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Research work

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Quantum mechanics on non-simply connected manifolds

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Introduction

Quantum mechanics on non-simply connected manifolds covers a large group of quantum models where the quantum properties of the systems are observable. There exists more than one equivalent model, how to describe the quantum mechanics on non-simply connected manifolds, some of them are discussed in [12]. The Hilbert space of U-equivariant functions defined on the universal covering $\tilde{M}$ of the manifold M is used in one of them. Schulman ansatz proposes the straightforward connection between the kernel of the propagator of the Hamiltonian defined on $L^2(\tilde{M})$ and the kernel of the propagator corresponding to formally the same Hamiltonian acting on U-equivariant functions. The advantage of this processus is that in some cases, it is easier to find the kernel of the propagator on the simply-connected manifold $\tilde{M}$ than in case of U-equivariant functions. The aim of this work was to proof the Schulman ansatz not only formally, to discuss its field of validity, existence of the kernel for U-equivariant functions as well as the uniqueness.

In the first chapter the basic definitions and theorems are pointed out, mainly those which will be used in the following parts of this thesis.

The proof of the existence and uniqueness of the kernel is based on the Schwartz kernel theorem and its reformulation for U-equivariant function, both of which are done in chapter 2. It is necessary to introduce the connection between $C_0^\infty(\tilde{M})$ and Hilbert space of U-equivariant functions, to prove that this map is well defined and to explore its properties. The second part of the chapter concerns the main properties of the derived kernels.

In chapter 3, the Schulman ansatz is introduced. The ansatz is rigorously proved for identity operator; and also some necessary properties are discussed.

The next two chapters describe two different models of quantum mechanics on non-simply connected manifolds, the Aharonov-Bohm effect with one solenoid and the quantum model for two and more anyons. In both cases the Schulman ansatz is used to find the kernel of the free propagator.

Chapter 1

Basic definitions

1.1 Covering space

Definition 1 (Path and homotopy class). *Path in a space X is a continuous map $f : I \to X$ where I is the unit interval $[0;1]$. A homotopy of paths in X is a family $f_t : I \to X$, $0 \leq t \leq 1$, such that*

- *the endpoints $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t*
- *the associated map $F : I \times I \to X$ defined by $F(s,t) = f_t(s)$ is continuous*

When two paths f_0 and f_1 are connected in this way by a homotopy f_t, they are said to be homotopic. This property will be denoted by $f_0 \simeq f_1$.

Proposition 1. *The relation of homotopy on paths with fixed endpoints is an equivalence relation.*

Definition 2. *Composition of two paths $f, g : I \to X$, such that $f(1) = g(0)$ is defined by the formula*

$$f.g(s) = \left\{ \begin{array}{cc} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1 \end{array} \right. . \tag{1.1}$$

Path with the same starting and ending point $f(0) = f(1) = x_0$ are called the loops, x_0 is called the basepoint.

Definition 3 (Fundamental group). *Set of all homotopy classes $[f]$ of loops at the base point x_0 is called the fundamental group of X and is denoted by $\pi_1(X, x_0)$.*

If X is path-connected, the group $\pi_1(X, x_0)$ is, up to isomorphism, independent of the choice of the basepoint x_0 . In this case the notation $\pi_1(X, x_0)$ is often abbreviated to $\pi_1(X)$.

Definition 4. *Space is called simply-connected if it is path-connected and its fundamental group is trivial.*

Definition 5. *A covering space of a space X is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ satisfying the following condition: There exists an open cover (U_α) of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped by p homeomorphically onto U_α .*

Definition 6. *Two covering spaces $(\tilde{X}_0, p_0), (\tilde{X}_1, p_1)$ are isomorphic if there exists a homeomorphism $f : \tilde{X}_0 \to \tilde{X}_1$ such that $p_1 \circ f = p_0$.*

Proposition 2. *A covering space of a connected , locally path-connected topological space is connected.*

Definition 7 (Universal covering). *Universal covering of X is $(\tilde{X}, \pi, x_0)$ where $\tilde{X} = \{(p, \gamma)/\sim$, where γ is a path from x_0 to $p \in X\}$ and $(p_1, \gamma_1) \sim (p_2, \gamma_2)$ iff $p_1 = p_2$ and $\gamma_1.\gamma_2^{-1}$ are homotopically equivalent to a point, $\pi : \tilde{X} \to X$ defined by $\pi((p, \gamma)) = p$.*

Proposition 3. *The universal covering manifold $\tilde{X}$ is a principal fibre bundle over X with group $\pi_1(X)$ and projection p.*

Proposition 4. *The universal covering space of a connected manifold X is simply connected, and it is the only one covering space of X with this property (up to the isomorphism).*

Definition 8 (Induced homeomorphism). *Let $p : X \to Y$ be a continuous map, $y_0 \in Y$, $x_0 \in X$ base points such that $y_0 = p(x_0)$. Then p induces a homeomorphism $p_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by composing loops $f : I \to X$ based at x_0 with p, it means $p_*[f] = [p \circ f]$.*

1.2 Properly discontinuous action

Definition 9 (properly discontinuous 1). *Let G be an action of a group on a manifold X. Action is called properly discontinuous, if $\forall x \in X$ there exists a neighborhood U such that for varying $g \in G$ all the images $g.U$ are disjoint, it means that $g_1.U \bigcap g_2.U \neq 0$ implies $g_1 = g_2$.*

Definition 10 (properly discontinuous 2). *Action of the group on X is called properly discontinuous if for every point $x \in X$ there exists a neighborhood U such that $U \cap g.U$ is nonempty only for finitely many $g \in G$.*

Proposition 5. *If G acts freely, then the previous two definitions are equivalent.*

Definition 11. *Let $p : \tilde{X} \to X$, $\tilde{X}$ is a covering of X. $G(\tilde{X}, p, X)$ is a group defined by*

$$G(\tilde{X}, p, X) = \{h : \tilde{X} \to \tilde{X}, h \text{ homeomorphism such that } p \circ h = p, \} \quad (1.2)$$

Proposition 6. *If $p_*\pi_1(\tilde{X}, \tilde{x}_0)$ is normal in $\pi_1(X, x_0)$ then*

$$G(\tilde{X}, p, X) \cong \pi_1(X, x_0)/p_*\pi_1(\tilde{X}, \tilde{x}_0), \quad (1.3)$$

where $x_0 = p(\tilde{x}_0)$

If we take as $\tilde{X}$ the universal covering of X, because it is simply connected ($\pi_1(\tilde{X}, \tilde{x}_0)$ is trivial) $p_*\pi_1(\tilde{X}, \tilde{x}_0)$ is also trivial, so normal and $G(\tilde{X}, p, x)$ is isomorphic to $\pi_1(X, x)$.

Theorem 1. *If $\tilde{X}$ is continuous and locally linearly continuous, then the action of the group $G(\tilde{X}, p, X)$ on $\tilde{X}$ is properly discontinuous.*

As the direct result we obtain that if the universal covering space is continuous and locally linearly continuous, then the action of $\pi_1(X, x)$ is properly discontinuous. So the sufficient request is X continuous and locally linearly continues.

For proofs see [11].

1.3 Quadratic form

Definition 12 (quadratic form). *Let $\mathcal{H}$ be a Hilbert space. A quadratic form is a map $q : Q(q) \times Q(q) \to C$, where $Q(q)$ is a dense linear subset of $\mathcal{H}$ called the form domain, such that $q(., \psi)$ is conjugate linear, $q(\varphi, .)$ is linear for $\varphi, \psi \in Q(q)$. q is symmetric if $q(\varphi, \psi) = \overline{q(\psi, \varphi)}$. If $q(\varphi, \varphi) \geq 0$ for all $\varphi \in Q(q)$ then q is called positive. If there exists M such that $q(\varphi, \varphi) \geq -M\|\varphi\|^2$ then q is semibounded.*

Definition 13. *Let q be a semibounded quadratic form, $q(\psi, \psi) \geq -M\|\psi\|^2$. q is called closed if $Q(q)$ is complete under the norm*

$$\|\psi\|_{+1} = \sqrt{q(\psi, \psi) + (M + 1)\|\psi\|^2}, \quad (1.4)$$

where $\|.\|$ is the norm generated by the scalar product. If q is closed and $D \subset Q(q)$ is dense in $Q(q)$ in the $\|.\|_{+1}$ norm, then D is called a form core for q.

Remark 1: $\|.\|_{+1}$ comes from the inner product

$$(\psi, \varphi)_{+1} = q(\psi, \varphi) + (M+1)(\psi, \varphi). \tag{1.5}$$

Proposition 7. *q is closed if and only if for $\forall \psi_n \in Q(q)$, such that $\psi_n \to \psi$ and $q(\psi_n - \psi_m, \psi_n - \psi_m) \to 0$ for $n, m \to \infty$, then $\psi \in Q(q)$ and $q(\psi_n - \psi, \psi_m - \psi) \to 0$.*

Theorem 2. *If q is a closed semibounded quadratic form, then q is the quadratic form of a unique self-adjoint operator.*

Proof. in [9]. □

Theorem 3 (the Fridrichs extension). *Let A be a positive symmetric operator and let $q(\varphi, \psi) = (\varphi, A\psi)$ for $\varphi, \psi \in Dom(A)$. Then q is a closable quadratic form and its closure $\hat{q}$ is the quadratic form of a unique self-adjoint operator $\hat{A}$. $\hat{A}$ is a positive extension of A, and the lower bound of its spectrum is the lower bound of q. Further, $\hat{A}$ is the only self-adjoint extension of A whose domain is contained in the form domain of $\hat{q}$.*

Remark: It is sufficient for A to be bounded from below.

Proof. in [9]. □

Chapter 2

The Schwartz Kernel theorem

2.1 The Schwartz kernel theorem

Theorem 4 (Schwartz kernel theorem). *Let $\mathcal{K} \in \mathcal{D}'(X_1 \times X_2)$. Then by the equation*

$$\langle K\phi, \psi\rangle = \mathcal{K}(\psi \otimes \phi), \psi \in C_0^\infty(X_1), \phi \in C_0^\infty(X_2), \tag{2.1}$$

is defined a linear map $K : C_0^\infty(X_2) \to \mathcal{D}'(X_1)$ which is continuous in the sense that $K\phi_j \to 0$ in $\mathcal{D}'(X_1)$ if $\phi_j \to 0$ in $C_0^\infty(X_2)$. Conversely, to every such linear map K there is one and only one distribution $\mathcal{K}$ such that 2.1 is valid. One calls $\mathcal{K}$ the kernel of K.

Proof. in [10]. □

For the rigorous proof of Schulman ansatz we use a following reformulation of the Schwart kernel theorem:

Theorem 5. *Let $B \in \mathcal{B}(\tilde{\mathcal{H}})$, $\tilde{\mathcal{H}} = L^2(\tilde{M}, d\mu)$, then there exists one and only one $\beta \in \mathcal{D}'(\tilde{M} \times \tilde{M})$ such that $\beta(\bar{\phi_1} \otimes \phi_2) = \langle \phi_1, B\phi_2\rangle$ for $\forall \phi_1, \phi_2 \in \mathcal{D}(\tilde{M})$. Moreover the map $B \to \beta$ is an injection.*

Proof. The proof comes directly from the fact, that B restricted to $C_0^\infty(\tilde{M})$ is continuous as the function $B : C_0^\infty(\tilde{M}) \to L^2(\tilde{M},\ \mathrm{d}\mu)$. Because $C_0^\infty(\tilde{M})$ is dense in $L^2(\tilde{M}, d\mu)$ and $I : L^2(\tilde{M}, d\mu) \to \mathcal{D}'(\tilde{M})$, where I is the identity map, is continuous, so $B : C_0^\infty(\tilde{M}) \to \mathcal{D}'(\tilde{M})$ is continuous and there exists unique β from the previous theorem. □

Distribution δ is β from previous definition for $B = I$. So

$$\delta(\phi_1 \otimes \phi_2) = \langle \bar{\phi_1}, \phi_2\rangle = \int_{\tilde{M}} \phi_1(x)\phi_2(x)\ \mathrm{d}\mu(x). \tag{2.2}$$

In the following, symbol δ will be use in previous meaning, so $\delta(x-y)$ in the standard meaning.

Definition 14. *Let Γ be a group acting freely and transitively on $\tilde{M}$, $M = \Gamma/\tilde{M}$ and let U be a one dimensional unitary representation of Γ. The U-equivariant function is $\psi : \tilde{M} \to C$, $\psi(g.x) = U(g)\psi(x)$ and $\int_D |\psi|^2 \ d\mu < \infty$, where D is a fundamental domain of $\tilde{M}$. The scalar product is defined by $\langle \psi, \phi \rangle = \int_D \psi \bar{\phi} \ d\mu$.*

Because the metric is Γ-invariant and $\psi(g.x)\bar{\phi}(g.x) = \psi(x)\phi(x)$ the scalar product is independent on the choice of D. For $\psi : \tilde{M} \to C$, $\psi(g.x) = U(g)\psi(x)$, there exists a unique $\phi : M \to R$, such that $|\psi| = p^*\phi = \phi \circ p$. So $\int_D |\psi|^2 \ \mathrm{d}\mu = \int_M \phi^2 \ \mathrm{d}\mu$.

Let $\psi : \tilde{M} \to C$, we can define $\Phi : C_0^\infty(\tilde{M}) \to U$-equivariant functions by

$$\Phi\psi(x) = \sum_{g \in \Gamma} U^{-1}(g)\psi(g.x), \tag{2.3}$$

for $\forall \psi \in C_0^\infty(\tilde{M})$.

Lemma 1. *Let $\mathcal{H}_U$ be a Hilbert space of U-ekvivariant function defined on $\tilde{M}$, $M = \Gamma/\tilde{M}$, $\Gamma = \pi_1(M)$, M continuous and locally linearly continuous, D fundamental sheet of $\tilde{M}$, Φ is the map from the previous. Then $\Phi : C_0^\infty(\tilde{M}) \to \mathcal{H}_U$ is well defined linear map and is continuous. Let L_g be the left action of Γ on $\tilde{M}$:*

$$L_g : \tilde{M} \to \tilde{M} : x \to g.x$$
$$L_g^* : C_0^\infty(\tilde{M}) \to C_0^\infty(\tilde{M}), (L_g^*\varphi)(x) = \varphi(g.x).$$

Then $\Phi \circ L_g^* = U(g)\Phi$.

Proof. Because the properties of M, Γ is properly discontinuous. For $\forall x \in \tilde{M}$ there exists a neighborhood V_x of x such that for all $y \in V_x$ the set of g such that $\psi(g.y) \neq 0$ is finite and independent on the choice of y. Because suppψ is compact, for all $x \in \tilde{M}$ the set of g such that $\psi(g.x)$ is finite and can be taken the same for all x (will be denoted by Γ').

First of all we will proof that $\Phi\psi \in \mathcal{H}_U$. $\Phi\psi$ is U-equivariant:

$$\Phi\psi(g'.x) = \sum_{g \in \Gamma'} U^{-1}(g)\psi(g.g'.x) = \sum_{g \in \Gamma'} U(g')U^{-1}(g.g')\psi(g.g'.x) =$$
$$U(g') \sum_{g \in \Gamma'} U^{-1}(g)\psi(g.x) = U(g')\Phi\psi(x).$$

For $\psi \in C_0^\infty$, there exists ϕ such that $|\Phi\psi| = p^*\phi$, $\mathrm{supp}(\phi) \subset p(\mathrm{supp})(\psi)$. So

$$\|\Phi\psi\|^2 = \int_M \phi^2 \, \mathrm{d}\mu < \infty. \tag{2.4}$$

$\Phi : C_0^\infty(\tilde{M}) \to \mathcal{H}_U$ is continuous. $\psi_j \to 0$ in C_0^∞ if

$$\begin{aligned} &\exists S \subset \tilde{M}, S \text{ compact, such that } \mathrm{supp}\psi_j \subset S, \\ &\qquad\qquad \forall \alpha, \partial_\alpha \psi_j \to 0 \text{ regularly.} \end{aligned} \tag{2.5}$$

So $\Phi\psi_j \to 0$ locally regularly. Because Γ' can be chosen independently on j, $\Phi\psi_j \to 0$ converge regularly. Moreover there exists a unique ϕ_j such that $|\Phi\psi_j| = p^*\phi_j$, $\phi_j \in C(M)$, $\mathrm{supp}\phi_j \subset p(S)$, $\forall j$. $\phi_j \to 0$ regularly on M, and $\|\Phi\psi_j\|^2_{\mathcal{H}_U} = \int_M \phi_j^2 \, \mathrm{d}\mu \to 0$.

Finally $L_g^* = U(g)\Phi$:

$$\begin{aligned} \Phi \circ L_g^*\phi(x) = \sum_{g\in\Gamma} U^{-1}(g')\phi(g'.g.x) = \sum_{g'\in\Gamma} U(g)U^{-1}(g'.g)\phi(g'.g.x) = \\ U(g) \sum_{g'\in\Gamma} U^{-1}(g')\phi(g'.x) = U(g)\Phi\phi(x). \end{aligned} \tag{2.6}$$

□

Theorem 6 (Schwartz kernel theorem for $U-$equivariant functions). *Let $\mathcal{H}_U$ be a Hilbert space of U-ekvivariant function defined on $\tilde{M}$ and $B \in \mathcal{B}(\mathcal{H}_U)$, $M = \Gamma/\tilde{M}$, $\Gamma = \pi_1(M)$, M continuous and locally linearly continuous, D fundamental sheet of M. Then there exists one and only one distribution $\beta_U \in \mathcal{D}'(\tilde{M} \times \tilde{M})$, such that*

$$\beta_U(\bar{\phi}_1 \otimes \phi_2) = \langle \sum_{g\in\Gamma} U^{-1}(g)\phi_1(g.x), B \sum_{g\in\Gamma} U^{-1}(g)\phi_2(g.x)\rangle, \forall \phi_1, \phi_2 \in \mathcal{D}(\tilde{M}),$$

where $\langle \phi_1, \phi_2 \rangle = \int_D \bar{\phi}_1(x)\phi_2(x) \, d\mu(x)$.

Proof. The proof comes from the theorem 4 and the lemma 1. $B\Phi : C_O^\infty \to \mathcal{H}_U$ is continuous and linear, $I : \mathcal{H}_U \to \mathcal{D}'(\tilde{M})$, where I is the identity map, is continuous. The uniquness is because $\Phi(C_0^\infty(\tilde{M}))$ is dense in $\mathcal{H}_U$. □

2.2 Properties of β and β_U

First of all let us mention that for F diffeomorphism, $f \in \mathcal{D}'$ the following is valid:

$$\langle f(F(x)), \psi(x) \rangle = \langle f(x), \frac{\mathrm{d}\mu(F^{-1}(x))}{\mathrm{d}\mu(x)} \psi(F^{-1}(x)) \rangle. \tag{2.7}$$

In this case the mesure is Γ-invariant, so $\frac{\mathrm{d}\mu(g^{-1}.x)}{\mathrm{d}\mu(x)} = 1$.

- $\beta_U(g.x,y) = U(g).\beta_U(x,y)$, where $g \in \Gamma$.

  Proof: Using the lemma 1

$$\beta_U(g.x,y)(\phi \otimes \varphi) = \langle \Phi L^*_{g^{-1}} \bar{\phi}(x), B\Phi\varphi(x)\rangle =$$
$$\langle U(g^{-1})\Phi\bar{\phi}(x), B\Phi\varphi(x)\rangle = U(g)\langle \Phi\bar{\phi}(x), B\Phi\varphi(x)\rangle. \tag{2.8}$$

- $\beta_U(x,g.y) = U(g^{-1})\beta_U(x,y)$, where $g \in \Gamma$.

  Proof:

$$\beta_U(x,g.y)(\phi \otimes \varphi) = \langle \Phi\bar{\phi}(x), B\Phi L^*_{g^{-1}}\varphi(x)\rangle =$$
$$\langle \Phi\bar{\phi}(x), BU(g^{-1})\Phi\varphi(x)\rangle = U(g^{-1})\langle \Phi\bar{\phi}(x), B\Phi\varphi(x)\rangle$$
$$= U(g^{-1})\beta_U(x,y)(\phi \otimes \varphi). \tag{2.9}$$

Remark 1 Kernel obtained from the Schwartz kernel theorem has the similar meaning as the kernel known from physical applications: C_0^∞ is dense in $\mathcal{D}'$ in the sense of limit in $\mathcal{D}'$. It means that in case of $B \in \mathcal{H}(\tilde{M})$ there exists $\beta_k(x,y) \in C_0^\infty(\tilde{M} \times \tilde{M})$ such that

$$\lim_{k\to\infty} \beta_k(\phi \otimes \varphi) = \langle \bar{\phi}, B\varphi\rangle. \tag{2.10}$$

We also know, that

$$\lim_{k\to\infty} \beta_k(\bar{\phi} \otimes \varphi) = \lim_{k\to\infty} \int_{\tilde{M}} \int_{\tilde{M}} \beta_k(x,y)\phi(x)\varphi(y)\ \mathrm{d}\mu(x)\ \mathrm{d}\mu(y) =$$
$$\lim_{k\to\infty} \int_{\tilde{M}} \phi(x)(\int_{\tilde{M}} \beta_k(x,y)\varphi(y)\ \mathrm{d}\mu(y))\ \mathrm{d}\mu(x) =$$
$$\int_{\tilde{M}} \phi(x)(B\varphi)(x)\ \mathrm{d}\mu(x), \tag{2.11}$$

in the sense of limit in $\mathcal{D}'(\tilde{M} \times \tilde{M})$. It means that

$$\lim_{k\to\infty} \int_{\tilde{M}} \phi(x)(\int_{\tilde{M}} \beta_k(x,y)\varphi(y)\ \mathrm{d}\mu(y) - (B\varphi)(x))\ \mathrm{d}\mu(x) =$$
$$\lim_{k\to\infty} \langle \phi, \int_{\tilde{M}} \beta_k(.,y)\varphi(y)\ \mathrm{d}\mu(y) - (B\varphi)\rangle = 0, \tag{2.12}$$

for all $\phi \in \mathcal{C}_0^\infty(\tilde{M})$. So

$$B\varphi = \lim_{k\to\infty} \int_{\tilde{M}} \beta_k(.,y)\varphi(y)\ \mathrm{d}\mu(y) \text{ in } \mathcal{D}'(\tilde{M}). \tag{2.13}$$

In case of $B \in \mathcal{B}(\mathcal{H}_U)$ the situation is more complicated: There exists $\beta_{U,k}(x,y) \in C_0^\infty(\tilde{M} \times \tilde{M})$ such that

$$\lim_{k\to\infty} \beta_{U,k}(\phi \otimes \varphi) = \langle \Phi(\bar{\phi}), B\Phi(\varphi)\rangle, \tag{2.14}$$

where $\Phi\phi(x) = \sum_{g\in\Gamma} U(g^{-1})\phi(g.x)$. It means that

$$\begin{aligned}\lim_{k\to\infty} \beta_{U,k}(\phi \otimes \varphi) = \lim_{k\to\infty} \int_{\tilde{M}} \int_{\tilde{M}} \beta_{U,k}(x,y)\phi(x)\varphi(y) \ \mathrm{d}\mu(x) \ \mathrm{d}\mu(y) = \\ \lim_{k\to\infty} \int_{\tilde{M}} \phi(x)(\int_{\tilde{M}} \beta_{U,k}(x,y)\varphi(y) \ \mathrm{d}\mu(y)) \ \mathrm{d}\mu(x) = \\ \int_D \sum_{g\in\Gamma} U(g^{-1})\phi(g.x) B(\sum_{g\in\Gamma} U(g^{-1})\varphi(g.x)) \ \mathrm{d}\mu(x).\end{aligned} \tag{2.15}$$

Remark 2 Direct consequence of theorem 5 is that $\forall\varphi(x) \in \mathcal{D}'(\tilde{M})$ the function $\langle\beta(x,y), \varphi(y)\rangle$ is regular distribution. This property will be used in the following.

Remark 3 $\langle\beta_U(x,y), \psi(y)\rangle$ from the theorem 6 is also regular distribution:

$$\begin{aligned}\beta_U(\phi \otimes \varphi) = \langle \Phi\bar{\phi}(x), B\Phi\varphi(x)\rangle = \\ \sum_{g\in\Gamma} U(g^{-1}) \int_D \phi(g.x) B(\sum_{g\in\Gamma} U(g^{-1})\varphi(g.x)) \ \mathrm{d}\tilde{V}(x) = \\ \int_{\tilde{M}} \phi(x) f(x) B(\sum_{g\in\Gamma} U(g^{-1})\varphi(g.x)) \ \mathrm{d}\tilde{V}(x),\end{aligned} \tag{2.16}$$

where $f(x) = U(g^{-1})$, g is such that $g^{-1}.x \in D$.

Kernel of identity on U-equivariant functions will be denoted by δ_U. Then

$$\begin{aligned}\delta_U(\phi_1 \otimes \phi_2) = \langle \Phi\bar{\phi}_1(x), \Phi\phi_2(x)\rangle = \\ \int_D \sum_{g\in\Gamma} \overline{U^{-1}(g)}\phi_1(g.x) \sum_{g'\in\Gamma} U^{-1}(g')\phi_2(g'.x) \ \mathrm{d}\mu.\end{aligned} \tag{2.17}$$

Because the sums are finite

$$\delta_U(\phi_1 \otimes \phi_2) = \sum_{g\in\Gamma} \sum_{g'\in\Gamma} U^{-1}(g'.g^{-1}) \int_D \phi_1(g.x)\phi_2(g'.x) \ \mathrm{d}\mu(x). \tag{2.18}$$

Chapter 3

Schulman ansatz

Schulman ansatz gives the relation between two operators which are formally the same, but are defined on two different Hilbert spaces. Let $\tilde{M}$ be simply connected Riemann manifold and Γ be a discrete group acting freely on $\tilde{M}$[1] and the Riemann metric is Γ-invariant. Let $M = \Gamma/\tilde{M}$, then $\pi_1(M) = \Gamma$. One of the Hilbert spaces is $\tilde{\mathcal{H}} := L^2(\tilde{M})$, H is Hamiltonian on $\tilde{\mathcal{H}}$ which is Γ-invariant[2], $\mathcal{U}(t)$ is corresponding propagator, it means

$$\mathcal{U}(t) = \exp(-\frac{i}{\hbar}Ht). \tag{3.1}$$

$\mathcal{U}(t)$ is bounded operator on $\tilde{\mathcal{H}}$ and it is possible to use Schwartz kernel theorem to find the kernel of $\mathcal{U}(t)$. The second Hilbert spaces is defined by

$$\mathcal{H}_U = \{\phi \text{ mesurable on } \tilde{M}, \forall g \in \Gamma, \phi(g.x) = U(g)\phi(x), \int_D |\phi(x)|^2 \, \mathrm{d}\mu < \infty\}$$

and let H_U be formally the same Hamiltonianon $\mathcal{H}_U$, $\mathcal{U}_U(t)$ its propagator. Also in this case Kernel theorem for U-equivariant functions may be used.

Remark: $\mathcal{U}(t)$ is in both cases unitary operator,$\{\mathcal{U}(t)\}_t$ is one parametric group with properties

$$\begin{aligned} \mathcal{U}(t)\mathcal{U}(s) &= \mathcal{U}(t+s), \\ \mathcal{U}(t)^{-1} &= \mathcal{U}(-t), \\ \mathcal{U}(0) &= 1. \end{aligned} \tag{3.2}$$

Let $\mathcal{K}_t$, resp. $\mathcal{K}_t^U$ be a kernel of operator $\mathcal{U}(t)$, resp. $\mathcal{U}_U(t)$. Schulman ansatz proposes that

$$\mathcal{K}_t^U(x,y) = \sum_{g\in\Gamma} U(g^{-1})\mathcal{K}_t(g.x,y). \tag{3.3}$$

[1]It means if $g.x = x$ for some x, then g =identity in Γ.

[2]H is Γ-invariant Hamiltonian if $L^*_{g^{-1}} H L^*_g = H$ for all $g \in \Gamma$. If $H = -\Delta_{LB} + V$, where V is $\Gamma-$invariant potential, then H is Γ-invariant.

Using the notation from the previous chapter, Schulman ansatz reads:

$$\langle \Phi\psi(x), \mathcal{U}_U(t)\Phi\varphi(x)\rangle_{\mathcal{H}_U} = \sum_{g\in\Gamma} U(g^{-1})\langle \psi(x), \mathcal{U}(t)\varphi(g^{-1}.x)\rangle_{L^2(\tilde{M})}, \tag{3.4}$$

for $\psi, \varphi \in C_0^\infty(\tilde{M})$.

3.0.1 Schulman ansatz for identity operator

Let us proof that $\delta_U(x,y) = \sum_{g\in\Gamma} U(g^{-1})\delta(g.x,y)$ It means to prove that

$$\begin{aligned}\sum_{g\in\Gamma}\sum_{g'\in\Gamma} U^{-1}(g.g') \int_D \phi_1(g.x)\phi_2(g'.x)\ \mathrm{d}\mu(x) = \\ \sum_{g\in\Gamma} U(g^{-1}) \int_{\tilde{M}} \phi_1(g^{-1}.x)\phi_2(x)\ \mathrm{d}\mu(x).\end{aligned} \tag{3.5}$$

Really

$$\begin{aligned}\sum_{g\in\Gamma} U(g^{-1}) \int_{\tilde{M}} \phi_1(g^{-1}.x)\phi_2(x)\ \mathrm{d}\mu(x) = \\ \sum_{g\in\Gamma} U(g^{-1}) \sum_{g'\in\Gamma} \int_{g'.D} \phi_1(g^{-1}.x)\phi_2(x)\ \mathrm{d}\mu(x) = \\ \sum_{g\in\Gamma} U(g^{-1}) \sum_{g'\in\Gamma} \int_D \phi_1(g^{-1}.g'.x)\phi_2(g'.x)\ \mathrm{d}\mu(x) = \\ \sum_{g\in\Gamma}\sum_{g'\in\Gamma} U^{-1}(g'^{-1}.g) \int_D \phi_1(g.x)\phi_2(g'.x)\ \mathrm{d}\mu(x).\end{aligned} \tag{3.6}$$

All the relations are correct, because for $\forall\phi \in \mathcal{D}(\tilde{M})$ fixed, the subset of Γ such that $\phi(g.x) \neq 0, \forall x \in D$, is finite.

3.1 Properties of $\mathcal{K}_t$ and $\mathcal{K}_t^U$

3.1.1 Properties of $\mathcal{K}_t$

- $\mathcal{K}_t(g.x, g.y) = \mathcal{K}_t(x,y)$ it means $\mathcal{K}_t(g.x, y) = \mathcal{K}_t(x, g^{-1}.y)$

  Proof:

  $$\begin{aligned}K_t(g.x, g.y)(\phi\otimes\varphi) = \langle \bar{\phi}(g^{-1}.x), \mathcal{U}(t)\varphi(g^{-1}.x)\rangle = \\ \langle \phi(x), \mathcal{U}(t)\varphi(x)\rangle = \mathcal{K}_t(x,y)(\phi\otimes\varphi),\end{aligned} \tag{3.7}$$

  because if H is Γ-invariant, also $\mathcal{U}(t)$ is Γ-invariant and $\langle U(g)\bar{\phi}(x), \mathcal{U}(t)U(g)\varphi(x)\rangle = \langle \bar{\phi}(x), \mathcal{U}(t)\varphi(x)\rangle$

- $\lim_{t\to 0_+} \mathcal{K}_t(x,y) = \delta(x,y)$

  Proof:

$$\lim_{t\to 0_+} \mathcal{K}_t(x,y)(\phi(x)\otimes\varphi(y)) = \lim_{t\to 0_+} \langle \phi, \mathcal{U}(t)\varphi\rangle = \langle \phi, \varphi\rangle = \delta(x,y)(\phi(x)\otimes\varphi(y)). \tag{3.8}$$

- $\overline{\mathcal{K}_t(y,x)} = \mathcal{K}_{-t}(x,y)$

  Proof: First of all let us mention that for $f \in \mathcal{D}'$, $\varphi \in \mathcal{D}$, $\langle \bar{f}, \varphi\rangle = \overline{\langle f, \bar{\varphi}\rangle}$.

$$\overline{\mathcal{K}_t(y,x)}(\phi(x)\otimes\varphi(y)) = \overline{\langle \varphi, \mathcal{U}(t)\bar{\phi}\rangle} = \langle \mathcal{U}(t)\bar{\phi}, \varphi\rangle = \langle \bar{\phi}, \mathcal{U}(-t)\varphi\rangle = \mathcal{K}_{-t}(x,y)(\phi(x)\otimes\varphi(y)). \tag{3.9}$$

- $(i\frac{\partial}{\partial t} + \Delta_{LB})\vartheta(t)\mathcal{K}_t(x,y) = i\delta(t)\delta(x,y)$, where $\vartheta(t)$ is the Heaviside step function.

  Proof:

$$(i\frac{\partial}{\partial t} + \Delta_{LB})\vartheta(t)\mathcal{K}_t(x,y) = i\vartheta(t)\frac{\partial}{\partial t}\mathcal{K}_t(x,y) + i\delta(t)\mathcal{K}_0(x,y) + \vartheta(t)\Delta_{LB}\mathcal{K}_t(x,y). \tag{3.10}$$

  Because $\mathcal{K}_0(x,y) = \delta(x,y)$ it is sufficient to prove that $i\frac{\partial}{\partial t}\mathcal{K}_t(x,y) + \Delta_{LB}\mathcal{K}_t(x,y) = 0$, for $t > 0$:

$$(i\frac{\partial}{\partial t}\mathcal{K}_t(x,y) + \Delta_{LB}\mathcal{K}_t(x,y))(\phi(x)\otimes\varphi(y)) = \langle\overline{\phi}, (i\frac{\partial}{\partial t}\mathcal{U}(t) + \Delta_{LB}\mathcal{U}(t))\varphi\rangle = 0, \tag{3.11}$$

  because $\mathcal{U}(t)$ is a solution of Schrodinger equation.

- $\int_{\tilde{M}} \mathcal{K}_s(x,y)\mathcal{K}_t(y,z)\,\mathrm{d}\tilde{V}(y) = \mathcal{K}_{s+t}(x,z)$ in the following sense[3]:

$$\int_{\tilde{M}} (\mathcal{K}_s(x,y), \phi(x))(\mathcal{K}_t(y,z)\varphi(z))\,\mathrm{d}\tilde{V}(y) = \mathcal{K}_{s+t}(\phi\otimes\varphi). \tag{3.12}$$

  Proof:

$$\int_{\tilde{M}} (\mathcal{K}_s(x,y), \phi(x))(\mathcal{K}_t(y,z)\varphi(z))\,\mathrm{d}\tilde{V}(y) = \int_{\tilde{M}} \overline{(\mathcal{U}(-s)\phi)(x)}(\mathcal{U}(t)\varphi)(x)\,\mathrm{d}\tilde{V}(x) = \langle \mathcal{U}(-s)\phi, \mathcal{U}(t)\varphi\rangle = \langle \phi, \mathcal{U}(s)\mathcal{U}(t)\varphi\rangle = \langle\phi, \mathcal{U}(s+t)\varphi\rangle = \mathcal{K}_{s+t}(x,y)(\phi(x)\otimes\varphi(y)). \tag{3.13}$$

[3] $\langle \mathcal{K}_t(x,y), \phi(x)\rangle$ and $\langle \mathcal{K}_t(y,x), \phi(x)\rangle$, for $\phi \in C_0^\infty(\tilde{M})$, are regular distributions.

3.1.2 Properties of $\mathcal{K}_t^U$

Kernel of the propagator defined on $\mathcal{H}_U$ must fulfil following properties (all the relations are defined as in previous):

- $K_t^U(g.x, y) = U(g)\mathcal{K}_t^U(x, y)$
- $\mathcal{K}_t^U(x, g.y) = U(g^{-1})\mathcal{K}_t^U(x, y)$
- $\overline{\mathcal{K}_t^U(x, y)} = \mathcal{K}_{-t}^U(y, x)$
- $\int_D \mathcal{K}_s^U(x, y)\mathcal{K}_t^U(y, z)\ \mathrm{d}V(y) = \mathcal{K}_{s+t}^U(x, z)$
- $(i\frac{\partial}{\partial t} + \Delta_{LB})\vartheta(t)\mathcal{K}_t^U(x, y) = i\delta(t)\delta^U(x, y)$

Remark All the properties are well defined, because $\mathcal{K}_t^U(x, y)\varphi(y)$ is regular distribution.

We will prove that $\mathcal{K}_t^U$ defined by Schulman ansatz formally fulfil the previous relations.

- $K_t^U(g'.x, y) = U(g')\mathcal{K}_t^U(x, y)$:

  Proof:

$$\begin{aligned}
\mathcal{K}_t^U(g'.x, y) &= \sum_{g\in\Gamma} U(g^{-1})\mathcal{K}_t(g.g'.x, y) \\
&= U(g') \sum_{g\in\Gamma} U(g'^{-1})U(g^{-1})\mathcal{K}_t(g.g'.x, y) \\
&= U(g') \sum_{g\in\Gamma} U(g^{-1})\mathcal{K}_t(g.x, y) = U(g')\mathcal{K}_t^U(x, y).
\end{aligned}$$

- $\mathcal{K}_t^U(x, g'.y) = U(g'^{-1})\mathcal{K}_t^U(x, y)$:

  Proof:

$$\begin{aligned}
\mathcal{K}_t^U(x, g'.y) &= \sum_{g\in\Gamma} U(g^{-1})\mathcal{K}_t(g.x, g'.y) \\
&= \sum_{g\in\Gamma} U(g^{-1})\mathcal{K}_t(g'^{-1}g.x, y) \\
&= U(g'^{-1}) \sum_{g\in\Gamma} U(g')U(g^{-1})\mathcal{K}_t(g'^{-1}.g.x, y) \\
&= U(g'^{-1})\mathcal{K}_t^U(x, y).
\end{aligned}$$

- $\overline{\mathcal{K}_t^U(x,y)} = \mathcal{K}_{-t}^U(y,x)$:

  Proof:

$$\begin{aligned}
\overline{\mathcal{K}_t^U(x,y)} &= \sum_{g\in\Gamma} \overline{U(g^{-1})\mathcal{K}_t(g.x,y)} = \sum_{g\in\Gamma} U(g)\mathcal{K}_{-t}(y,g.x) \\
&= \sum_{g\in\Gamma} U(g)\mathcal{K}_{-t}(g^{-1}.y,x) = \mathcal{K}_{-t}^U(y,x).
\end{aligned}$$

- $\int_D \mathcal{K}_s^U(x,y)\mathcal{K}_t^U(y,z)\ \mathrm{d}V(y) = \mathcal{K}_{s+t}^U(x,z)$:

  Proof:

$$\begin{aligned}
&\int_D \mathcal{K}_s^U(x,y)\mathcal{K}_t^U(y,z)\ \mathrm{d}\tilde{V}(y) \\
=\ & \int_D \sum_{g\in\Gamma} U(g^{-1})\mathcal{K}_s(g.x,y) \sum_{g'\in\Gamma} U(g'^{-1})\mathcal{K}_t(g'.y,z)\ \mathrm{d}\tilde{V}(y) \\
=\ & \sum_{g\in\Gamma}\sum_{g'\in\Gamma} U(g^{-1})U(g'^{-1}) \int_D \mathcal{K}_s(g.x,y)\mathcal{K}_t(g'.y,z)\ \mathrm{d}\tilde{V}(y) \\
=\ & \sum_{h\in\Gamma}\sum_{g'\in\Gamma} U(h^{-1}) \int_D \mathcal{K}_s(g'^{-1}.h.x,y)\mathcal{K}_t(g'.y,z)\ \mathrm{d}\tilde{V}(y) \\
=\ & \sum_{h\in\Gamma} U(h^{-1}) \sum_{g'\in\Gamma} \int_D \mathcal{K}_s(h.x,g'.y)\mathcal{K}_t(g'.y,z)\ \mathrm{d}\tilde{V}(y) \\
=\ & \sum_{h\in\Gamma} U(h^{-1}) \int_{\tilde{M}} \mathcal{K}_s(h.x,y)\mathcal{K}_t(y,z)\ \mathrm{d}\tilde{V}(y) \\
=\ & \sum_{h\in\Gamma} U(h^{-1})\mathcal{K}_{s+t}(h.x,z) \\
=\ & \mathcal{K}_{t+s}^U(x,z).
\end{aligned}$$

  In the third equality the substitution $h = g^{-1}g'^{-1}$ is used.

- $(i\frac{\partial}{\partial t} + \Delta_{LB})\vartheta(t)\mathcal{K}_t^U(x,y) = i\delta(t)\delta^U(x,y)$:

  Proof:

$$\begin{aligned}
(i\frac{\partial}{\partial t} + \Delta_{LB})\vartheta(t)\mathcal{K}_t^U(x,y) &= (i\frac{\partial}{\partial t} + \Delta_{LB})\vartheta(t)\sum_{g\in\Gamma} U(g^{-1})\mathcal{K}_t(g.x,y) \\
&= \sum_{g\in\Gamma} U(g^{-1})(i\frac{\partial}{\partial t} + \Delta_{LB})\vartheta(t)\mathcal{K}_t^U(x,y) \\
&= \sum_{g\in\Gamma} U(g^{-1})i\delta(t)\tilde{\delta}(x,y) \\
&= i\delta(t)\delta^U(x,y).
\end{aligned}$$

Chapter 4

Aharonov-Bohm Effect

Aharonov-Bohm effect with one solenoid was first published in 1959 and it helped to understand the role of potentials in quantum mechanics and geometric properties of fields. Aharonov-Bohm effect with one solenoid is based on the following model: Non relativistic quantum particle is moving in external field with the flux concentrated in one infinitely thin solenoid. Outer the solenoid the flux vanishes. Even though the particles do not pass throw the field, they are influenced by this field.

Configuration space of each particle is $M = R^2 - (0,0)$ (solenoid is located in (0,0)). On M the flux is null, but the electromagnetic potential $\overrightarrow{A}$ is non-null.

In nonrelativistic quantum mechanics, Hamiltonian of charged particle in electromagnetical field is given by

$$H = \frac{1}{2\mu}\left(\overrightarrow{p} - e\overrightarrow{A}\right)^2 + e\phi, \tag{4.1}$$

where $\overrightarrow{p}$ is an impulse operator of a free particle, $\overrightarrow{A}$, resp. ϕ is a vector, resp. a scalar potential of electromagnetic field. In case of Aharonov-Bohm effect with one solenoid $\overrightarrow{A}$ is given by

$$\overrightarrow{A} = \frac{\Phi}{2\pi(x_1^2 + x_2^2)}(-x_2, x_1), \tag{4.2}$$

and Hamiltonian is in the form

$$H = -\frac{\hbar^2}{2\mu}\left[\left(\partial_1 - i\frac{e}{\hbar}A_1\right)^2 + \left(\partial_2 - i\frac{e}{\hbar}A_2\right)^2\right], \tag{4.3}$$

where $Dom(H) = \{\psi \in AC^2(R^2)$, the second derivatives quadratically integrable$\}$. In polar coordinates

$$x_1 = r\cos\varphi$$

$$x_2 \;=\; r\sin\varphi$$

and with $\alpha = -\frac{e\Phi}{2\pi\hbar}$ the Hamiltonian can be expressed as

$$H = -\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial}{\partial\varphi} + i\alpha\right)^2\right). \tag{4.4}$$

Solving the problem of finding the eigen-functions of the operator H we obtain the equation

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial}{\partial\varphi} + i\alpha\right)^2 + k^2\right)\psi = 0.$$

The generalized solution is a linear combination of Bessel functions

$$\psi = \sum_{m=-\infty}^{+\infty} \exp(im\varphi)[a_m J_{m+\alpha}(kr) + b_m J_{-(m+\alpha)}(kr)], \tag{4.5}$$

where a_m, b_m are some arbitrary constants.

Eliminating the functions with singularities in $r = 0$ we obtain the solution

$$\psi = \sum_{m=-\infty}^{+\infty} \exp(im\varphi)c_m(k)J_{|m+\alpha|}(kr), \tag{4.6}$$

where k^2 is fixed value of the energy. Spectrum of the operator H is the interval $[0,\infty[$ and is absolutely continues and infinitely degenerated. The wave function is

$$\psi = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{+\infty} \exp(im\varphi) \int_0^\infty d_m(k)J_{|m+\alpha|}(kr)k \; \mathrm{d}k. \tag{4.7}$$

Finally, $\{J_\beta(kr); k > 0, \beta > 0\}$ is generalized set of eigen-functions in $L^2(R, r\;\mathrm{d}r)$, so the map

$$L^2(R_+, k\;\mathrm{d}k) \to L^2(R_+, r\;\mathrm{d}r) : a(k) \to \hat{a}(r) = \int_0^\infty a(k)J_\beta(kr)k\;\mathrm{d}k \tag{4.8}$$

is unitary and

$$\begin{aligned}
\|\psi\|^2 &= \sum_{-\infty}^{\infty} \int_0^\infty |d_m(k)|^2 k\;\mathrm{d}k,\\
H\psi &= \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{+\infty} \exp(im\varphi) \int_0^\infty k^2 d_m(k)J_{|m+\alpha|}(kr)k\;\mathrm{d}k.
\end{aligned} \tag{4.9}$$

4.1 Propagator for the Aharonov-Bohm effect with one solenoid

In this paragraph we will find the propagator for Aharonov-Bohm effect with one solenoid using the Schulman ansatz. From the previous chapter, $M = R^2 - (0,0)$ and $\Gamma = \pi_1(M) = Z$, $U(n) = \exp(2\pi i n\alpha)$, where $\alpha \in (0,1)$. $\tilde{M}$ is complete with the point A, a copy of $(0,0)$. Using polar coordinates (r,φ), we can identify fundamental domain with $R^+ \times (0,2\pi)$. Universal covering space $\tilde{M}$ can be identified with $R^+ \times R$. The action of Γ is given by $n.(r,\varphi) = (r, \varphi + 2\pi n)$.

First of all we must find free propagator on $\tilde{M}$. The geometry of $\tilde{M}$ is special. Two points $x = (r,\varphi), x_0 = (r_0,\varphi_0)$ can be connected by a geodesic curve only if $|\varphi - \varphi_0| < \pi$. Let us define the function

$$\begin{aligned} \chi(x,x_0) &= 1 \text{ if } |\varphi-\varphi_0| < \pi, \\ &= 0 \text{ otherwise.} \end{aligned} \tag{4.10}$$

The free propagator on $\tilde{M}$ (for free particle with mass equal to μ) is

$$\mathcal{K}_t(x,x_0) = \chi(x,x_0)\frac{\mu}{2\pi \mathrm{i}\hbar t}\exp(\frac{\mathrm{i}\mu}{2\hbar t}\mathrm{dist}^2(x,x_0)) +$$
$$\frac{\mu}{2\pi \mathrm{i}\hbar t}\int_{-\infty}^{\infty}\frac{\mathrm{d}s}{2\pi}\left(\frac{1}{\Phi-\pi+\mathrm{i}s} - \frac{1}{\Phi+\pi+\mathrm{i}s}\right)\exp(\mathrm{i}\mu R^2(s)/2\hbar t), \tag{4.11}$$

where $R^2(s) = r^2 + r_0^2 + 2rr_0\coth(s)$ and $\Phi = \varphi - \varphi_0$.

To derive the above expression we use the fact that the complete set of generalized eigenfunctions for the free particle on $\tilde{M}$ are

$$\{(\sqrt{2\pi}\hbar)^{-1}J_{|\nu|}(pr/\hbar)\exp \mathrm{i}\nu\varphi = \beta_{\nu,p}, \nu\in R, p>0\}. \tag{4.12}$$

Each function $\psi \in L^2(\tilde{M})$ can be writen as $\psi = \int_R \,\mathrm{d}\nu \int_{R^+}\,\mathrm{d}p\langle\psi,\beta_{\nu,p}\rangle\beta_{\nu,p}$ in sense of equality in $\mathcal{D}'$. From the definition of the propagator

$$\begin{aligned} \mathcal{U}(t)\psi = \int \mathcal{K}_t\psi &= \\ \int\int \exp(-\mathrm{i}t\mu p^2/2\hbar)\beta_{\nu,p}(pr_0/\hbar)\langle\psi,\beta_{\nu,p}\rangle\,\mathrm{d}\nu p\,\mathrm{d}p &= \\ \int\int \exp(-\mathrm{i}t\mu p^2/2\hbar)\beta_{\nu,p}(r_0,\varphi_0)(\int\psi(r,\varphi)\beta_{\nu,p}(r,\varphi)r\,\mathrm{d}r\,\mathrm{d}\varphi)\,\mathrm{d}\nu p\,\mathrm{d}p &= \\ \int(\int\int \exp(-\mathrm{i}t\mu p^2/2\hbar)\beta_{\nu,p}(r_0,\varphi_0)\beta_{\nu,p}(r,\varphi)\,\mathrm{d}\nu p\,\mathrm{d}p)\psi(r,\varphi)r\,\mathrm{d}r\,\mathrm{d}\varphi. & \end{aligned} \tag{4.13}$$

So

$$\mathcal{K}_t(x,x_0) =$$

$$\frac{1}{2\pi\hbar^2}\int_0^\infty p\ \mathrm{d}p\int_{-\infty}^\infty \mathrm{d}\nu J_{|\nu|}(pr/\hbar)J_{|\nu|}(pr_0/\hbar)\exp(\mathrm{i}\nu\Phi)\exp(-\mathrm{i}t\mu p^2/2\hbar) \quad =$$
$$\frac{\mu}{2\pi\mathrm{i}\hbar t}\exp(\mathrm{i}\mu(r^2+r_0^2)/2\hbar t)\times$$
$$\int_{-\infty}^\infty \mathrm{d}\nu\exp(-\mathrm{i}|\nu|\pi/2)J_{|\nu|}(rr_0/2\hbar t)\exp(i\nu\Phi). \quad (4.14)$$

Using the identity

$$\exp(-\mathrm{i}\nu\pi/2)J_\nu(z) =$$
$$\frac{1}{\pi}\int_0^\pi \exp(-\mathrm{i}z\cos(s))\cos(\nu s)\ \mathrm{d}s - \frac{\sin(\pi\nu)}{\pi}\int_0^\infty \exp(\mathrm{i}z\cosh(s)-\nu s)\ \mathrm{d}s,$$

for ν, z positive, we obtain the expression (4.11).

From this expression one can derives formula (4.17), which is useful in case of Aharonov-Bohmov with more solenoids.

For three points $x_1, x_2, x_3 \in \tilde{M}\cup\{A\}$ such that $\chi(x_1,x_2)=\chi(x_2,x_3)=1$ and for two positive times t_1, t_2 we put

$$V\left(\begin{array}{c} x_3, x_2, x_1 \\ t_2, t_1 \end{array}\right) = \frac{i\hbar}{\mu}\left(\frac{1}{\theta-\pi+\mathrm{i}s} - \frac{1}{\theta+\pi+\mathrm{i}s}\right), \quad (4.15)$$

where θ is the oriented angle $x_1x_2x_3$, $u=\ln(t_2r_1/t_1r_2)$ and $r_1=\mathrm{dist}(x_1,x_2)$, $r_2=\mathrm{dist}(x_2,x_3)$. Using the substitution

$$s=\ln(t_1r_0/t_0r), \qquad \mathrm{d}s = t\delta(t_1+t_0-t)\vartheta(t_1)\vartheta(t_0)(t_1t_0)^{-1}\ \mathrm{d}t_1\ \mathrm{d}t_0, \quad (4.16)$$

we obtain

$$\mathcal{K}_t(x,x_0) = Z_t(x,x_0) +$$
$$\int\int \mathrm{d}t_1\ \mathrm{d}t_0\delta(t_1+t_0-t)V\left(\begin{array}{c} x, A, x_0 \\ t_1, t_0 \end{array}\right)Z_{t_1}(x,A)Z_{t_0}(A,x_0),$$

where

$$Z_t(x,x_0) = \vartheta(t)\chi(x,x_0)\frac{\mu}{2\pi\mathrm{i}\hbar t}\exp\left(\frac{\mathrm{i}\mu}{2\hbar t}\mathrm{dist}^2(x,x_0)\right). \quad (4.17)$$

4.1.1 Proof of the kernel

Not only we can derive the above mentioned formula using the eigen-functions, but we can also prove by the direct computation that the formula (4.17) for the kernel on $\tilde{M}$ is correct. It means to prove that

$$(\mathrm{i}\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2\mu}\Delta)\vartheta(t)\mathcal{K}_t(x,x_0) = \mathrm{i}\hbar\delta(t)\delta(x,x_0). \quad (4.18)$$

In other words it means

$$\lim_{t\to 0_+} \mathcal{K}_t(x,x_0) = \delta(x,x_0) \tag{4.19}$$

$$(\mathrm{i}\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2\mu}\Delta)\mathcal{K}_t(x,x_0) = 0, \text{ for } t>0. \tag{4.20}$$

First of all let us mention that $Z_t(x,x_0)$ is kernel of the free propagator on R^2. It is true that

$$(\mathrm{i}\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2\mu}\Delta)Z_t(x,x_0) = \frac{\mathrm{i}\hbar}{4\pi t}\exp\left(\frac{\mathrm{i}\mu}{2\hbar t}\text{ dist }(x,x_0)^2\right)(\partial_{\overrightarrow{n}}\delta_{L_+} + \partial_{\overrightarrow{n}}\delta_{L_-}) \tag{4.21}$$

where δ_{L_+}, resp. δ_{L_-} are defined by

$$\langle\delta_{L_+},\psi(r,\varphi)\rangle = \int_0^\infty \psi(r,\pi_+)r\ \mathrm{d}r, \tag{4.22}$$

resp.

$$\langle\delta_{L_-},\psi(r,\varphi)\rangle = \int_0^\infty \psi(r,\pi_-)r\ \mathrm{d}r. \tag{4.23}$$

It holds true for any $f(t)\in C^1(\bar{R}_+)$,

$$\begin{aligned}(i\partial_t + \frac{1}{4}(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2)) \times \\ \int_0^t \frac{1}{\theta + \mathrm{i}\ln(\frac{(t-s)r_0}{sr})}\frac{1}{t-s}\exp(\frac{\mathrm{i}r^2}{t-s})f(s)\ \mathrm{d}s = \\ \frac{\pi r_0}{2r^2(r+r_0)}\exp(\mathrm{i}(r+r_0)\frac{r}{t})\Big[f(\frac{tr_0}{r+r_0})\delta'(\theta) - \\ \mathrm{i}\frac{r}{r+r_0}((1+\mathrm{i}\frac{r_0(r+r_0)}{t})f(\frac{tr_0}{r+r_0}) + \frac{tr_0}{r+r_0}f'(\frac{tr_0}{r+r_0}))\delta(\theta)\Big].\end{aligned} \tag{4.24}$$

After few transformations we obtain

$$\begin{aligned}(i\hbar\partial_t + \frac{\hbar^2}{2\mu}(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2)) \times \\ \int_0^t \frac{1}{\theta + \mathrm{i}\ln(\frac{(t-s)r_0}{sr})}\frac{\hbar}{t-s}\exp(\frac{\mathrm{i}r^2\mu}{2\hbar(t-s)})f(\frac{2\hbar s}{\mu})\ \mathrm{d}s = \\ \frac{\pi r_0\hbar^2}{r^2\mu(r+r_0)}\exp(\mathrm{i}(r+r_0)\frac{\mu r}{2\hbar t})\Big[f(\frac{2\hbar tr_0}{\mu(r+r_0)})\delta'(\theta) - \\ \mathrm{i}\frac{r}{r+r_0}((1+\mathrm{i}\frac{\mu r_0(r+r_0)}{2\hbar t})f(\frac{2\hbar tr_0}{\mu(r+r_0)}) + \frac{2\hbar tr_0}{\mu(r+r_0)}f'(\frac{2\hbar tr_0}{\mu(r+r_0)}))\delta(\theta)\Big].\end{aligned} \tag{4.25}$$

Setting $f(u) = \frac{1}{u}\exp\left(\frac{\mathrm{i}r_0^2}{u}\right)$ we obtain

$$(i\hbar\partial_t + \frac{\hbar^2}{2\mu}(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2)) \times$$
$$\int_0^t \frac{1}{\theta + \mathrm{i}\ln(\frac{(t-s)r_0}{sr})}\frac{\hbar}{t-s}\exp(\frac{\mathrm{i}r^2\mu}{2\hbar(t-s)})\frac{\mu}{2\hbar s}\exp\left(\frac{\mathrm{i}r_0^2\mu}{2\hbar s}\right)\,\mathrm{d}s =$$
$$\frac{\pi r_0\hbar^2}{r^2\mu(r+r_0)}\exp(\mathrm{i}(r+r_0)\frac{\mu r}{2\hbar t})[\frac{\mu(r+r_0)}{2\hbar t r_0}\exp\left(\frac{\mathrm{i}r_0^2\mu(r+r_0)}{2\hbar t r_0}\right)\delta'(\theta) -$$
$$\mathrm{i}\frac{r}{r+r_0}\{(1+\mathrm{i}\frac{\mu r_0(r+r_0)}{2\hbar t})\frac{\mu(r+r_0)}{2\hbar t r_0}\exp\left(\frac{\mathrm{i}r_0^2\mu(r+r_0)}{2\hbar t r_0}\right) +$$
$$\frac{2\hbar t r_0}{\mu(r+r_0)}(-\frac{\mu(r+r_0)^2}{4\hbar^2 t^2 r_0^2} - \frac{\mathrm{i}\mu^3 r_0^2(r+r_0)^3}{8\hbar^3 t^3 r_0^3})\exp\left(\frac{\mathrm{i}\mu r_0^2(r+r_0)}{2\hbar t r_0}\right)\}\delta(\theta)] =$$
$$\frac{\pi}{2r^2 t}\exp\left(\frac{\mathrm{i}\mu}{2\hbar t}(r+r_0)^2\right)\delta'(\theta). \quad (4.26)$$

Using this identity we obtain

$$(\mathrm{i}\hbar\partial_t + \frac{\hbar^2}{2\mu}(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2)) \times$$
$$\int\int \mathrm{d}t_1\,\mathrm{d}t_0\delta(t_1+t_0-t)V\begin{pmatrix} x, A, x_0 \\ t_1, t_0 \end{pmatrix} Z_{t_1}(x,A)Z_{t_0}(A,x_0) =$$
$$(\mathrm{i}\hbar\partial_t + \frac{\hbar^2}{2\mu}(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2)) \times$$
$$\frac{-\mathrm{i}\mu}{4\hbar\pi^2}\int_0^t [\frac{1}{\theta - \pi + \mathrm{i}\ln(\frac{(t_0-t)r_1}{t_0 r_2})} - \frac{1}{\theta + \pi + \mathrm{i}\ln(\frac{(t_0-t)r_1}{t_0 r_2})}] \times$$
$$\frac{1}{(t_0-t)t_0}\exp(\frac{\mathrm{i}\mu}{2\hbar t}(\frac{r^2}{t_0-t} + \frac{r_0^2}{t_0}))\,\mathrm{d}t_0 =$$
$$\frac{\mathrm{i}}{4r^2 t\pi}\exp\left(\frac{\mathrm{i}\mu}{2\hbar t}(r+r_0)^2\right)(\delta'(\theta-\pi) - \delta'(\theta+\pi)). \quad (4.27)$$

Finally using the formula [1]

$$\frac{1}{r^2}\delta'(\theta \mp \pi) = \pm\partial_{\overrightarrow{n}}\delta_{L_\pm} \quad (4.28)$$

we obtain (4.20).

Before the justification of (4.19), let us prove the following proposition:

[1] $\int_{R_+\times R}\frac{1}{r^2}\delta'(\theta\mp\pi)\varphi(r,\theta)r\,\mathrm{d}r\,\mathrm{d}\theta = -\int_{R_+}\frac{1}{r}\partial_\theta\varphi(r,\pm\pi)\,\mathrm{d}r = -\int_{L_\pm}\frac{\partial\varphi}{\partial\overrightarrow{n}}\,\mathrm{d}l$

Proposition 8. *Let $\psi(x) \in \mathcal{D}(R)$. Then*

$$\int_0^{+\infty} \frac{1}{it} \exp(\frac{i}{t}x)\psi(x)\ dx \to \psi(0),\ \textit{for } t \to \infty,$$

so

$$\lim_{t\to 0} \frac{1}{it} \exp(\frac{i}{t}x)\vartheta(x) = \delta(x)\ \textit{in } \mathcal{D}'.$$

Proof.

$$\lim_{t\to 0} \int_0^{+\infty} \frac{1}{\mathrm{i}t} \exp(\frac{\mathrm{i}}{t}x)\psi(x)\ \mathrm{d}x =$$

$$-\lim_{t\to 0}[\exp(\frac{\mathrm{i}}{t}x)\psi(x)]_0^{+\infty} + \lim_{t\to 0} \int_0^{+\infty} \exp(\frac{\mathrm{i}}{t}x)\psi'(x)\ \mathrm{d}x =$$

$$\psi(0) + \lim_{t\to 0}[-\mathrm{i}t \exp(\frac{\mathrm{i}}{t}x)\psi(x)]_0^{+\infty} + \lim_{t\to 0} \int_0^{+\infty} t \exp(\frac{\mathrm{i}}{t}x)\psi''(x)\ \mathrm{d}x = \psi(0).$$

□

The equation (4.19) comes from the fact that $Z_t(x, x_0)$ is a free propagator, so

$$\lim_{t\to 0} Z_t(x, x_0) = \delta(x, x_0). \tag{4.29}$$

This can be also easily proof using the previous proposition. It remains to prove that

$$\lim_{t\to 0} \int\int\ \mathrm{d}t_1\ \mathrm{d}t_0 \delta(t_1 + t_0 - t) V \begin{pmatrix} x, A, x_0 \\ t_1, t_0 \end{pmatrix} Z_{t_1}(x, A) Z_{t_0}(A, x_0) = 0$$

in $\mathcal{D}'$:

$$\lim_{t\to 0} \int\int\ \mathrm{d}t_1\ \mathrm{d}t_0 \delta(t_1 + t_0 - t) V \begin{pmatrix} x, A, x_0 \\ t_1, t_0 \end{pmatrix} Z_{t_1}(x, A) Z_{t_0}(A, x_0) =$$

$$-\lim_{t\to 0} \int_{R^2}\ \mathrm{d}x \int_0^t\ \mathrm{d}t_0 \frac{\exp(\frac{\mathrm{i}\mu}{2\hbar}(\frac{r^2}{t-t_0} + \frac{r_0^2}{t_0}))}{\theta - \pi + \mathrm{i}\ln(\frac{(t-t_0)r_0}{t_0 r})} \frac{\mu^2}{4\pi^2\hbar^2(t-t_0)t_0} \psi(x) +$$

$$\lim_{t\to 0} \int_{R^2}\ \mathrm{d}x \int_0^t\ \mathrm{d}t_0 \frac{\exp(\frac{\mathrm{i}\mu}{2\hbar}(\frac{r^2}{t-t_0} + \frac{r_0^2}{t_0}))}{\theta + \pi + \mathrm{i}\ln(\frac{(t-t_0)r_0}{t_0 r})} \frac{\mu^2}{4\pi^2\hbar^2(t-t_0)t_0} \psi(x),$$

where $r = \mathrm{dist}^2(x, (0,0))$ and $r_0 = \mathrm{dist}^2(x_0, (0,0))$.

$$\lim_{t\to 0} \int_{R^2}\ \mathrm{d}x \int_0^t\ \mathrm{d}t_0 \frac{\exp(\frac{\mathrm{i}\mu}{2\hbar}(\frac{r^2}{t-t_0} + \frac{r_0^2}{t_0}))}{\theta - \pi + \mathrm{i}\ln(\frac{(t-t_0)r_0}{t_0 r})} \frac{\mu^2}{4\pi^2\hbar^2(t-t_0)t_0} \psi(x) =$$

$$\lim_{t\to 0}\int_0^t \frac{\mu}{2\pi\hbar t_0}\exp(\frac{\mathrm{i}\mu}{2\hbar}\frac{r_0^2}{t_0})\times$$

$$(\int_0^{2\pi}\int_R \frac{\exp(\frac{\mathrm{i}\mu r^2}{2\hbar(t-t_0)})}{\theta-\pi+\mathrm{i}\ln(\frac{(t-t_0)r_0}{rt_0})}\frac{\mu}{2\pi\hbar(t-t_0)}\psi(r,\varphi)r\ \mathrm{d}r\ \mathrm{d}\varphi)\ \mathrm{d}t_0$$

$$=\lim_{t\to 0}\int_0^t \frac{\mu}{2\pi\hbar t_0}\exp(\frac{\mathrm{i}\mu}{2\hbar}\frac{r_0^2}{t_0})F(t-t_0)\ \mathrm{d}t_0,$$

where $F(t-t_0)=\int_0^{2\pi}\int_R \frac{1}{\theta-\pi+\mathrm{i}\ln(\frac{(t-t_0)r_0}{rt_0})}\frac{\mu}{2\pi\hbar(t-t_0)}\exp(\frac{\mathrm{i}\mu r^2}{2\hbar(t-t_0)}))\psi(r,\varphi)r\ \mathrm{d}r\ \mathrm{d}\varphi$. It is easy to see from the proposition 8 that $F(t-t_0)\in C^0([0,1])$ and that $\lim_{t\to t_0}F(t-t_0)=0$.

$$\lim_{t\to 0}\int_0^t \frac{1}{t_0}\cos(\frac{\mathrm{i}}{t_0})F(t-t_0)\ \mathrm{d}t_0=\lim_{t\to 0}\int_{1/t}^{+\infty}\frac{1}{s}\cos(\mathrm{i}s)F(t-\frac{1}{s})\ \mathrm{d}s=$$

$$(\lim_{t\to\infty}\int_{1/t}^{\lceil\frac{1}{2\pi t}+\frac{1}{4}\rceil}+\sum_{k=\lceil\frac{1}{2\pi t}+\frac{1}{4}\rceil}^{\infty}\int_{-\frac{\pi}{2}+2k\pi}^{\frac{\pi}{2}+2k\pi}+\int_{\frac{\pi}{2}+2k\pi}^{\frac{3\pi}{2}+2k\pi})\frac{1}{s}\cos(\mathrm{i}s)F(t-\frac{1}{s})\ \mathrm{d}s=$$

$$\lim_{t\to 0}\int_{1/t}^{\lceil\frac{1}{2\pi t}+\frac{1}{4}\rceil}\frac{1}{s}\cos(\mathrm{i}s)F(t-\frac{1}{s})\ \mathrm{d}s+$$

$$\lim_{t\to 0}\sum_{k=\lceil\frac{1}{2\pi t}+\frac{1}{4}\rceil}^{\infty}\int_{-\frac{\pi}{2}+2k\pi}^{\frac{\pi}{2}+2k\pi}\cos(\mathrm{i}s)\left(\frac{F(t-\frac{1}{s})}{s}-\frac{F(t-\frac{1}{s-\pi})}{s-\pi}\right)\ \mathrm{d}s=0.(4.30)$$

Because

$$\lim_{t\to 0}\int_1^{\lceil\frac{1}{2\pi}+\frac{1}{4}\rceil}|\frac{1}{s}F(t(1-\frac{1}{s}))|\ \mathrm{d}s=0,$$

because there exists integrable majorant and we can change the limit and the integral.

$$\lim_{t\to 0}\sum_{k=\lceil\frac{1}{2\pi t}+\frac{1}{4}\rceil}^{\infty}\int_{-\frac{\pi}{2}+2k\pi}^{\frac{\pi}{2}+2k\pi}|\cos(\mathrm{i}s)\left(\frac{F(t-\frac{1}{s})-F(t-\frac{1}{s-\pi})}{s-\pi}-\frac{F(t-\frac{1}{s})\pi}{s(s-\pi)}\right)|\le$$

$$\lim_{t\to 0}\int_{\lceil\frac{1}{2\pi t}+\frac{1}{4}\rceil}^{+\infty}\frac{C}{s(s-\pi)^2}+\frac{C}{s^2(s-\pi)^2}\ \mathrm{d}s=0,$$

where we have used that $|F(t-\frac{1}{s})-F(t-\frac{1}{s-\pi})|\le C|\frac{1}{s(s-\pi)}|$, because $F\in C^0([0,1])$.

4.1.2 Use of Schulman ansatz

The summation in Schulman ansatz is based on the formula

$$\sum_{n=-\infty}^{\infty}\exp(-2\pi\mathrm{i}\alpha n)\left(\frac{1}{2\Phi+2\pi n-\pi+is}-\frac{1}{2\Phi+2\pi+\pi+is}\right)=$$

$$-sin(\pi\alpha)\frac{\exp(-\alpha(s-\mathrm{i}\Phi))}{1+\exp(-s+\mathrm{i}\Phi)}, \quad (4.31)$$

for $\alpha \in (0,1)$, $\Phi, s \in R$, $s \neq 0$. We obtain the propagator

$$\mathcal{K}_t^U(x,x_0) = \exp(-2\pi i\alpha n)\frac{\mu}{2\pi\mathrm{i}\hbar t}\exp(i\mu|x-x_0|^2/2\hbar t) -$$
$$\frac{sin(\pi\alpha)}{\pi}\int_{-\infty}^{\infty} \mathrm{d}s\frac{\mu}{2\pi\mathrm{i}\hbar t}\exp(i\mu R^2(s)/2\hbar t)\frac{\exp(-\alpha(s-\mathrm{i}\Phi))}{1+\exp(-s+\mathrm{i}\Phi)}, \quad (4.32)$$

where n is such that $\Phi + 2\pi n \in (-\pi, \pi)$.

Chapter 5

Models for More Anyons

Anyons are indistinguishable particles with the configuration space of one particle R^2. In this section, we will describe model for two and more anyons, which enables us to use Schulman Ansatz (chapter 3).

Let $\delta_N \subset R^{2N}$ be a diagonal (position of at least two points coincide) and let S_N be a group of permutations acting naturally on R^{2N}. Configuration space of N indistinguishable particles which cannot penetrate each other is

$$M_N = (R^{2N} - \delta_N)/S_N. \tag{5.1}$$

Fundamental group $\pi_1(M_N)$ of the manifold M_N is a braid group denoted as B_N[1] and it becomes the structure group of its universal covering $\widetilde{M}_N$.

Different quantizations corresponds to different choices of a unitary representation U of the group B_N. Properties of the group B_N imply that

$$U(\sigma_1) = \ldots = U(\sigma_{N-1}) = \exp(2\pi i\alpha).$$

Let us suppose that $\alpha \in (0, 1)$.

[1]The braid group B_N is an infinite group with $N-1$ generators $\sigma_1, \ldots, \sigma_{N-1}$ which satisfy two following properties:

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},$$

for $i = 1, 2, \ldots, n-2$ and

$$\sigma_i\sigma_j = \sigma_j\sigma_i,$$

for $|i-j| \geq 2$. In case of S_N also the third property is valid:

$$\sigma_i\sigma_i = \text{identity},$$

which is not valid for B_N.

The free Hamiltonian is given by the structure of M_N, resp. $\tilde{M}_N$ following from the Euclidean geometry on R^{2N}. Suppose we have a potential V defined on M_N and let $\tilde{V}$ be its lift to $\tilde{M}_N$. Hamiltonian

$$\tilde{H}_U := \tilde{\Delta} + \tilde{V}, \tag{5.2}$$

acts in the Hilbert space

$$\mathcal{H}_U = \{\psi, \text{quadratical integrable}, \psi(g.x) = U(g)\psi(x) \forall g \in B_N\}, \tag{5.3}$$

The integration in the scalar product is over any fundamental domain[2] of the action of B_N and $\tilde{\Delta}$ is Laplce-Beltram operator on $\tilde{M}_N$.

Next we cut the manifold M_N to get a simply connected domain D. D can also be taken as one sheet of universal covering $\tilde{M}_N$. Let us define the coordinates of a point $x \in R^2$ by $x = (x^1, x^2)$ and let E_{jk} be hyperplanes determinated by the equations $x_j^2 = x_k^2$, where $x \equiv (x_1, x_2, \ldots, x_N) \in R^{2N}$. Let $E := \cup_{j<k} E_{jk} \supset \delta_N$, then $R^{2N} - E$ be a subset of open simply connected domains. Fix one of them, D, defined by

$$x_1^2 < x_2^2 < \ldots < x_N^2.$$

D can be identified with a subset of M_N. Split $\partial D \backslash \delta_N$ to $2(N-1)$ cells $C_{j,\epsilon}, j = 1, \ldots, N-1, \epsilon = \pm$ given by the equations

$$x_1^2 < \ldots < x_j^2 = x_{j+1}^2 < \ldots < x_N^2, \epsilon(x_j^1 - x_{j+1}^1) < 0. \tag{5.4}$$

Then D is a fundamental domain of $\tilde{M}_N$.

One unitary equivalent description of quantum mechanics can be obtained by the restriction from $\tilde{M}_N$ to D (the inverse process is evident because the Hilbert space on $\tilde{M}_N$ are U-equivariant functions). Hilbert space on D is $L^2(D)$ with the following boundary conditions: Let $x \in C_{j,+}$, then from (5.3) one obtains the boundary conditions

$$\psi(\ldots, x_{j+1}, x_j, \ldots) = \exp(-2\pi i \alpha \,\mathrm{sgn}(x_{j+1}^1 - x_j^1))\psi(\ldots, x_j, x_{j+1}, \ldots),$$
$$\text{in the limit } (x_{j+1}^2 - x_j^2) \downarrow 0. \tag{5.5}$$

Hamiltonian is also obtained by restriction from $\tilde{M}_N$ to D.

It is useful to separate the center of mass by appropriate choice of coordinates:

$$z = \frac{1}{N}(x_1 + \ldots + x_N) \text{ a } y_1 = x_{j+1} - x_j, j = 1, \ldots, N-1.$$

[2]Fundamental domain $D \subset \tilde{M}$ is a simply connected domain such that $g.D$ are disjoin ($g \in \pi_1(M)$) and $\bigcup_g g.D$ is equal to $\tilde{M}$ up to zero set measure.

Diagonal does not depend to the configuration space, so

$$y_j + y_{j+1} + \ldots + y_k \neq 0, \text{ pro } j \leq k. \tag{5.6}$$

Generators of the group S_N act in the following way:

$$\begin{aligned} \sigma_j(y) = y', \text{ with } y'_k &= y_k, \text{ pro } |k-j| > 1 \\ &= y_k + y_j, \quad |k-j| = 1 \\ &= -y_j, \quad k-j = 0. \end{aligned} \tag{5.7}$$

The action of S_N on z is trivial.

Laplacian expressed in new coordinates is in the form:

$$\Delta_x = \frac{1}{N}\Delta_z + 2\Delta_y - 2\sum_{j=1}^{N-2} \nabla_{y_j} \cdot \nabla_{y_{j+1}}. \tag{5.8}$$

New coordinates can be also defined on D: $D \cong R^2 \times D^{red}$, $D^{red} \equiv R_+^2 \times \ldots \times R_+^2$, where $R_+^2 := R \times R_+$ a $R_+ :=]0, +\infty[$, $\partial D \cong R^2 \times \partial D^{red}$.

In the following we will consider only the non-trivial part. Coordinates of D^{red}, $(\xi, \eta) \equiv (\xi_1, \ldots, \xi_{N-1}, \eta_1, \ldots, \eta_{N-1})$, $\xi_j \in R$, $\eta_j > 0$, $\forall j$, satisfy boundary conditions

$$\psi(\xi', \eta)|_{\eta_j = 0_+} = \exp(-2\pi i \alpha \xi_j)\psi(\xi, \eta)|_{\eta_j = 0_+}, \quad j = 1, \ldots, N-1, \tag{5.9}$$

where

$$\begin{aligned} \xi'_k &= \xi_k, \text{ pro } k \neq j-1, j, j+1, \\ \xi_{j\pm1} &= \xi_{j\pm1} + \xi_j \text{ a } \xi'_j = -\xi_j. \end{aligned}$$

5.1 Propagator for two free anyons

In this case $M \cong R^2 \times M^{red}$ with variables $z := (x_1 + x_2)/2, y := x_2 - x_1$. Then $\nabla_x = \frac{1}{2}\nabla_z + 2\nabla_y$. $\tilde{M}^{red} = R_+ \times R$ in the polar coordinates, $D^{red} = R_+ \times (0, \pi)$. The group $\Gamma = \pi_1(M)$ is an infinite group with one generator h, $h.(r, \phi) = (r, \phi - \pi)$, $U(h) = \exp(2\pi \mathrm{i}\alpha)$. Boundary condition (5.9) is

$$\psi(r, \pi_-) = \exp(-2\pi \mathrm{i}\alpha)\psi(r, 0_+). \tag{5.10}$$

We will focus only on the propagator in $L^2(D^{red})$ with the previous boundary condition, with the mass 1/4. Hamiltonian is given by the equation

$$H = -4\hbar^2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} \right), \tag{5.11}$$

where

$$\psi(r,\phi-\pi)=\exp(2\pi\mathrm{i}\alpha)\psi(r,\phi). \tag{5.12}$$

We try to find eigenfunction in the separated form

$$\psi(r,\phi)=\exp(\mathrm{i}m\phi)\beta(r).$$

It means to solve the equation

$$-4\hbar^2\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right)\exp(\mathrm{i}m\phi)\beta(r)=4\hbar^2k^2\beta(r).$$

Finally we obtain the equation

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{m^2}{r^2}-k^2\right)\beta(r)=0.$$

The solutions are Bessel functions

$$\beta(r)=a_mJ_m(kr)+b_mJ_{-m}(kr). \tag{5.13}$$

The boundary condition (5.12) implies that

$$\exp(\mathrm{i}m(\phi-\pi))=\exp(\mathrm{i}m\phi+2\pi\mathrm{i}\alpha). \tag{5.14}$$

Solving this equation and eliminating functions with singularity in $r=0$ we obtain the complete set of generalized eigen-functions

$$\{\pi^{-1/2}J_{2|n+\alpha|}(pr)\exp(-2\mathrm{i}(n+\alpha)\phi),n\in Z,p>0\}. \tag{5.15}$$

Consequently, as in the section 4.1,

$$\mathcal{K}_t^U(y,y_0)=$$
$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\int_0^{\infty}\exp(-2\mathrm{i}p^2t)J_{2|n+\alpha|}(pr)J_{2|n+\alpha|}(pr_0)\exp(-2\mathrm{i}(n+\alpha)(\phi-\phi_0))p\ \mathrm{d}p.$$

The same result may be obtained by using Schulman's ansatz. The propagator $\mathcal{K}_t(.,.)$ on $\tilde{M}^{red}$ is given by

$$\mathcal{K}_t(x,x_0)=\chi(x,x_0)\frac{1}{8\pi\mathrm{i}t}\exp(\mathrm{i}\mathrm{dist}^2(x,x_0)/8t)+$$
$$\frac{1}{8\pi\mathrm{i}t}\int_{-\infty}^{\infty}\frac{\mathrm{d}s}{2\pi}\left(\frac{1}{\Phi-\pi+\mathrm{i}s}-\frac{1}{\Phi+\pi+\mathrm{i}s}\right)\exp(\mathrm{i}R^2(s)/8t).$$

Applying the Schulman ansatz (chapter 3.3) we obtain

$$\sum_{n=-\infty}^{\infty} \exp(2\pi \mathrm{i}\alpha)\mathcal{K}_t(g.x, x_0) =$$

$$\sum_{n=-\infty}^{\infty} \exp(2\pi \mathrm{i}\alpha)(\chi(g.x, x_0)\frac{1}{8\pi \mathrm{i}t}\exp(\mathrm{i}\mathrm{dist}^2(g.x, x_0)/8t) +$$

$$\frac{1}{8\pi \mathrm{i}t}\int_{-\infty}^{\infty}\frac{\mathrm{d}s}{2\pi}\left(\frac{1}{\Phi - \pi n + \pi + \mathrm{i}s} - \frac{1}{\Phi + \pi n + \pi + \mathrm{i}s}\right)\exp(\mathrm{i}R^2(s)/8t)), \quad (5.16)$$

where $R^2(s) = r^2 + r_0^2 + 2rr_0\cosh(s)$. Finally, using the identity

$$\frac{1}{2}\sum_{n=-\infty}^{\infty} \exp(2\pi \mathrm{i}\alpha n)\left(\frac{1}{\Phi + \pi n - \pi + \mathrm{i}s} - \frac{1}{\Phi + \pi n + \pi + \mathrm{i}s}\right) =$$

$$-2\sin(2\pi\alpha)\frac{\exp(2\alpha(s - i\Phi))}{1 - \exp(2(s - i\Phi))}, \qquad (5.17)$$

valid for $\alpha \in (0, 1), s \neq 0$ we obtain K_t^U. Restriction of K_t^U on D^{red} (denoted by $\mathcal{K}_t^D$, used only for simplification in notation) is in form

$$\mathcal{K}_t^D(y, y_0) = \frac{1}{8\pi \mathrm{i}t}\exp(i|y - y_0|^2/8t) + \exp(\mp 2\pi \mathrm{i}\alpha)\frac{1}{8\pi \mathrm{i}t}\exp(i|y + y_0|^2/8t)$$

$$-\frac{\sin(2\pi\alpha)}{\pi}\int_{-\infty}\infty\, \mathrm{d}s\frac{1}{4\pi \mathrm{i}t}\exp(iR^2(s)/8t)\frac{2\alpha(s - i\Phi)}{1 - \exp(2(s - \mathrm{i}\Phi))},$$

where $\Phi \in (0, \pi)$, resp. $\Phi \in (-\pi, 0)$.

Summary

One of the possibilities how to describe quantum mechanics on non-simply connected manifold is to use the Hilbert space of U-equivariant functions. In this paper, there is mainly investigated the proof of the Schulman ansatz, which permits to derive the kernel of the propagator in this case. For this purpose, the Schwartz kernel theorem was introduced and its reformulation for U-equivariant was derived in case of continuous and locally linearly continuous manifolds (section 2.1). In the next section the properties of the kernel are explored. Schulamn ansatz is then proved for identity operator in chapter 3, where the idea of the rigorous proof is also formulated. We have proved that the left hand side of the equality (3.4) is well-defined, it remains to future investigation to proof the convergence of the sum on the right hand side of this equality.

On the example of two anyons and Aharonov-Bohm effect with one solenoid we can see the use of Schulman ansatz which brings the result very quickly in case of knowledge of the kernel of the propagator defined on $L^2(\tilde{M})$. Nevertheless this knowledge is the limitative factor for this method. For example in case of tree anyons the kernel of the free propagator on $L^2(\tilde{M})$ is not known and we can conclude only the existence and uniqueness of the kernel for U-equivariant functions. To find this kernel remains one of the open question of quantum mechanics on non-simply connected manifold.

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