České vysoké učení technické v Praze
Fakulta jaderná a fyzikálně inženýrská

DIPLOMOVÁ PRÁCE

České vysoké učení technické v Praze
Fakulta jaderná a fyzikálně inženýrská
Katedra matematiky

Kvantová mechanika na násobně souvislých varietách

Quantum mechanics on multiply connected manifolds

Petra Kocábová

Školitel: Prof. Ing. Pavel Šťovíček, DrSc., KM, FJFI, ČVUT v Praze
Akademický rok: 2006/2007

Název práce:	Kvantová mechanika na násobně souvislých varietách
Autor:	Petra Kocábová
Obor:	Matematické modelování
Druh práce:	Diplomová práce
Vedoucí práce:	Prof. Ing. Pavel Šťovíček, DrSc., KM, FJFI, ČVUT v Praze

Abstrakt:

Tato práce pojednává o kvantové mechanice na násobně souvislých varietách. Diskutovali jsme rozklad levé regularní reprezentace na ireducibilní reprezentace. Tento rozklad lze vytvořit pomocí von Neumannova direktního integrálu. Pokud provedeme tento rozklad vzhledem k centru algebry generované levou regulární reprezentací, lze zformulovat zobecněnou Peter-Weyl Plancherelovu větu. Dále jsem v této práci ukázala, že Blochova analýza může být zobecněna na souvislé a lokálně lineárně souvislé variety s fundamentální grupou Typu I. V případě násobně souvislých variet, Schulmannův ansatz se používá k odvození jádra propagátoru v prostoru ekvivariantních funkcí. Při platnosti předchozích předpokladů jsem diskutovala a dokázala platnost Schulmannova ansatzu.

Klíčové slova: Blochova analýza, Schulmanův ansatz, násobně souvislá varieta

Title:	Quantum mechanics on multiply connected manifolds
Author:	Petra Kocábová

Abstract:

This work analysis quantum mechanics in multiply connected spaces. The decomposition of left regular representations to irreducible representation is discussed. This decomposition is made using the von Neumann direct integral. Using this decomposition with respect to the center of the algebra generated by the left regular representation, the generalized Peter-Weyl Plancherel theorem is formulated. It is shown, that the Bloch analysis may be generalized to the multiply connected and locally linearly connected manifolds with the fundamental group of Type I. In case of multiply connected manifolds, Schulman ansatz may be used to compute the kernel of the propagator on the Hilbert space of equivariant functions. This formula is discussed and is proved in case of the previous assumptions.

Key words: Bloch analysis, Schulman ansatz, multiply connected manifolds

Prohlašuji, že jsem svou diplomovou práci vypracovala samostatně a použil jsem pouze podklady (literaturu, projekty, SW atd.) uvedené v přiloženém seznamu.

V Praze dne 10.května 2007 ....

podpis

Chtěla bych poděkovat všem lidem, kteří mi při mé práci jakkoliv pomohli a především pak svému školiteli, Prof. Ing. Pavlu Šťovíčkovi, DrSc., za jeho cenné rady a připomínky, bez kterých bych tuto práci jen stěží dokončila.

Contents

Introduction

Quantum mechanics on non-simply connected manifolds covers a large group of quantum models where the quantum properties of the systems are observable. There exists more than one equivalent model, how to describe the quantum mechanics on non-simply connected manifolds, some of them are discussed in [Koc1]. The Hilbert space of Λ-equivariant functions, where Λ is a unitary representation of the fundamental group, defined on the universal covering $\tilde{M}$ of the manifold M is used in one of them. Schulman ansatz proposes the straightforward connection between the kernel of the propagator of the Hamiltonian defined on $L^2(\tilde{M})$ and the kernel of the propagator corresponding to formally the same Hamiltonian acting on Λ-equivariant functions. The advantage of this processus is that in some cases, it is easier to find the kernel of the propagator on the simply-connected manifold $\tilde{M}$ than in case of Λ-equivariant functions. The aim of this work was to prove the Schulman ansatz not only formally and to discuss its field of validity.

In the first chapter the basic definitions and theorems are pointed out, mainly those which will be used in the following parts of this thesis.

If the fundamental group is not abelian or finite, von Neumann direct integral is used to find the irreducible representations of the fundamental group of the configuration manifold. In the second chapter, the main theorems are discussed and some of the properties are mentioned. In case of general group, we may obtain the infinite dimensional irreducible representation and some of the properties valid for the abelian or finite groups are not satisfied. But still the Fourier-Stieltjes transformation can be defined and the generalize Peter-Weyl Plancherel theorem holds.

Fourier-Stieltjes transformation for general locally compact group is defined in the third chapter, in case of abelian groups and the construction of the Plancherel measure is discussed.

The following two chapters are the main chapters of this work. In both of them, the assumption that the fundamental group is of Type I is made. This property is necessary if we want to formulate the Peter-Weyl Plancherel with the irreducible unitary representations. In the fourth chapter I generalized the Bloch analysis, ordinary done for abelian groups, to the locally compact groups of Type I. Also the term of unitary equivariant function is generalized in case of irreducible representation (in case of abelian groups, only the representation of the dimension 1 occurs) and also the definition of the Hamiltonian on such spaces is discussed and the decomposition of such operator with respect to the von Neumann direct integral to the unitary equivariant function is made.

The fifth chapter follows the theory of generalized Bloch analysis and using the inverse Fourier-Stieltjes transformation Schulman ansatz is formulated and proved in models, where the configuration spaces are the manifolds with the locally compact fundamental group of Type I.

The last chapter describes the well known Aharonov-Bohm effect with two solenoids. Finally the Schulman ansatz is used to find the kernel of the free propagator of this model.

List of Symbols

$A = \int_Y A_y \, \mathrm{d}y$	decomposition of A with respect to the direct integral	p. 32
$\mathcal{B}(\tilde{\mathcal{M}})$	bounded operators on $\tilde{\mathcal{H}}$	p. 33
$Dom(A)$	domain of A	p. 13
C	complex numbers	p. 18
$C_0(\Gamma)$		p. 26
$C_0^*(\Gamma)$		p. 26
D	fundamental domain of $\tilde{M}$	p. 11
$\mathcal{D}'(X)$	set of all distribution on X	p. 38
Ff	Fourier-Stieltjes transform of f	p. 27
F_y	Fourier-Stieltjes transform in $y \in Y$	p. 27
g	element of Γ	p. 11
H	Hamiltonian	p. 37
$\mathrm{Hom}_\Gamma(V, V')$		p. 29
$\mathcal{H}$	Hilbert space	p. 12
$\mathcal{H}_y$	Hilbert space of R_y-equivariant functions	p. 32
$\tilde{\mathcal{H}}_y$	Hilbert space of Λ_y-equivariant function	p. 34
$\mathcal{H}_1 \otimes \mathcal{H}_2$		p. 32
K_t	kernel of $U(t)$	p. 39
K_t^y	kernel of $\tilde{U}_y(t)$	p. 39
$\mathcal{I}_2(\mathcal{L}_y)$	Hilbert-Schmidt operators on $\mathcal{L}_y$	p. 30
I	identity operator	p. 39
$\mathcal{L}_y'$		p. 30
$\mathcal{L}_y$		p. 30
$\tilde{M}$	covering space of M	p. 11
$\mathcal{M}(\Gamma)$		p. 26
$Q(q)$	form domain of q	p. 12
q	quadratic form	p. 12
R_y	decomposition of R by the direct integral	p. 20
$R(g)$	left regular representation of Γ	p. 16
$U(t)$	propagator on $L^2(\tilde{M})$	p. 38
$\tilde{U}_y(t)$	propagator on $\tilde{\mathcal{H}}_y$	p. 39

$tr[.]$	trace, generalized trace	p. 23
W	weakly closed algebra generated by $R(g), g \in \Gamma$	p. 20
W_y	weakly closed algebra generated by $R_y(g)$	p. 20
Y		p. 19
Z	center of W	p. 23
Γ	group, often fundamental group	p. 12
$\hat{\Gamma}$	dual object to Γ	p. 14
$\Gamma/\tilde{M}$	right factorization of $\tilde{M}$ by Γ	p. 41
$\Lambda_y(g)$	irreducible representation on $\mathcal{L}_y$	p. 30
$\pi_1(M, x_0)$	fundamental group of X with base point x_0	p. 10
$\pi : \tilde{M} \to M$		p. 13
Φ		p. 35
$\tilde{\Phi}$		p. 39
$\int_Y \mathcal{H}_y \, \mathrm{d}y$	direct integral of Hilbert space	p. 19
1	group identity	p. 23
$[.]$	homotopy class	p. 10
*	adjoint operation	p. 18
Δ_{LB}	Laplace-Beltrami operator	p. 36
$\#$	number of elements	p. 31

Chapter 1

Basic Definitions

1.1 Covering space

Definition 1 (Path and homotopy class, [Hat] chapter 1.1)**.** *Path in a space X is a continuous map $f : [0, 1] \to X$. A homotopy of paths in X is a family $f_t : [0, 1] \to X$, $0 \leq t \leq 1$, such that*

- *the endpoints $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t*
- *the associated map $F : I \times I \to X$ defined by $F(s, t) = f_t(s)$ is continuous*

When two paths f_0 and f_1 are connected in this way by a homotopy f_t, they are said to be homotopic. This property will be denoted by $f_0 \simeq f_1$.

Proposition 1 ([Hat] proposition 1.2)**.** *The relation of homotopy of paths with fixed endpoints is an equivalence relation.*

Definition 2. *Composition of two paths $f, g : [0, 1] \to X$, such that $f(1) = g(0)$ is defined by the formula*

$$f \circ g(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases} .$$

Path with the same starting and ending point $f(0) = f(1) = x_0$ are called the loops, x_0 is called the basepoint.

Definition 3 (Fundamental group, [Hat] proposition 1.3)**.** *Let f be a path in a space X. Set of all homotopy classes $[f]$ of loops at the base point x_0 is called the fundamental group of X and is denoted by $\pi_1(X, x_0)$.*

If X is path-connected, the group $\pi_1(X, x_0)$ is, up to isomorphism, independent of the choice of the basepoint x_0. In this case the notation $\pi_1(X, x_0)$ is often abbreviated to $\pi_1(X)$.

Definition 4 ([Hat] proposition 1.6)**.** *A space is called simply-connected if it is path-connected and its fundamental group is trivial. A space is called multiply-connected if it is path-connected and its fundamental group is not trivial.*

Definition 5 ([Hat] chapter 1.3)**.** *A covering space of a space X is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ satisfying the following condition: There exists an open cover (U_α) of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped by p homomorphically onto U_α .*

Definition 6. *Two covering spaces $(\tilde{X}_0, p_0), (\tilde{X}_1, p_1)$ are isomorphic if there exists a homeomorphism $f : \tilde{X}_0 \to \tilde{X}_1$ such that $p_1 \circ f = p_0$.*

Proposition 2. *A covering space of a connected, locally path-connected topological space is connected.*

Definition 7 (Universal covering)**.** *Universal covering of X is $(\tilde{X}, \pi, x_0)$ where $\tilde{X} = \{(x_1, \gamma)/\sim$, where γ is a path from x_0 to $x_1 \in X\}$ and $(x_1, \gamma_1) \sim (x_2, \gamma_2)$ if and only if $x_1 = x_2$ and $\gamma_1.\gamma_2^{-1}$ is homotopically equivalent to a point, $\pi : \tilde{X} \to X$ defined by $\pi((x, \gamma)) = x$.*

Proposition 3. *The universal covering manifold $\tilde{X}$ is a principal fibre bundle over X with group $\pi_1(X, x_0)$ and projection π.*

Proposition 4. *The universal covering space of a connected manifold X is simply connected, and it is the only one covering space of X with this property (up to an isomorphism).*

Definition 8. *A fundamental domain D of $\tilde{X}$ is a simply connected domain of X such that $\pi^{-1}(D) = \tilde{X}$ up to zero set measure and $g.D \cap D = \emptyset$, $g \neq 1$, $g \in \pi_1(X)$.*

Definition 9 (Induced homeomorphism)**.** *Let $p : X \to Y$ be a continuous map, $y_0 \in Y$, $x_0 \in X$ base points such that $y_0 = p(x_0)$. Then p induces a homomorphism $p_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by composing loops $f : I \to X$ based at x_0 with p, it means $p_*[f] = [p \circ f]$.*

1.2 Properly discontinuous action

Definition 10 (properly discontinuous 1, [Hat] p. 72)**.** *Let* Γ *be a group. Action of the group on a manifold* X *is called properly discontinuous, if* $\forall x \in X$ *there exists a neighborhood* U *such that for varying* $g \in \Gamma$ *all the images* $g.U$ *are disjoint, it means that* $g_1.U \bigcap g_2.U \neq 0$ *implies* $g_1 = g_2$*.*

Definition 11 (properly discontinuous 2, [Hat] p. 73)**.** *Action of the group* Γ *on* X *is called properly discontinuous if for every point* $x \in X$ *there exists a neighborhood* U *such that* $U \cap g.U$ *is nonempty only for finitely many* $g \in \Gamma$*.*

Proposition 5. *If* Γ *acts freely, then the previous two definitions are equivalent.*

Definition 12. *Let* $p : \tilde{X} \to X$*,* $\tilde{X}$ *be a covering of* X*.* $G(\tilde{X}, p, X)$ *is a group defined by*

$G(\tilde{X}, p, X) = \{h : \tilde{X} \to \tilde{X}, h$ *is a homeomorphism such that* $p \circ h = p, \}$

Proposition 6 ([Koz])**.** *If* $p_*\pi_1(\tilde{X}, \tilde{x}_0)$ *is normal subgroup of* $\pi_1(X, x_0)$ *then*

$$G(\tilde{X}, p, X) \cong \pi_1(X, x_0)/p_*\pi_1(\tilde{X}, \tilde{x}_0),$$

where $x_0 = p(\tilde{x}_0)$*.*

If we take as $\tilde{X}$ the universal covering of X, because it is simply connected ($\pi_1(\tilde{X}, \tilde{x}_0)$ is trivial) $p_*\pi_1(\tilde{X}, \tilde{x}_0)$ is also trivial, so normal and $G(\tilde{X}, p, x)$ is isomorphic to $\pi_1(X, x)$.

Theorem 1 ([Koz])**.** *If* $\tilde{X}$ *is connected and locally linearly connected, then the action of the group* $G(\tilde{X}, p, X)$ *on* $\tilde{X}$ *is properly discontinuous.*

As a direct corollary one obtains that if the universal covering space is connected and locally linearly connected, then the action of $\pi_1(X, x)$ is properly discontinuous. So a sufficient requirement is X connected and locally linearly connected.

1.3 Quadratic form

Definition 13 (quadratic form)**.** *Let* $\mathcal{H}$ *be a Hilbert space. A quadratic form is a map* $q : Q(q) \times Q(q) \to C$*, where* $Q(q)$ *is a dense linear subset of* $\mathcal{H}$ *called the form domain, such that* $q(., \psi)$ *is conjugate linear,* $q(\varphi, .)$ *is linear for* $\varphi, \psi \in Q(q)$*,* q *is symmetric if* $q(\varphi, \psi) = \overline{q(\psi, \varphi)}$*. If* $q(\varphi, \varphi) \geq 0$ *for all* $\varphi \in Q(q)$ *then* q *is called positive. If there exists* M *such that* $q(\varphi, \varphi) \geq -M\|\varphi\|^2$ *then* q *is semibounded.*

Definition 14. *Let q be a semibounded quadratic form, $q(\psi, \psi) \geq -M\|\psi\|^2$. q is called closed if $Q(q)$ is complete under the norm*

$$\|\psi\|_{+1} = \sqrt{q(\psi, \psi) + (M+1)\|\psi\|^2}, \tag{1.1}$$

where $\|.\|$ is the norm generated by the scalar product. If q is closed and $D \subset Q(q)$ is dense in $Q(q)$ in the $\|.\|_{+1}$ norm, then D is called a form core for q.

Remark 1: $\|.\|_{+1}$ comes from the inner product

$$(\psi, \varphi)_{+1} = q(\psi, \varphi) + (M+1)(\psi, \varphi). \tag{1.2}$$

Proposition 7. *q is closed if and only if for $\forall \psi_n \in Q(q)$, such that $\psi_n \to \psi$ and $q(\psi_n - \psi_m, \psi_n - \psi_m) \to 0$ for $n, m \to \infty$, then $\psi \in Q(q)$ and $q(\psi_n - \psi, \psi_m - \psi) \to 0$.*

Theorem 2 ([Ree])**.** *If q is a closed semibounded quadratic form, then q is the quadratic form of a unique self-adjoint operator.*

Theorem 3 (the Fridrichs extension)**.** *Let A be a positive symmetric operator and let $q(\varphi, \psi) = (\varphi, A\psi)$ for $\varphi, \psi \in Dom(A)$. Then q is a closable quadratic form and its closure $\bar{q}$ is the quadratic form of a unique self-adjoint operator $\bar{A}$. $\bar{A}$ is a positive extension of A, and the lower bound of its spectrum is the lower bound of q. Further, $\bar{A}$ is the only self-adjoint extension of A whose domain is contained in the form domain of $\bar{q}$.*

Remark: It is sufficient for A to be bounded from below.

1.4 Theory of irreducible representations

Definition 15 ([Ross1] definition 21.1)**.** *Let Γ be a group. A map $g \to V(g)$, where $g \in \Gamma$ and $V(g)$ are unitary operators on a Hilbert space $\mathcal{H}$, is called unitary representation of Γ if*

$$\begin{aligned} V(g)V(g') &= V(g.g') \\ V(1) &= I. \end{aligned} \tag{1.3}$$

This representation is called irreducible if $\{0\}$ and $\mathcal{H}$ are the only closed subspaces of $\mathcal{H}$ that are invariant under the operators $V(g)$ for all $g \in \Gamma$.

Definition 16 ([Fou] chapter 2.1)**.** *If V is a representation of Γ, its character χ_V is the complex-valued function on the group defined by*

$$\chi_V(g) = tr(V(g)). \tag{1.4}$$

In particular $\chi_V(hgh^{-1}) = \chi(g)$.

Definition 17. *Let Γ be a locally compact group. Let $\mathcal{V}(\Gamma)$ be the set of all continuous irreducible unitary representations V of Γ. Let us factorize this set by the equivalence defined by linear isometries of the representations. This set will be denoted by $\hat{\Gamma}$ and we will call it the dual object of Γ. Each $V \in \hat{\Gamma}$ is a set of representation of Γ equivalent to some fixed representation.*

Definition 18 ([Ross1] definition 24.1)**.** *For fixed $g \in \Gamma$, let g' be the function on $\hat{\Gamma}$ such that*

$$g'(V) = V(g) \tag{1.5}$$

for all $V \in \hat{\Gamma}$ and let τ be the mapping

$$\tau(g) = g'. \tag{1.6}$$

Let $\hat{\hat{\Gamma}}$ be the dual object of $\hat{\Gamma}$. In the case of abelian groups , $\hat{\hat{\Gamma}}$ is called the second character group of Γ.

Theorem 4 ([Ross1] theorem 24.8)**.** *Let Γ be a finite group. The mapping τ is a topological isomorphism of Γ into $\hat{\hat{\Gamma}}$.*

Theorem 5 (Schur's lemma, [Jung] theorem 3.6)**.** *Let V be a unitary representation of Γ on $\mathcal{H}$. Suppose that $A : \mathcal{H} \to \mathcal{H}$ is a bounded linear transformation. Then V is irreducible if and only if*

$$AV(g) = V(g)A \tag{1.7}$$

implies, that $A = \lambda I$.

Theorem 6. *Let Γ be an Abelian group. Then every irreducible representation of Γ on the Hilbert space $\mathcal{H}$ is one dimensional.*

Proof. For every element of $g \in \Gamma$, g acts on $\mathcal{H}$ and it is a homomorphism. In case of Abelian group it is a homomorphism and it commutes with every element of Γ. Using the Schur's lemma, g acts by multiple of identity. Every subspace of $\mathcal{H}$ is invariant, so $\mathcal{H}$ must be one dimensional. □

Therefore in case of Abelian groups, we can identify $\hat{\Gamma}$ with the group of all characters.

Theorem 7 (Gel'fand-Raikov, [Ross1] theorem 22.12)**.** *Let Γ be a locally compact group. For every $g \in \Gamma$ different from 1 there exists a continuous irreducible unitary representation V of Γ such that $V(g) \neq I$.*

Theorem 8 ([Ross1] theorem 22.13)**.** *Every irreducible continuous representation V of a compact group Γ by unitary operators on a Hilbert space $\mathcal{H}$ is finite-dimensional.*

In the next few paragraphs we will focus on the definition of Haar measure and some basic theorems about it:

Definition 19. *A finitely additive measure λ (or simply a measure λ) is a nonnegative extended real-valued set function defined on a ring $\mathcal{S}$ of subsets of a nonempty set Y such that*

- $\lambda(\emptyset) = 0$
- $\lambda(\cup_{k=1}^{n} A_k) = \sum_{k=1}^{n} \lambda(A_k)$ *if each $A_k \in \mathcal{S}$ and the A_k's are pairwise disjoint.*

If λ is a complex-valued set function satisfying the above properties, than λ is called a complex measure.

Definition 20 ([Ross1] definition 11.28)**.** *Let Γ be a group and λ a measure on Γ. A subset A of Γ is said to be λ-measurable if for every subset S of Γ, the inequality*

$$\lambda(S) \geq \lambda(S \cap A) + \lambda(S \cap (\Gamma - A))$$

is valid.

Theorem 9 ([Ross1] theorem 11.29)**.** *Using the notation from the previous definition, the family of all λ-measurable subsets of Γ is a σ-algebra of subsets of Γ and the set function λ is countably additive on the family of all λ-measurable subsets of Γ.*

Theorem 10 ([Ross1] p.193)**.** *Let Γ be a group. There exists a nonnegative, extended real-valued set function defined for all subsets of Γ, and it is a measure on the σ-algebra of all λ-measurable subsets of Γ. The set function has the following properties:*

- $0 < \lambda(U)$ *for all nonempty sets U,*

- $\lambda(U) < \infty$ *for some open set* U,
- $\lambda(gU) = \lambda(U)$ *for all* $U \subset \Gamma$ *and* $g \in \Gamma$ *(*λ *is left invariant).*

λ *is called the left invariant Haar measure on* Γ *(similarly for right invariant Haar measure).*

Definition 21 ([Ross1] definition 11.1)**.** *Let* Y *be a topological space and* $\mathcal{O}$ *is the family of all open subsets of* Y*, then the family of Borel sets in* Y *is defined as the smallest* σ*-algebra of sets containing* $\mathcal{O}$*.*

Theorem 11 (Uniquness of Haar measure, [Ross1] p. 126)**.** *Let* $\mathcal{S}$ *denote the family of Borel sets in* Γ*. Let* μ *be any measure on* $\mathcal{S}$ *such that:*

- $\mu(F) < \infty$ *if* F *is compact,*
- $\mu(U) > 0$ *for some open set* U,
- $\mu(gB) = \mu(B)$ *for all* $B \in \mathcal{S}$ *and* $g \in \Gamma$,
- *for every open set* U, $\mu(U) = sup\ \{\mu(F)|F \text{ is compact and } F \subset V\}$,
- *for all* $A \in \mathcal{S}$, $\mu(A) = inf\ \{\mu(V)|V \text{ is open and } V \supset A\}$.

Then if λ *is any left Haar measure, then there is a positive number* c *such that* $\mu(B) = c\lambda(B)$ *for all* $B \in \mathcal{S}$*.*

For simplicity, we will denote the Haar measure by dg, even thought this notation is little bit inexact.

Definition 22. *Let* dg *be the left invariant Haar measure. Left regular representation of the group* Γ *is the unitary representation acting on* $L^2(\Gamma,\ dg)$ *in the following way:*

$$R(g)f(h) = f(g^{-1}.x),\ \text{for } \forall g \in \Gamma.$$

Remark 1. *In case of left invariant Haar measure* $R(g)$ *is an unitary representation of* Γ *on the Hilbert space* $L^2(\Gamma,\ dg)$*.*

1.5 Fubini theorem

Theorem 12 (Fubini)**.** *Let* $(X, \mathcal{S}, \mu)$ *and* $(Y, \mathcal{T}, \lambda)$ *be spaces with* $\sigma-$*finite measures and let* f *be* $\mathcal{T} \times \mathcal{S}$*-measurable function on* $X \times Y$*. Then if* $0 \leq f \leq \infty$ *and if*

$$\phi(x) = \int_Y f_x\ d\lambda, \quad \psi(y) = \int_X f_y\ d\mu,$$

then ϕ *(resp.* ψ*) is* $\mathcal{S}$*-measurable (resp.* $\mathcal{T}$*-measurable) and*

$$\int_X \phi \ d\mu = \int_{X\times Y} f \ d(\mu \times \lambda) = \int_Y \psi \ d\lambda. \tag{1.8}$$

If f *is a complex function and*

$$\int_X \phi^* \ d\mu < \infty, \ \textit{where } \phi^*(x) = \int_Y |f|_x \ d\lambda,$$

then $f \in L^1(X \times Y)$*.*

Assume that $f \in L^1(X \times Y)$*. Then up to zero measure set for all* $x \in X$*,* $f_x \in L^1(Y)$ *and up to zero measure set for all* $y \in Y$*,* $f_y \in L^1(X)$*. The functions* ϕ *and* ψ *are defined almost everywhere, they are from* $L^1(X)$*,* $L^1(Y)$ *respectively and the equation (1.8) is valid.*

Chapter 2

Direct integral of Hilbert spaces

First of all we will give a basic definitions concerning the operator algebras and then discuss the existence of direct integral and its properties.

2.1 Selfadjoint and *-algebras

Definition 23 ([Bra] p. 19)**.** *Let $\mathcal{V}$ be a vector space over C, the space $\mathcal{V}$ is called the algebra if it is equipped with the multiplication, the multiplication must be associative and distributive. The algebra is called comumtative (abelian), if the product is commutative. A mapping * is called the adjoint operation of $A \in \mathcal{V}$ if it has the following properties:*

$$\begin{aligned} A^{**} &= A \\ (AB)^* &= B^* A^* \\ (\alpha A + \beta B)^* &= \bar{\alpha} A^* + \bar{\beta} B^*, \end{aligned}$$

where $\alpha, \beta \in C$. An algebra with adjoint operation is called a *-algebra. Subset $\mathcal{W}$ of the algebra is called selfadjoint, if $A \in \mathcal{W}$ implies that $A^* \in \mathcal{W}$.

The algebra is normed if there exists a norm $\|.\|$ satisfying the property:

$$\|AB\| \leq \|A\|\|B\|. \tag{2.1}$$

An Algebra complete under such norm is called a Banach algebra, moreover if $\|A\| = \|A^*\|$, then we speak about Banach *-algebra.

A Banach *-algebra which satisfies

$$\|AA^*\| = \|A\|^2, \tag{2.2}$$

for $\forall A \in \mathcal{V}$ is called C^*-algebra.

2.2 Direct integral with respect to a weakly closed commutative self-adjoint algebra

In the following, $\mathcal{H}$ will denote a Hilbert space:

Definition 24. *Let P be a weakly closed commutative self-adjoint algebra of bounded operators , then P' is a weakly closed self-adjoint algebra of bounded operators which commutes with every element of P.*

Definition 25 ([Mau1], paragraph 1)**.** *Let $s(y)$ be a distribution of a variable y (a non-decreasing right-semicontinuous bounded real valued function), for $y \in Y$, $\mathcal{H}_y$ be a Hilbert space (finite or infinite dimensional). Suppose that for each $y \in Y$ the Hilbert space $\mathcal{H}_y$ of finite or infinite dimension $k(y)$ is given such that $k(y)$ is a s-measurable function of y. We call two function equivalent if they differ on a set of measure zero. The Hilbert space $\mathcal{H}$ of all equivalence classes of complex valued s-measurable functions $f_n(y)$, $n = 1, 2, \ldots, k(y)$ for which the Lesbeque-Stieltjes integral $\int_{-\infty}^{\infty} \sum_{n=1}^{k(y)} |f_n(y)|^2 \; ds(y)$ is finite, is the generalized direct integral of the space $\mathcal{H}_y$ with respect to the weight function $s(y)$. We write*

$$\mathcal{H} = \int_Y \mathcal{H}_y \; ds(y).$$

To every $f \in \mathcal{H}$ corresponds a vector valued function $f_y \in \mathcal{H}_y$ and we denote it by $f = \int_Y f_y \; ds(y)$. Furthermore one has for any two elements $f, f' \in \mathcal{H}$

$$\langle f, f' \rangle = \int_Y \langle f_y, f'_y \rangle_{\mathcal{H}_y} \; ds(y).$$

An operator defined on $\mathcal{H}$ is called decomposable, if it is possible to write it in the form

$$Af = \int_Y A_y f_y \; ds(y), \; \text{for } \forall f \in \mathcal{H}. \tag{2.3}$$

Such A will be denoted by $A = \int_Y A_y \; ds(y)$.

Because a change of A_y on an arbitrary set of s-measure zero will not change A, A corresponds to a class of operator valued functions A_y.

Theorem 13. *Let A, B be two decomposable operators, $A = \int_Y A_y \; dy$, $B = \int_Y B_y \; dy$. Then*

$$\begin{aligned} A + B &= \int_Y (A_y + B_y) \; dy \\ \alpha A &= \int_Y (\alpha A_y) \; dy \end{aligned}$$

In the next chapters we will deal with the decomposition of a Hilbert space with respect to a weakly closed commutative self-adjoint algebra, later with respect to the center (special case of decomposition with respect to a weakly closed commutative self-adjoint algebra). In the following, some of the main results are recalled:

Theorem 14 ([Mau1] paragraph 3, (more details in [Neu1] theorem IV))**.** *Let P be a weakly closed commutative self-adjoint algebra of bounded operators defined on $\mathcal{H}$, then there exists a direct integral $\mathcal{H} = \int_Y \mathcal{H}_y \; dy$, which is up to equivalence uniquely defined by the requirement, that P consists of those operators, which are decomposable by $\mathcal{H} = \int \mathcal{H}_y \; dy$ in the following way: for $B \in P$,*

$$Bf = \int_Y b_y f_y \; dy, \forall f \in \mathcal{H},$$

where b_y is essentially bounded y-measurable complex function.

Theorem 15 ([Neu1] theorem V)**.** *Let $\mathcal{H} = \int_Y H_y \; dy$ is formed with respect to P. Then a bounded operator A is decomposable if and only if $A \in P'$, i.e $AB = BA$ for all $B \in P$.*

Similarities with the decomposition for abelian groups and for general locally compact group is explained by the following theorems:

Theorem 16 ([Mau1] theorem 1.2)**.** *Let F be any self-adjoint family of bounded operators in a separable Hilbert space $\mathcal{H}$. Form with respect to any weakly closed commutative self-adjoint subalgebra P of F' the direct integral $\mathcal{H} = \int_Y \mathcal{H}_y \; dy$. Choose for every $A \in F$ any one representative A_y from the class of operator valued functions into which A decomposes and denote the family of operators thus obtained in $\mathcal{H}_y$ by F_y. Then whatever the choice of representatives, almost all the $\mathcal{H}_y$ will be irreducible under F_y if and only if P is a maximal commutative self-adjoint subalgebra of F'.*

Remark 2. *In case of discrete group it means that $R_y(.)$ can be taken in such way, that it is the representation of Γ. W_y will denote the weakly closed algebra generated by the operators $R_y(g)$ for varying $g \in \Gamma$.*

In case of discrete groups, we can summarize that if W is the weakly closed algebra generated by $R(g)$ then we can decompose $L^2(\Gamma)$ as in theorem 16. Further $R_y(g)$, $g \in \Gamma$ form the irreducible unitary representation in $\mathcal{H}_y$ up to a zero measure set. This decomposition may be composed from infinite dimensional Hilbert spaces and in case of free group with two generators almost all the Hilbert space obtained from the above decomposition will be infinite dimensional ([Mau1], p.16).

The following example illustrates the decomposition discussed in the theorem 16:

Example: Symmetric group S_3

To illustrate the previous chapter, we use the symmetric group S_3, it's elements we will represented by the matrixes 3×3:

$$1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, d = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, e = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then the elements of regular representation in the above basis are:

$$R(1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, R(a) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$R(b) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, R(c) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$R(d) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, R(e) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$R(g)$, $g \in S_3$, generates the algebra W. This algebra can be decomposed to irreducible representations using the new basis. The transition matrix between the bases is

$$U = \begin{pmatrix} \frac{1}{6}\sqrt{6} & \frac{1}{6}\sqrt{6} & \frac{1}{6}\sqrt{6} & \frac{1}{6}\sqrt{6} & \frac{1}{6}\sqrt{6} & \frac{1}{6}\sqrt{6} \\ \frac{1}{6}\sqrt{6} & -\frac{1}{6}\sqrt{6} & -\frac{1}{6}\sqrt{6} & -\frac{1}{6}\sqrt{6} & \frac{1}{6}\sqrt{6} & \frac{1}{6}\sqrt{6} \\ 1/3\sqrt{3} & 1/3\sqrt{3} & -\frac{1}{6}\sqrt{3} & -\frac{1}{6}\sqrt{3} & -\frac{1}{6}\sqrt{3} & -\frac{1}{6}\sqrt{3} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1/3\sqrt{3} & -1/3\sqrt{3} & \frac{1}{6}\sqrt{3} & \frac{1}{6}\sqrt{3} & -\frac{1}{6}\sqrt{3} & -\frac{1}{6}\sqrt{3} \end{pmatrix}.$$

Then the representation in the new base is

$$R(1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad R(a) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$R(b) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad R(c) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$

$$R(d) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad R(e) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

We made a decomposition of $L^2(S_3)$ with respect to a maximal commutative self-adjoint subalgebra of W'. The decomposition can be written in the form

$$L^2(S_3) \equiv S_3 = span[\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}] \oplus span[\begin{pmatrix} 0 & -2 & -2 \\ -2 & 0 & -2 \\ -2 & -2 & 0 \end{pmatrix}] \oplus$$

$$span[\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -2 \\ -1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}] \oplus span[\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ -2 & 2 & 0 \end{pmatrix}]$$

in the matrix representation used on the beginning of the section.

2.2.1 Decomposition with respect to the center

So far, we have defined a direct integral with respect to a weakly closed self-adjoint commutative algebra P and summarize the properties of such decomposition in case when P is maximal such subalgebra of bounded operators. In the following, we have a self-adjoint weakly closed algebra W of bounded operators defined on a Hilbert space $L^2(\Gamma)$ generated by $R(g)$ and we do the decomposition of $L^2(\Gamma)$ with respect to its center Z, it means with respect to the maximal self-adjoint weakly closed subalgebra of W which commutes with every element of W (so Z is commutative weakly closed subalgebra of bounded operator and we can form the direct integral with respect of Z).

Because $W \subset Z'$, every operator in W is decomposable. If we are dealing with a finite dimensional space $\mathcal{H}$ and a self-adjoint algebra of operators in it then the component spaces, which one obtains by the decomposition with respect to its center need not be irreducible, but they correspond to blocks of equivalent irreducible representations, i.e. each of these components obtained from the decomposition with respect to Z is a direct sum of irreducible spaces which transform equivalently. In case of infinite Hilbert space the situation is more complicated, the representation thus obtained need not be irreducible, even they do not generally correspond to the blocs of irreducible representation. But in case of discrete groups, the generalized trace can be defined on the spaces W_y, the generalize Fourier-Stieltjes transform can be defined on such spaces and the generalized Peter-Weyl Plancherel theorem holds (see chapter 3).

2.3 Generalized trace for discrete groups

Theorem 17. *If Γ is a locally compact discrete group, Then the weakly closed self adjoint algebra W generated by $R(g)$, $g \in \Gamma$ is the algebra of all bounded operators in $L^2(\Gamma)$ consisting of exactly those operators A which are defined by a function $a(g) \in L^2(\Gamma)$ such that*

$$A : x(h) \to \sum_{\Gamma} a(g) x(g^{-1}h) \tag{2.4}$$

and which are bounded. Furthermore we can define a trace on the algebra W:

$$tr[A] = a(1), \tag{2.5}$$

where 1 is the identity on Γ.

As it was mentioned in the previous chapter, $L^2(\Gamma)$ can be decomposed with respect to the center Z. W thus decomposes to the algebras W_y. The trace of W_y can be defined in the following way: Let $e_1 \in L^2(\Gamma)$ such that

$e_1(g) = \delta_{1g}$. Then $e_1 = \int_Y e_{1,y}\, \mathrm{d}y$. If $A \in W$ then $A = \int_Y A_y\, \mathrm{d}y$. Then the trace on W_y can be defined by

$$tr_y[A_y] = \langle A_y e_{1,y}, e_{1,y} \rangle_{\mathcal{H}_y}. \tag{2.6}$$

This trace satisfies the following properties:

- let A be a bounded operator, then for almost all $y \in Y$, $tr_y[A_y]$ is finite,
- $tr_y[A_y + B_y] = tr_y[A_y] + tr_y[B_y]$,
- $tr_y[A_y A_y^*] \geq 0$, because $tr_y[A_y A_y^*] = \langle A_y A_y^* e_{1,y}, e_{1,y} \rangle = \|A_y^* e_{1,y}\| \geq 0$,
- $tr_y[A_y] = \overline{tr_y[A_y^*]}$,
- $tr_y[A_y B_y] = tr_y[B_y A_y]$,

Chapter 3

Generalized Fourier-Stieltjes transformation

3.1 Fourier-Stieltjes transformation for locally compact abelian groups

Throughout this section Γ will be a locally compact abelian group.

3.1.1 Normalization of Haar measures to Plancherel measure

Definition 26 ([Ross1], definition 23.23). *Let Γ be a locally compact abelian group with the character group $\hat{\Gamma}$. For an arbitrary nonempty subset F of Γ, let $A(\hat{\Gamma}, \Gamma)$ be the subset of $\hat{\Gamma}$ consisting of all χ such that $\chi(F) = \{1\}$. The set $A(\hat{\Gamma}, \Gamma)$ is called the annihilator of F in $\hat{\Gamma}$.*

In the following paragraph, we will briefly summarize the facts necessary for the normalization of the measures on the group and on its dual ([Ross2], p.208): Every locally compact abelian group Γ is in the form $R^n \times F$, where n is a nonnegative integer and F is locally compact abelian group that contains a compact open subgroup J. If F is compact, we take $J = F$, otherwise we choose J arbitrary. Let $\hat{F}$ be the dual group to F. Let $A(\hat{F}, J)$ is the annihilator, then $A(\hat{F}, J)$ is a compact subset of $\hat{\Gamma}$. Moreover, we can identify $\hat{\Gamma}$ with $R^a \times \hat{F}$. The following Plancherel's theorem is valid for the Haar measure chosen as follows: let μ be the necessarily unique Haar measure on the subgroup F such that $\mu(J) = 1$ and the measure on A is such that $\hat{\mu}(A) = 1$. Let μ^n be the Lesbegue measure on R^n. Every Haar measure on

Γ is in the form

$$\lambda = c(2\pi)^{-n/2}\mu_n \times \mu, \tag{3.1}$$

where c is a positive number. The measure on $\hat{\Gamma}$ will then be in the form

$$\hat{\lambda} = \frac{1}{c}(2\pi)^{-n/2}\mu_n \times \hat{\mu}. \tag{3.2}$$

3.1.2 Fourier-Stieltjes transformation

Definition 27 ([Ross1] theorem 14.4). *Let $C_0(\Gamma)$ be the set of complex valued functions on Γ such that $\forall\varepsilon$ there exists a compact subset F on Γ, such that*

$$|f(x)| < \varepsilon, \forall x \in (X - F). \tag{3.3}$$

Let $C_0^*(\Gamma)$ be the set of all bounded linear functionals defined on $C_0(\Gamma)$. Then to every $\varphi \in C_0^*(\Gamma)$ there exists a complex valued measure μ defined on the σ-algebra $\mathcal{M}_\mu$ of subsets of Γ containing the Borel sets in Γ such that

$$\varphi(f) = \int_\Gamma f \ d\mu \text{ for all } f \in C_0(\gamma). \tag{3.4}$$

The set of all this measures will be denoted by $\mathcal{M}(\Gamma)$.

Definition 28 ([Ross1] definition 31.2). *For a complex measure $\mu \in \mathcal{M}(\Gamma)$, let $\hat{\mu}$ be the complex-valued function on $\hat{\Gamma}$ such that*

$$\hat{\mu}(\chi) = \int_\Gamma \chi \ d\mu(g). \tag{3.5}$$

Then $\hat{\mu}$ is called the Fourier-Stieltjes transform of μ. If $f \in L^1(\Gamma)$, so that $d\mu = f \ dg$, where dg is the normalized Haar measure, than $\hat{f}$ is called the Fourier transform of the function f, where $\hat{f}$ is determinated by the equation

$$\hat{f} \ d\chi = f \ dg,$$

and $d\chi$ is the normalized measure corresponding to dg.

Definition 29. *For a measure μ in $\mathcal{M}(\hat{\Gamma})$, let $\check{\mu}$ the complex-valued function on Γ such that $\check{\mu}(g) = \int_{\hat{\Gamma}} \bar{\chi}(g) \ d\mu$. Then this function is called the inverse Fourier-Stieljes transformation of μ. If $f \in L^1(\hat{\Gamma})$, so that $d\mu = \int f \ d\chi$, than $\check{f}$ is called the inverse Fourier transform of the function f.*

Theorem 18 (Plancherel, [Ross1] theorem 31.18). *The Fourier transformation is a linear isometry of $L^2(\Gamma)$ onto $L^2(\hat{\Gamma})$, and the inverse Fourier transformation is a linear isometry of $L^2(\hat{\Gamma})$ onto $L^2(\Gamma)$. These two transformations are inverse to each other.*

3.2 Fourier-Stieltjes transformation for locally compact groups

In case of non-abelian locally compact groups, the situation is more complicated and the construction and the proofs of all the theorems are based on von Neumann work and his generalized direct integral. For simplicity we suppose that the group is locally compact with two-side invariant Haar-measure.

Let W denote the weakly closed self-adjoint algebra of bounded operators on $L^2(\Gamma)$ generated by the operators $R(g)$, $g \in \Gamma$ (left regular representation). Let Z be the center of W ($Z = W' \cap W$, Z maximal commutative subalgebra). Let $L^2(\Gamma)$ be the direct integral of the spaces $\mathcal{H}_y$, $y \in Y$ with respect to Z:

$$L^2(\Gamma) = \int_Y \mathcal{H}_y \, \mathrm{d}y.$$

Definition 30 (Generalized Fourier transform, [Mau3] p. 371). *Let $f(g) \in L^2(\Gamma)$ satisfying*

$$f(g) = \bar{f}(g^{-1}). \tag{3.6}$$

Then $f(g)$ determinates an operator $F'f : L^2(\Gamma,\ dg) \to L^2(\Gamma,\ dg)$ by means of convolution with the function $f \in L^2(\Gamma)$:

$$F'f(h(g')) = \int_\Gamma f(g)h(g^{-1}.g') \ dg.$$

$F'f$ is defined on certain dense subspace of $L^2(\Gamma)$ and there exists a self adjoint extension of this operator denoted by Ff. Ff is called a Fourier-Stieltjes transformation of f. A general $f \in L^2(\Gamma)$ can be written as $f(g) = a(g)+ib(g)$, where a, b satisfies the condition (3.6). Let Fa, Fb be the Fourier-Stieltjes transformation of a, b, then the Fourier-Stieltjes transform of f is defined by

$$Ff = Fa + iFb$$

and it can be shown, that Ff is decomposable by the direct integral and $F_y f$ is densely defined operator on $\mathcal{H}_y$ for almost every $y \in Y$.

Let us remark that in case of function, where the following has a good sense, the above definition reformulates

$$F_y f = \int_\Gamma f(g) R_y(g) \, \mathrm{d}g,$$

where $R_y(g)$ is the decomposition of the left regular representation on the Γ:

$$R_y(g) f(g') = f(g^{-1}.g'),$$

where $f \in L^2(\Gamma)$.

Theorem 19 ([Mau3] p. 373). *On the algebra W_y there exists a relative trace tr_y. For a fixed $y \in Y$, tr_y is a complex-valued linear functional defined on certain linear subset of W_y. Taking the fixed normalization of tr_y for each y, then the measure on Y can be taken in such way, that the generalized Peter-Weyl Plancherel theorem holds:*

$$\int_\Gamma f_1(g)\bar{f}_2(g)\ dg = \int_Y tr_y[F_y f_1 F_y^* f_2]\ dy, \tag{3.7}$$

for $f_j(g) \in L^2(\Gamma)$, for $j = 1, 2$. Though $tr_y[F_y f_1 F_y^* f_2]$ has a meaning for almost every y.

Let A_y, B_y be those operator in W_y such that the $tr_y[A_y A_y^*]$ exists for almost every y and such that $\int_Y (tr_y[A_y A_y^*])\ \mathrm{d}y$ exists and is finite. These operators with the scalar product defined by

$$\int_Y (tr_y[A_y B_y^*])\ \mathrm{d}y$$

form a incomplete space X with scalar product.

Using the previous notation one obtains

$$\langle f_1, f_2 \rangle_{L^2(\Gamma)} = \langle F f_1, F f_2 \rangle_X.$$

Let us remark that the Fourier-Stieltjes transformation is an injection, because it is a linear isomorphism.

The following theorem completes the proof, that the generalized Fourier-Stieltjes transform is a bijection, so that there exists an inverse Fourier-Stieltjes transform which preserves the scalar product.

Theorem 20 ([Mau3] p. 380). *The Fourier-Stieltjes transform maps $L^2(\Gamma)$ on the completion of X.*

We can conclude, that there exists the inverse Fourier-Stieltjes transform $F^* = F^{-1}$ which maps the completion of X ($\bar{X}$) onto $L^2(\Gamma,\ \mathrm{d}g)$ and it preserves the norm.

Because F is the unitary operator mapping $L^2(\Gamma) \to \bar{X}$, one can ask, how the inverse Fourier-Stieljes transformation F^* looks like. This question will be partly answered in the chapter 4 for a special type of locally compact groups.

Chapter 4

Generalized Bloch analysis and Schwartz kernel theorem

4.1 Preliminary

Definition 31 ([Jung] p. 17). *Let V, V' be representations of Γ in $\mathcal{H}$ resp. $\mathcal{H}'$. An interwinning operator is a bounded linear map $A : \mathcal{H} \to \mathcal{H}'$ such that*

$$AV(g) = V'(g)A, \forall g \in \Gamma. \tag{4.1}$$

The set of all interwinning operators is denoted by $Hom_\Gamma(V, V')$. $Hom_\Gamma(V, V')$ is a linear space, if $V = V'$ then $Hom_\Gamma(V, V')$ is an algebra.

Definition 32 ([Jung], p. 20). *A representation V of Γ is called a factor representation, if the center of $Hom_\Gamma(V, V)$ consists purely of multiples of the identity, i.e. $Z(Hom_\Gamma(V, V)) = \{\lambda 1\}$, $\lambda \in C$.*

Definition 33. *A factor representation is called Type I if it contains an irreducible subrepresentation. A group Γ is called Type I when all its factor representations are Type I.*

Remark 3.

- *If Γ is compact then Γ is Type I.*
- *If Γ locally compact and abelian then Γ is Type I.*
- *If Γ is semi-simple Lie group then Γ is Type I.*
- *Let Γ be a countably infinite discrete group, then Γ is Type I if and only if there exists $\Gamma' \subset \Gamma$, Γ' abelian and normal such that Γ/Γ' is finite ([Tho] Satz 6).*
- *If Γ is connected and nilpotent then Γ is Type I.*

- *The Free group with two generators is not Type I.*

Theorem 21 ([Spr]). *Let Γ be a locally compact separable group of Type I. Let $R(g)$, $g \in \Gamma$ be the regular representation generating the weakly closed algebra W. Then the decomposition with respect to the center Z of W decomposes $L^2(\Gamma)$ to a direct integral of Hilbert space $\mathcal{L}'_y$, $y \in Y$. Then $R(g) = \int_Y R_y(g)\ dy$ and $R_y(g)$ corresponds to the blocs of equivalent irreducible representations $\Lambda_y(g)$.*

Remark 4. *The assumtion that the group is of Type I assures that the decomposition with respect to the center will have the same properties as in the case of finite groups, discussed at the end of chapter 2.*

Theorem 22 ([Tho] Kollorar 1.). *If Γ is discrete, separable group of Type I, then all irreducible representations are finite dimensional and the dimensions are uniformly bounded. For every $y \in Y$, tr_y is defined in the ordinary way, as the norm on the space of Hilbert-Schmidt operators.*

Remark 5. *In case of group of Type I, Y corresponds to the dual object $\hat{G}$, but still the letter Y will be used in the following.*

Also in this case $L^2(\Gamma)$ can be decomposed to $\mathcal{L}'_y$. $\mathcal{L}'_y$ further decomposes to a direct integral of Hilbert spaces $\mathcal{L}_y$ which are all mutually equivalent, and also $R_y(g)$ decomposes to irreducible equivalent representations $\Lambda_y(g)$ acting on $\mathcal{L}_y$.

To clarify the construction let us repeat it in the previous notation

$$L^2(\Gamma) = \int_{Y'} \mathcal{L}'_y \ \mathrm{d}y'$$

with respect to the center of W. $\mathcal{L}'_y = \bigoplus_{i=1}^k \mathcal{L}_y$. One can identify $\mathcal{L}'_y$ with the Hilber space $\mathcal{I}_2(\mathcal{L}_y)$ formed by Hilbert-Schmidt operators on $\mathcal{L}_y$. Let F_y be the Fourier-Stieltjes transformation of $L^2(\Gamma) \to \mathcal{I}_2(\mathcal{L}_y)$ defined similarly as in the previous chapter. Then the Peter-Weyl Plancherel theorem holds:

Theorem 23. *Let Γ be a locally compact separable group of Type I. Then there exists a unique positive measure dy on Y satisfying the condition*

$$\int_\Gamma |f(g)|^2 \ dg = \int_Y tr_y[F_y f (F_y f)^*] \ dy, \tag{4.2}$$

where $F_y f$ is the Fourier-Stieltjes transform of f. Further F is a unitary operator $L^2(\Gamma) \to \int_Y \mathcal{I}_2(\mathcal{L}_y)\ dy$.

Example: Symmetric group S_3

As it is well known, in the case of a finite group, the left regular representation decomposes to irreducible representations, its multiplicity is equal to its dimension. As it was shown,the left regular representation of S_3 decomposes to two one dimensional representations and one two dimensional representation with multiplicity two. If we normalize the Haar measure on S_3 in such way, that the measure of Γ is equal to one, then the Plancherel measure of one point set $\{y\}$ is equal to the dimension of the Hilbert space divided by the number of the elements of the group, i.e. $\mathrm{d}y = \frac{\dim \mathcal{H}_y}{\#\Gamma}$.

In this case: Y=$\{1,2,3\}$, $\dim H_1 = 1$, $\dim H_2 = 1$, $\dim H_3 = 2$. Just for illustration, let us discribe how Λ_3 looks like:

$$\Lambda_3(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \Lambda_3(a) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \Lambda_3(b) = \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$

$$\Lambda_3(c) = \begin{pmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \Lambda_3(d) = \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}, \Lambda_3(e) = \begin{pmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}. \quad (4.3)$$

Then using the orthonormal base $e_h(g) = \delta(g, h)$,

$$F_y e_h = \sum_{g \in S_3} e_h(g)\Lambda_y(g) = \Lambda_y(h) \quad (4.4)$$

then for all $g \in S_3$ it holds true:

$$\sum_{y=1,2,3} tr_y[\Lambda_y(g)\Lambda_y^*(g)] = 1\frac{1}{6} + 1\frac{1}{6} + 2\frac{1}{3} = 1 = \|e_g\|^2, \quad (4.5)$$

so the Peter-Weyl Plancherel theorem holds.

4.1.1 Inverse Fourier-Stieljes transformation

Theorem 24. *Let F be the Fourier-Stieljes trasformation with the same notation as in the previous paragraph. Let $A \in \int_Y \mathcal{I}_2(\mathcal{L}_y)\ dy$, then*

$$F_g^* A = \int_Y tr_y[\Lambda_y(g)^* A_y]\ dy. \quad (4.6)$$

Proof. Let $\{e_h\}_{h\in\Gamma}$ be the orthonormal base defined by $e_h(g) = \delta_{hg}$. Then

$$\begin{aligned} F_g^* F(e_h) &= \int_Y tr_y[\Lambda_y(g)^*(F_y e_h)]\ \mathrm{d}y = \\ &\int_Y tr_y[\Lambda_y(g)^*(\int_\Gamma e_h(g')\Lambda_y(g')\ \mathrm{d}g')]\ \mathrm{d}y \\ = \int_Y tr_y[\Lambda_y(g)^*\Lambda_y(h)]\ \mathrm{d}y &= \int_Y tr_y[(F_y e_g)^* F_y e_h]\ \mathrm{d}y \\ &= \langle e_g, e_h\rangle = \delta_{gh} = e_h(g). \end{aligned}$$

Thus F^* is the inverse Fourier-Stieljes transformation of F. $\square$

Let M be a connected and locally linearly connected manifold, $\pi_1(M) = \Gamma$ the fundamental group of M, $\tilde{M}$ the universal covering space of M and D a fundamental domain of $\tilde{M}$. It is well known, that the fundamental group is discrete and at most countable group ([Lee]). We further suppose that Γ is of Type I.

Let $\mathcal{H}_y$ be the space of R_y-equivariant functions defined on $\tilde{M}$ with values in the Hilbert-Schmidt operators on $\mathcal{L}_y$, it means

$$\begin{aligned} \mathcal{H}_y = \{A_y : \tilde{M} \to \mathcal{I}_2(\mathcal{L}_y), \\ \text{such that } R_y(g)[A_y(x)] = \Lambda_y(g)A_y(x) = A_y(g.x), \forall g \in \Gamma \\ \int_M tr_y[A_y(x)A_y^*(x)] \, \mathrm{d}\mu < \infty\}. \end{aligned}$$

4.2 Generalized Bloch analysis

Before the main theorem of generalized Bloch analysis, let us introduce several lemmas and definitions:

Definition 34. *Let $\mathcal{L}_y$ be a Hilbert space, then $\mathcal{L}_y^*$ is the space of continues antilinear fuctionals defined on $\mathcal{L}_y$.*

The Riesz lemma claims that there exists a conjugate linear isometric bijection ρ such that

$$\rho : \mathcal{L}_y \to \mathcal{L}_y^* : u \to \phi_u, \phi_u(v) = \langle u, v\rangle. \tag{4.7}$$

So $\mathcal{L}_y^*$ is the Hilbert space with the scalar product given by

$$\langle \phi_1, \phi_2\rangle_{\mathcal{L}_y^*} = \langle \rho^{-1}\phi_2, \rho^{-1}\phi_1\rangle_{\mathcal{L}_y}. \tag{4.8}$$

Definition 35. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two Hilbert spaces, then by $\mathcal{H}_1 \otimes \mathcal{H}_2$ we denote the Hilbert spanned by vectors $\phi_1 \otimes \phi_2$, where $\phi_1 \in \mathcal{H}_1$, $\phi_2 \in \mathcal{H}_2$. The scalar product is given by $\langle \phi_1 \otimes \phi_2, \phi_1' \otimes \phi_2'\rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} = \langle \phi_1, \phi_1'\rangle_{\mathcal{H}_1}\langle \phi_2, \phi_2'\rangle_{\mathcal{H}_2}$. Such space will be preHilbert. $\mathcal{H}_1 \otimes \mathcal{H}_2$ denotes the closure of such space.*

Lemma 1. *Let $\mathcal{I}_2(\mathcal{L}_y)$ be the space of Hilbert-Schmidt operators defined on the Hilbert space $\mathcal{L}_y$. Then $\mathcal{I}_2(\mathcal{L}_y) \equiv \mathcal{L}_y^* \otimes \mathcal{L}_y$.*

Proof. There exists $\Psi : \mathcal{L}_y^* \otimes \mathcal{L}_y \to \mathcal{I}_2(\mathcal{L}_y)$ and it is an unitary operator. Let $\phi \in \mathcal{L}_y^*, u \in \mathcal{L}_y$, we define Ψ by:

$$\Psi(\phi_1 \otimes u)v = \phi_1(v)u. \tag{4.9}$$

- Ψ is an isometry:

  $$\langle \Psi\phi_1 \otimes u_1, \Psi\phi_2 \otimes u_2\rangle_{\mathcal{I}_2(\mathcal{L}_y)} = \sum_n \langle \Psi(\psi_1 \otimes u_1)v_n, (\Psi\phi_2 \otimes u_2)v_n\rangle_{\mathcal{L}_y},$$

  where $\{v_n\}_{n\in N}$ is an orthonormal basis of $\mathcal{L}_y$. Then

  $$\begin{aligned}\sum_n \langle \Psi(\psi_1 \otimes u_1)v_n, (\Psi\phi_2 \otimes u_2)v_n\rangle_{\mathcal{L}_y} \\ = \sum_n \phi_1(v_n)\phi_2(v_n)\langle u_1, u_2\rangle_{\mathcal{L}_y} \\ = \sum_n \langle \rho^{-1}\phi_1, v_n\rangle\langle \rho^{-1}\phi_2, v_n\rangle\langle u_1, u_2\rangle = \langle \phi_1, \phi_2\rangle_{\mathcal{L}_y^*}\langle u_1, u_2\rangle_{\mathcal{L}_y} \\ = \langle \phi_1 \otimes u_1, \phi_2 \otimes u_2\rangle_{\mathcal{L}_y^* \otimes \mathcal{L}_y}.\end{aligned}$$

- Ψ is onto: Because Ψ is an isometry, RanΨ is closed. Let $A \in \mathcal{I}_2(\mathcal{L}_y)$, $A \perp \Psi(\phi \otimes u)$ $\forall \phi \in \mathcal{L}_y^*$, $u \in \mathcal{L}_y$

  $$0 = \sum_n \langle Av_n, \Psi(\phi \otimes u)v_n\rangle_{\mathcal{L}_y} = \sum_n \langle Av_n, \phi(v_n)u\rangle. \tag{4.10}$$

  Let $\phi = \rho(v_k)$, $u = v_j$. Then $\forall k, j$ the sum reduces to $\langle Av_k, v_j\rangle = 0$ and it implies that $\forall k$

  $$Av_k = 0.$$

  Because span$\{v_k\}_k$ is dense in $\mathcal{L}_y$, $A = 0$ and Ψ is onto.

Thus Ψ is an unitary operator and we can identify

$$\mathcal{I}_2(\mathcal{L}_y) \equiv \mathcal{L}_y^* \otimes \mathcal{L}_y. \tag{4.11}$$

□

Lemma 2. *Let $A \in \mathcal{B}(\mathcal{L}_y)$, $B \in \mathcal{I}_2(\mathcal{L}_y)$ such that $B = \phi \otimes v$ (using the notation from the previous lemma). Let $L_A : \mathcal{I}_2(\mathcal{L}_y) \to \mathcal{I}_2(\mathcal{L}_y)$ defined by $L_A[B] = AB$. Then*

$$AB = \phi \otimes (Av), \tag{4.12}$$

so L_A can be written in the form

$$L_A = I \otimes A. \tag{4.13}$$

Proof. The proof follows directly from the previous lemma. □

Using the previous two lemmas, $\mathcal{H}_y$ can be decompose to

$$\mathcal{H}_y = \mathcal{L}_y^* \otimes \tilde{\mathcal{H}}_y, \tag{4.14}$$

where $\tilde{\mathcal{H}}_y$ are Λ_y-equivariant functions with values in $\mathcal{L}_y$, it means:

$$\tilde{\mathcal{H}}_y = \{v_y : \tilde{M} \to \mathcal{L}_y, \text{ such that } \Lambda_y(g)v_y(x) = v_y(g.x), \forall g \in \Gamma,$$
$$\int_M \|v_y(x)\|^2_{\mathcal{L}_y} \, \mathrm{d}\mu < \infty\} \tag{4.15}$$

Let us describe some properties of $f \in L^2(\tilde{M})$, where $\tilde{M}$ is the universal covering space of M with the fundamental group Γ. Locally the fibration $\pi : \tilde{M} \to M$ is trivial, it means $\pi^{-1}(U) = U \times \Gamma$.

- $\tilde{M} = \cup_{x \in M} \pi^{-1}(x)$, where π is the projection from the definition 7. Then the Fubini theorem implies that

$$\|f\|^2_{L^2(\tilde{M})} = \int_{\tilde{M}} |f|^2 \, \mathrm{d}\mu = \int_M \int_{\pi^{-1}(x)} |f(x)|^2 \, \mathrm{d}g \, \mathrm{d}\mu \tag{4.16}$$

  and for almost all $x \in \tilde{M}$

$$f|_{\pi^{-1}(x)} \in L^2(\pi^{-1}(x)). \tag{4.17}$$

- If $f_x(g) = f(g^{-1}.x)$ then $f_x \in L^2(\Gamma)$.
- Let L_h be the left action of Γ on $\tilde{M}$:

$$L_h x = h.x. \tag{4.18}$$

  Then $L_h^* : L^2(\tilde{M}) \to L^2(\tilde{M})$ is defined by

$$L_h^* f(x) = f(h.x). \tag{4.19}$$

Theorem 25. *Let M, $\tilde{M}$, Γ, $R_y(g)$ is defined as above. Let*

$$\mathcal{H}_y = \{A_y : \tilde{M} \to \mathcal{I}_2(\mathcal{L}_y),$$
$$\textit{such that } R_y(g)[A_y(x)] = \Lambda_y(g)A_y(x) = A_y(g.x), \forall g \in \Gamma$$
$$\int_M tr_y[A_y(x)A_y^*(x)] \; d\mu < \infty\}.$$

Let $\Phi : L^2(\tilde{M}) \to \int_Y \mathcal{H}_y \; dy$ be defined by

$$\Phi_y f(x) = F_y f_x, \tag{4.20}$$

where F_y is the Fourier-Stieljes transformation in y. Then for $f \in C_0^\infty(\tilde{M})$:

$$\Phi_y f(x) = \sum_\Gamma f(g^{-1}.x)\Lambda_y(g). \tag{4.21}$$

Further Φ is a well defined unitary operator.

Proof. The proof of this theorem will be done in several steps:

Let $f \in C_0^\infty(\tilde{M})$, then the sum in 4.21 is finite because the action of Γ on $\tilde{M}$ is properly discontinuous. Then the definition of the Fourier-Stieltjes transformation gives directly the relation (4.21).

- Let $f \in L^2(\tilde{M})$, then $\Phi_y f \in \mathcal{H}_y$:

$$F_y(L_h^* f_x) = \Lambda_y^{-1}(h) F_y f_x$$

  and

$$\Phi_y f(g.x) = F_y f_{g.x} = F_y(L_{g^{-1}}^* f_x) = \Lambda_y(g) F_y f_x = \Lambda_y(g)\Phi_y f(x). \tag{4.22}$$

- Φ is an isometry:

$$\begin{aligned}\int_Y \|\Phi_y f\|^2 \;\mathrm{d}y &= \int_Y \int_M \|(\Phi_y f)(x)\|^2_{\mathcal{I}_2(\mathcal{L}_y)} \;\mathrm{d}\mu \;\mathrm{d}y \\ = \int_M \int_Y \|F_y f_x\|^2_{\mathcal{I}_2(\mathcal{L}_y)} \;\mathrm{d}y \;\mathrm{d}\mu &= \int_M \int_\Gamma \|f_x(g)\|^2 \;\mathrm{d}g \\ &= \int_M \int_{\pi^{-1}(x)} |f(x)|^2 \;\mathrm{d}\mu = \|f\|^2_{L^2(\tilde{M})}.\end{aligned} \tag{4.23}$$

- Φ is surjective: Because Φ is a bounded operator and $L^2(\tilde{M})$ is closed, $\mathrm{Ran}\Phi$ is closed. Let $A \in \int_Y \mathcal{H}_y \;\mathrm{d}y$, $A = \int_Y A_y \;\mathrm{d}y$, $A \perp \mathrm{Ran}\Phi$. We will prove that $A = 0$:

$$\begin{aligned}0 = \int_Y \langle A_y, \Phi_y f\rangle_{\mathcal{H}_y} \;\mathrm{d}y, \forall f \in L^2(\tilde{M}),\\ \int_Y \langle A_y(x), \Phi_y f(x)\rangle_{\mathcal{H}_y} \;\mathrm{d}y\\ = \int_M \int_Y \langle A_y(x), F_y f_x\rangle_{\mathcal{I}_2(\mathcal{L}_y)} \;\mathrm{d}y \;\mathrm{d}\mu.\end{aligned} \tag{4.24}$$

Because the generalized Fourier-Stieljes transformation is a unitary operator between $L^2(\Gamma)$ and $\int_Y \mathcal{I}_2(\mathcal{L}_y)\ \mathrm{d}y$, for almost all $x \in \tilde{M}$, there exists $a_x \in L^2(\Gamma)$ such that $F_y a_x = A_y(x)$. Then

$$\int_M \int_Y \langle A_y(x), F_y f_x\rangle_{\mathcal{I}_2(\mathcal{L}_y)}\ \mathrm{d}y\ \mathrm{d}\mu = \int_M \langle a_x, f_x\rangle_{L^2(\Gamma)}\ \mathrm{d}\mu = 0.$$

Let $U \subset M$, U open such that

$$\pi^{-1}(U) = U \times \Gamma. \tag{4.25}$$

Let $f \in L^2(\tilde{M})$ such that $\mathrm{supp} f \in \pi^{-1}(U)$ and $f(x) = f_1(u) f_2(g)$, $u \in U, g \in \Gamma$, $f_1 \in C_0^\infty(U)$, $f_2 \in L^2(\Gamma)$, $x = (u, g)$. Then

$$0 = \int_U f_1(u)\langle a_{(u,g)}, f_2\rangle_{L^2(\Gamma)}\ \mathrm{d}\mu, \tag{4.26}$$

and it implies that for almost all $u \in U$ and $g \in \Gamma$, $\langle a_{(u,g)}, f_2\rangle = 0$ and so $a_{(u,g)}(h) = 0$ for all $h \in \Gamma$. Because $\tilde{M}$ can be covered by countably many open sets U, such that (4.25) is valid, $a_x(g) = 0$ for almost all $x \in \tilde{M}$ and all $g \in \Gamma$. Then for almost all $x \in \tilde{M}$, $A_y(x) = 0$ and so $A = 0$.

□

Definition 36. *Let g be a metric tensor. Then Laplace-Beltrami operator is defined by*

$$\Delta_{LB} = \frac{1}{\sqrt{g}} \partial_i g^{ij} \sqrt{g} \partial_j. \tag{4.27}$$

Remark 6. $(-\Delta_{LB})$ *is positive operator on* $C_0^\infty(\tilde{M})$.

Remark 7. *Let $\varphi \in C_0^\infty(\tilde{M})$, Φ from the previous theorem, $\Phi_y\varphi \in \mathcal{H}_y$ and $\Phi_y\varphi = \sum_{g\in\Gamma} \varphi(g^{-1}.x)\Lambda_y(g)$ and the sum is finite. We can define $(-\Delta_{LB})_y$ on the dense subset of $\mathcal{H}_y$ by the relation*

$$(-\Delta_{LB})_y \Phi_y \varphi(x) = \Phi_y(-\Delta_{LB})\varphi(x). \tag{4.28}$$

- $(-\Delta_{LB})_y$ *is a symmetric operator:*

$$\begin{aligned} &\langle(-\Delta_{LB})_y \Phi_y \psi_1(x), \Phi_y \psi_2(x)\rangle_{\mathcal{H}_y} \\ &= \langle \sum_\Gamma (-\Delta_{LB})_y L^*_{g^{-1}} \bar{\psi}_1(x) \Lambda_y(g), \sum_\Gamma \psi_2(g'^{-1}.x)\Lambda_y(g')\rangle_{\mathcal{H}_y} \\ &= \sum_\Gamma \sum_\Gamma \int_D L^*_{g^{-1}}(-\Delta_{LB})\bar{\psi}_1(x)\psi_2(g'^{-1}.x)\langle \Lambda_y(g), \Lambda_y(g')\rangle_{\mathcal{I}_2(\mathcal{L}_y)} \\ &= \sum_{g\in\Gamma} \sum_{g'\in\Gamma} \int_{g^{-1}.D} (-\Delta_{LB})\bar{\psi}_1(x) L^*_{g'^{-1}g}\psi_2(x)\varphi(g'^{-1}g), \end{aligned}$$

where $D \subset \tilde{M}$ *is a fundamental domain and*

$$\varphi(gg'^{-1}) = \langle \Lambda_y(g), \Lambda_y(g') \rangle_{\mathcal{I}_2(\mathcal{L}_y)} = \langle \Lambda_y(g'^{-1}.g), I \rangle_{\mathcal{I}_2(\mathcal{L}_y)}.$$

Then

$$\begin{aligned}\sum_{g\in\Gamma}\sum_{g'\in\Gamma}\int_{g^{-1}.D} &(-\Delta_{LB})\bar{\psi}_1(x) L^*_{g'^{-1}g}\psi_2(x)\varphi(g'^{-1}g)\ d\mu \\ &= \sum_{g\in\Gamma}\sum_{h\in\Gamma}\int_{g^{-1}.D}(-\Delta_{LB})\bar{\psi}_1(x)L^*_h\psi_2(x)\varphi(h)\ d\mu \\ &= \sum_{h\in\Gamma}\int_{\tilde{M}}(-\Delta_{LB})\bar{\psi}_1(x)L^*_h\psi_2(x)\varphi(h)\ d\mu \\ &= \sum_{h\in\Gamma}\int_{\tilde{M}}\bar{\psi}_1(x)(-\Delta_{LB})L^*_h\psi_2(x)\varphi(h)\ d\mu \\ &= \sum_{h\in\Gamma}\int_{\tilde{M}}\bar{\psi}_1(x)L^*_h(-\Delta_{LB})\psi_2(x)\varphi(h)\ d\mu \\ &= \langle \Phi_y\psi_1, (-\Delta_{LB})_y\Phi_y\psi_2\rangle_{\mathcal{H}_y}. \end{aligned} \tag{4.29}$$

- $(-\Delta_{LB})$ *is positive operator: Let* $\psi \in C_0^\infty(\tilde{M})$ *such that its support is in* D*. The functions* $\Phi\psi$ *are dense in* $\mathcal{H}_y$*. Then*

$$\begin{aligned}&\langle \Phi_y\psi, (-\Delta_{LB})\Phi_y\psi\rangle_{\mathcal{H}_y} \\ &= \int_D \bar{\psi}(x)(-\Delta_{LB})\psi(x)\langle\Lambda_y(1),\Lambda_y(1)\rangle_{\mathcal{I}_2(\mathcal{L}_y)} \geq 0.\end{aligned} \tag{4.30}$$

Theorem 26. *Let* $H = -\Delta_{LB} + V(x)$ *is a Hamiltonian with a* Γ*-invariant potential acting on* $L^2(\tilde{M})$*. Then* H *decomposes under* Φ*:*

$$\Phi H \Phi^* = \int_Y H_y\ dy, \tag{4.31}$$

where $H_y = (-\Delta_{LB})_y + V(x)$ *is formally the same operator acting on* $\mathcal{H}_y$*.*

Proof. Let $\varphi(x) \in C_0^\infty(\tilde{M})$, $\varphi_x(g) = \varphi(g^{-1}.x)$, $\varphi_x \in L^2(\Gamma)$ for almost all x. Let $\{e_h\}$ be the orthonormal basis of $L^2(\Gamma)$, $e_h(g) = \delta_{hg}$. Then

$$\begin{aligned}\varphi_x &= \sum_{h\in\Gamma}\varphi(h^{-1}.x)e_h \\ F\varphi_x &= \sum_{h\in\Gamma}\varphi(h^{-1}.x)Fe_h.\end{aligned} \tag{4.32}$$

Further

$$\Phi_y\varphi(x) = F_y\varphi_x = \sum_{h\in\Gamma}\varphi(h^{-1}.x)F_ye_h = \sum_{h\in\Gamma}L^*_{h^{-1}}\varphi(x)F_ye_h. \tag{4.33}$$

The sum in (4.33) is finite, uniformally for $x \in U \subset \bar{U} \subset \tilde{M}$.

Δ_{LB} is Γ invariant, so

$$\Delta_{LB} L_g^* = L_g^* \Delta_{LB} \tag{4.34}$$

Moreover

$$\begin{aligned}(-\Delta_{LB})_y \Phi_y f &= (-\Delta_{LB})_y \sum_{h\in\Gamma} L_{h^{-1}}^* f(x) F_y e_h \\ &= \sum_{h\in\Gamma} (-\Delta_{LB}) L_{h^{-1}}^* f(x) F_y e_h \\ = \sum_{h\in\Gamma} L_{h^{-1}}^* (-\Delta_{LB}) f(x) F_y e_h &= \Phi_y(-\Delta_{LB} f(x)),\end{aligned} \tag{4.35}$$

because $(-\Delta_{LB})_y$ acts in the variable x and for $f \in C_0^\infty(\tilde{M})$, the sum is finite.

$(-\Delta_{LB})_y$ is the operator defined as in the previous chapter and there exists selfadjoint extension of this operator (see Theorem 3).

$V(x)$ is Γ invariant, so $VL_h^* = L_h^* V$. $V(x)$ is a bounded operator and similarly as in the previous case, $\Phi V \Phi^* = \int_Y V_y \, \mathrm{d}y$, where V_y is the formally the same operator acting on $\mathcal{H}_y$. $\square$

Then also $U(t) = \exp(-\frac{\mathrm{i}}{\hbar} Ht)$ is decomposable under the direct integral and $\Phi U(t) \Phi^* = \int_Y U_y(t) \, \mathrm{d}y$. U_y is the propagator corresponding to H_y for almost all y.

4.3 The Schwartz kernel theorem

Theorem 27 (Schwartz kernel theorem, [Hor]). *Let $\mathcal{K} \in \mathcal{D}'(X_1 \times X_2)$. Then by the equation*

$$\langle K\phi, \psi \rangle = \mathcal{K}(\psi \otimes \phi), \psi \in C_0^\infty(X_1), \phi \in C_0^\infty(X_2), \tag{4.36}$$

is defined a linear map $K : C_0^\infty(X_2) \to \mathcal{D}'(X_1)$ which is continuous in the sense that $K\phi_j \to 0$ in $\mathcal{D}'(X_1)$ if $\phi_j \to 0$ in $C_0^\infty(X_2)$. Conversely, to every such linear map K there is one and only one distribution $\mathcal{K}$ such that (4.36) is valid. One calls $\mathcal{K}$ the kernel of K.

For the existence of the kernel for the propagator defined on $L^2(\tilde{M}, \, \mathrm{d}\mu)$we use the following reformulation of the Schwartz kernel theorem:

Theorem 28. *Let $B \in \mathcal{B}(L^2(\tilde{M}, \, d\mu))$, then there exists one and only one $\beta \in \mathcal{D}'(\tilde{M} \times \tilde{M})$ such that $\beta(\bar{\phi_1} \otimes \phi_2) = \langle \phi_1, B\phi_2 \rangle$ for $\forall \phi_1, \phi_2 \in C_0^\infty(\tilde{M})$. Moreover the map $B \to \beta$ is an injection.*

Proof. The proof follows directly from the fact, that B restricted to $C_0^\infty(\tilde{M})$ is continuous as the function $B : C_0^\infty(\tilde{M}) \to L^2(\tilde{M},\ \mathrm{d}\mu)$. Because $C_0^\infty(\tilde{M})$ is dense in $L^2(\tilde{M}, d\mu)$ and $I : L^2(\tilde{M}, d\mu) \to \mathcal{D}'(\tilde{M})$, where I is the identity map, is continuous, $B : C_0^\infty(\tilde{M}) \to \mathcal{D}'(\tilde{M})$ is continuous and there exists unique β from the previous theorem satisfying the above equation. □

Distribution $\delta(x-y)$ is the kernel for the identity perator I:

$$\delta(x-y)(\phi_1(x) \otimes \phi_2(y)) = \langle \bar{\phi_1}, \phi_2 \rangle = \int_{\tilde{M}} \phi_1(x)\phi_2(x)\ \mathrm{d}\mu(x). \tag{4.37}$$

To obtain the kernel in case of $U_y(t)$, we use the following mapping $\tilde{\Phi}$:

$$\tilde{\Phi}_y : L^2(\tilde{M}) \otimes \mathcal{L}_y \to \tilde{\mathcal{H}}_y : \tilde{\Phi}_y = \sum_{g\in\Gamma} L^*_{g^{-1}} \otimes \Lambda(g), \tag{4.38}$$

where

$$\tilde{\mathcal{H}}_y = \{v_y : \tilde{M} \to \mathcal{L}_y,\ \text{such that } \Lambda_y(g)v_y(x) = v_y(g.x), \forall g \in \Gamma, \\ \int_M \|v_y(x)\|^2_{\mathcal{L}_y}\ \mathrm{d}\mu < \infty\} \tag{4.39}$$

The generalized kernel for $U_y(t)$ would be

$$K_t^y(x,z) \in \mathcal{D}'(\tilde{M} \times \tilde{M}) \otimes \mathcal{B}(\mathcal{L}_y) \tag{4.40}$$

defined by

$$\langle u_1, (K_t^y(x,z), \bar{\psi}_1(x)\psi_2(z))u_2\rangle_{\mathcal{L}_y} = \langle \tilde{\Phi}_y(\psi_1 \otimes u_1), \tilde{U}_y(t)\tilde{\Phi}_y(\psi_2 \otimes u_2)\rangle_{\tilde{\mathcal{H}_y}}, \tag{4.41}$$

where $\Phi U(t) \Phi^* = \int_Y U_y(t) = \int_Y \mathcal{L}_y^* \otimes \tilde{U}_y(t)$.

Let $\{u_n^y\}_{n\in N}$, $\{\phi_n^y\}_{n\in N}$ be mutually dual orthonormal basis of $\mathcal{L}_y$,$\mathcal{L}_y^*$ respectively. Then if $A \in \mathcal{H}_y$, then $A(x) = \sum_n \phi_n^y \otimes A(x)u_n^y = \sum_n \phi_n^y \otimes A_n^y(x)$ and $A_n^y \in \tilde{\mathcal{H}}_y$. Then if $f \in C_0^\infty(\tilde{M})$ we have

$$\Phi_y f = \sum_n \phi_n^y \otimes (\tilde{\Phi}_y f \otimes u_n^y). \tag{4.42}$$

Then

$$(\mathcal{K}_t(x,z), \bar{\psi}_1(x)\psi_2(z)) = \langle \psi_1, U(t)\psi_2 \rangle$$

$$
\begin{aligned}
&= \langle \Phi\psi_1, \Phi U(t)\Phi^{-1}\Phi\psi_2\rangle_{\int_Y \mathcal{H}_y} \\
= \langle \Phi\psi_1, \int_Y I_y \otimes \tilde{U}_y(t)\ \mathrm{d}y\Phi\psi_2\rangle &= \int_Y \langle \Phi_y\psi_1, I_y \otimes \tilde{U}_y(t)\Phi_y\psi_2\rangle_{\mathcal{H}_y}\ \mathrm{d}y \\
&= \int_Y \langle \sum_n \phi_n^y \otimes \tilde{\Phi}_y(\psi_1 \otimes u_n^y), \sum_k I_y\phi_k^y \otimes \tilde{U}_y(t)\tilde{\Phi}_y(\psi_2 \otimes u_n^y)\rangle\ \mathrm{d}y \\
&= \int_Y \sum_n \langle \tilde{\Phi}_y\psi_1 \otimes u_n^y, \tilde{U}_y(t)\tilde{\Phi}_y\psi_2 \otimes u_n^y\rangle\ \mathrm{d}y \\
&= \int_Y \sum_n \langle u_n^y, (\mathcal{K}_t^y(x,z), \bar{\psi}_1(x)\psi_2(z))u_n^y\rangle_{\mathcal{L}_y}\ \mathrm{d}y \\
&= \int_Y tr_y[\mathcal{K}_t^y(x,z), \bar{\psi}_1(x)\psi_2(z)]\ \mathrm{d}y. \qquad (4.43)
\end{aligned}
$$

Chapter 5

Schulman ansatz

Schulman ansatz gives a relation between two operators which are formally the same, but are defined on two different Hilbert spaces. Let $\tilde{M}$ be simply connected and locally linearly connected Riemann manifold with Γ-invariant Riemann metric. Let Γ be at most countable discrete group acting freely on $\tilde{M}$.[1] We also suppose that the action of Γ is properly discontinuous. Let $M = \Gamma\backslash\tilde{M}$, then $\pi_1(M) = \Gamma$. Let us further suppose that Γ is of Type I.

One of the Hilbert spaces is the space of $L^2(\tilde{M},\ \mathrm{d}\mu)$, H is a Hamiltonian on $L^2(\tilde{M})$ which is Γ-invariant[2], $U(t)$ is the corresponding propagator, it means

$$U(t) = \exp(-\frac{i}{\hbar}Ht). \tag{5.1}$$

$U(t)$ is a bounded operator on $L^2(\tilde{M})$ and it is possible to use the Schwartz kernel theorem to find the kernel $K_t(x, y)$ of $U(t)$. The second Hilbert spaces is the space of Λ_y-equivariant functions with values in $\mathcal{L}_y$ defined in the previous chapter as $\tilde{\mathcal{H}}_y$. Let H_y be formally the same Hamiltonian on $\tilde{\mathcal{H}}_y$, $\tilde{U}_y(t)$ its propagator(see the previous chapter).

Remark 8. *$U(t)$, resp. $\tilde{U}_y(t)$ is in both cases unitary operator,$\{U(t)\}_t$ is one parametric group with properties*

$$\begin{aligned} U(t)U(s) = &\quad U(t+s), \\ U(t)^{-1} = &\quad U(-t), \\ U(0) = &\quad 1. \end{aligned} \tag{5.2}$$

[1] It means if $g.x = x$ for some x, then g =identity in Γ.

[2] H is Γ-invariant Hamiltonian if $L^*_{g^{-1}}HL^*_g = H$ for all $g \in \Gamma$. If $H = -\Delta_{LB} + V$, where V is $\Gamma-$invariant potential, then H is Γ-invariant.

Let K_t, resp. K_t^y be the kernel of operator $U(t)$, resp. $\tilde{U}_y(t)$.

Formally in case of abelian groups, Schulmann ansatz proposes, that

$$K_t^\chi(x,z) = \int_\Gamma \chi(g^{-1}) K_t(g.x,z) \,\mathrm{d}g. \tag{5.3}$$

It means that for $\varphi, \psi \in C_0^\infty(\tilde{M})$

$$\langle \Phi\psi(x), \tilde{U}_\chi(t)\Phi\varphi(x)\rangle_{\mathcal{H}_\chi} = \int_\Gamma \chi(g^{-1}) \langle \psi(x), U(t)\varphi(g.x)\rangle_{L^2(\tilde{M})} \,\mathrm{d}g,$$

for almost all χ.

If Γ is a locally compact group of Type I the Schulman ansatz reformulates

$$(K_t^y(x,z), \bar{\psi}_1(x)\psi_2(z)) = \sum_{g\in\Gamma} \Lambda(g^{-1})(K_t(g.x,z), \bar{\psi}_1(x)\psi_2(z)). \tag{5.4}$$

5.1 Proof of the Schulman ansatz

Lemma 3. $tr_y[(K_t^y(x,z), \bar{\psi}_1(x)\psi_2(z))] \in L^1(Y,\ dy)$

Proof. From the previous:

$$\begin{aligned} &|tr_y[(K_t^y(x,z), \bar{\psi}_1(x)\psi_2(x))]| \\ &= |\langle \Phi_y\psi_1, I\otimes \tilde{U}_y(t)\Phi_y\psi_2\rangle_{\mathcal{H}_y}| \le \|\Phi_y\psi_1\|_{\mathcal{H}_y}\|\Phi\psi_2\|_{\mathcal{H}_y}. \end{aligned} \tag{5.5}$$

Then

$$\begin{aligned} \int_Y |tr_y[K_t^y(x,z), \bar{\psi}_1(x)\psi_2(z)]| \,\mathrm{d}y &\le \int_Y \|\Phi_y\psi_1\|_{\mathcal{H}_y}\|\Phi\psi_2\|_{\mathcal{H}_y} \,\mathrm{d}y \\ &\le (\int_Y \|\Phi_y\psi_1\|^2_{\mathcal{H}_y} \,\mathrm{d}y)^{\frac{1}{2}} (\int_Y \|\Phi_y\psi_2\|^2_{\mathcal{H}_y} \,\mathrm{d}y)^{\frac{1}{2}} \\ &= \|\Phi\psi_1\|\|\Phi\psi_2\| = \|\psi_1\|_{L^2(\tilde{M})}\|\psi_2\|_{L^2(\tilde{M})} < \infty. \end{aligned} \tag{5.6}$$

□

Lemma 4. $K_t^y(g.x,z) = \Lambda(g)K_t^y(x,z)$

Proof.

$$\begin{aligned} &\langle u_1, (K_t^y(g.x,z), \bar{\psi}_1(x)\psi_2(z))u_2\rangle_{\mathcal{L}_y} \\ &= \langle u_1, (K_t^y(x,z), \bar{\psi}_1(g^{-1}.x)\psi_2(z))u_2\rangle_{\mathcal{L}_y} \\ &= \langle \tilde{\Phi}_y L^*_{g^{-1}}\psi_1 \otimes u_1, \tilde{U}_y(t)\tilde{\Phi}_y\psi_2\otimes u_2\rangle_{\tilde{\mathcal{H}_y}} \\ &= \langle \tilde{\Phi}_y\psi_1 \otimes \Lambda(g^{-1})u_1, \tilde{U}_y(t)\tilde{\Phi}_y\psi_2\otimes u_2\rangle_{\tilde{\mathcal{H}_y}} \\ &= \langle \Lambda(g^{-1})u_1, (K_t^y(x,z), \bar{\psi}_1(x)\psi_2(z))u_2\rangle_{\mathcal{L}_y} \\ &= \langle u_1, \Lambda(g)(K_t^y(x,z), \bar{\psi}_1(x)\psi_2(z))u_2\rangle_{\mathcal{L}_y}, \end{aligned} \tag{5.7}$$

for $\forall u_1, u_2 \in \mathcal{L}_y$, $\forall \psi_1, \psi_2 \in C_0^\infty(\tilde{M})$. □

Lemma 5. $K(g) = (\mathcal{K}_t(g^{-1}.x, z), \bar{\psi}_1(x)\psi_2(z)) \in L^2(\Gamma,\ dg)$

Proof. Let $\psi \in C_0^\infty(\tilde{M})$, because the action of Γ on $\tilde{M}$ is properly discontinuous, then $\psi = \sum_{j=1}^n \eta_j$, where

$$\mathrm{supp}(L_g^*\eta_j) \cap \mathrm{supp}\eta_j = \emptyset, \tag{5.8}$$

for $\forall g \in \Gamma, g \neq 1$. Without lost of generality, we can suppose,that ψ_1 satisfies equation (5.8). Then

$$\|L_g^*\psi_1\|_{L^2(\tilde{M})} = \|\psi_1\|_{L^2(\tilde{M})}. \tag{5.9}$$

So $\{\frac{1}{\|\psi_1\|}L_g^*\psi\}_{g\in\Gamma}$ is an orthonormal system in $L^2(\tilde{M})$. Then

$$\begin{aligned} K(g) = (K_t(g^{-1}.x, y), \bar{\psi}_1(x)\psi_2(z)) &= (\mathcal{K}_t(x, z), \overline{L_g^*\psi_1(x)}\psi_2(z)) \\ &= \langle L_g^*\psi_1, U(t)\psi_2\rangle_{L^2(\tilde{M})} \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} \sum_{g\in\Gamma} |K(g)|^2 &= \|\psi_1\|^2 \sum_{g\in\Gamma} |\langle \frac{1}{\|\psi_1\|} L_g^*\psi_1, U(t)\psi_2\rangle|^2 \\ &\leq \|\psi_1\|^2 \|U(t)\psi_2\|^2 = \|\psi_1\|^2\|\psi_2\|^2 < \infty, \end{aligned} \tag{5.11}$$

where we have used the Bessel inequality and the fact, that $U(t)$ is an unitary operator. $\square$

Lemma 6. *Let $L(y) = (K_t^y(x, z), \bar{\psi}_1(x)\psi_2(z))$. Then $L \in \int_Y \mathcal{I}_2(\mathcal{L}_y)\ dy$.*

Proof.

$$\|L(y)\|^2_{\mathcal{I}_2(\mathcal{L}_y)} = \sum_j \sum_k |\langle u_j, (K_t^y(x, y), \bar{\psi}_1(x)\psi_2(z))u_k\rangle_{\mathcal{L}_y}|^2, \tag{5.12}$$

where $\{u_j\}$ is an arbitrary orthonormal base in $\mathcal{L}_y$. Then

$$\begin{aligned} \|L(y)\|^2_{\mathcal{I}_2(\mathcal{L}_y)} &= \sum_j \sum_k |\langle \tilde{\Phi}_y\psi_1 \otimes u_j, \tilde{U}_y(t)\tilde{\Phi}_y\psi_2 \otimes u_k\rangle_{\tilde{H}_y}|^2 \\ &\leq \sum_j \sum_k \|\tilde{\Phi}_y\psi_1 \otimes u_j\|^2_{\tilde{\mathcal{H}}_y} \|\tilde{\Phi}_y\psi_2 \otimes u_k\|^2_{\tilde{\mathcal{H}}_y}. \end{aligned} \tag{5.13}$$

Let ψ_1 and ψ_2 be such that

$$\mathrm{supp}L_g^*\psi_i \cap \mathrm{supp}\psi_i = \emptyset, g \neq 1. \tag{5.14}$$

Then

$$\tilde{\Phi}_y(\psi_i \otimes u(x)) = \sum_{\Gamma} \psi_i(g^{-1}.x)\Lambda(g)u, \tag{5.15}$$

but the sum is non zero only for at most one $g \in \Gamma$. Thus

$$\|\tilde{\Phi}_y(\psi_i \otimes u)(x)\|^2_{\mathcal{L}_y} = \sum_{g\in\Gamma} |\psi_i(g.x)|^2 \|\Lambda(g)u\|^2_{\mathcal{L}_y},$$

$$\begin{aligned} \|\tilde{\Phi}_y \psi_i \otimes u\|^2_{\tilde{\mathcal{H}}_y} &= \int_D \sum_{g\in\Gamma} |\psi_i(g.x)|^2 \|u\|^2_{\mathcal{L}_y} \, \mathrm{d}\mu \\ &= \|\psi_i\|_{L^2(\tilde{M})} \|u\|^2_{\mathcal{L}_y}. \end{aligned} \tag{5.16}$$

Then the inequality (5.13) leads to

$$\begin{aligned} \|L(y)\|^2_{\mathcal{I}_2(\mathcal{L}_y)} &\leq \sum_j \sum_k \|u_j\|^2 \|u_k\|^2 \|\psi_1\|^2_{L^2(\tilde{M})} \|\psi_2\|^2_{L^2(\tilde{M})} \\ &= (\dim\, \mathcal{L}_y)^2 \|\psi_1\|^2_{L^2(\tilde{M})} \|\psi_2\|^2_{L^2(\tilde{M})}. \end{aligned} \tag{5.17}$$

The assumption that Γ is of Type I assures, that $\dim\mathcal{L}_y$ is finite and it is uniformally bounded for $y \in Y$. Further the measure of Y is finite, so

$$\|L\|^2 = \int_Y \|L(y)\|^2_{\mathcal{I}_2(\mathcal{L}_y)} \, \mathrm{d}y \leq \text{const.measure of } Y < \infty. \tag{5.18}$$

□

Equation (4.43) implies that

$$\begin{aligned} K(g) = (K_t(g^{-1}x, z), \bar{\psi}_1(x)\psi_2(z)) &= \int_Y tr_y[K^y_t(g^{-1}.x, z), \bar{\psi}_1(x)\psi_2(z)] \, \mathrm{d}y \\ &= \int_Y tr_y[\Lambda(g^{-1})K^y_t(x, z), \bar{\psi}_1(x)\psi_2(z)] \, \mathrm{d}y = F^*_g L. \end{aligned}$$

We have thus prooved that

$$K(g) = F^*_g L \tag{5.19}$$

and because of the two previous lemmas, we can conclued that the Schulman ansatz in our notation reformulates

$$L(y) = F_y K. \tag{5.20}$$

It means

$$(K^y_t(x, y), \bar{\psi}_1(x)\psi_2(z)) = \sum_{g\in\Gamma} \Lambda(g)(K_t(g^{-1}.x, z), \bar{\psi}_1(x)\psi_2(z)) \tag{5.21}$$

for almost all y.

Chapter 6

Aharonov-Bohm Effect with two vortices

We consider a non-relativistic particle moving in the external gauge field with the flux concentrated in two infinitely thin parallel solenoids. Outer the solenoids the flux vanishes. Even though the particles do not pass throw the field, they are influenced by this field. In the idealized setup, the configuration space is $M = R^2 - \{a, b\}$ and the gauge field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ vanishes on M. All gauge fields A_μ with vanishing field strength on some configuration space M are in correspondence with unitary representation Λ on the group $\Gamma = \pi_1(M, x_1)$.

For the computation of the kernel, we will use the universal covering space $\tilde{M}$, $\Gamma\backslash\tilde{M} = M$. $\tilde{\mathcal{H}}_y$ is the Hilbert space of Λ equivariante functions defined on $\tilde{M}$.

The goal of this chapter is to prove the formula for the propagator on $\tilde{\mathcal{H}}_y$ using the Schulman ansatz.

First of all, we must find the formula for the propagator on the universal covering space $\tilde{M}$.

The Aharonov-Bohm Hamiltonian acts in $L^2(R^2,\ \mathrm{d}^2x)$ and it is the self adjoint operator

$$H = -\Delta_{LB} \tag{6.1}$$

with the domain determined by the boundary conditions on the cuts L_a, L_b introduced below.

Without lost of generality we chose $a = (0, 0)$, $b = (\rho, 0)$, and let L_a, L_b be two half lines $L_a = (1, 0)t, t \in (-\infty, 0]$, $L_b = (1, 0)t,\ t \in [b, \infty)$. Then $M - \{L_a, L_b\}$ is a fundamental domain. We introduce two polar coordinates with respect to the points a and b, then $\varphi_a = \pm\pi$, resp. $\varphi_b = \pm\pi$ corresponds to L_a, resp. L_b. The boundary of the fundamental domain consists of points

a, b, four half lines (two sides of L_a, two sides of L_b) and two ∞. The fundamental domain will be denoted by $D = (a, b)$. Copies of a, b will be denoted by $A., B$. One obtains the universal covering spaces by patching together the fundamental domains along four halflines with four other sheets, $\tilde{M} = \cup_{i,j} D(A_j, B_i)$ where $\mathrm{dist}(A_j, B_i) = \rho$. For simplicity we will add to the universal covering space the copies of a and b, the set of all these points will be denoted by $\mathcal{C}$ and this set is at most countable

In the following $\chi(x_1, x_2)$ is the function characterizing the property that x_1, x_2 can be connected with a finite geodesic. Two points can be connected with a finite geodesic if $|\varphi_a^1 - \varphi_a^2| < \pi$ and $|\varphi_b^1 - \varphi_b^2| < \pi$. $\chi(A, x) = 1$ if $x \in \bigcup_i D(A, B_i)$ and similarly for the copies of b.

For simplicity we define

$$Z_t(x, x_0) = \vartheta(t)\chi(x, x_0)\frac{\mu}{2\pi \mathrm{i}\hbar t} \exp\left(\frac{\mathrm{i}\mu}{2\hbar t}\mathrm{dist}^2(x, x_0)\right), \tag{6.2}$$

$$V\left(\begin{array}{c} x_3, x_2, x_1 \\ t_2, t_1 \end{array}\right) = \frac{\mathrm{i}\hbar}{\mu}\left(\frac{1}{\theta - \pi + \mathrm{i}\ln(\frac{t_2 r_1}{t_1 r_2})} - \frac{1}{\theta + \pi + \mathrm{i}\ln(\frac{t_2 r_1}{t_1 r_2})}\right), \tag{6.3}$$

where ϑ is the Heaviside step function, $\theta = \angle x_1 x_2 x_3$ is the oriented angle, $u = \ln(t_2 r_1 / t_1 r_2)$ and $r_1 = \mathrm{dist}(x_1, x_2), r_2 = \mathrm{dist}(x_2, x_3)$. Let γ be the sequence $(C_1, C_2 \ldots C_n)$, where $\mathrm{dist}(C_i, C_{i+1}) = \rho$ and $c_i \in \mathcal{C}$. In the next two section, we will prove that

$$\begin{aligned} &K_t(x, x_0) = \\ &\sum_{\gamma, n \geq 0} \int_{R^{n+1}} \mathrm{d}t_n \ldots \mathrm{d}t_0 \delta(t_n + \ldots + t_0 - t) V\left(\begin{array}{c} x, C_n, C_{n-1} \\ t_n, t_{n-1} \end{array}\right) \times \\ &V\left(\begin{array}{c} C_n, C_{n-1}, C_{n-2} \\ t_{n-1}, t_{n-2} \end{array}\right) \times \ldots \times V\left(\begin{array}{c} C_2, C_1, x_0 \\ t_1, t_0 \end{array}\right) \times \\ &Z_{t_n}(x, C_n) Z_{t_{n-1}}(C_n, C_{n-1}) \times Z_{t_0}(C_1, x_0), \end{aligned} \tag{6.4}$$

where γ runs over all possible sequences of length n, is the kernel of the propagator on $\tilde{M}$. First of all, we will prove that

$$\begin{aligned} &K_t(x, x_0) = Z_t(x, x_0) + \\ &\int_{R^2} \mathrm{d}t_1 \, \mathrm{d}t_0 \delta(t_1 + t_0 - t) V\left(\begin{array}{c} x, C, x_0 \\ t_1, t_0 \end{array}\right) Z_{t_1}(x, A) Z_{t_0}(A, x_0) \end{aligned} \tag{6.5}$$

is the kernel of the propagator in one solenoid case.

6.1 Kernel on $R^+ \times R$

In this section we will find the kernel of the propagator on the universal covering space corresponding to Aharonov-Bohm effect with one solenoid.

In this case $M = R^2 - (0,0)$ and $\Gamma = \pi_1(M) = Z$. $\tilde{M}$ is completed with the point A, a copy of $(0,0)$. Using polar coordinates (r,φ), we can identify fundamental domain with $R^+ \times (0,2\pi)$. Universal covering space $\tilde{M}$ can be identified with $R^+ \times R$. The action of Γ is given by $n.(r,\varphi) = (r,\varphi + 2\pi n)$.

We want to prove that

$$K_t(x,x_0) = Z_t(x,x_0) + \int_{R^2} \mathrm{d}t_1 \, \mathrm{d}t_0 \delta(t_1+t_0-t) V\begin{pmatrix} x, C, x_0 \\ t_1, t_0 \end{pmatrix} Z_{t_1}(x,A) Z_{t_0}(A,x_0) \tag{6.6}$$

is the kernel of the free propagator on the universal covering space $\tilde{M}$. It means to prove that

$$(\mathrm{i}\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2\mu}\Delta_{LB})\vartheta(t) K_t(x,x_0) = \mathrm{i}\hbar\delta(t)\delta(x,x_0). \tag{6.7}$$

In other words it means

$$\lim_{t\to 0_+} K_t(x,x_0) = \delta(x,x_0) \tag{6.8}$$

$$(\mathrm{i}\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2\mu}\Delta_{LB}) K_t(x,x_0) = 0, \text{ for } t > 0 \tag{6.9}$$

on $R^+ \times R$.

First of all let us mention that $Z_t(x,x_0)$, up to the term $\chi(x,x_0)$, is kernel of the free propagator on R^2. It is true that

$$(\mathrm{i}\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2\mu}\Delta_{LB}) Z_t(x,x_0) = \frac{\mathrm{i}\hbar}{4\pi t} \exp\left(\frac{\mathrm{i}\mu}{2\hbar t} \text{ dist } (x,x_0)^2\right) (\partial_{\overrightarrow{n}}\delta_{L_+} + \partial_{\overrightarrow{n}}\delta_{L_-}) \tag{6.10}$$

where δ_{L_+}, resp. δ_{L_-} are distributions defined by

$$\langle \delta_{L_+}, \psi(r,\varphi)\rangle = \int_0^\infty \psi(r,\pi_+) r \, \mathrm{d}r, \tag{6.11}$$

resp.

$$\langle \delta_{L_-}, \psi(r,\varphi)\rangle = \int_0^\infty \psi(r,\pi_-) r \, \mathrm{d}r.$$

It holds true for any $f(t) \in C^1(\bar{R}_+)$,

$$(i\partial_t + \frac{1}{4}(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2)) \times$$

$$\int_0^t \frac{1}{\theta + \mathrm{i}\ln(\frac{(t-s)r_0}{sr})} \frac{1}{t-s} \exp\left(\frac{\mathrm{i}r^2}{t-s}\right) f(s)\ \mathrm{d}s$$
$$= \frac{\pi r_0}{2r^2(r+r_0)} \exp\left(\mathrm{i}(r+r_0)\frac{r}{t}\right) \Big[f\left(\frac{tr_0}{r+r_0}\right)\delta'(\theta) -$$
$$\mathrm{i}\frac{r}{r+r_0}\left(\left(1+\mathrm{i}\frac{r_0(r+r_0)}{t}\right) f\left(\frac{tr_0}{r+r_0}\right) + \frac{tr_0}{r+r_0} f'\left(\frac{tr_0}{r+r_0}\right)\right)\delta(\theta)\Big]. \quad (6.12)$$

After few transformations we obtain

$$(i\hbar\partial_t + \frac{\hbar^2}{2\mu}(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2)) \times$$
$$\int_0^t \frac{1}{\theta + \mathrm{i}\ln(\frac{(t-s)r_0}{sr})} \frac{\hbar}{t-s} \exp\left(\frac{\mathrm{i}r^2\mu}{2\hbar(t-s)}\right) f\left(\frac{2\hbar s}{\mu}\right)\ \mathrm{d}s$$
$$= \frac{\pi r_0 \hbar^3}{r^2\mu(r+r_0)} \exp\left(\mathrm{i}(r+r_0)\frac{\mu r}{2\hbar t}\right) \Big[f\left(\frac{2\hbar t r_0}{\mu(r+r_0)}\right)\delta'(\theta) - \frac{\mathrm{i}r}{r+r_0} \times$$
$$\left(\left(1+\mathrm{i}\frac{\mu r_0(r+r_0)}{2\hbar t}\right) f\left(\frac{2\hbar t r_0}{\mu(r+r_0)}\right) + \frac{2\hbar t r_0}{\mu(r+r_0)} f'\left(\frac{2\hbar t r_0}{\mu(r+r_0)}\right)\right)\delta(\theta)\Big].$$

Setting $f(u) = \frac{1}{u}\exp\left(\frac{\mathrm{i}r_0^2}{u}\right)$ we obtain

$$(i\hbar\partial_t + \frac{\hbar^2}{2\mu}(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2)) \times$$
$$\int_0^t \frac{1}{\theta + \mathrm{i}\ln\left(\frac{(t-s)r_0}{sr}\right)} \frac{\hbar}{t-s} \exp\left(\frac{\mathrm{i}r^2\mu}{2\hbar(t-s)}\right) \frac{\mu}{2\hbar s} \exp\left(\frac{\mathrm{i}r_0^2\mu}{2\hbar s}\right)\ \mathrm{d}s$$
$$= \frac{\pi r_0 \hbar^3}{r^2\mu(r+r_0)} \exp\left(\mathrm{i}(r+r_0)\frac{\mu r}{2\hbar t}\right) \Big[\frac{\mu(r+r_0)}{2\hbar t r_0} \exp\left(\frac{\mathrm{i}r_0^2\mu(r+r_0)}{2\hbar t r_0}\right)\delta'(\theta) -$$
$$\mathrm{i}\frac{r}{r+r_0}\Big\{(1+\mathrm{i}\frac{\mu r_0(r+r_0)}{2\hbar t})\frac{\mu(r+r_0)}{2\hbar t r_0} \exp\left(\frac{\mathrm{i}r_0^2\mu(r+r_0)}{2\hbar t r_0}\right) +$$
$$\frac{2\hbar t r_0}{\mu(r+r_0)}\left(-\frac{\mu^2(r+r_0)^2}{4\hbar^2 t^2 r_0^2} - \frac{\mathrm{i}\mu^3 r_0^2 (r+r_0)^3}{8\hbar^3 t^3 r_0^3}\right) \exp\left(\frac{\mathrm{i}\mu r_0^2(r+r_0)}{2\hbar t r_0}\right)\Big\}\delta(\theta)\Big]$$
$$= \frac{\pi\hbar^2}{2r^2 t} \exp\left(\frac{\mathrm{i}\mu}{2\hbar t}(r+r_0)^2\right)\delta'(\theta).$$

Using this identity we obtain

$$(\mathrm{i}\hbar\partial_t + \frac{\hbar^2}{2\mu}(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2)) \times$$
$$\int\int\ \mathrm{d}t_1\ \mathrm{d}t_0 \delta(t_1+t_0-t) V\left(\begin{array}{c} x, A, x_0 \\ t_1, t_0 \end{array}\right) Z_{t_1}(x,A) Z_{t_0}(A,x_0)$$

$$= (\mathrm{i}\hbar\partial_t + \frac{\hbar^2}{2\mu}(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2)) \times$$
$$\frac{-\mathrm{i}\mu}{4\hbar\pi^2}\int_0^t [\frac{1}{\theta - \pi + \mathrm{i}\ln(\frac{(t_0-t)r_1}{t_0 r_2})} - \frac{1}{\theta + \pi + \mathrm{i}\ln(\frac{(t_0-t)r_1}{t_0 r_2})}] \times$$
$$\frac{1}{(t_0 - t)t_0}\exp\left(\frac{\mathrm{i}\mu}{2\hbar t}(\frac{r^2}{t_0 - t} + \frac{r_0^2}{t_0})\right)\,\mathrm{d}t_0$$
$$= \frac{\mathrm{i}\hbar}{4r^2 t\pi}\exp\left(\frac{\mathrm{i}\mu}{2\hbar t}(r + r_0)^2\right)(\delta'(\theta - \pi) - \delta'(\theta + \pi)). \tag{6.13}$$

Finally using the formula [1]

$$\frac{1}{r^2}\delta'(\theta \mp \pi) = \pm\partial_{\overrightarrow{n}}\delta_{L_\pm} \tag{6.14}$$

we obtain (6.9).

Before the justification of (6.8), let us prove the following proposition:

Proposition 8. *Let $\psi(x) \in C_0^\infty(R)$. Then*

$$\int_0^{+\infty} \frac{1}{it}\exp(\frac{i}{t}x)\psi(x)\ dx \to \psi(0),\ \textit{for } t \to \infty,$$

so

$$\lim_{t\to 0}\frac{1}{it}\exp(\frac{i}{t}x)\vartheta(x) = \delta(x)\ \textit{in } \mathcal{D}'(R).$$

Proof.

$$\lim_{t\to 0}\int_0^{+\infty}\frac{1}{\mathrm{i}t}\exp(\frac{\mathrm{i}}{t}x)\psi(x)\ \mathrm{d}x$$
$$= -\lim_{t\to 0}[\exp(\frac{\mathrm{i}}{t}x)\psi(x)]_0^{+\infty} + \lim_{t\to 0}\int_0^{+\infty}\exp(\frac{\mathrm{i}}{t}x)\psi'(x)\ \mathrm{d}x$$
$$= \psi(0) + \lim_{t\to 0}[-\mathrm{i}t\exp(\frac{\mathrm{i}}{t}x)\psi'(x)]_0^{+\infty} + \lim_{t\to 0}\int_0^{+\infty}\mathrm{i}t\exp(\frac{\mathrm{i}}{t}x)\psi''(x)\ \mathrm{d}x = \psi(0).$$

□

The equation (6.8) follows from the fact that $Z_t(x, x_0)$ is the free propagator, so

$$\lim_{t\to 0} Z_t(x, x_0) = \delta(x, x_0). \tag{6.15}$$

[1] $\int_{R_+\times R}\frac{1}{r^2}\delta'(\theta \mp \pi)\varphi(r,\theta)r\ \mathrm{d}r\ \mathrm{d}\theta = -\int_{R_+}\frac{1}{r}\partial_\theta\varphi(r, \pm\pi)\ \mathrm{d}r = -\int_{L_\pm}\frac{\partial\varphi}{\partial\overrightarrow{n}}\ \mathrm{d}l$

This can be also easily proved using the previous proposition. It remains to prove that

$$\lim_{t\to 0}\int\int \mathrm{d}t_1\ \mathrm{d}t_0\delta(t_1+t_0-t)V\left(\begin{array}{c} x,A,x_0\\ t_1,t_0\end{array}\right)Z_{t_1}(x,A)Z_{t_0}(A,x_0)=0$$

in $\mathcal{D}'(R)$:

$$\lim_{t\to 0}\int\int \mathrm{d}t_1\ \mathrm{d}t_0\delta(t_1+t_0-t)V\left(\begin{array}{c} x,A,x_0\\ t_1,t_0\end{array}\right)Z_{t_1}(x,A)Z_{t_0}(A,x_0)$$

$$=-\lim_{t\to 0}\int_{R^2}\mathrm{d}x\int_0^t \mathrm{d}t_0\frac{\exp(\frac{\mathrm{i}\mu}{2\hbar}(\frac{r^2}{t-t_0}+\frac{r_0^2}{t_0}))}{\theta-\pi+\mathrm{i}\ln(\frac{(t-t_0)r_0}{t_0 r})}\frac{\mu^2}{4\pi^2\hbar^2(t-t_0)t_0}\psi(x)+$$

$$\lim_{t\to 0}\int_{R^2}\mathrm{d}x\int_0^t \mathrm{d}t_0\frac{\exp(\frac{\mathrm{i}\mu}{2\hbar}(\frac{r^2}{t-t_0}+\frac{r_0^2}{t_0}))}{\theta+\pi+\mathrm{i}\ln(\frac{(t-t_0)r_0}{t_0 r})}\frac{\mu^2}{4\pi^2\hbar^2(t-t_0)t_0}\psi(x),$$

where $r=\mathrm{dist}(x,(0,0))$ and $r_0=\mathrm{dist}(x_0,(0,0))$.

$$\lim_{t\to 0}\int_{R^2}\mathrm{d}x\int_0^t \mathrm{d}t_0\frac{\exp(\frac{\mathrm{i}\mu}{2\hbar}(\frac{r^2}{t-t_0}+\frac{r_0^2}{t_0}))}{\theta-\pi+\mathrm{i}\ln(\frac{(t-t_0)r_0}{t_0 r})}\frac{\mu^2}{4\pi^2\hbar^2(t-t_0)t_0}\psi(x)$$

$$=\lim_{t\to 0}\int_0^t\frac{\mu}{2\pi\hbar t_0}\exp(\frac{\mathrm{i}\mu}{2\hbar}\frac{r_0^2}{t_0})\times$$

$$(\int_{-\pi}^{\pi}\int_R\frac{\exp(\frac{\mathrm{i}\mu r^2}{2\hbar(t-t_0)})}{\theta-\pi+\mathrm{i}\ln(\frac{(t-t_0)r_0}{rt_0})}\frac{\mu}{2\pi\hbar(t-t_0)}\psi(r,\theta)r\ \mathrm{d}r\ \mathrm{d}\theta)\ \mathrm{d}t_0$$

$$=\lim_{t\to 0}\int_0^t\frac{\mu}{2\pi\hbar t_0}\exp(\frac{\mathrm{i}\mu}{2\hbar}\frac{r_0^2}{t_0})F(t-t_0)\ \mathrm{d}t_0,$$

where $F(t-t_0)=\int_{-\pi}^{\pi}\int_R\frac{1}{\theta-\pi+\mathrm{i}\ln(\frac{(t-t_0)r_0}{rt_0})}\frac{\mu}{2\pi\hbar(t-t_0)}\exp(\frac{\mathrm{i}\mu r^2}{2\hbar(t-t_0)})\psi(r,\theta)r\ \mathrm{d}r\ \mathrm{d}\theta$. It is easy to see from proposition 8 that $F(s)\in C^0([0,1])$ and that $\lim_{t\to t_0}F(t-t_0)=0$.

$$\lim_{t\to 0}\int_0^t\frac{1}{t_0}\cos(\frac{\mathrm{i}}{t_0})F(t-t_0)\ \mathrm{d}t_0=\lim_{t\to 0}\int_{1/t}^{+\infty}\frac{1}{s}\cos(\mathrm{i}s)F(t-\frac{1}{s})\ \mathrm{d}s=$$

$$(\lim_{t\to\infty}\int_{1/t}^{\lceil\frac{1}{2\pi t}+\frac{1}{4}\rceil}+\sum_{k=\lceil\frac{1}{2\pi t}+\frac{1}{4}\rceil}^{\infty}\int_{-\frac{\pi}{2}+2k\pi}^{\frac{\pi}{2}+2k\pi}+\int_{\frac{\pi}{2}+2k\pi}^{\frac{3\pi}{2}+2k\pi})\frac{1}{s}\cos(\mathrm{i}s)F(t-\frac{1}{s})\ \mathrm{d}s$$

$$=\lim_{t\to 0}\int_{1/t}^{\lceil\frac{1}{2\pi t}+\frac{1}{4}\rceil}\frac{1}{s}\cos(\mathrm{i}s)F(t-\frac{1}{s})\ \mathrm{d}s+$$

$$\lim_{t\to 0}\sum_{k=\lceil\frac{1}{2\pi t}+\frac{1}{4}\rceil}^{\infty}\int_{-\frac{\pi}{2}+2k\pi}^{\frac{\pi}{2}+2k\pi}\cos(\mathrm{i}s)\left(\frac{F(t-\frac{1}{s})}{s}-\frac{F(t-\frac{1}{s-\pi})}{s-\pi}\right)\ \mathrm{d}s=0.\quad(6.16)$$

Because

$$\lim_{t\to 0}\int_1^{\lceil\frac{1}{2\pi}+\frac{1}{4}\rceil}|\frac{1}{s}F(t(1-\frac{1}{s}))|\ \mathrm{d}s=0,$$

and because there exists an integrable majorant, we can change the limit and the integral. The second term

$$\lim_{t\to 0}\sum_{k=\lceil\frac{1}{2\pi t}+\frac{1}{4}\rceil}^{\infty}\int_{-\frac{\pi}{2}+2k\pi}^{\frac{\pi}{2}+2k\pi}|\cos(\mathrm{i}s)\left(\frac{F(t-\frac{1}{s})-F(t-\frac{1}{s-\pi})}{s-\pi}-\frac{F(t-\frac{1}{s})\pi}{s(s-\pi)}\right)|\leq$$

$$\lim_{t\to 0}\int_{\lceil\frac{1}{2\pi t}+\frac{1}{4}\rceil}^{+\infty}\frac{C}{s(s-\pi)^2}+\frac{C}{s^2(s-\pi)^2}\ \mathrm{d}s=0,$$

where we have used that $|F(t-\frac{1}{s})-F(t-\frac{1}{s-\pi})|\leq C|\frac{1}{s(s-\pi)}|$, because $F\in C^0([0,1])$.

6.2 Kernel on the universal covering space in case of two solenoid Aharonov-Bohm effect

For simplicity, we will use the following notation:

$$K_t(x,x_0)=\sum_{\gamma}K_t^{\gamma}(x,x_0), \tag{6.17}$$

where we assume that γ runs over all possible sequences $(C_1,C_2\dots C_n)$, where $\mathrm{dist}(C_i,C_{i+1})=\rho$ and all possible length n (will be denoted by $|\gamma|$),

$$K_t^{\gamma}(x,x_0)=\int_{R^{n+1}}\mathrm{d}t_n\dots\ \mathrm{d}t_0\delta(t_n+\dots+t_0-t)V\begin{pmatrix}x,C_n,C_{n-1}\\ t_n,t_{n-1}\end{pmatrix}\times$$
$$V\begin{pmatrix}C_n,C_{n-1},C_{n-2}\\ t_{n-1},t_{n-2}\end{pmatrix}\times\dots\times V\begin{pmatrix}C_2,C_1,x_0\\ t_1,t_0\end{pmatrix}\times$$
$$Z_{t_n}(x,C_n)Z_{t_{n-1}}(C_n,C_{n-1})\dots Z_{t_0}(C_1,x_0),$$

and γ can be also the empty sequence, in that case $K_t^{\gamma}(x,x_0)=Z_t(x,x_0)$. We want to proof that $K_t(x,x_0)$ is the kernel on $\tilde{M}$ in the case of two solenoid Aharonov-Bohm effect.

Let $x\in\tilde{M}$, then $D(x)=\{y\in\tilde{M}|\chi(x,y)=1\}$.The border of $D(x)$ consists of four halflines, each two starting in A (will be noted by $\partial D(x,A)$) resp. B (noted by $\partial D(x,B)$). If x is one of the copies of a or of b, $x=C_1$ then the border of $D(C_1)$ will consist of those $D(C_1,C)$ such that $\mathrm{dist}(C_1,C)=\rho$, so $\partial D(C_1)=\bigcup_{C\in\mathcal{C},\mathrm{dist}(C_1,C)=\rho}\partial D(C_1,C)$.

Remark 9. *If $G = \{x \in R^2, r > 0, \theta \in (-\pi, \pi)\}$ then on the border*

$$\frac{\partial}{\partial \overrightarrow{n}} = \pm \frac{\partial}{r \partial \theta}, \tag{6.18}$$

$\theta = \pm\pi$ and $\overrightarrow{n}$ is the normalized vector.

Remark 10. *If G is an open set with piecewise smooth boundary and $f \in C^2(G) \cap C^2(\tilde{M} - \overline{G}) \cap C^1(\overline{M - \overline{G}}) \cap C^1(\overline{G})$ then in the sense of distributions*

$$\Delta_{LB} f = \{\Delta_{LB} f\} + [\frac{\partial f}{\partial \overrightarrow{n}}]_{\partial G} \delta_{\partial G} + \frac{\partial}{\partial \overrightarrow{n}}([f]_{\partial G} \delta_{\partial G}). \tag{6.19}$$

We will verify that (6.4) is the kernel on the universal covering space in two steps:

$$\left(\mathrm{i}\hbar \partial_t + \frac{\hbar^2}{2\mu} \Delta_{LB} \right) K_t(x, x_0) = 0, \text{ for } (t, x) \in (0, \infty) \times \tilde{M}, \tag{6.20}$$

$$\lim_{t \to 0_+} K_t(x, x_0) = \delta(x, x_0). \tag{6.21}$$

Verification of (6.20):

Proof for $|\gamma| = 0$:

Then $\gamma = ()$.

$$\left(\mathrm{i}\hbar \partial_t + \frac{\hbar^2}{2\mu} \Delta_{LB} \right) \tilde{K}_t^{()}(x, x_0) = 0, \tag{6.22}$$

for $x \in D(x_0)$. Moreover

$$\frac{\partial}{\partial \overrightarrow{n}} Z_t(x, x_0) = 0 \tag{6.23}$$

for $x \in \partial D(x_0)$, because $Z_t(x, x_0)$ does not depend on the angle.

So

$$\begin{aligned} &\left(\mathrm{i}\hbar \partial_t + \frac{\hbar^2}{2\mu} \Delta_{LB} \right) \tilde{K}_t^{()}(x, x_0) \\ &= \left\{ \left(\mathrm{i}\hbar \partial_t + \frac{\hbar^2}{2\mu} \Delta_{LB} \right) \tilde{K}_t^{()}(x, x_0) \right\} \\ &- \frac{\hbar^2 \partial}{2\mu \partial \overrightarrow{n}} \tilde{K}_t^{()}(x, x_0) \delta_{\partial G} - \frac{\hbar^2 \partial}{2\mu \partial \overrightarrow{n}} (K_t^{()}(x, x_0) \delta_{\partial D(x_0)}) \\ &= - \frac{\hbar^2 \partial}{2\mu \partial \overrightarrow{n}} (K_t^{()}(x, x_0) \delta_{\partial D(x_0)}) = T_1. \end{aligned}$$

Proof for $|\gamma| = 1$:

Hence $\gamma = (C)$, $C \in \mathcal{C}$. We have

$$K_t^\gamma(x,x_0) = \int_R V\begin{pmatrix} x, C, x_0 \\ t-s, s \end{pmatrix} Z_{t-s}(x,C)Z_s(C,x_0)\ \mathrm{d}s$$
$$= \frac{\mu}{4\pi^2 \mathrm{i}\hbar}\chi(x,C)\chi(C,x_0)$$
$$\int_0^t \left(\frac{1}{\theta - \pi + \mathrm{i}\ln(\frac{(t-s)r_0}{sr})} - \frac{1}{\theta + \pi + \mathrm{i}\ln(\frac{(t-s)r_0}{sr})} \right) \times$$
$$\frac{1}{(t-s)s}\exp(\frac{\mathrm{i}\mu}{2\hbar}(\frac{r^2}{t-s} + \frac{r_0^2}{s}))\ \mathrm{d}s,$$

where $r = \mathrm{dist}(x,C)$, $r_0 = \mathrm{dist}(C,x_0)$, $\theta = \angle x_0 C x$. From one solenoid case, we already know, that just the boundary terms will occur in the final term. The boundary will occur from derivation of $\chi(x,C)$ and from the singularities in $\frac{1}{\theta-\pi+\mathrm{i}\ln(\frac{(t-s)r_0}{sr})}$, resp. $\frac{1}{\theta+\pi+\mathrm{i}\ln(\frac{(t-s)r_0}{sr})}$:

$$(\mathrm{i}\hbar\partial_t + \frac{\hbar^2}{2\mu}\Delta_{LB})K_t^\gamma(x,x_0) = T_2 + T_3, \tag{6.24}$$

where

$$T_2 = \frac{\hbar^2 \partial}{2\mu\partial\overrightarrow{n}}\left(K_t^{\gamma'}(x,x_0)\delta_{\partial D(x_0,C)}\right) \tag{6.25}$$

and

$$T_3 = -\frac{\hbar^2}{2\mu}\left(\frac{\partial}{\partial\overrightarrow{n}}K_t^\gamma(x,x_0)\right)\delta_{\partial D(G)} - \frac{\hbar^2\partial}{2\mu\partial\overrightarrow{n}}\left(K_t^\gamma(x,x_0)\delta_{\partial D(C)}\right). \tag{6.26}$$

The formula for T_3 follows directly from the generalized equation for Δ_{LB}. Using the formula for the one solenoid case (6.13) we obtain the final term (T_2) coming from the singularities:

$$(\mathrm{i}\hbar\partial_t + \frac{\hbar^2}{2\mu}(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2)) \times$$
$$\frac{-\mathrm{i}\mu}{4\hbar\pi^2}\int_0^t [\frac{1}{\theta - \pi + \mathrm{i}\ln(\frac{(t_0-t)r_1}{t_0 r_2})} - \frac{1}{\theta + \pi + \mathrm{i}\ln(\frac{(t_0-t)r_1}{t_0 r_2})}] \times$$
$$\frac{1}{(t_0-t)t_0}\exp(\frac{\mathrm{i}\mu}{2\hbar t}(\frac{r^2}{t_0-t} + \frac{r_0^2}{t_0}))\ \mathrm{d}t_0$$
$$= \frac{\mathrm{i}\hbar}{4r^2 t\pi}\exp\left(\frac{\mathrm{i}\mu}{2\hbar t}(r+r_0)^2\right)(\delta'(\theta-\pi) - \delta'(\theta+\pi)), \tag{6.27}$$

considering only the cases for which $\chi(C, x_0) = 1$. For $\theta = \pm\pi$ we have $r + r_0 = \mathrm{dist}(x, x_0)$ so after these substitutions one obtains the term:

$$T_2 = \chi(C, x_0)\frac{\mathrm{i}\hbar}{4r^2\pi t}\frac{\partial}{\partial\theta}\left(\exp\left(\frac{\mathrm{i}\mu}{2\hbar t}\mathrm{dist}(x, x_0)^2\right)(\delta(\theta - \pi) - \delta(\theta + \pi))\right) = \frac{\hbar^2}{2\mu}\frac{\partial}{\partial\overrightarrow{n}}\left(K_t^{\gamma'}(x, x_0)\delta_{\partial D(x_0, C)}\right), \quad (6.28)$$

where

$$K_t^{\gamma'}(x, x_0) = \chi(x, x_0)\frac{\mu}{2\hbar\pi \mathrm{i}t}\left(\exp\left(\frac{\mathrm{i}\mu}{2\hbar t}\mathrm{dist}^2(x, x_0)\right)\right), \gamma' = (). \quad (6.29)$$

Proof for $|\gamma| \geq 2$:

$\gamma = (C_1, C_2, \ldots, C_n)$, $n \geq 2$, $\gamma' = (C_1, C_2, \ldots, C_{n-1})$. Then

$$\begin{aligned} K_t^{\gamma}(x, x_0) &= \int_{R^n} \mathrm{d}t_{n-1} \ldots \mathrm{d}t_0 V\begin{pmatrix} x, C_n, C_{n-1} \\ t-\tau, t_{n-1} \end{pmatrix} Z(t-\tau, x, C_n) F_\gamma(t_0, \ldots, t_{n-1}, x_0) \\ &= \frac{1}{2\pi}\chi(x, C_n)\int_{R^n} \mathrm{d}t_{n-1} \ldots \mathrm{d}t_0 \times \\ &\left(\frac{1}{\theta - \pi + \mathrm{i}\ln\left(\frac{(t-\tau)\rho}{t_{n-1}r}\right)} - \frac{1}{\theta + \pi + \mathrm{i}\ln\left(\frac{(t-\tau)\rho}{t_{n-1}r}\right)}\right) \\ &\frac{1}{t-\tau}\exp\left(\frac{\mathrm{i}r^2}{t-\tau}\right)F_\gamma(t_0, \ldots, t_{n-1}, x_0), \end{aligned}$$

where $\tau = t_{n-1} + \ldots + t_0$, $r = \mathrm{dist}(x, C_n)$, $\theta = \angle C_{n-1}C_n x$, $C_0 = x_0$ and

$$F_\gamma(t_0, \ldots, t_{n-1}, x_0) = \prod_{j=0}^{n-2} V\begin{pmatrix} C_{j+2}, C_{j+1}, C_j \\ t_{j+1}, t_j \end{pmatrix} \prod_{j=0}^{n-1} Z_{t_j}(C_{j+1}, C_j). \quad (6.30)$$

Applying $\left(\mathrm{i}\hbar\partial_t + \frac{\hbar^2}{2\mu}\Delta_{LB}\right)$ on $K_t^\gamma(x, x_0)$ we obtain several boundary terms from derivation of the singularities as in the previous case and from the discontinuity on the boundary of $D(x, C_n)$.

Using the identity

$$\left(\mathrm{i}\hbar\partial_t + \frac{\hbar^2}{2\mu}\Delta_{LB}\right)\frac{1}{\theta - \pi + \mathrm{i}\ln\left(\frac{(t-\tau)r_0}{\tau r}\right)}\frac{1}{t-\tau}\exp\left(\frac{\mathrm{i}\mu r^2}{2\hbar(t-\tau)}\right)$$

$$= \frac{\hbar^2\pi}{\mu r^2}\exp\left(\frac{\mathrm{i}\mu r^2}{2\hbar(t-\tau)}\right)\times$$

$$\left(\delta\left(t-\frac{(r+r_0)\tau}{r_0}\right)\delta'(\theta)-\mathrm{i}\frac{\tau r}{r_0}\delta'\left(t-\frac{(r+r_0)\tau}{r_0}\right)\delta(\theta)\right), \tag{6.31}$$

and the substitution $\tau' = t_{n-2}+\ldots+t_0$, we obtain

$$\left(\mathrm{i}\hbar\partial_t+\frac{\hbar^2}{2\mu}\Delta_{LB}\right)K_t^\gamma(x,x_0) = M_1+M_2+M_3+M_4, \tag{6.32}$$

where

$$M_1 = -\frac{\hbar^2}{2\mu}\left(\frac{\partial}{\partial\overrightarrow{n}}K_t^\gamma(x,x_0)\right)\delta_{\partial D(C_n)} - \frac{\hbar^2\partial}{2\mu\partial\overrightarrow{n}}(K_t^\gamma(x,x_0)\delta_{\partial D(C_n)}),$$

$$M_2 = \frac{\hbar^2\rho}{2\mu r^2(r+\rho)}\int_{R^n} \mathrm{d}t_{n-1}\ldots\ \mathrm{d}t_0\exp\left(\frac{\mathrm{i}\mu r^2}{2\hbar(t-\tau)}\right)F_\gamma(t_0,\ldots,t_{n-1},x_0)\times$$
$$\delta\left(t_{n-1}-\frac{\rho(t-\tau')}{r+\rho}\right)\delta'(\theta-\pi),$$

$$M_3 = \frac{\hbar^2\rho}{2\mu r^2(r+\rho)}\int_{R^n} \mathrm{d}t_{n-1}\ldots\ \mathrm{d}t_0\exp\left(\frac{\mathrm{i}\mu r^2}{2\hbar(t-\tau)}\right)F_\gamma(t_0,\ldots,t_{n-1},x_0)\times$$
$$\frac{\mathrm{i}rt_{n-1}}{(r+\rho)}\delta'(t_{n-1}-\frac{\rho(t-\tau')}{r+\rho})\delta(\theta-\pi),$$

$$M_4 = \frac{\hbar^2\rho}{2\mu r^2(r+\rho)}\int_{R^n} \mathrm{d}t_{n-1}\ldots\ \mathrm{d}t_0\exp\left(\frac{\mathrm{i}\mu r^2}{2\hbar(t-\tau)}\right)F_\gamma(t_0,\ldots,t_{n-1},x_0)\times$$
$$\delta(t_{n-1}-\frac{\rho(t-\tau')}{r+\rho})\delta'(\theta+\pi),$$

$$M_5 = \frac{\hbar^2\rho}{2\mu r^2(r+\rho)}\int_{R^n} \mathrm{d}t_{n-1}\ldots\ \mathrm{d}t_0\exp\left(\frac{\mathrm{i}\mu r^2}{2\hbar(t-\tau)}\right)F_\gamma(t_0,\ldots,t_{n-1},x_0)\times$$
$$\frac{\mathrm{i}rt_{n-1}}{r+\rho}\delta'(t_{n-1}-\frac{\rho(t-\tau')}{r+\rho})\delta(\theta+\pi).$$

In the following we will use the identities:

- For $t_{n-1}=\rho\frac{t-\tau'}{r+\rho}$ and $\theta=\pm\pi$:

$$t-\tau = \frac{r(t-\tau')}{r+\rho} \tag{6.33}$$

- For $t_{n-1} = \rho\frac{t-\tau'}{r+\rho}$ and $\theta = \pm\pi$:

$$\frac{\rho}{r+\rho}\exp\left(\frac{\mathrm{i}\mu r^2}{2\hbar(t-\tau)}\right)\frac{\mu}{2\pi\mathrm{i}\hbar t_{n-1}}\exp\left(\frac{\mathrm{i}\mu\rho^2}{2\hbar t_{n-1}}\right)$$
$$= \frac{\mu}{2\pi\mathrm{i}\hbar(t-\tau')}\exp\left(\frac{\mathrm{i}\mu(r+\rho)^2}{2\hbar(t-\tau')}\right) = Z(t-\tau', x, C_{n-1}). \qquad (6.34)$$

- For $t_{n-1} = \rho\frac{t-\tau'}{r+\rho}$ and $\theta = \pm\pi$:

$$V\left(\begin{array}{c} C_n, C_{n-1}, C_{n-2} \\ \frac{\rho}{r+\rho}s_2, s_1 \end{array}\right) = V\left(\begin{array}{c} x, C_{n-1}, C_{n-2} \\ s_2, s_1 \end{array}\right). \qquad (6.35)$$

-

$$\frac{\partial}{\partial s}\left(\exp\left(\frac{\mathrm{i}\mu r^2}{2\hbar(t-\tau'-s)}\right)\exp\left(\frac{\mathrm{i}\mu\rho^2}{2\hbar s}\right)\right)|_{s=\frac{\rho(t-\tau')}{r+\rho}} = 0$$

  and

$$\exp\left(\frac{\mathrm{i}\mu r^2}{2\hbar(t-\tau'-s)}\right)\exp\left(\frac{\mathrm{i}\mu\rho^2}{2\hbar s}\right)|_{s=\frac{\rho(t-\tau')}{r+\rho}} = \exp\left(\frac{\mathrm{i}\mu(r+\rho)^2}{2\hbar(t-\tau')}\right). (6.36)$$

- For $\theta = \pi$:

$$\frac{\partial}{\partial s}V\left(\begin{array}{c} C_n, C_{n-1}, C_{n-2} \\ s, t_{n-2} \end{array}\right)|_{s=\frac{\rho(t-\tau')}{r+\rho}} = \frac{\mathrm{i}(r+\rho)}{\rho(t-\tau')}\frac{\partial}{\partial\theta'}V\left(\begin{array}{c} x, C_{n-1}, C_{n-2} \\ t-\tau', t_{n-2} \end{array}\right).$$

Then

$$M_2 = \frac{\hbar^2}{2\mu r}\frac{\partial}{\partial\theta}\Big(\frac{1}{r}\delta(\theta-\pi)\int_{R^{n-2}} \mathrm{d}t_{n-2}\ldots\ \mathrm{d}t_0 Z_{t-\tau'}(x, C_{n-1}) \times$$
$$V(\begin{array}{c} x, C_{n-1}, C_{n-2} \\ t-\tau', t_{n-2} \end{array}) \times F_{\gamma'}(t_0, \ldots, t_{n-2}, x_0)\Big)$$
$$= \frac{\hbar^2}{2\mu}\frac{\partial}{\partial\overrightarrow{n}}(K_t^{\gamma'}(x, x_0)\delta_{\partial D(C_{n-1}, C_n+)}) \qquad (6.37)$$

and similarly

$$M_4 = \frac{\hbar^2}{2\mu}\frac{\partial}{\partial\overrightarrow{n}}(K_t^{\gamma'}(x, x_0)\delta_{\partial D(C_{n-1}, C_n-)}). \qquad (6.38)$$

$$
\begin{aligned}
M_3 = \frac{\hbar^2\rho}{2\mu\pi(r+\rho)^2}\delta(\theta-\pi)\int_{R^{n-1}} \, \mathrm{d}t_{n-1}\ldots \, \mathrm{d}t_0 \times \\
\frac{\partial}{\partial s}(\exp\left(\frac{\mathrm{i}\mu r^2}{2\hbar(t-\tau'-s)}\right)\exp\left(\frac{\mathrm{i}\mu\rho^2}{2\hbar s}\right) \times \\
V\left(\begin{array}{c} C_{n-1}, C_{n-2}, C_{n-3} \\ s, t_{n-2}\end{array}\right) F'_\gamma(t_0,\ldots,t_{n-2},x_0))\big|_{s=\frac{\rho(t-\tau')}{r+\rho}} \\
= -\frac{\hbar^2}{2\mu}\frac{\mu}{2\pi\mathrm{i}\hbar(t-\tau')}\frac{1}{r(r+\rho)}\delta(\theta-\pi)\frac{\partial}{\partial\theta'} \times \\
\big(\int_{R^{n-1}} \, \mathrm{d}t_{n-1}\ldots \, \mathrm{d}t_0 \exp\left(\frac{\mathrm{i}\mu(r+\rho)^2}{2\hbar(t-\tau')}\right) \times \\
V\left(\begin{array}{c} x, C_{n-1}, C_{n-2} \\ t-\tau', t_{n-2}\end{array}\right) F_{\gamma'}(t_0,\ldots,t_{n-2},x_0)\big) \\
= \frac{\hbar^2}{2\mu}\delta_{\partial D(C_{n-1},C_n+)}\frac{\partial}{\partial\overrightarrow{n}}K_t^{\gamma'}(x,x_0). \qquad (6.39)
\end{aligned}
$$

Similarly

$$
M_5 = \frac{\hbar^2}{2\mu}\delta_{\partial D(C_{n-1},C_n-)}\frac{\partial}{\partial\overrightarrow{n}}K_t^{\gamma'}(x,x_0). \qquad (6.40)
$$

Finally

$$
\begin{aligned}
\left(\mathrm{i}\hbar\partial_t + \frac{\hbar^2}{2\mu}\Delta_{LB}\right)K_t^\gamma(x,x_0) \\
= -\frac{\hbar^2}{2\mu}\left(\frac{\partial}{\partial\overrightarrow{n}}K_t^\gamma(x,x_0)\right)\delta_{\partial D(C_n)} - \frac{\hbar^2\partial}{2\mu\partial\overrightarrow{n}}(K_t^\gamma(x,x_0)\delta_{\partial D(C_n)}) + \\
\frac{\hbar^2}{2\mu}\delta_{\partial D(C_{n-1},C_n)}\frac{\partial}{\partial\overrightarrow{n}}K_t^{\gamma'}(x,x_0) + \frac{\hbar^2}{2\mu}\frac{\partial}{\partial\overrightarrow{n}}(K_t^{\gamma'}(x,x_0)\delta_{\partial D(C_{n-1},C_n)}). \qquad (6.41)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\left(\mathrm{i}\hbar\partial_t + \frac{\hbar^2}{2\mu}\Delta_{LB}\right)K_t(x,x_0) \\
= \sum_{n\geq 2,|\gamma|=n}\big[-\frac{\hbar^2}{2\mu}\left(\frac{\partial}{\partial\overrightarrow{n}}K_t^\gamma(x,x_0)\right)\delta_{\partial D(C_n)} - \frac{\hbar^2\partial}{2\mu\partial\overrightarrow{n}}(K_t^\gamma(x,x_0)\delta_{\partial D(C_n)}) +
\end{aligned}
$$

$$\frac{\hbar^2}{2\mu}\delta_{\partial D(C_{n-1},C_n)}\frac{\partial}{\partial\overrightarrow{n}}K_t^{\gamma'}(x,x_0)+\frac{\hbar^2}{2\mu}\frac{\partial}{\partial\overrightarrow{n}}(K_t^{\gamma'}(x,x_0)\delta_{\partial D(C_{n-1},C_n)})\Big]+$$
$$\sum_{|\gamma|=1}\Big[\frac{\hbar^2\partial}{2\mu\partial\overrightarrow{n}}\left(K_t^{\gamma_0}(x,x_0)\delta_{\partial D(x_0,C)}\right)-\frac{\hbar^2}{2\mu}\left(\frac{\partial}{\partial\overrightarrow{n}}K_t^{\gamma_1}(x,x_0)\right)\delta_{\partial D(G)}-$$
$$\frac{\hbar^2\partial}{2\mu\partial\overrightarrow{n}}\left(K_t^{\gamma_1}(x,x_0)\delta_{\partial D(C)}\right)\Big]-\frac{\hbar^2\partial}{2\mu\partial\overrightarrow{n}}(K_t^{()}(x,x_0)\delta_{\partial D(x_0)})=0,$$

where we have used that

$$\delta_{\partial D(x_0)}=\bigcup_{C\in\mathcal{C},\chi(x_0,C)=1}\delta_{\partial D(x_0,C)} \tag{6.42}$$

and that

$$\partial D(C)=\bigcup_{X\in\mathcal{C},\mathrm{dist}(C,X)=\rho}\partial D(C,X). \tag{6.43}$$

6.3 Kernel in case of two solenoid Aharonov-Bohm effect

Applying the Schulman ansatz to the kernel on $\tilde{M}$

$$K_t(x,x_0)=\sum_{\gamma,n\geq 0}\int_{R^{n+1}}\mathrm{d}t_n\ldots\,\mathrm{d}t_0\delta(t_n+\ldots+t_0-t)V\begin{pmatrix}x,C_n,C_{n-1}\\ t_n,t_{n-1}\end{pmatrix}\times$$
$$V\begin{pmatrix}C_n,C_{n-1},C_{n-2}\\ t_{n-1},t_{n-2}\end{pmatrix}\times\ldots\times V\begin{pmatrix}C_2,C_1,x_0\\ t_1,t_0\end{pmatrix}\times$$
$$Z_{t_n}(x,C_n)Z_{t_{n-1}}(C_n,C_{n-1})\times\ldots\times Z_{t_0}(C_1,x_0),$$

we obtain

$$\sum_{g\in\Gamma}\Lambda(g^{-1},y)K_t(g.x,x_0)$$
$$=\sum_{g\in\Gamma}\sum_{\gamma,n\geq 0}\int_{R^{n+1}}\mathrm{d}t_n\ldots\,\mathrm{d}t_0\delta(t_n+\ldots+t_0-t)\Lambda(g^{-1},y)\times$$
$$V\begin{pmatrix}g.x,C_n,C_{n-1}\\ t_n,t_{n-1}\end{pmatrix}V\begin{pmatrix}C_n,C_{n-1},C_{n-2}\\ t_{n-1},t_{n-2}\end{pmatrix}\times\ldots\times V\begin{pmatrix}C_2,C_1,x_0\\ t_1,t_0\end{pmatrix}\times$$
$$Z_{t_n}(g.x,C_n)Z_{t_{n-1}}(C_n,C_{n-1})\times\ldots\times Z_{t_0}(C_1,x_0).$$

The group Γ has two generators, g_a rotation around the point a and g_b rotation around the point b. $\chi(g.x,C_n)$, where C_n is the copy of the point a, is non-null if and only if $g = g_a^n$, $n \in Z$ and similarly with the copies of b. So the sum reduces to

$$\sum_{\gamma, n\geq 0}\sum_{m\in Z}\int_{R^{n+1}} \mathrm{d}t_n \ldots \mathrm{d}t_0 \delta(t_n + \ldots + t_0 - t)\Lambda(g_{C_n}^{-m}, y) \times$$
$$V\begin{pmatrix} g_{C_n}^m.x, C_n, C_{n-1} \\ t_n, t_{n-1}\end{pmatrix} V\begin{pmatrix} C_n, C_{n-1}, C_{n-2} \\ t_{n-1}, t_{n-2}\end{pmatrix} \times \ldots \times V\begin{pmatrix} C_2, C_1, x_0 \\ t_1, t_0\end{pmatrix} \times$$
$$Z_{t_n}(g_{C_n}^m.x, C_n) Z_{t_{n-1}}(C_n, C_{n-1}) \times \ldots \times Z_{t_0}(C_1, x_0).$$

Using the equalities

$$\sum_{k\in Z}\exp(2\pi \mathrm{i}\alpha k)\left(\frac{1}{\theta + 2k\pi - \pi + \mathrm{i}s} - \frac{1}{\theta + 2\pi k + \pi + \mathrm{i}s}\right)$$
$$= -\sin(\pi\alpha)\int \mathrm{d}\tau \frac{\exp((\theta + \mathrm{i}s)\tau)}{\sin(\pi(\alpha + \mathrm{i}\tau))} \tag{6.44}$$

and

$$\int_{-\infty}^{\infty} \mathrm{d}\tau \frac{\exp((\theta + \mathrm{i}s)\tau)}{\sin(\pi(\alpha + \mathrm{i}\tau))} = 2\frac{\exp(-\alpha(s - \mathrm{i}\theta))}{1 + \exp(-s + \mathrm{i}\theta)}, 0 < \alpha < 1, |\theta| < \pi, \tag{6.45}$$

one obtains the formula for the kernel of the Aharonov-Bohm effect with two solenoids on the space of Λ equivariant functions. For simplicity, we use only one dimensional unitary representation:

$$\Lambda(g_a) = \exp(2\pi \mathrm{i}\alpha), \Lambda(g_b) = \exp(2\pi \mathrm{i}\beta).$$

If we want to write the result on the fundamental domain D without using the copies of a, b, we sum the result over are possible sequences γ, such that its projection on the manifold D is the same.

We thus obtain the final result:

$$K_t^y(x, x_0) = \left\{\begin{array}{c} 1 \\ \exp(2\pi \mathrm{i}\alpha) \\ \exp(2\pi \mathrm{i}\alpha)\end{array}\right\} \frac{\mu}{2\pi \mathrm{i}\hbar t}\exp(\mathrm{i}\mu|x - x_0|^2/2\hbar t) - \frac{\sin(\pi\alpha)}{\pi} \times$$
$$\int_{-\infty}^{\infty} \mathrm{d}s \frac{\mu}{2\pi \mathrm{i}\hbar t}\exp[\mathrm{i}\mu(r^2 + r_0^2 + 2rr_0\cosh(s))/2\hbar t]\frac{\exp(-\alpha(s - \mathrm{i}\Theta))}{1 + \exp(-s + \mathrm{i}\Theta)} +$$
$$\left\{\begin{array}{c} 1 \\ \exp(2\pi \mathrm{i}\beta) \\ \exp(2\pi \mathrm{i}\beta)\end{array}\right\} \frac{\mu}{2\pi \mathrm{i}\hbar t}\exp(\mathrm{i}\mu|x - x_0|^2/2\hbar t) - \frac{\sin(\pi\beta)}{\pi} \times$$
$$\int_{-\infty}^{\infty} \mathrm{d}s \frac{\mu}{2\pi \mathrm{i}\hbar t}\exp[\mathrm{i}\mu(r^2 + r_0^2 + 2rr_0\cosh(s))/2\hbar t]\frac{\exp(-\beta(s - \mathrm{i}\Theta))}{1 + \exp(-s + \mathrm{i}\Theta)} -$$

$$
\frac{\mu}{2\pi \mathrm{i}\hbar t} \exp(\frac{\mathrm{i}\mu}{2\hbar t} \mathrm{dist}(x,x_0)) +
$$

$$
\frac{\mu}{2\pi \mathrm{i}\hbar} \sum_{\gamma, n\geq 2} (-1)^n \int_0^\infty \frac{\mathrm{d}t_n}{t_n} \ldots \int_0^\infty \frac{\mathrm{d}t_0}{t_0} \delta(t_n + \ldots + t_0 - t) \times
$$

$$
\exp\left(\frac{\mathrm{i}\mu}{2\hbar}\left(\frac{r_n^2}{t_n} + \ldots + \frac{r_0^2}{t_0}\right)\right) S_\gamma(s,\varphi,\varphi_0), \tag{6.46}
$$

$$
S_\gamma(s,\varphi,\varphi_0) = \frac{\sin(\pi\sigma_n)}{\pi} \frac{\exp[-\sigma_n(s_n - \mathrm{i}\varphi)]}{1+\exp(-s_n + \mathrm{i}\varphi)} \frac{\sin \pi\sigma_{n-1}}{\pi} \frac{\exp(-\sigma_{n-1}s_{n-1})}{1+\exp(-s_n)}
$$

$$
\times \ldots \times \frac{\sin(\pi\sigma_1)}{\pi} \frac{\exp[-\sigma_1(s_1 - \mathrm{i}\varphi_0)]}{1+\exp(-s_1 + \mathrm{i}\varphi_0)}, \tag{6.47}
$$

where $s_j = \ln(t_j r_{j-1}/t_{j-1} r_j), 1 \leq j \leq n$, $(r_n, \varphi), (r_0, \varphi_0)$ are the polar coordinates of x, x_0 with respect to the center c_n, c_1, γ runs over all finite sequences $\gamma = \{c_n, \ldots, c_1\}$, $c_j = a$ or $b, c_j \neq c_{j+1}$, $\sigma = \alpha$ ir $\sigma = \beta$ provided $c_j = a$ or $c_j = b$. The value in the composite brackets depends on whether $\Phi = \varphi - \varphi_0$ belongs to the interval $(-\pi, \pi)$ or $(\pi, 2\pi)$ or $(-2\pi, -\pi)$.

Summary

One of the possibilities how to describe quantum mechanics on non-simply connected manifold is to use the Hilbert space of equivariant functions. In this work, there is mainly investigated the proof of the Schulman ansatz, which allowes to derive the kernel of the propagator in this case. For this purpose, the generalize Bloch analysis is formulated and all necessary theorems are proved in case of locally linearly connected manifolds with the fundamental group of Type I. In this case, von Neumann integral decomposes the left regular representation to blocks of equivalent irreducible representations and the Peter-Weyl Plancherel theorem gives the relation between square integrable functions on the group and Hilbert-Schmidt operators defined on the Hilbert spaces corresponding to the irreducible representations. This theorem is necessary for the construction of the Bloch analysis. In the case of locally compact groups of Type I, the inverse Fourier-Stieltejes transformation can be defined and using this mapping I proved the Schulman ansatz with the generalized assumptions.

If we do not suppose that the group is of Type I, several properties are not valid: irreducible representations of such group are not finite dimensional, Peter-Weyl Plancherel theorem holds, but not necessarily only irreducible representations are present in the decompositions, Fourier transformations gives the relation between the $L^2(\Gamma)$ and the operators, where the generalized trace can be defined, but generally it is not the ordinary trace which is used in the proof of the theorems of Bloch analysis. Even though, we may suppose, that generalized Bloch analysis may be formulated for some special cases of group, which are not of Type I and in similar cases, Schulman ansatz may be used.

On the example of Aharonov-Bohm effect with two solenoids we can see the use of Schulman ansatz which brings the result very quickly in case of knowledge of the kernel of the propagator defined on $L^2(\tilde{M})$. Nevertheless this knowledge is the limitative factor of this method.

Bibliography

[Ahr] Y. Aharonov, D. Bohm: Significance of Electromagnetic Potentials in the Quantum Theory, *Phys. Rev.*, Vol. 115, No. 3, 485-491 (1959)

[Bra] O. Bratteli, D. W. Robinson: *Operator Algebras and Quantum Statistical Mechanics 1*, Springer-Verlag, Second Edition (1987)

[Dix] J. Dixmier: *Les algebre d'opérateur dans l'espace hilbertien*, Gauthier-Villars (1957)

[Fou] W. Foulton, J. Harris: *Representation Theory, A First Course*, Springer-Verlag (1991)

[Fran] T. Frankel: *The Geometry of Physics:An Introduction*,Cambridge University Press (2003)

[Hat] A. Hatcher: *Algebraic Topology*, Cambridge University Press (2002)

[Hor] L. Hormander: *The Analysis of Linear Partial Differential Operatators I.*, Springer-Verlag, Berlin (1983)

[Jung] G. Jungman: *Notes on Groups and Representations*

[Koz] Č. Koznevski: *Načalnyj kurs algebraičcskoj topologii*, Izdavatělstvo MIR, Moskva (1983)

[Koc1] P. Kocábová: Kvantová Mechanika na Nejednoduše souvislých Varietách, Rešeršní práce, FJFI, ČVUT v Praze (2005)

[Koc2] P. Kocábová: Quantum Mechanics on Non-simply Connected Manifolds, Výzkumný úkol, FJFI, ČVUT v Praze (2006)

[Lee] J. M. Lee, *Introduction to Topological Manifolds*, Springer-Verlag (2000)

[Mau1] F. I. Mautner, Unitary Representations of Locally Compact Groups I, *Annals of Mathematics*, Vol. 51, No. 1, p. 1-25 (1950)

[Mau2] F. I. Mautner, Uniraty Representations of Locally Compact Groups II, *Annals of Mathematics*, Vol. 52, No. 3, p. 528-556 (1950)

[Mau3] F. I. Mautner, Note of the Fourier inversion formula on groups, *Transactions of the American Mathematical Society*, Vol. 78, No. 3, p. 371-384 (1955)

[Neu1] J. v. Neumann, On Rings of Operators. Reduction theory, *Annals of Mathematics*, Vol. 50, No. 2, p. 401-485 (1949)

[Ree] M. Reed, B. Simon: *Methods of Modern Mathematical Physics I: Functional analysis*, Academic Press, Inc. (1980)

[Ross1] E. Hewitt, K. A. Ross: *Absract Harmonica Analysis I*, Springer-Verlag, Second Edition (1979)

[Ross2] E. Hewitt, K. A. Ross: *Abstract Harmonic Analysis II*, Springer-Verlag (1997)

[Sch] L. Schulman: *Techniques and Applications of Path Integration*, Wiley Classics Library (1996)

[Sto1] P. Šťovíček: The Green Function for the Two-Solenoid Aharonov-Bohma Effect, *Phys. Lett. A* 142,5-10 (1989)

[Sto2] P. Šťovíček: Krein's formula approach to the multisolenoid Aharonov-Bohm effect, *Journal of Mathematical Physics*, Vol. 32 (1991)

[Spr] *Encyclopaedia of Mathematics*, Springer-Verlag, (2002)

[Tho] E. Thoma: Uber unitare Darstellungen abzahlbarer, diskreter Gruppen, *Math. Annalen* 153, p.11-138 (1964)