathrm{d}}{\mathrm{d}t}U(t,0) &= H(t)U(t,0) \\ i\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{it(n-1)\omega}\, U(t,0)) &= -RH(-t))R(\mathrm{e}^{it(n-1)\omega}\, U(t,0)), \\ i\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{it(n-1)\omega}\, RU(t,0)R) &= -H(-t)\,\mathrm{e}^{it(n-1)\omega}\, RU(t,0)R, \\ i\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-it(n-1)\omega}\, RU(-t,0)R) &= H(t)\,\mathrm{e}^{-it(n-1)\omega}\, RU(-t,0)R. \end{aligned}$$

Using well-known property of monodromy operator $\forall n \in \mathbb{Z}, \quad U(t+nT, s+nT) = U(t,s)$ one can derive also symmetries of Floquet operator.

$$\boxed{\forall t \in \mathbb{R}, \quad U(t,0) = \mathrm{e}^{-it(n-1)\omega}\, RU(-t,0)R}, \tag{7.14}$$

$$\boxed{U(T,0) = \mathrm{e}^{iT(n-1)\omega}\, RU(T,0)^{-1}R}. \tag{7.15}$$

$$\tag{7.16}$$

This can be applied to obtain following

$$\boxed{\forall t \in \mathbb{R}, \quad U(0,t)H_0U(t,0) = (n-1)\omega - RU(-t,0)H_0U(-t,0)R}$$

One can derive perturbed unitary propagator generated by $H_{(1)}(t)$ defined in (7.3).

$$U_{(1)}(t,s) = \mathrm{e}^{-itH_0}\, \mathrm{e}^{-i(t-s)\widetilde{H_{(1)}}}\, \mathrm{e}^{isH_0}, \tag{7.17}$$

One can observe that for $\omega \neq \Omega$ is propagator trivial $U(t,s) = \mathrm{e}^{-i(t-s)H_0}$, $U(T,0) = 1$.

Using symmetries listed in (7.10) for non-trivial case $\Omega = \omega$ one can see that

$$\begin{aligned} RU_{(1)}(t,0)R &= R\,\mathrm{e}^{-itH_0}\, RR\,\mathrm{e}^{-it\widetilde{H_{(1)}}}\, R = \mathrm{e}^{-it(n-1)\omega}\, \mathrm{e}^{+itH_0}\, \mathrm{e}^{it\widetilde{H_{(1)}}} = \mathrm{e}^{-it(n-1)\omega}\, U_{(1)}(-t,0) =, \\ &= \mathrm{e}^{-it(n-1)\omega}\, \mathrm{e}^{i2tH_0}\, \mathrm{e}^{i\frac{T}{2}H_0}\, U_{(1)}(t,0)\, \mathrm{e}^{-i\frac{T}{2}H_0} \end{aligned}$$

Thus if $n = 2m-1$ and $\Omega = \omega$ then

$$\boxed{\forall t \in \mathbb{R}, \langle U_{(1)}(t,0)e_m | H_0\, U_{(1)}(t,0)e_m\rangle = \frac{n-1}{2}\omega}.$$

Thus in analogy with (7.14) for non-trivial case $\Omega = \omega$ one gets

$$\boxed{\forall t \in \mathbb{R}, \quad U_{(1)}(t,0) = \mathrm{e}^{-it(n-1)\omega}\, \mathrm{e}^{-i2tH_0}\, \mathrm{e}^{-i\frac{T}{2}H_0}\, RU_{(1)}(t,0)R\,\mathrm{e}^{i\frac{T}{2}H_0}}, \tag{7.18}$$

$$\boxed{\forall t \in \mathbb{R}, \quad U_{(1)}(t,0) = \mathrm{e}^{-it(n-1)\omega}\, RU(-t,0)R}, \tag{7.19}$$

$$\boxed{U_{(1)}(T,0) = \mathrm{e}^{-i\frac{T}{2}H_0}\, RU_{(1)}(T,0)R\,\mathrm{e}^{i\frac{T}{2}H_0}}, \tag{7.20}$$

$$\boxed{U_{(1)}(T,0) = RU_{(1)}^{-1}(T,0)R} \tag{7.21}$$

These symmetries has interesting implications on the spectrum and eigenvectors of $U_{(1)}(T,0)$ particularly since it has for non-trivial case $\Omega = \omega$ non-degenerate spectrum.

Lemma 8 (symmetry)**.** *Let $A = \sum_{\mathrm{e}^{i\lambda} \in \sigma(A)} \mathrm{e}^{i\lambda}\, P^{(\lambda)}$ be a unitary operator, $\{P_{(\lambda)}\}_{\mathrm{e}^{i\lambda}\in\sigma(A)}$ a system of orthogonal projectors that defines operator. Assume that R is matrix defined in definition 17.*

Then

$$A = RA^{-1}R \quad \Leftrightarrow \quad \left(\forall\, \mathrm{e}^{i\lambda} \in \sigma(A), \quad (\mathrm{e}^{-i\lambda} \in \sigma(A) \wedge P^{(-\lambda)} = RP^{(\lambda)}R\right)$$

Proof.

$$A^{-1} = \sum_{\mathrm{e}^{i\lambda}\in\sigma(A)} \mathrm{e}^{-i\lambda}\, P^{(\lambda)} = \sum_{\mathrm{e}^{i\lambda}\in\sigma(A)} \mathrm{e}^{i\lambda}\, RP^{(-\lambda)}R = RAR$$

Since $\{RP^{(\lambda)}R\}_{\lambda\in\sigma(A)}$ is also a system of orthogonal projectors that defines operator. □

Lemma 9.

$$\forall\varepsilon : 0 < \varepsilon < \pi, u - \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau U_{(1)}(0,t)H_0U_{(1)}(t,0)\mathrm{d}t = \begin{cases} \frac{(n-1)\omega}{2} & \omega = \Omega \\ H_0 & \omega \neq \Omega \end{cases}$$

Proof. The proof is based on the first symmetry in 7.18 and the fact that for $\forall 0 < \varepsilon < \pi$ $U_{(1)}$ has non-degenerate spectrum. That is a result of the section 7.5 also equation 7.23 and 7.24 will be used in following. One can define

$$\begin{aligned} B_\tau &:= \frac{1}{\tau}\int_0^\tau U_{(1)}(0,t)H_0U_{(1)}(t,0)\mathrm{d}t, \\ B_\infty &:= \sum_{l=1}^n |v^{(l)}\rangle\langle v^{(l)}|B_T|v^{(l)}\rangle\langle v^{(l)}| = \sum_{l=1}^n |v^{(l)}\rangle\langle v^{(l)}| \quad \langle v^{(l)}|B_T v^{(l)}\rangle. \end{aligned}$$

The T-mean theorem 14 states that

$$\lim_{\tau\to\infty}\|B_\tau - B_\infty\| = 0$$

since the dimension is finite.

Now due to the first symmetry in 7.18 and lemma 8 one has following symmetries

$$\begin{aligned} |v^{(l)}\rangle\langle v^{(l)}| &= \mathrm{e}^{-i\frac{T}{2}H_0} R|v^{(l)}\rangle\langle v^{(l)}|R\,\mathrm{e}^{i\frac{T}{2}H_0} \\ \mathrm{e}^{-i\frac{T}{2}H_0} RB_TR\,\mathrm{e}^{i\frac{T}{2}H_0} &= (n-1)\omega - B_T. \end{aligned}$$

Thus

$$\langle v^{(l)}|B_Tv^{(l)}\rangle = \langle v^{(l)}|R\,\mathrm{e}^{i\frac{T}{2}H_0}\,\mathrm{e}^{-i\frac{T}{2}H_0}\,RB_TR\,\mathrm{e}^{-i\frac{T}{2}H_0}\,\mathrm{e}^{i\frac{T}{2}H_0}\,Rv^{(l)}\rangle = (n-1)\omega - \langle v^{(l)}|B_Tv^{(l)}\rangle.$$

□

Lemma 10. *Let*

$$\begin{aligned} \mu_{(1)}[\infty](\omega j) : &= u - \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau U_{(1)}(0,t)P^{H_0}(\omega j)U_{(1)}(t,0)\mathrm{d}t = \\ &= \begin{cases} \sum_{l=1}^n |v^{(l)}\rangle\langle v^{(l)}|P^{H_0}(\omega j)|v^{(l)}\rangle\langle v^{(l)}| & \omega = \Omega, \\ P^{H_0}(\omega j) & \omega \neq \Omega \end{cases} \end{aligned}$$

Then

$$\left\|\mu_{(1)}[\infty](\omega j)\right\|_2 = \begin{cases} \sqrt{\frac{3}{2(n+1)}} & \omega = \Omega, \\ 1 & \omega \neq \Omega, \end{cases}$$

where $\|.\|_2$ *is Hilbert-Schmidt operator norm.*

Proof.

$$\begin{aligned} (\left\|\mu_{(1)}[\infty](\omega j)\right\|_2)^2 &= \mathrm{Tr}\left(\mu_{(1)}[\infty](\omega j)\right)^* \mu_{(1)}[\infty](\omega j) = \\ &= \begin{cases} \sum_{l=1}^n \mathrm{Tr}\,|\langle v^{(l)}|P^{H_0}(\omega j)v^{(l)}\rangle|^2 & \omega = \Omega, \\ \mathrm{Tr}\,P^{H_0}(\omega j) & \omega \neq \Omega \end{cases} \end{aligned}$$

Rest of the proof is straightforward using definition of $v^{(l)}$ from (7.23) and well-known trigonometric equations. □

7.5 Approximated propagator

In this section following notation will be used:

$$a_n(\lambda) := \det(\frac{2}{\epsilon}\widetilde{H_{(1)}} + \lambda) \text{ in Hilbert space } \dim \mathcal{H} = \dim \mathbb{C}^n = n.$$

It is easily shown using Laplace formula for determinants twice and one can see that

$$a_n(\lambda) = \lambda a_{n-1}(\lambda) - a_{n-2}(\lambda). \tag{7.22}$$

Note that $a_0 = 1$ is consistent with the recurrence relation. The first few determinants can be computed directly from definition:

$$a_1 = \lambda \quad a_2 = \lambda^2 - 1, \quad a_3 = \lambda(\lambda^2 - 2), \quad a_4 = (\lambda^2 - \frac{3+\sqrt{5}}{2})(\lambda^2 - \frac{3-\sqrt{5}}{2}), \quad a_5 = \lambda(\lambda^2-1)(\lambda^2-3).$$

With proper initial conditions difference equation (7.22) has following solution for $\lambda \neq \pm 2$.

$$\begin{aligned} a_n &= \frac{(\lambda + \sqrt{\lambda^2-4})^{n+1} - (\lambda - \sqrt{\lambda^2-4})^{n+1}}{2^{n+1}\sqrt{\lambda^2-4}} = \\ &= \frac{1}{2^{n+1}\sqrt{\lambda^2-4}} \sum_{k=0}^{n+1} \binom{n+1}{k} \lambda^{n+1-k} \sqrt{\lambda^2-4}^{\,k} (1 - (-1)^k) \\ &= \frac{1}{2^n} \sum_{k=0}^{[\frac{n}{2}]} \binom{n+1}{2k+1} \lambda^{n-2k} (\lambda^2-4)^k = \left(\frac{\lambda}{2}\right)^n \sum_{k=0}^{[\frac{n}{2}]} \binom{n+1}{2k+1} \lambda^{-2k} (\lambda^2-4)^k \end{aligned}$$

It is now easy to see that roots of polynomial a_n are located in open interval $(-2, 2)$. Thus for finding all roots we can use substitution $\lambda = 2\cos\varphi$.

$$\begin{aligned} a_n(2\cos\varphi) &= \frac{(2\cos\varphi + \sqrt{4\cos^2\varphi - 4})^{n+1} - (2\cos\varphi - \sqrt{4\cos^2\varphi - 4})^{n+1}}{2^{n+1}\sqrt{\lambda^2-4}} = \\ &= \frac{\sin((n+1)\varphi)}{\sin\varphi} \end{aligned}$$

Thus eigenvalue system is $\sigma(\widetilde{H_{(1)}}) = \{-\epsilon \cos\frac{\pi l}{n+1}\}_{l=1}^{n} \subset (-\epsilon, \epsilon)$. Thus spectrum of $\widetilde{H_{(1)}}$ is non-degenerate. Corresponding eigenvector problem can be solved easily solving recurrence relation:

$$v_0^{(l)} = 0, v_1^{(l)} = 1, \quad \forall j \in \{1, 2, \dots n-2\}, \quad -iv_j^{(l)} - 2\cos\frac{\pi l}{n+1} v_{j+1}^{(l)} + iv_{j+2}^{(l)} = 0.$$

Solution after normalization to 1 i.e. $\sum_{j=1}^{n} |v_j^{(k)}|^2 = 1$ is

$$\boxed{\langle e_j | v^{(l)} \rangle = v_j^{(l)} = (-i)^j \sin(\frac{\pi l j}{n+1}) \sqrt{\frac{2}{n+1}}}. \tag{7.23}$$

Thus one gets:

$$\boxed{\widetilde{H_{(1)}} = \sum_{l=1}^{n} -\varepsilon \cos\frac{\pi l}{n+1} \quad |v^{(l)}\rangle\langle v^{(l)}|} \tag{7.24}$$

Explicit formula for propagator $U_{(1)}(t) = \mathrm{e}^{-itH_0}\,\mathrm{e}^{-it\widetilde{H_{(1)}}}$ can now be written down

$$\boxed{\langle e_j | U_{(1)}(t) e_k\rangle = \mathrm{e}^{-itH_0}\,\mathrm{e}^{-it\widetilde{H_{(1)}}} = \sum_{l=1}^{n} \mathrm{e}^{-it[(j-1)\omega - \varepsilon\cos\frac{\pi l}{n+1}]}(-i)^{j-k}\sin(\frac{\pi lj}{n+1})\sin(\frac{\pi lk}{n+1})\frac{2}{n+1}} \tag{7.25}$$

One can use mentioned symmetries in case of the first-order approximated time evolution to obtain interesting results as given in corollary (9).

7.6 The simplest case

Let $\dim\mathcal{H} = \dim\mathbb{C}^2 = 2$ and

$$U(T,0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is easy to show using (7.25) that for $\omega = \Omega$ one has

$$U_{(1)}(t,0) = \begin{pmatrix} \cos(\frac{\epsilon t}{2}) & -\sin(\frac{\epsilon t}{2}) \\ \mathrm{e}^{-i\omega t}\sin(\frac{\epsilon t}{2}) & \mathrm{e}^{-i\omega t}\cos(\frac{\epsilon t}{2}) \end{pmatrix}.$$

From unitarity and symmetry (7.15) one can prove strong restriction in such low dimension on $U(T,0)$. From $U^{-1}(T,0) = U^*(T,0) = \mathrm{e}^{i(2-1)\omega T}\,RU(T,0)R$ one obtains

$$\frac{1}{\det U(T,0)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} = \mathrm{e}^{i\omega T}\begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

Using well known lemma [3] from theory of linear differential equations that implies that $i\frac{d}{dt}\mathrm{det}U(t,0) = \mathrm{Tr}(H)\mathrm{det}U(t,0)$ one gets $\mathrm{det}U(t,0) = \mathrm{e}^{-i\omega t}$. Thus $\det U(T,0) = \mathrm{e}^{-i\omega T}$, $c = \mathrm{e}^{-i\omega T}\,\overline{c} = -b$ and $\mathrm{e}^{-i\omega T}\,\overline{a} = d$. It is straightforward that one can find $\phi, \eta \in [0, 2\pi)$ so that

$$U(T,0) = \mathrm{e}^{-i\frac{\omega T}{2}}\begin{pmatrix} \cos(\phi)\,\mathrm{e}^{i\eta} & -\sin(\phi) \\ \sin(\phi) & \cos(\phi)\,\mathrm{e}^{-i\eta} \end{pmatrix}.$$

Due to the reasons given in 7.7 the author expects that for $U(T,0) = U_{(1)}(T,0) + O(\varepsilon)$ and also

$$\phi(\epsilon) = \begin{cases} \frac{\epsilon}{2} + O(\varepsilon^2) & \text{for } \omega = \Omega, \\ 0 & \text{elsewhere.} \end{cases},$$
$$\eta(\epsilon) = 0.$$

7.7 Remarks and suggestions

This section is devoted to the possibilities for further research and proposals of theorems. Main idea is a proposal to study asymptotic behavior of $(H_0)_\infty$ as $n \to \infty$ expecting that at least for the finite dimensional cases it can be proven that $(H_0)_\infty$ is somehow close to it's first order approximation. It appears from 14 that the best would probably be to study at first asymptotic behavior of $\mu[\infty]$ and then try to prove some theorems regarding $(H_0)_\infty$.

Following definitions are in fact recapitulation of the definition of the problem done in 7.2 in the finite dimension and inductive generalization of the problem to the infinite case. Also notation defined in this section is used in numerical analysis in the section 7.8.

Definition 18.

Let $\mathcal{H} := \overline{\mathrm{span}\{e_j : j \in \mathbb{N}\}}$ *be a complex separable Hilbert space,*
$\mathcal{E} := \{e_j\}_{j\in\mathbb{N}}$ *be orthonormal basis of* $\mathcal{H}$*,*
$\mathcal{H}^{(n)} := \mathrm{span}\{e_j : (j \in \mathbb{N} : j \le n)\}$ *be a complex finite dimensional Hilbert space,*
$\mathcal{E}^{(n)} := \{e_j\}_{j\in\mathbb{N}:j\le n}$ *be orthonormal basis of* $\mathcal{H}^{(n)}$*,*

$$E_{\mathcal{H}^{(n)}} := \sum_{j\in\mathbb{N}:j\le n} |e_j\rangle\langle e_j|.$$

$$H := \mathrm{diag}_{\mathcal{E}}(0,\omega,2\omega,\dots) + \varepsilon\sin(\Omega t)\mathrm{jac}_{\mathcal{E}}(1,1)$$
$$H^{(n)} := E_{\mathcal{H}^{(n)}} H E_{\mathcal{H}^{(n)}} = \mathrm{diag}_{\mathcal{E}^{(n)}}(0,\omega,2\omega,\dots,(n-1)\,\omega) + \varepsilon\sin(\Omega t)\mathrm{jac}_{\mathcal{E}^{(n)}}(1,1)$$
$$\widetilde{H_{(1)}} := \begin{cases} \frac{i\epsilon}{2}\mathrm{jac}_{\mathcal{E}}(-1,1) & \textit{for } \omega = \Omega \\ 0 & \textit{elsewhere.} \end{cases}$$

$$H_{(1)}(t) := H_0 + \mathrm{e}^{itH_0}\,\widetilde{H_{(1)}}\,\mathrm{e}^{-itH_0} = \begin{cases} H_0 + \frac{i\epsilon}{2}\mathrm{jac}_{\mathcal{E}}(-\mathrm{e}^{i\omega t},\mathrm{e}^{-i\omega t}) & \omega = \Omega, \\ H_0 & \omega \ne \Omega, \end{cases}$$
$$\widetilde{H_{(1)}^{(n)}} := E_{\mathcal{H}^{(n)}}\widetilde{H_{(1)}}E_{\mathcal{H}^{(n)}} = \begin{cases} \frac{i\epsilon}{2}\mathrm{jac}_{\mathcal{E}^{(n)}}(-1,1) & \textit{for } \omega = \Omega, \\ 0 & \textit{elsewhere,} \end{cases}$$
$$H_{(1)}^{(n)}(t) := E_{\mathcal{H}^{(n)}} H_{(1)}(t) E_{\mathcal{H}^{(n)}} =$$
$$= H_0^{(n)} + \mathrm{e}^{itH_0^{(n)}}\,\widetilde{H_{(1)}^{(n)}}\,\mathrm{e}^{-itH_0^{(n)}} = \begin{cases} H_0^{(n)} + \frac{i\epsilon}{2}\mathrm{jac}_{\mathcal{E}^{(n)}}(-\mathrm{e}^{i\omega t},\mathrm{e}^{-i\omega t}) & \omega = \Omega, \\ H_0^{(n)} & \omega \ne \Omega. \end{cases}$$

$U(t,s)$ *be jointly continuous unitary propagator generated by* H*,*
$U_{(1)}(t,s) = \mathrm{e}^{-itH_0}\,\mathrm{e}^{-i\varepsilon(t-s)\widetilde{H_{(1)}}}\,\mathrm{e}^{isH_0}$ *be jointly continuous unitary propagator generated by* $H_{(1)}$*,*
$U^{(n)}(t,s)$ *be jointly continuous unitary propagator generated by* $H^{(n)}$*,*
$U_{(1)}^{(n)}(t,s) = \mathrm{e}^{-itH_0^{(n)}}\,\mathrm{e}^{-i\varepsilon(t-s)\widetilde{H_{(1)}^{(n)}}}\,\mathrm{e}^{isH_0^{(n)}}$ *be jointly continuous unitary propagator generated by* $H_{(1)}^{(n)}$*.*

$$\mu[\infty]_\lambda := P_j^{U(T,0)}\frac{1}{T}\int_0^T U(0,t)P_\lambda^{H_0}U(t,0)\mathrm{d}tP_j^{U(T,0)}$$
$$\mu_{(1)}[\infty]_\lambda := P_j^{U_{(1)}(T,0)}\frac{1}{T}\int_0^T U_{(1)}(0,t)P_\lambda^{H_0}U_{(1)}(t,0)\mathrm{d}tP_j^{U_{(1)}(T,0)},$$
$$\mu^{(n)}[\infty]_\lambda := P_j^{U^{(n)}(T,0)}\frac{1}{T}\int_0^T U^{(n)}(0,t)P_\lambda^{H_0^{(n)}}U(t,0)\mathrm{d}tP_j^{U^{(n)}(T,0)},$$
$$\mu_{(1)}^{(n)}[\infty]_\lambda := P_j^{U_{(1)}^{(n)}(T,0)}\frac{1}{T}\int_0^T U_{(1)}^{(n)}(0,t)P_\lambda^{H_0^{(n)}}U_{(1)}^{(n)}(t,0)\mathrm{d}tP_j^{U_{(1)}^{(n)}(T,0)}.$$

The operator $H_{(1)}^{(n)}(t)$ comes from the first order perturbation of the operator $-i\partial_t + H^{(n)}(t)$ as shown in the section 7.3. Particularly since $U(0,0) = 1$ one has from theorem 4 rather formally

$$\langle t = 0|\,\mathrm{e}^{-iT(-i\partial_t + H^{(n)}(t))}\,|t = 0\rangle = U(T,0) \text{ and } \langle t = 0|\,\mathrm{e}^{-iT(-i\partial_t + H_{(1)}^{(n)}(t))}\,|t = 0\rangle = U_{(1)}(T,0).$$

Thus it is expectable that, using some existing perturbation theory, one can prove convergence of the eigenprojectors of the operator $\mathrm{e}^{-iT(-i\partial_t + H^{(n)}(t))}$ to the corresponding eigenprojectors of the operator $\mathrm{e}^{-iT(-i\partial_t + H^{(n)}_{(1)}(t))}$ and that it also holds true for "diagonal element" for "generalized" vector $|t = 0\rangle$. Observing the results of numerical analysis from section 7.8 especially the contour plots in the figures 7.7,7.8,7.9, taking into account fact that H_1 gets "effectively smaller" with respect to the H_0 as $n \to \infty$ i.e. $\frac{\left\|H_1^{(n)}\right\|}{\left\|H_0^{(n)}\right\|} \xrightarrow{n\to\infty} 0$ thus ε gets effectively "smaller", the author came to the the proposal of the following theorem.

Proposal of theorem 1. *There is non-increasing real function $f(x)$ such that*

$$\left\|\mu^{(n)}[\infty]_\lambda - \mu^{(n)}_{(1)}[\infty]_\lambda\right\| \leq \varepsilon f(n).$$

Perturbation theory from paper [19] might not be impossible to apply the infinite dimensional version of the problem since all eigenvalues of K_0 has infinite degeneration and K_1 has continuous spectrum as can seen from the eigenvalues of $H_1^{(n)}$. However one can study the strong-resolvent operator limit of the sequence $H^{(n)}_{(1)}$, which is $H_{(1)}$. It can be also seen that assumptions of the lemma 1 holds true, thus one has also strong convergence of the $\mathrm{st-}\lim_{n\to\infty} U^{(n)}_{(1)}(T,0) = U_{(1)}(T,0)$. Observing asymptotic properties of eigenvectors of the sequence $U^{(n)}_{(1)}(T,0)$ given in (7.23) one sees that $U_{(1)}(T,0)$ has absolutely continuous spectrum i.e. $\sigma_{sc}(U_{(1)}(T,0)) = \emptyset$, $\sigma_{sc}(U_{(1)}(T,0)) = \emptyset$. Another hint comes from the lemma 10, which implies

$$\lim_{n\to\infty} \left\|\mu^{(n)}_{(1)}[\infty]_\lambda\right\|_2 = 0.$$

Thus if some analogue of previous proposal of theorem holds true, one might be able to prove also the following one.

Proposal of theorem 2.

$$\exists \varepsilon_0 > 0, \forall \varepsilon > 0 : \varepsilon < \varepsilon_0, \forall \lambda \in \mathbb{R}, \mu[\infty]_\lambda = \begin{cases} 0 & \omega = \Omega \\ P_\lambda^{H_0} & \omega \neq \Omega. \end{cases}$$

7.8 Numerical analysis

This section is devoted to numerical analysis of the studied problem, which was done in program Wolfram Mathematica 8. For convenience $\Omega = 1$ will be assumed. Notation used in the previous section and in the theorem 14 will be used in this section. Note that following notation will be used:

$$h(t) := \langle U(t)e_1 | H_0\, U(t) e_1\rangle,$$
$$h_{(1)}(t) := \langle U_{(1)}(t)e_1 | H_0\, U_{(1)}(t) e_1\rangle.$$

From figures 7.1,7.2, 7.3, 7.4, 7.5, 7.4 one observes that numerical evaluation of the function $h(t)$ obtained by solving Schrödiger equation of the problem is well approximated by function $h_{(1)}(t)$ i.e. even in large time scale $h(t)$ appears to only oscillate around $h_{(1)}(t)$ with amplitude dependent on the parameter ε. In fact in the figures two functions are so similar that one can effectively see only one of them. From further calculations one can see that amplitude of oscillation grows with ε.

It is reasonable also to compare mean energy over infinite time period with it's first order approximation. This can be due to 14 done evaluating mean value of $H^{(n)}_\infty$ and $(H^{(n)}_\infty)_{(1)}$ for some vector v. For convenience only $v = e_1$ is studied in this paper. Due to lemma 9 one sees that

$$\langle e_1 | (H^{(n)}_\infty)_{(1)} e_1\rangle = \begin{cases} \frac{(n-1)\omega}{2} & \omega = \Omega, \\ 0 & \omega \neq \Omega. \end{cases}$$

Thus it will be enough to visualize only the three parameter function $\langle e_1 | H^{(n)}_\infty e_1\rangle$ of parameters ω, ε, n. For this purpose contour plot has been depicted in the figures 7.7,7.8,7.9 for dimensions $n = 1, 5, 10$. One

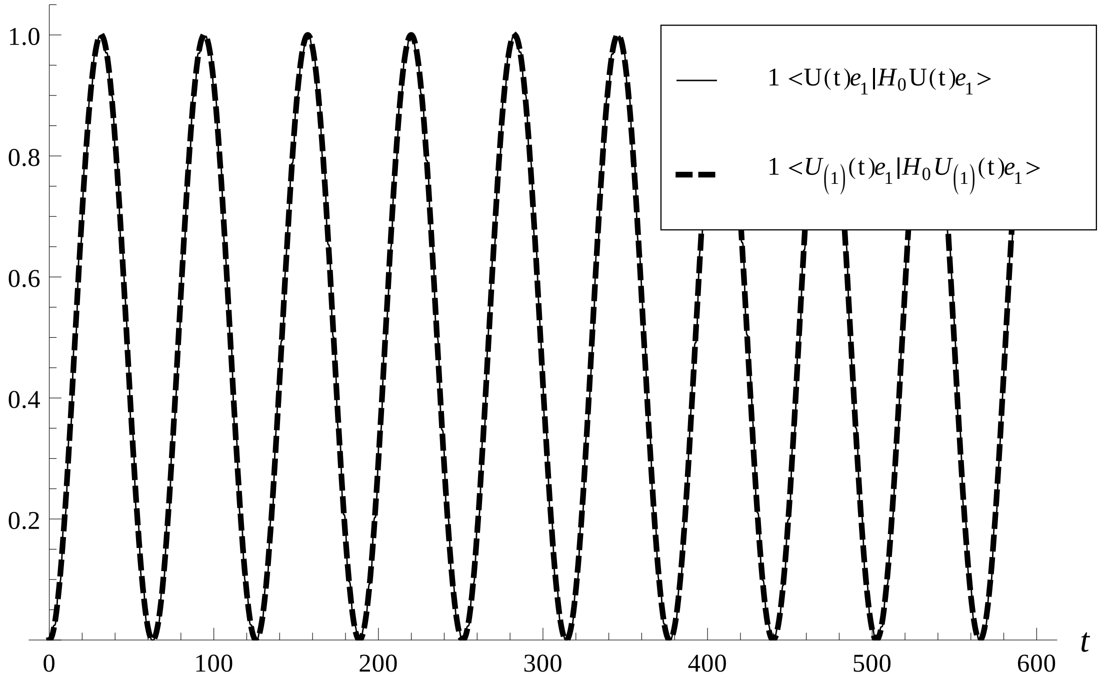

Figure 7.1: Numerical comparison of time evolution and first order approximation of time evolution of energy with initial state e_1 and parameters equal to $\varepsilon = 0.1, \omega = \Omega = 1, n = 2$.

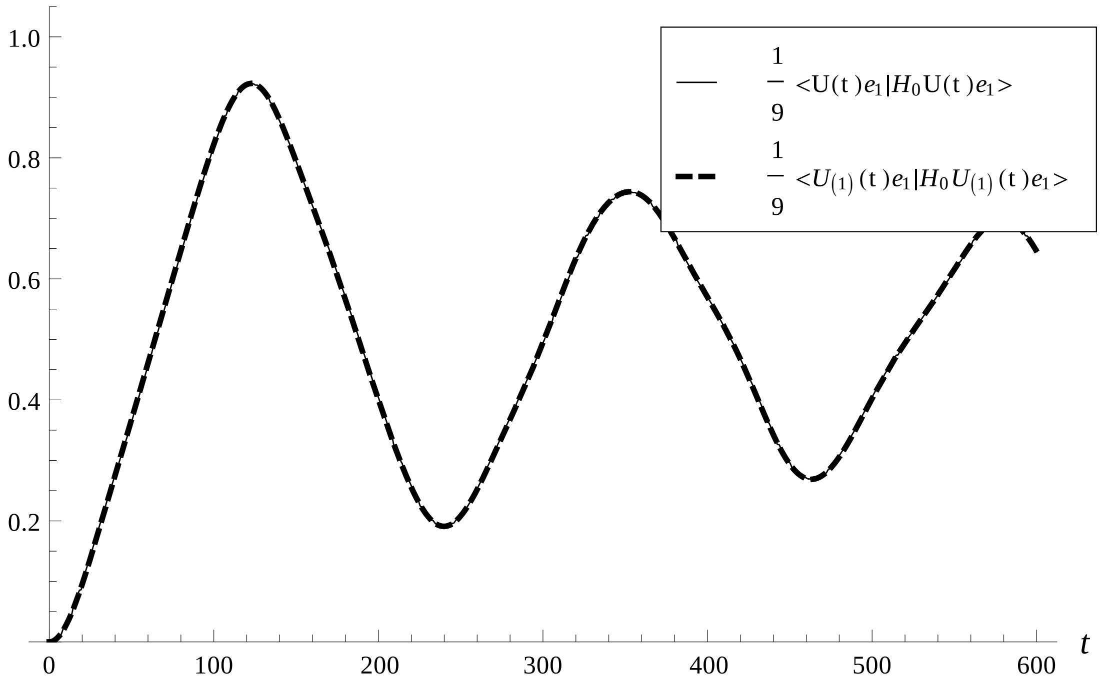

Figure 7.2: Numerical comparison of time evolution and first order approximation of time evolution of energy with initial state e_1 and parameters equal to $\varepsilon = 0.1, \omega = \Omega = 1, n = 10$.

sees that for $\varepsilon \to 0$ first order approximation well approximates function $\langle e_1 | H_\infty^{(n)} e_1 \rangle$ and unexpectedly this appears to hold also for limit $n \to \infty$. This may be because H_1 gets effectively smaller with respect to the H_0 as $n \to \infty$ i.e. $\frac{\left\| H_1^{(n)} \right\|}{\left\| H_0^{(n)} \right\|} \xrightarrow{n \to \infty} 0$ thus ε gets effectively "smaller". Note that the numerical

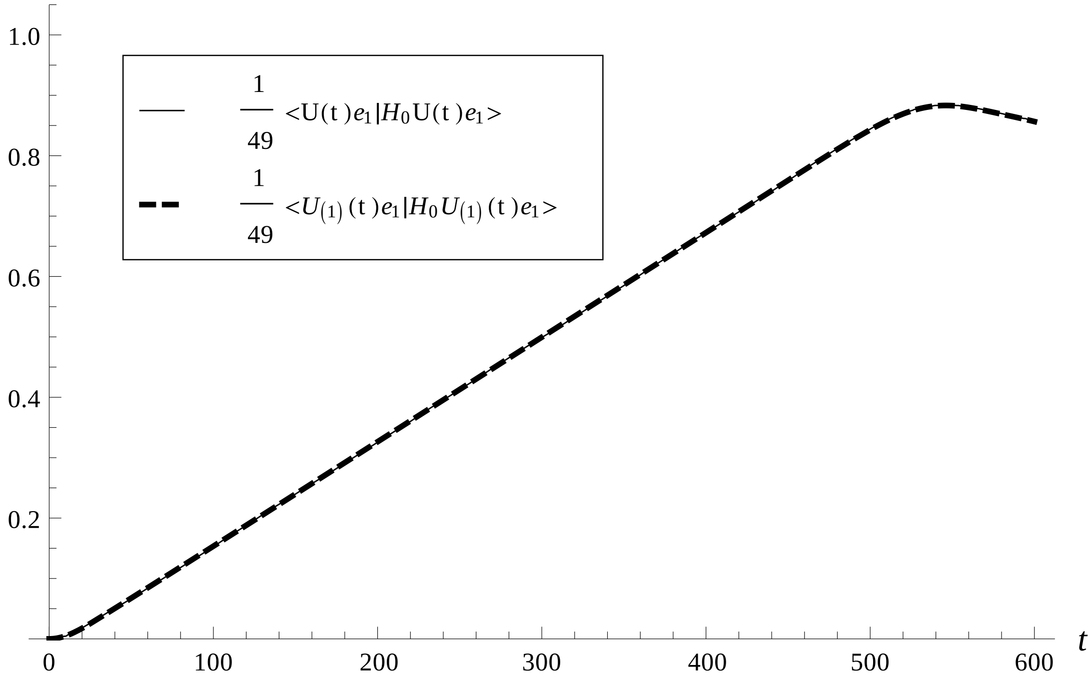

Figure 7.3: Numerical comparison of time evolution and first order approximation of time evolution of energy with initial state e_1 and parameters equal to $\varepsilon = 0.1, \omega = \Omega = 1, n = 50$.

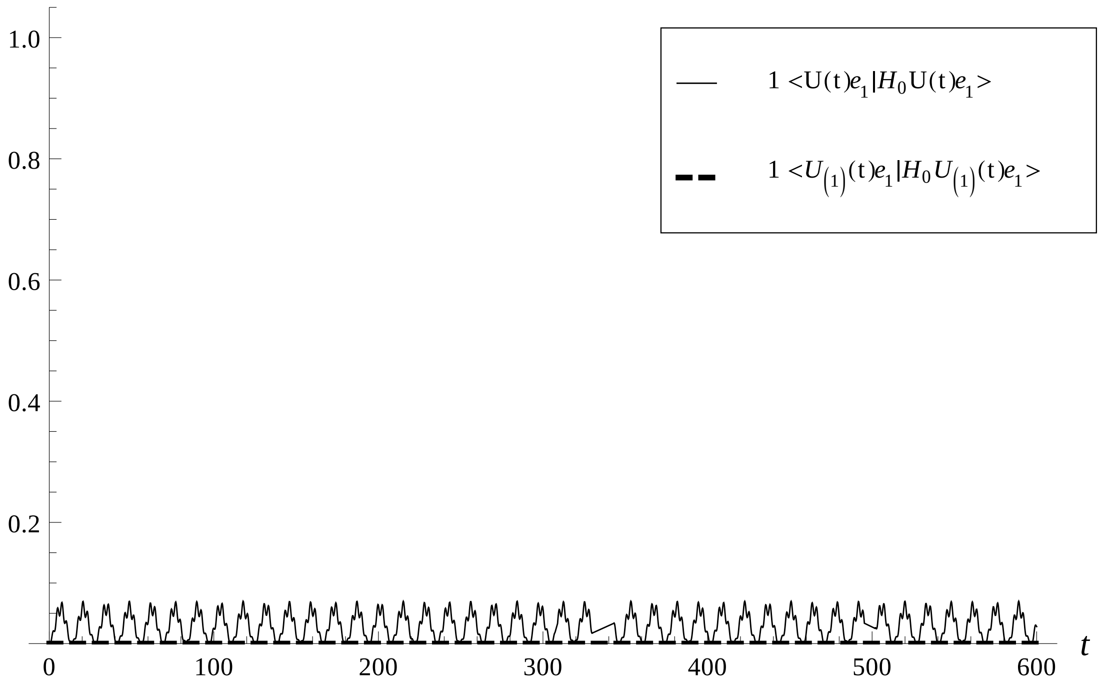

Figure 7.4: Numerical comparison of time evolution and first order approximation of time evolution of energy with initial state e_1 and parameters equal to $\varepsilon = 0.1, \omega = 1.44 \neq \Omega = 1, n = 2$.

computation is done by solving Schrödiger equation of the problem obtaining $\{U(t) : t \in [0, T]\}$. Then $U(T)$ is used to compute $P_j^{U(T)}$ and then integral in $(H_0)_1^{(n)}$ is approximated by a sum in order to speed up the computation in the following way:

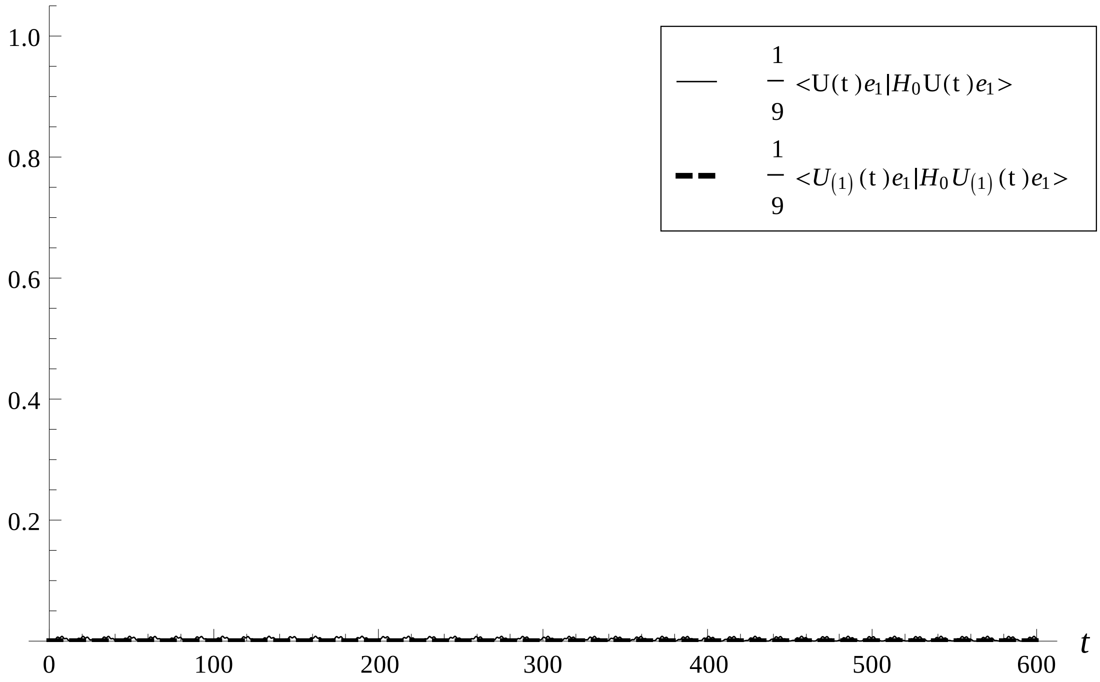

Figure 7.5: Numerical comparison of time evolution and first order approximation of time evolution of energy with initial state e_1 and parameters equal to $\varepsilon = 0.1, \omega = 1.44 \neq \Omega = 1, n = 10$.

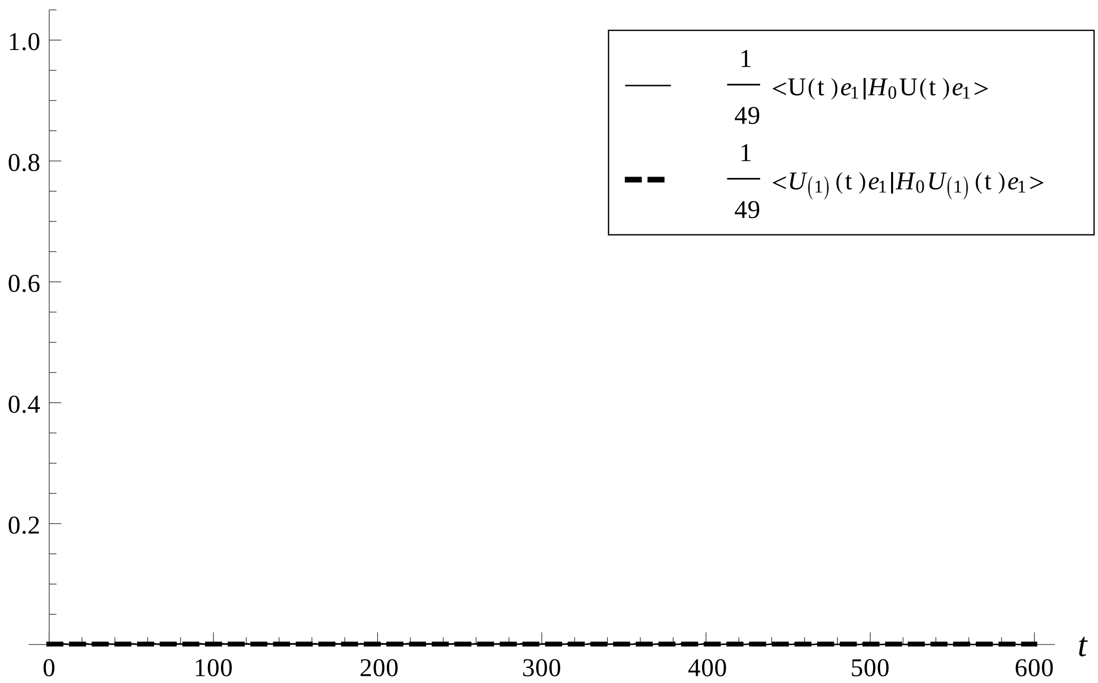

Figure 7.6: Numerical comparison of time evolution and first order approximation of time evolution of energy with initial state e_1 and parameters equal to $\varepsilon = 0.1, \omega = 1.44 \neq \Omega = 1, n = 50$.

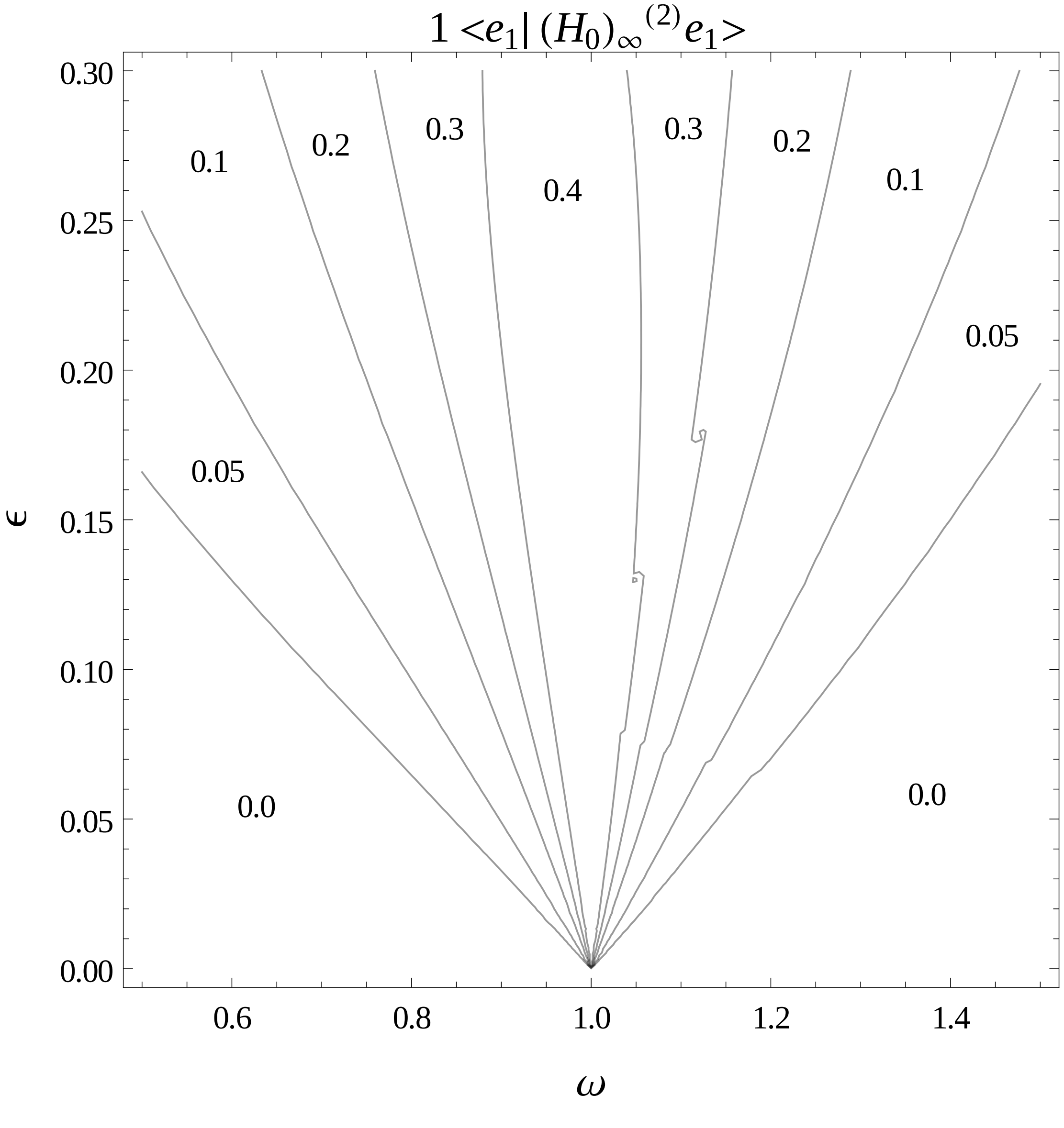

Figure 7.7: Numerical computation of mean energy over infinite time period with initial state e_1 in the dimension $n = 2$ with $\Omega = 1$. The numbers in the bordered areas represent the infimum of the function in that area.

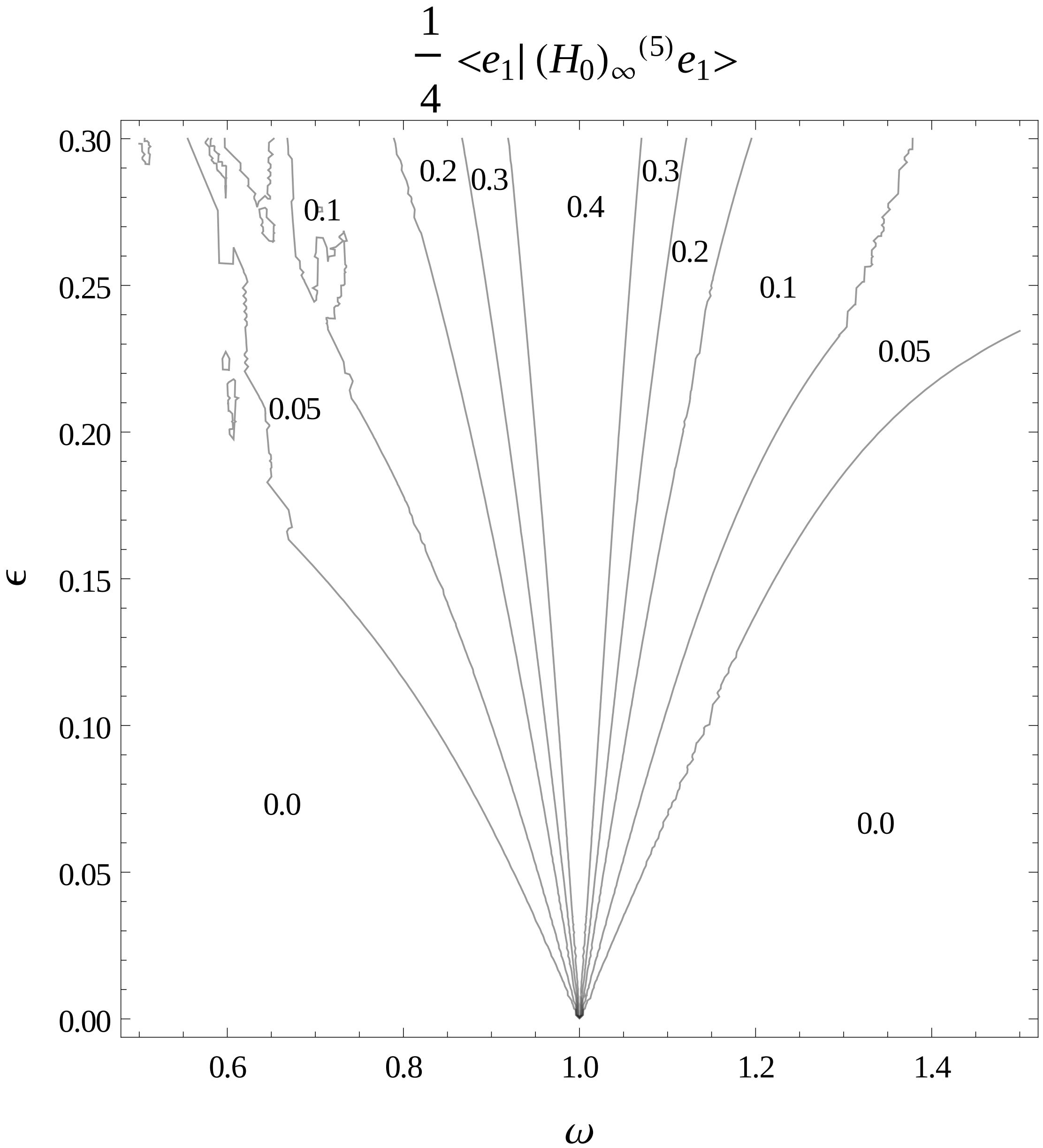

Figure 7.8: Numerical computation of mean energy over infinite time period with initial state e_1 in the dimension $n = 5$ with $\Omega = 1$. The numbers in the bordered areas represent the the infimum of the function in that area.

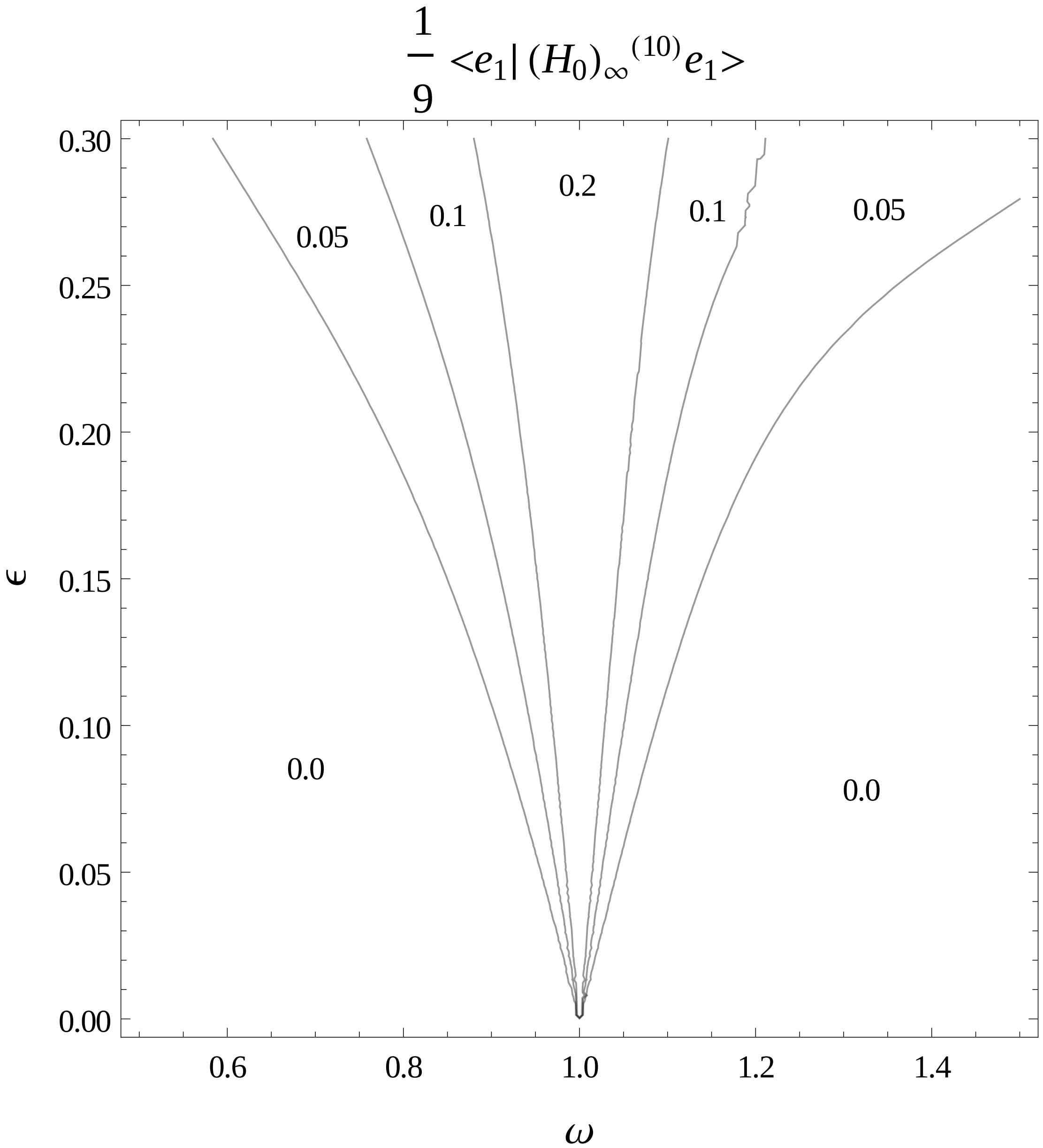

Figure 7.9: Numerical computation of mean energy over infinite time period with initial state e_1 in the dimension $n = 10$ with $\Omega = 1$. The numbers in the bordered areas represent the infimum of the function in that area.

Conclusion

The author studied existing theory of time-dependent Hamiltonians, particullary theoretical problems appearing in the cases, where gaps between points in spectra of the Hamiltonian are constant. The author studied analytically and numerically a simple case of the system and using gathered knowledge attempted to enrich existing theory proposing a new approach based on time-mean of Hamilton operator using theory of positive operator measures and integration with respect to positive operator measure. The author then proposed a possible way for continuation of the research regarding the studied simple case.

Bibliography

[1] M. Reed and B. Simon, Methods of modern mathematical physics, vol. I, Academic Press, (1972).

[2] M. Reed and B. Simon, Methods of modern mathematical physics, vol. II, Academic Press, (1975).

[3] E. A. Conddington, N. Levinson, Theory of ordinary differential equations, McGraw-Hill, (1955).

[4] P. Duclos E. Soccorsi P. Šťovicek, Rev. Math. Phys. **20** (2008) 725.

[5] J.S. Howland: Floquet operators with singular spectrum I, Ann. Inst. Henri Poincaré **49** (1989) 309-323.

[6] J.S. Howland: Floquet operators with singular spectrum II, Ann. Inst. Henri Poincaré **49** (1989) 325-334.

[7] J. S. Howland, Scattering theory for Hamiltonians periodic in time, Indiana J. Math. **28** (1979) 471-494.

[8] J. S. Howland, Floquet operators with singular spectrum, III, Ann. Inst. H. Poincar´e, **69** (1998) 265-273.

[9] J. S. Howland, Stationary Scatering Theory for Time-dependent Hamiltonians, Math. Ann. **207**, (1974) 317-335

[10] G. Nenciu: Floquet operators with absolutely continuous spectrum, Ann. Inst. Henri Poincaré **59** (1993) 91-97.

[11] A. Joye: Absence of absolutely continuous spectrum of Floquet operators, J. Stat. Phys.**75** (1994) 929-952.

[12] J. Bellissard: Stability and instability in quantum mechanics, in: Albeverio and Blanchard, eds., Trends and Developments in Eighties, World Scientific, Singapore, (1985), pp. 1-106.

[13] M. Combescure: The quantum stability problem for time-periodic perturbations of the harmonic oscillator, Ann. Inst. Henri Poincaré **47** (1987) 62-82; Erratum, Ann. Inst. Henri Poincaré **47** (1987) 451-454.

[14] P.M. Bleher, H.R. Jauslin, J.L. Lebowitz: Floquet spectrum for two-level systems in quasi-periodic time dependent fields, J. Stat. Phys. **68** (1992) 271.

[15] D. Bambusi, S. Graffi: Time quasi-periodic unbounded perturbations of the Schrödinger operators and KAM methods, Commun. Math. Phys. **219** (2001) 465-480.

[16] M. A. Naimark, Spectral functions of a symmetric operator, Izv. Akad. Nauk. SSSR Ser. Mat., **4** (1940), 277-318.

[17] M. M. Malamud, S. M. Malamud, Spectral theory of operator measures in Hilbert space, St. Petersburg Math. J. Vol. 15 (2004), No. **3**, Pages 323–373.

[18] V. Enss and K. Veseliȉ, Bound states and propagating states for time-dependent Hamiltonians, Ann. Inst. H. Poincaré **39** (1983), 159–191.

[19] T. Kato, Perturbation Theory for Linear Operators, Springer, (1980)

[20] S. G. Krein, M. I. Khazan, Differential equations in a Banach space, Ref. Zh. Matematica, (1968-1982)

[21] P. Exner, M. Havlí ček, Linear operators in quantum physics, Karolinum, (1993)

[22] P. Duclos, O. Lev, P. Šťovíček, M. Vittot, Weakly regular Floquet Hamiltonians with pure point spectrum, Rev. Math. Phys. **14** (2002) 531-568