Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Department of Mathematics

A Resonant Effect for a Periodically Time-Dependent Singular Flux Tube and a Homogeneous Magnetic Field

DOCTORAL THESIS

November 14, 2012

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by

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Dissertation submitted to
the Faculty of Nuclear Sciences and Physical Engineering of
Czech Technical University in Prague
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

in the branch of study Mathematical Physics
of the study program Mathematical Engineering

Prague, October 2012

Contents

List of Symbols

$\mathbb{Z}$	integer numbers, $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
$\mathbb{N}$	natural numbers, $\mathbb{N} = \{1, 2, 3, \dots\}$
$\mathbb{N}_0$	non-negative integer numbers, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$
$\mathbb{R}$	real numbers
$\mathbb{R}_+$	positive real numbers
$\mathbb{C}$	complex numbers
$\operatorname{Im} z$	imaginary part of a complex number z
$\operatorname{Re} z$	real part of a complex number z
$[a,\, b]$	closed interval with endpoints $-\infty < a < b < +\infty$
$]a, b[$	open interval with endpoints $-\infty \leq a < b \leq +\infty$
S_1	unit circle
$\mathbb{T}^d$	d-dimensional torus
$C(J)$	Banach space of continuous functions on closed interval J
$C_0^\infty(J)$, $\mathscr{D}(J)$	smooth functions with compact support in J
$\mathscr{D}'(J)$	the space of generalized functions on J
$L^2(M, \mathrm{d}\mu)$	square-integrable functions on $(M, \mathrm{d}\mu)$
$L^1(M, \mathrm{d}\mu)$	integrable functions on $(M, \mathrm{d}\mu)$
$L^1_{\mathrm{loc}}(M, \mathrm{d}\mu)$	locally integrable functions on $(M, \mathrm{d}\mu)$
$\ell^2(\mathbb{N}, \mathbb{C})$	complex sequences with natural indices satisfying $\sum_n \lvert\xi_n\rvert^2 < \infty$
$\langle\psi,\, \varphi\rangle$	scalar (inner) product
$\operatorname{dom} A$	domain of a linear operator A
$x^\perp$	a point $x \in \mathbb{R}^2$ rotated 90° counter-clockwise
$\mathscr{F}[f(\varphi)]_k$	kth Fourier coefficient of f
$\operatorname{supp} \mathscr{F}[f(\varphi)]$	the set of non-zero Fourier coefficients
${}_2F_1$	hypergeometric functions
$L_n^{(p)}$	generalized Laguerre polynomials
$\delta(x)$	Dirac delta function

Introduction

This thesis is concerned with a time evolution of a certain non-autonomous dynamical model. In particular we are interested in a model of a charged massive particle moving on a plane and influenced by a homogeneous magnetic field and a time-periodic singular flux tube. Illustration of this setting is presented in Figure 1. We study this model in the frameworks of non-relativistic classical and quantum mechanics. Although it is not possible to solve the Hamiltonian equations of motion or the Schrödinger equation analytically, it turns out that we are able to employ approximative methods of perturbation theory to exhibit a curious acceleration effect. More precisely, for some particular choices of values of parameters we claim that the particle gains energy during its time evolution. Both in classical and quantum framework we are able to estimate the rate of energy growth in those resonant situations. It turns out that the energy grows linearly with time.

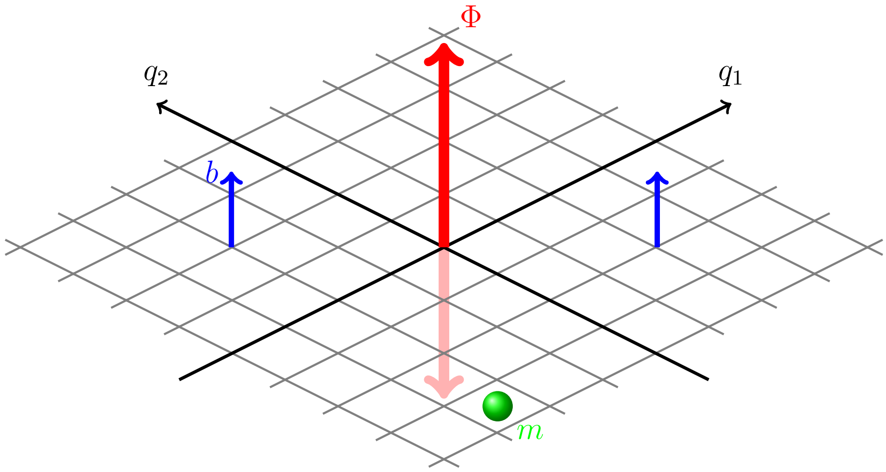

Figure 1: The description of the model: a massive charged particle moving on a plane and influenced by a homogeneous magnetic field of magnitude b and a time-periodic singular flux tube.

Without loss of generality we can assume that the singular flux $\Phi(t)$ pierces the plane at the origin of Cartesian coordinate system. The Cartesian coordinates in the plane are denoted by $q = (q_1, q_2) \in \mathbb{R}^2 \setminus \{0\}$. We will see that the hole in the plane plays an interesting role during the resonance. Note that the configuration space of this model is not simply connected.

From the viewpoint of classical non-relativistic mechanics the model is described by Hamilton's function

$$H(q,p,t) = \frac{1}{2m}\Big(p - eA(q,t)\Big)^2, \quad (q,p) \in \mathbb{P} = \Big(\mathbb{R}^2 \setminus \{0\}\Big) \times \mathbb{R}^2,$$

where

$$A(q,t) = \Big(-\frac{b}{2} + \frac{\Phi(t)}{2\pi|q|^2}\Big)(-q_2, q_1)$$

is the vector potential. The dynamics of the system is then governed by the set of Hamilton's equations of motion

$$q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q}, \quad (q(0), p(0)) \in \mathbb{P}.$$

In our present case these equations constitute four non-autonomous coupled ordinary differential equations. Due to the obvious rotational symmetry of the system we perform our analysis in polar coordinates. However, in order to describe the resonant behavior it is better to employ guiding-center coordinates [21]. This analysis and discussion is carried out in Chapter 1, which essentially contains the results of [5]. The main result of the cited article is that the energy $\mathcal{E}(t)$ of the particle grows with a rate

$$\gamma_{\mathrm{acc}} = \lim_{t\to\infty} \frac{\mathcal{E}(t)}{t} = -\frac{e\omega_c}{4\pi}\,\Phi'(c) > 0, \tag{1}$$

where ω_c is the cyclotron frequency and c is a constant which depends on initial conditions.

In the framework of quantum mechanics the dynamics of the system is governed by the Hamiltonian operator

$$H = \frac{1}{2m}\Big(- i\hbar\nabla - eA(q,t)\Big)^2$$

acting in the Hilbert space $L^2(\mathbb{R}^2 \setminus \{0\}, \mathrm{d}q)$. We circumvent the ambiguity of this formal differential expression by taking our Hamiltonian to be the Friedrichs extension of the corresponding minimal differential operator. This choice corresponds to the standard Aharonov-Bohm Hamiltonian. Similarly to the classical case we work in polar coordinates. Our main result, again, is a formula for the rate of energy growth. The main dynamical object in this setting is the unitary propagator, an object that is out of reach for a full analytical analysis. However, using the perturbation method analogous to that employed in the classical model we are able to construct an approximation to the propagator and study its dynamical properties. We find out that in the resonant situation the mean value of energy of the system grows with rate

$$\gamma_{\mathrm{acc}} = \frac{|\varepsilon| e\omega_c\Omega}{2}\,\frac{\int_0^\pi \sin(\theta)|\rho(\theta)|\,\mathrm{d}\theta}{\int_0^\pi |\rho(\theta)|\,\mathrm{d}\theta} = \frac{e\omega_c}{4\pi}\,\frac{\int_0^\pi \Phi'(\theta)|\rho(\theta)|\,\mathrm{d}\theta}{\int_0^\pi |\rho(\theta)|\,\mathrm{d}\theta}, \tag{2}$$

where ρ is related to the initial radial function $\psi \in L^2(\mathbb{R}_+, r\,\mathrm{d}r)$ and the flux function is taken to be $\Phi(\Omega t) = \Phi_0 - \varepsilon\cos(\Omega t)$. Note the similarity between Equation (1) and Equation (2). This analysis of the quantum system is carried out in Chapter 2, which is based on [17].

We remark, that our results could be of interest in accelerator physics. While the betatron principle uses a linearly time dependent flux tube to accelerate particles on cyclotron orbits around the flux [19], the resonance effect we observe in the present work has the feature that acceleration can be achieved with arbitrarily small field strength. A second aspect is that, in contrast to the case of a linearly increasing flux, cyclotron orbits which do not encircle the flux tube are accelerated as well. In fact, for the linear case it was shown in [6] that outside the flux tube one has the usual drift of the guiding center, without acceleration, along the field lines of the averaged potential.

The text is organized into two Chapters and three Appendices. The first and second Chapter contains the results of [5] and [17], respectively. Additional material closely related to that in the Chapters is presented in Appendices.

Let me summarize the notation used throughout the text. Chapters are numbered by Arabic numerals and appendices by capital Latin letters. Chapters and appendices are further divided into Sections. Equations are numbered within Chapters, so (2.3) denotes the equation number 3 in Chapter 2. Similar convention holds for Theorems, Propositions and other environments. For the convenience of the reader we also present a short List of Symbols at the beginning of the document.

Chapter 1

Classical Mechanics

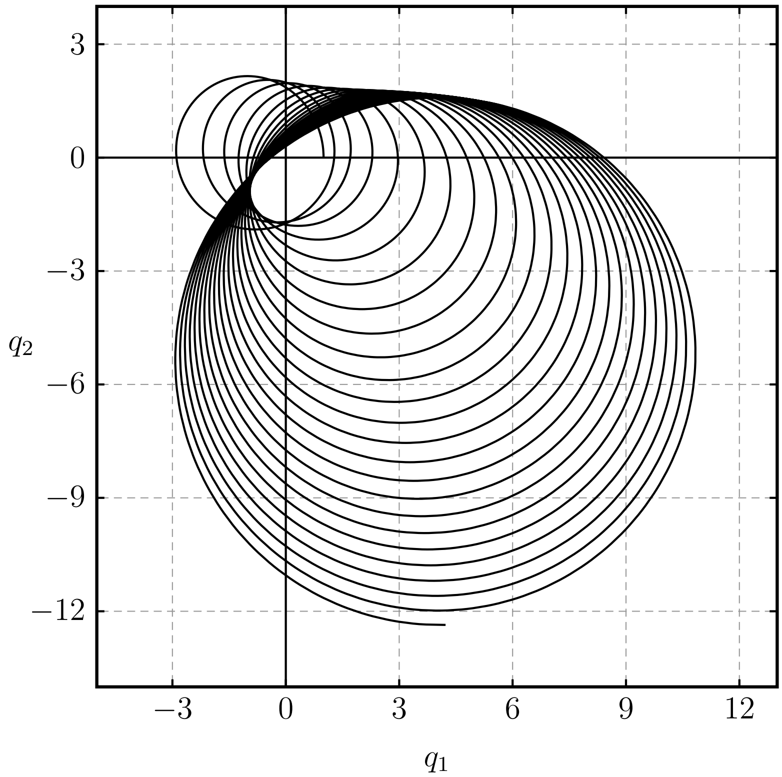

Figure 1.1: The numerical solution $q(t)$ of the equations of motion in the plane for $t \in [0, 150]$, with $\Phi(t) = 2\pi\varepsilon f(\Omega t)$, $f(t) = \sin(t) - (1/3)\cos(2t)$, for the values of parameters $\varepsilon = 0.35$, $b = 1$, $\Omega = 1$, and with the initial conditions $q(0) = (1, 0)$, $q'(0) = (0, 1.617)$.

Let us consider a classical point particle of mass m and charge e moving on the punctured plane $\mathbb{R}^2 \setminus \{0\}$ in the presence of a homogeneous magnetic field of magnitude b. Suppose further that a singular magnetic flux line whose strength $\Phi(t)$ is oscillating with frequency Ω intersects the plane at the origin. The equations of motion in phase space $\mathbb{P} = (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$ are generated by the time-dependent Hamiltonian

$$H(q, p, t) = \frac{1}{2m} \left(p - eA(q, t)\right)^2, \text{ with } A(q, t) = \left(-\frac{b}{2} + \frac{\Phi(t)}{2\pi|q|^2}\right) q^{\perp}, \tag{1.1}$$

where $(q, p) \in \mathbb{P}$, $t \in \mathbb{R}$. Here and throughout this chapter we denote $x^{\perp} = (-x_2, x_1)$ for $x = (x_1, x_2) \in \mathbb{R}^2$. Our aim is to understand the dynamics of this system for large times. Of particular interest is the growth of energy as well as the drift of the guiding center.

1.1 Equations of motion

In view of the rotational symmetry of the system we prefer to work with polar coordinates $q = r(\cos\theta, \sin\theta)$, where $(\theta, r) \in S_1 \times \mathbb{R}_+$. Corresponding generalized momenta are

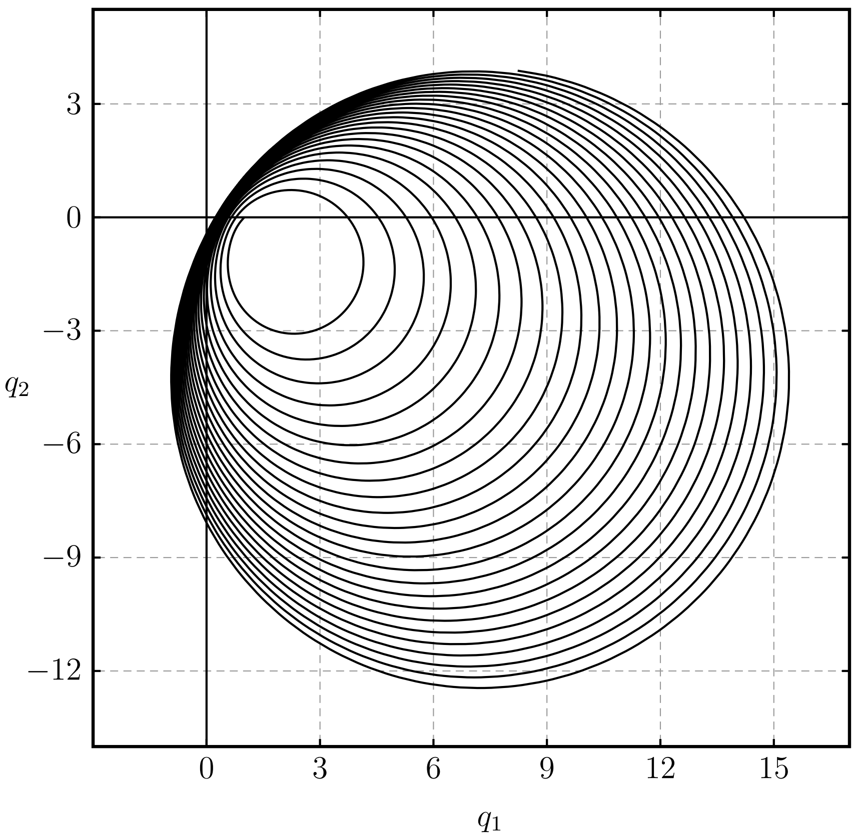

Figure 1.2: The numerical solution $q(t)$ of the equations of motion in the plane for $t \in [0, 150]$, with $\Phi(t) = 2\pi\varepsilon f(\Omega t)$, $f(t) = \sin(t) - (1/3)\cos(2t)$, for the values of parameters $\varepsilon = 0.35$, $b = 1$, $\Omega = 1$, and with the initial conditions $q(0) = (1, 0)$, $q'(0) = (-1, -1.617)$.

transformed in the following way

$$p_1 = p_r \cos\theta - \frac{p_\theta}{r}\sin\theta, \tag{1.2}$$

$$p_2 = p_r \sin\theta + \frac{p_\theta}{r}\cos\theta. \tag{1.3}$$

In fact, the coordinate transformation of the configuration space $\mathbb{R}^2 \smallsetminus \{0\} \to S_1 \times \mathbb{R}_+$ induces a canonical transformation of the phase space which can be deduced from the transformation of the canonical one-form

$$p_1\,\mathrm{d}q_1 + p_2\,\mathrm{d}q_2 = \Big(p_1\cos\theta + p_2\sin\theta\Big)\mathrm{d}r + r\Big(p_2\cos\theta - p_1\sin\theta\Big)\mathrm{d}\theta$$

In order for the transformation to be canonical one has to require

$$p_r = p_1\cos\theta + p_2\sin\theta \quad \text{and} \quad p_\theta = r\Big(p_2\cos\theta - p_1\sin\theta\Big).$$

This gives (1.2) and (1.3) immediately. The Hamiltonian of the studied model in polar coordinates (cf. (1.1)) then reads

$$H(r, \theta, p_r, p_\theta, t) = \frac{1}{2m}\left(p_r^2 + \left(\frac{1}{r}\left(p_\theta - \frac{e\Phi(t)}{2\pi}\right) + \frac{eb}{2}r\right)^2\right). \tag{1.4}$$

Since $\partial_\theta H = 0$, the generalized momentum p_θ is an integral of motion and thus the analysis of the system effectively reduces to a one-dimensional radial motion with time-dependent coefficients. Note that in the original coordinates this integral of motion is the angular momentum (the third component)

$$p_\theta = q_1 p_2 - q_2 p_1.$$

In order to simplify many expressions let us fix values of various physical constants. From now on we set $e = m = 1$, and so the cyclotron frequency equals b. We assume, without loss of generality, that b is positive. Finally we set

$$a(t) = p_\theta - \frac{1}{2\pi}\,\Phi(t). \tag{1.5}$$

In the polar Hamiltonian (1.4) one may omit the term $ba(t)/2$ not contributing to the equations of motion and thus one arrives at the expression for the radial Hamiltonian

$$H_{\mathrm{rad}}(r, p_r, t) = \frac{p_r^2}{2} + \frac{a(t)^2}{2r^2} + \frac{b^2 r^2}{8}. \tag{1.6}$$

Transformation to action-angle coordinates

First, we introduce the action-angle coordinates for a frozen time. Assume for a moment that $a(t) = a$ is a time-independent constant and denote

$$V(r) = \frac{a^2}{2r^2} + \frac{b^2 r^2}{8}.$$

Suppose a fixed energy level E is greater than the minimal value $V_{\min} = b\,|a|/2$, attained at $r_{\min} = \sqrt{b\,|a|/2}$. Then the motion is constrained to a bounded interval $[r_-, r_+]$, and one has

$$E - V(r) = \frac{b^2}{8r^2}(r_+^2 - r^2)(r^2 - r_-^2), \tag{1.7}$$

where

$$r_\pm^2 = \frac{2}{b^2}\left(2E - ab \pm \sqrt{(2E - ab)^2 - a^2 b^2}\right). \tag{1.8}$$

This situation is illuminated by Figure 1.3.

The action equals

$$I(E) = \frac{1}{\pi}\int_{r_-}^{r_+} \sqrt{2(E - V(r))}\,\mathrm{d}r = \frac{b}{4\pi}\int_{r_-^2}^{r_+^2} \frac{1}{x}\sqrt{(r_+^2 - x)(x - r_-^2)}\,\mathrm{d}x = \\ = \frac{b}{8}(r_+ - r_-)^2 = \frac{1}{b}(E - V_{\min}).$$

For more details concerning the construction of action-angle coordinates the reader is advised to confer [2]. Hence expressing E in terms of a and I and using (1.8) we arrive at

$$r_\pm = \frac{2}{\sqrt{b}}\left(I + \frac{|a|}{2} \pm \sqrt{I(I + |a|)}\right)^{1/2} = \sqrt{\frac{2}{b}}\left(\sqrt{I + |a|} \pm \sqrt{I}\right). \tag{1.9}$$

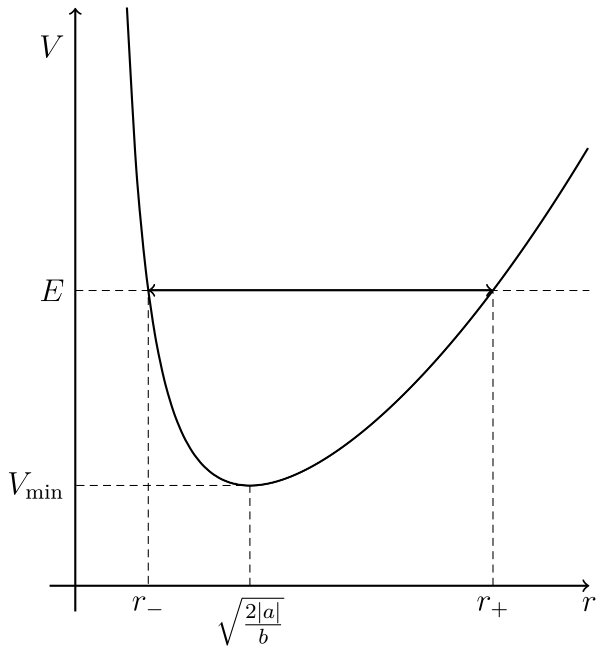

Figure 1.3: The potential V and a fixed energy level E. The motion in the radial coordinate is constrained to the interval $[r_-, r_+]$.

Using the generating function,

$$S(r,I) = \int_{r_-}^{r} \sqrt{2(E - V(\rho))}\mathrm{d}\rho = \frac{b}{2}\int_{r_-}^{r} \frac{1}{\rho}\sqrt{(r_+^2 - \rho^2)(\rho^2 - r_-^2)}\mathrm{d}\rho,$$

one derives the canonical transformation of variables between (r, p_r) and the action-angle variables (φ, I). One has

$$\begin{aligned}\frac{\partial S}{\partial I} &= \frac{b}{4}\int_{r_-}^{r} \frac{1}{\rho}\left(\sqrt{\frac{\rho^2 - r_-^2}{r_+^2 - \rho^2}}\frac{\mathrm{d}r_+(I)^2}{\mathrm{d}I} - \sqrt{\frac{r_+^2 - \rho^2}{\rho^2 - r_-^2}}\frac{\mathrm{d}r_-(I)^2}{\mathrm{d}I}\right)\mathrm{d}\rho \\ &= \frac{\pi}{2} - \arctan\left(\frac{r_+^2 + r_-^2 - 2r^2}{2\sqrt{(r_+^2 - r^2)(r^2 - r_-^2)}}\right).\end{aligned}$$

For the angle variable $\varphi = \partial S/\partial I - \pi/2$ one obtains

$$\sin(\varphi) = \frac{1}{\sqrt{I(I + |a|)}}\left(\frac{br^2}{4} - I - \frac{|a|}{2}\right).$$

Furthermore,

$$p_r = \frac{\partial S}{\partial r} = \frac{b}{2r}\sqrt{(r_+^2 - r^2)(r^2 - r_-^2)} = \frac{2}{r}\sqrt{I(I + |a|)}\cos(\varphi).$$

Finally one arrives at the relations

$$r = \frac{2}{\sqrt{b}}\left(I + \frac{|a|}{2} + \sqrt{I(I+|a|)}\,\sin(\varphi)\right)^{1/2}, \tag{1.10}$$

$$p_r = \frac{\sqrt{bI(I+|a|)}\,\cos(\varphi)}{\left(I + \frac{|a|}{2} + \sqrt{I(I+|a|)}\,\sin(\varphi)\right)^{1/2}}. \tag{1.11}$$

Note that, conversely,

$$I = \frac{1}{2b}\left(p_r^2 + \left(\frac{|a(t)|}{r} - \frac{br}{2}\right)^2\right) = \frac{1}{b}\left(H_{\rm rad}(r, p_r, t) - V_{\rm min}\right). \tag{1.12}$$

Let us now switch to the time-dependent case. The Hamiltonian transforms according to the rule

$$H_c(\varphi, I, t) = H_{\rm rad}\Big(r(\varphi, I, t), p_r(\varphi, I, t), t\Big) + \left.\frac{\partial S(u, I, t)}{\partial t}\right|_{u=r(\varphi,I,t)}.$$

One computes

$$\begin{aligned}\left.\frac{\partial S(u, I, t)}{\partial t}\right|_{u=r(\varphi,I,t)} &= \frac{b|a|'}{4}\int_{r_-}^{r(\varphi,I,t)} \frac{1}{\rho}\left(\sqrt{\frac{\rho^2 - r_-^2}{r_+^2 - \rho^2}}\,\frac{\partial r_+^2}{\partial |a|} - \sqrt{\frac{r_+^2 - \rho^2}{\rho^2 - r_-^2}}\,\frac{\partial r_-^2}{\partial |a|}\right) \mathrm{d}\rho \\ &= -\frac{|a|'}{2}\arctan\left(\frac{2\left(\sqrt{I(I+|a|)} + I\sin(\varphi)\right)\cos(\varphi)}{|a| + 2\left(\sqrt{I(I+|a|)} + I\sin(\varphi)\right)\sin(\varphi)}\right).\end{aligned}$$

Simplifying the expression and dropping those terms which are independent of φ and I one finally arrives at the equality

$$H_c(\varphi, I, t) = bI - |a(t)|'\arctan\left(\frac{\sqrt{I}\cos(\varphi)}{\sqrt{I + |a(t)|} + \sqrt{I}\sin(\varphi)}\right).$$

The Hamiltonian equations of motion take the form

$$\varphi' = b - \frac{\cos(\varphi)aa'}{2\sqrt{I(I+|a|)}\left(2I + |a| + 2\sqrt{I(I+|a|)}\,\sin(\varphi)\right)} \tag{1.13}$$

$$I' = -\frac{|a|'}{2}\left(1 - \frac{|a|}{2I + |a| + 2\sqrt{I(I+|a|)}\,\sin(\varphi)}\right). \tag{1.14}$$

For the sake of definiteness we shall focus on the case when $a(t)$ is a strictly positive function. More precisely, the angular momentum p_θ is supposed to be positive and greater than the amplitude of $\Phi(t)$. Let us stress, however, that this restriction on the sign of $a(t)$ is not essential for the resonance effect. In fact, notice that the radial Hamiltonian (1.6) depends on $a(t)^2$ and thus the sign of $a(t)$ is irrelevant for the motion in the radial direction. On the other hand, as discussed in Section 1.4, the sign of $a(t)$ determines whether the orbit encircles the singular magnetic flux or not.

Throughout this chapter, the function $\Phi(t)$ is supposed to be of the form

$$\Phi(t) = 2\pi\varepsilon f(\Omega t) \tag{1.15}$$

where $\Omega > 0$ and f is a 2π-periodic real function possibly obeying additional assumptions. Moreover, ε is supposed to be positive as well and playing the role of a small parameter. Thus one has $a(t) = p_\theta - \varepsilon f(\Omega t)$.

Basic properties of the radial dynamics

The Hamiltonian equations of motion for the radial Hamiltonian (1.6) have the form

$$r' = p_r, \quad p_r' = \frac{a(t)^2}{r^3} - \frac{b^2 r}{4} \tag{1.16}$$

This is equivalent to the nonlinear second-order differential equation

$$r'' + \frac{b^2}{4} r = \frac{a(t)^2}{r^3}. \tag{1.17}$$

Let us now look at the problem of zeros of $a(t)$ in more detail. If $a(t)$ has no zeros then the solutions of (1.16) are defined for all times $t \in \mathbb{R}$. In particular we have the following Proposition.

Proposition 1.1: Suppose $a(t)$ is a real continuously differentiable function defined on $\mathbb{R}$ having no zeros. Then for any initial condition $r(t_0) = r_0$, $r'(t_0) = r_1$, with $(t_0, r_0, r_1) \in \mathbb{R}\times]0, +\infty[\times\mathbb{R}$, there exists an unique solution of the differential equation (1.17) defined on the whole real line $\mathbb{R}$ and satisfying this initial condition.

Proof. Suppose $(r(t), p_r(t))$ is a solution of the Hamiltonian equations (1.16). Put $H(t) = H_{\mathrm{rad}}(r(t), p_r(t), t)$. Then

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} H(t)\right| = \frac{|a(t)a'(t)|}{r(t)^2} \leq \left|\frac{2a'(t)}{a(t)}\right| H(t).$$

From here one readily concludes that if $r(t)$ is a solution of (1.17) on a bounded interval $M \subset \mathbb{R}$, then there exists constants R_1, R_2, $0 < R_1 \leq R_2 < +\infty$, such that $R_1 \leq r(t) \leq R_2$ for all $t \in M$. From the general theory of ordinary differential equations it immediately follows that any solution $r(t)$ of the differential equation (1.17) can be continued to the whole real line. □

On the other hand, if there are zeros of $a(t)$, then it might happen that some solutions of (1.16) are defined only on a half-line. In this case it is possible that the particle hits the hole where the flux line pierces the plane.

Proposition 1.2: Let $r_1 > 0$ and suppose that $a(0) = 0$ and

$$\int_0^1 \frac{a(s)^2}{s^4} \mathrm{d}s < +\infty.$$

Then there is a solution of (1.17) defined on the interval $[0, \delta]$ for some $\delta > 0$ and satisfying initial condition $r(0) = 0$, $r'(0) = r_1$.

Proof. Our proof relies on a well known fixed point argument. The initial value problem

$$r'' + \frac{b^2}{4} = \frac{a(t)^2}{r^3}, \quad r(0) = 0, \quad r'(0) = r_1, \tag{1.18}$$

is equivalent to the integral equation

$$r(t) = r_1 t + \int_0^t (t-s)\left(\frac{a(s)^2}{r(s)^3} - \frac{b^2}{4} r(s)\right) \mathrm{d}s.$$

Let us fix α and β such that $0 < \alpha < r_1 < \beta$ and define

$$D_\delta = \Big\{ f : [0,\delta] \to \mathbb{R}\,;\ f \in C\big([0,\delta]\big),\ \alpha t \le f(t) \le \beta t \quad \text{for all}\, t \in [0,\delta] \Big\}.$$

The set D_δ is a closed[1] subspace of the Banach space[2] $C\big([0,\delta]\big)$. For $f \in D_\delta$ we can

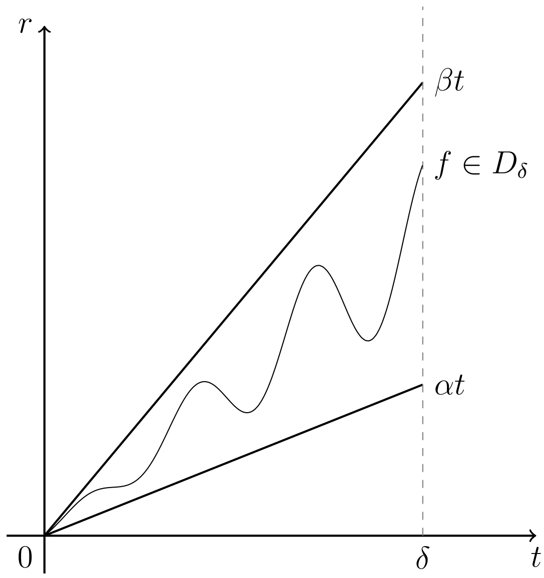

Figure 1.4: The D_δ space.

define

$$(Kf)(t) = r_1 t + \int_0^t (t-s)\left(\frac{a(s)^2}{f(s)^3} - \frac{b^2}{4} f(s)\right) \mathrm{d}s, \quad t \in [0,\delta].$$

First of all we have to choose δ so small, such that K maps D_δ into D_δ. More precisely, for any $f \in D_\delta$ one has to show that

$$\alpha t \le (Kf)(t) \le \beta t, \quad t \in [0,\delta]. \tag{1.19}$$

[1] The word "closed" has the topological meaning here.
[2] Continuous functions $f : [0,\delta] \to \mathbb{R}$ with supremum norm,

$$\|f\|_\infty = \sup_{x \in [0,\delta]} |f(x)|.$$

Inequalities in (1.19) are implied by

$$\frac{1}{\alpha^3}\int_0^t \frac{a(s)^2}{s^3}\,\mathrm{d}s \le \beta - r_1,$$
$$-\frac{1}{\beta^3}\int_0^t \frac{a(s)^2}{s^3}\,\mathrm{d}s + \frac{b^2}{4}\cdot\beta\frac{t^2}{6} + \frac{1}{\beta^3 t}\int_0^t \frac{a(s)^2}{s^2}\,\mathrm{d}s \le r_1 - \alpha, \quad t\in[0,\delta].$$

Since the left-hand-sides vanish as $t\to 0_+$ and $\alpha < r_1 < \beta$ it is clear that one can take $\delta > 0$ small enough and thus satisfy (1.19).

Furthermore, δ can be taken even smaller in order to make K a contraction on D_δ. If $f,g\in D_\delta$, then

$$\begin{aligned}\left|(Kf-Kg)(t)\right| &\le \left|\int_0^t (t-s)\left(\frac{a(s)^2}{f(s)^3} - \frac{a(s)^2}{g(s)^3} - \frac{b^2}{4}f(s) + \frac{b^2}{4}g(s)\right)\mathrm{d}s\right| \le \\ &\le \delta^2\cdot\|f-g\|_\infty + \|f-g\|_\infty \\ &\qquad\times \int_0^t (t-s)a(s)^2\frac{f(s)^3+f(s)g(s)+g(s)^2}{f(s)^3g(s)^3}\mathrm{d}s \le \\ &\le \left(\frac{b^2\delta^2}{4} + \frac{3\delta\beta^2}{\alpha^6}\int_0^\delta \frac{a(s)^2}{s^4}\mathrm{d}s\right)\cdot\|f-g\|_\infty, \quad t\in[0,\delta].\end{aligned}$$

This shows that one can find sufficiently small $\delta > 0$ in such a way that the inequality

$$\|Kf-Kg\|_\infty \le \gamma\|f-g\|_\infty, \quad f,g\in D_\delta,$$

is satisfied with $0<\gamma<1$.

According to the fixed point theorem[3] there is a function $r\in D_\delta$ such that $Kr=r$. This function solves our initial value problem (1.18). □

Let us conclude this Section with a preliminary qualitative characterization of trajectories in the resonant case. From equations (1.10) and (1.9) we see that

$$r^2 = \frac{1}{2}(r_+^2 + r_-^2) + \frac{1}{2}(r_+^2 - r_-^2)\sin(\varphi).$$

Thus if the angle φ increases then r oscillates between r_- and r_+ (though r_-, r_+ themselves are also time-dependent). Moreover, if $a(t)$ is bounded and $I\to\infty$ as $t\to\infty$ then $r_+(t)\to\infty$ and

$$r_-(t) = \frac{2|a(t)|}{br_+(t)} \to 0, \quad \text{as } t\to\infty.$$

Therefore in this case the trajectory in the q-plane periodically returns to the origin and then again escapes far away from it while its extremal distances to the origin converge respectively to zero and infinity. We refer again to Figure 1.1 for a typical trajectory in the q-plane in the case of resonant frequencies. In this example p_θ is positive and so the orbit encircles the singular magnetic flux located in the origin of coordinates, as discussed in Section 1.4. On the other hand, in Figure 1.2 p_θ is negative and therefore the trajectory does not encircle the singular magnetic flux.

[3]See [29], Theorem V.18.

1.2 The Poincaré-von Zeipel elimination method

Notation and a summary of basic formulas

We first study the model with the aid of the Poincaré-von Zeipel elimination method for this averaging method takes into account possible resonances, as explained in detail, for instance, in [3]. The main result of the current Section is a demonstration of the resonance effect for the dynamics generated by the first order averaged Hamiltonian. We start from introducing notation and recalling basic formulas.

Let $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ be the d-dimensional torus. For $f(\varphi) \in C(\mathbb{T}^d)$ and $k \in \mathbb{Z}^d$ we denote the kth Fourier coefficient of f by the symbol

$$\mathscr{F}[f(\varphi)]_k = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(\varphi) e^{-ik\cdot\varphi} \mathrm{d}\varphi.$$

We introduce $\operatorname{supp} \mathscr{F}[f(\varphi)]$ as the set of indices corresponding to non-zero Fourier coefficients of $f(\varphi)$,

$$\operatorname{supp} \mathscr{F}[f(\varphi)] = \Big\{k \in \mathbb{Z}^d\,;\, \mathscr{F}[f(\varphi)]_k \neq 0\Big\}.$$

For $f \in C(\mathbb{T}^d)$ and $\mathbb{L} \subset \mathbb{Z}^d$ put

$$\Big\langle f(\varphi) \Big\rangle_{\mathbb{L}} = \sum_{k\in\mathbb{L}} \mathscr{F}[f]_k e^{ik\cdot\varphi}. \tag{1.20}$$

Note that for $f \in C(\mathbb{T}^1)$ and $\nu \in \mathbb{N}$ one has

$$\Big\langle f(\varphi) \Big\rangle_{\mathbb{Z}\nu} = \frac{1}{\nu} \sum_{j=0}^{\nu-1} f\left(\varphi + \frac{2\pi}{\nu} j\right).$$

Indeed, for the proof it is sufficient to consider functions of the form $f(\varphi) = e^{i\ell\cdot\varphi}$ for some fixed $\ell \in \mathbb{Z}^d$. Then from the very definition (1.20) one has

$$\Big\langle f(\varphi) \Big\rangle_{\mathbb{Z}\nu} = \begin{cases} 0, & \ell \notin \mathbb{Z}\nu, \\ e^{i\ell\cdot\varphi}, & \ell \in \mathbb{Z}\nu. \end{cases}$$

On the other hand,

$$\frac{1}{\nu} \sum_{j=0}^{\nu-1} f\left(\varphi + \frac{2\pi}{\nu} j\right) = \frac{1}{\nu} \sum_{j=0}^{\nu-1} \exp\left(i\ell \cdot \left(\varphi + \frac{2\pi}{\nu} j\right)\right) = \begin{cases} \frac{1}{\nu} e^{i\ell\cdot\varphi} \frac{1-1}{1-e^{i(\ell_1+\cdots+\ell_d)\frac{2\pi}{\nu}}} = 0, & \ell \notin \mathbb{Z}\nu, \\ e^{i\ell\cdot\varphi}, & \ell \in \mathbb{Z}\nu. \end{cases}$$

Consider now a completely integrable Hamiltonian in action-angle coordinates,

$$K_0(I) = \omega \cdot I,$$

where I runs over a domain in $\mathbb{R}^d$, $\varphi \in \mathbb{T}^d$ and $\omega \in \mathbb{R}^d_+$ is a constant vector of frequencies. One is interested in a perturbed system with a small Hamiltonian perturbation so that the total Hamiltonian reads

$$K(\varphi, I, \varepsilon) = K_0(I) + \varepsilon K_*(\varphi, I, \varepsilon) = K_0(I) + \varepsilon K_1(\varphi, I) + \varepsilon^2 K_2(\varphi, I) + \ldots \tag{1.21}$$

where ε is a small parameter. The function $K_*(\varphi, I, \varepsilon)$ is assumed to be analytic in all variables.

Let $\mathbb{K}$ be the lattice of indices in $\mathbb{Z}^d$ corresponding to resonant frequencies and $\mathbb{K}^c$ be its complement, i.e.

$$\mathbb{K} = \{\omega\}^{\perp} \cap \mathbb{Z}^d, \quad \mathbb{K}^c = \mathbb{Z}^d \setminus \mathbb{K}. \tag{1.22}$$

One applies a formal canonical transformation of variables, $(I, \varphi) \mapsto (J, \psi)$, so that the Fourier series in the angle variables ψ of the resulting Hamiltonian $\mathcal{K}(\psi, J, \varepsilon)$ has non-zero coefficients only for indices from the lattice $\mathbb{K}$. The canonical transformation is generated by a function $S(\varphi, J, \varepsilon)$ regarded as a formal power series with coefficient functions $S_j(\varphi, J)$ and the absolute term $S_0(\varphi, J) = \varphi \cdot J$. Similarly, the new Hamiltonian $\mathcal{K}(\psi, J, \varepsilon)$ is sought in the form of a formal power series with coefficient functions $\mathcal{K}_j(\psi, J)$. One arrives at the system of equations $\mathcal{K}_0(J, \varphi) = K_0(J) = \omega \cdot J$ and

$$\mathcal{K}_j(\varphi, J) = \omega \cdot \frac{\partial S_j(\varphi, J)}{\partial \varphi} + P_j(\varphi, J), \quad j \geq 1,$$

where $P_1(\varphi, J) = K_1(\varphi, J)$ and the terms P_j for $j \geq 2$ are determined recursively. The formal von Zeipel Hamiltonian is defined by the equalities

$$\mathcal{K}_j(\psi, J) = \langle P_j(\psi, J) \rangle_{\mathbb{K}}$$

for $j \geq 1$. Coefficients $S_j(\varphi, J)$ are then solutions of the first order differential equations

$$\omega \cdot \frac{\partial S_j(\varphi, J)}{\partial \varphi} = - \langle P_j(\varphi, J) \rangle_{\mathbb{K}^c}, \quad j \geq 1.$$

In practice one truncates $\mathcal{K}(\psi, J, \varepsilon)$ at some order $m \geq 1$ of the parameter ε. Let us define the mth order averaged Hamiltonian

$$\mathcal{K}_{(m)}(\psi, J, \varepsilon) = \mathcal{K}_0(J) + \varepsilon \mathcal{K}_1(\psi, J) + \cdots + \varepsilon^m \mathcal{K}_m(\psi, J).$$

Similarly, let $S_{(m)}(\varphi, J, \varepsilon)$ be the truncated generating function. If $(\psi(t), J(t))$ is a solution of the Hamiltonian equations for $\mathcal{K}_{(m)}(\psi, J, \varepsilon)$, and if $(\varphi(t), I(t))$ is the same solution after the inverted canonical transformation generated by $S_{(m)}(\varphi, J, \varepsilon)$, then $(\varphi(t), I(t))$ is expected to approximate well the solution of the original system (governed by the Hamiltonian $K(\varphi, I, \varepsilon)$) for times of order $1/\varepsilon^m$ (see [3] for a detailed discussion).

The first-order averaged Hamiltonian

In this section we assume that $\Phi(t)$ is given by (1.15) where $\varepsilon > 0$ is regarded as a small parameter and the 2π-periodic real function $f(\varphi)$ fulfills

$$\sum_{k=1}^{\infty} k \left| \mathscr{F}[f(\varphi)]_k \right| < \infty. \tag{1.23}$$

This implies that $f \in C^1(\mathbb{T}^1)$.

In order to apply the von Zeipel method to our problem we first pass to the extended phase space by introducing a new phase $\varphi_2 = \Omega t$ and its conjugate momentum I_2. The old variables φ, I are denoted as φ_1, I_1. The Hamiltonian on the extended phase space is defined as

$$K(\varphi_1, \varphi_2, I_1, I_2) = \Omega I_2 + H_c(\varphi_1, I_1, \varphi_2/\Omega). \tag{1.24}$$

The systems of Hamiltonian equations for H_c and K are equivalent provided the initial conditions are properly matched (if $\varphi(0) = \varphi_0$ on the original phase space then $(\varphi_1(0), \varphi_2(0)) = (\varphi_0, 0)$ on the extended phase space). This procedure is a standard procedure how to pass from non-autonomous systems to autonomous. The price to be paid is the increase of number of degrees of freedom. In fact, let us assume that a non-autonomous classical system is described by a time-dependent Hamilton function $h(q, p, t)$, where $(q, p) \in \mathbb{P}$ are canonical coordinates in a phase space $\mathbb{P}$ and $t \in \mathbb{R}$. The corresponding Hamiltonian equations of motion read

$$q' = \frac{\partial h}{\partial p}(q, p, t), \quad p' = -\frac{\partial h}{\partial q}(q, p, t), \quad \big(q(0),\, p(0)\big) = (q_0, p_0) \in \mathbb{P}. \tag{1.25}$$

Dashes denote the derivative with respect to the time t. Note that for the total time derivative one has

$$\frac{\mathrm{d}}{\mathrm{d}t} h(q, p, t) = \frac{\partial h}{\partial t}(q, p, t).$$

Let us treat the time t as a new coordinate and introduce the energy E as the corresponding conjugate momentum and s as a new time parameter. The new Hamiltonian reads

$$k(q, p, t, E) = E + h(q, p, t).$$

The Hamilton equations are then

$$\begin{aligned} q' &= \frac{\partial k}{\partial p}(q, p, t, E), & p' &= -\frac{\partial k}{\partial q}(q, p, t, E), \\ t' &= \frac{\partial k}{\partial E}(q, p, t, E) = 1, & E' &= -\frac{\partial k}{\partial t}(q, p, t, E). \end{aligned} \tag{1.26}$$

with initial conditions

$$\big(q(0),\, p(0)\big) = (q_0, p_0) \quad \text{and} \quad \big(t(0),\, E(0)\big) = \big(t_0,\, h(q_0, p_0, t_0)\big).$$

An analogous procedure is possible in quantum mechanics and we will employ this idea in the next Chapter.

To adjust the notation to the general scheme, as introduced at the beginning of this Section, we also set $\omega_1 = b$, $\omega_2 = \Omega$. Thus one starts from the Hamiltonian on the extended phase space

$$K(\varphi, I, \varepsilon) = \omega_1 I_1 + \omega_2 I_2 + \varepsilon F(\varphi, I, \varepsilon) \tag{1.27}$$

where $(p_\theta > 0)$

$$F(\varphi, I, \varepsilon) = \omega_2 f'(\varphi_2) \arctan\left(\frac{\sqrt{I_1} \cos(\varphi_1)}{\sqrt{I_1 + p_\theta - \varepsilon f(\varphi_2)} + \sqrt{I_1} \sin(\varphi_1)} \right). \tag{1.28}$$

If the ratio ω_2/ω_1 is irrational then the lattice $\mathbb{K}$ is trivial, $\mathbb{K} = \{0\}$, and the von Zeipel method amounts to the ordinary averaging method in angle variables φ. Now we focus on the complementary case when

$$\lambda := \frac{\omega_2}{\omega_1} = \frac{\mu}{\nu}\,, \text{ with } \mu, \nu \in \mathbb{N} \text{ coprime.} \tag{1.29}$$

As we shall see, a resonance effect is exhibited already for the first order averaged Hamiltonian to which we restrict our discussion.

We have

$$K(\varphi, I, \varepsilon) = \omega_1 I_1 + \omega_2 I_2 + \varepsilon K_1(\varphi, I) + \varepsilon^2 \widetilde{K}(\varphi, I, \varepsilon)$$

where $\widetilde{K}(\varphi, I, \varepsilon)$ is an analytic function in ε,

$$K_1(\varphi, I) = \omega_2 f'(\varphi_2) F_1(\varphi_1, I_1), \quad F_1(\varphi_1, I_1) = \arctan\left(\frac{\sqrt{I_1}\cos(\varphi_1)}{\sqrt{I_1 + p_\theta} + \sqrt{I_1}\sin(\varphi_1)}\right).$$

One finds (cf. Appendix A, in particular Corollary A.3) that

$$\mathscr{F}[F_1(\varphi_1, I_1)]_k = \frac{i^{k-1}}{2k}\left(\frac{I_1}{I_1 + p_\theta}\right)^{|k|/2} \text{for } k \neq 0,\ \mathscr{F}[F_1(\varphi_1, I_1)]_0 = 0.$$

Obviously, the Fourier image of $K_1(\varphi, I)$ takes non-zero values only for indices (k, ℓ) such that $k \in \mathbb{Z} \setminus \{0\}$, $\ell \in \operatorname{supp} \mathscr{F}[f] \setminus \{0\}$, and

$$\mathscr{F}[K_1(\varphi, I)]_{(k,\ell)} = i\ell\omega_2\, \mathscr{F}[f(\varphi_2)]_\ell\, \mathscr{F}[F_1(\varphi_1, I_1)]_k.$$

Next we proceed to the von Zeipel canonical transformation of the first order. Set

$$\beta(J_1) = \sqrt{\frac{J_1}{J_1 + p_\theta}}\,. \tag{1.30}$$

The resonant lattice is given by $\mathbb{K} = \mathbb{Z}\,(\mu, -\nu)$, and one has

$$\begin{aligned} \mathcal{K}_1(\psi, J) &= \sum_{m \in \mathbb{K}} \mathscr{F}[K_1(\psi, J)]_m\, e^{im\cdot\psi} \\ &= -\frac{\omega_1}{2} \sum_{n \in \mathbb{Z}\setminus\{0\}} \mathscr{F}[f]_{-n\nu}\, i^{n\mu} \beta(J_1)^{|n|\mu}\, e^{in(\mu\psi_1 - \nu\psi_2)}. \end{aligned} \tag{1.31}$$

$S_1(\varphi, J)$ is a solution to the differential equation

$$\omega \cdot \frac{\partial S_1}{\partial \varphi} = -K_1 + \mathcal{K}_1.$$

Seeking $S_1(\varphi, J)$ in the form

$$S_1(\varphi, J) = 2\operatorname{Re}\left(\sum_{k=1}^{\infty} \mathscr{F}[f']_k G_k(\varphi_1, J_1)\, e^{ik\varphi_2}\right) \tag{1.32}$$

one finally arrives at the countable system of equations

$$\left(\frac{\partial}{\partial \varphi_1} + ik\lambda\right) G_k(\varphi_1, J_1) = \lambda \sum_{\substack{n \in \mathbb{Z}\setminus\{0\} \\ n \neq -k\lambda}} \frac{i^{n+1}}{2n}\, \beta(J_1)^{|n|}\, e^{in\varphi_1}, \quad k \geq 1.$$

For the solution we choose

$$G_k(\varphi_1, J_1) = \lambda \sum_{\substack{n\in\mathbb{Z}\smallsetminus\{0\}\\ n\neq -k\lambda}} \frac{i^n}{2n(n+k\lambda)}\, \beta(J_1)^{|n|}\, e^{in\varphi_1}. \tag{1.33}$$

Of course, if $k\lambda \notin \mathbb{Z}$ then the restriction $n \neq -k\lambda$ is void. On the other hand, if $k\lambda \in \mathbb{Z}$, and this happens if and only if $k \in \mathbb{Z}\nu$, then the solution $G_k(\varphi_1, J_1)$ is not unique.

Thus one finds the averaged Hamiltonian of the first order

$$\mathcal{K}_{(1)}(\psi, J) = \frac{\omega_1}{\nu}\,(\nu J_1 + \mu J_2) + \varepsilon\, \mathcal{K}_1(\psi, J), \tag{1.34}$$

with $\mathcal{K}_1(\psi, J)$ being given in (1.31).

The dynamics generated by the first-order Hamiltonian

Since μ and ν are coprime there exist $\tilde\mu, \tilde\nu \in \mathbb{Z}$ such that $\tilde\mu\mu + \tilde\nu\nu = 1$. Put

$$\mathbf{R} = \begin{pmatrix} \mu & -\nu \\ \tilde\nu & \tilde\mu \end{pmatrix} \tag{1.35}$$

and consider the canonical transformation $\chi = \mathbf{R}\psi$, $J = \mathbf{R}^{\mathrm{T}} L$. In particular,

$$\chi_1 = \mu\psi_1 - \nu\psi_2, \quad L_2 = \nu J_1 + \mu J_2, \quad J_1 = \mu L_1 + \tilde\nu L_2.$$

The momentum L_2 is an integral of motion for the Hamiltonian $\mathcal{K}_{(1)}(\psi, J)$. Let us define

$$\mathcal{Z}(\chi_1, J_1) = \varepsilon\mu\, \mathcal{K}_1(\mathbf{R}^{-1}\chi, J).$$

Then

$$\chi_1'(t) = \varepsilon\, \frac{\partial\mathcal{K}_1(\psi, J)}{\partial J_1}\, \frac{\partial J_1}{\partial L_1} = \frac{\partial\mathcal{Z}(\chi_1, J_1)}{\partial J_1},$$
$$J_1'(t) = -\varepsilon\, \frac{\partial\mathcal{K}_1(\psi, J)}{\partial\psi_1} = -\frac{1}{\mu}\, \frac{\partial\mathcal{Z}(\chi_1, J_1)}{\partial\chi_1}\, \frac{\partial\chi_1}{\partial\psi_1} = -\frac{\partial\mathcal{Z}(\chi_1, J_1)}{\partial\chi_1}.$$

Thus the time evolution in coordinates χ_1, J_1 is governed by the Hamiltonian $\mathcal{Z}(\chi_1, J_1)$.

Set

$$h(z) = -\varepsilon\mu\omega_1 \sum_{n=1}^{\infty} \mathscr{F}[f]_{-n\nu}\, i^{n\mu}\, z^n \tag{1.36}$$

and

$$\varrho(x) = \beta(x)^{\mu} = \left(\frac{x}{x+p_\theta}\right)^{\mu/2}, \quad x > 0.$$

Then, by assumption (1.23), $h(z)$ is holomorphic on the open unit disk $B_1 \subset \mathbb{C}$ and $h \in C^1\big(\overline{B_1}\big)$. One has $\mathcal{Z}(\chi_1, J_1) = \operatorname{Re}\Big[h\Big(\varrho(J_1)e^{i\chi_1}\Big)\Big]$. The Hamiltonian equations of motion read

$$\chi_1'(t) = \frac{\varrho'(J_1)}{\varrho(J_1)}\, \operatorname{Re}[z\, h'(z)], \quad J_1'(t) = \operatorname{Im}[z\, h'(z)], \quad \text{with } z = \varrho(J_1)e^{i\chi_1}. \tag{1.37}$$

Concerning the asymptotic behavior of Hamiltonian trajectories $(\chi_1(t), J_1(t))$, as $t \to +\infty$, one can formulate a proposition under somewhat more general circumstances.

Theorem 1.3: Let $h \in C^1\left(\overline{B_1}\right)$ and suppose $h(z)$ is a nonconstant holomorphic function on the open unit disk B_1. Let $\varrho : [0, +\infty[\to [0, 1[$ be a smooth function such that $\varrho'(x) > 0$ for $x > 0$, $\varrho(0) = 0$ and $\lim_{x\to+\infty} \varrho(x) = 1$. Let $\mathcal{Z}(\chi_1, J_1)$ be the Hamilton function on $\mathbb{R}\times\,]0, +\infty[$ defined by

$$\mathcal{Z}(\chi_1, J_1) = \mathrm{Re}\Big[h\Big(\varrho(J_1)e^{i\chi_1}\Big)\Big] .$$

Then for almost all initial conditions $(\chi_1(0), J_1(0))$ the corresponding Hamiltonian trajectory fulfills

$$\lim_{t\to+\infty} \chi_1(t) = \chi_1(\infty) \in \mathbb{R}, \lim_{t\to+\infty} J_1(t) = +\infty, \tag{1.38}$$

and

$$\lim_{t\to+\infty} J_1'(t) = \mathrm{Im}\Big[e^{i\chi_1(\infty)}\, h'\Big(e^{i\chi_1(\infty)}\Big)\Big] > 0. \tag{1.39}$$

Proof. Set $R(z) = \mathrm{Re}[h(z)]$, $z \in \overline{B_1}$. Then $\mathrm{d}R_z \equiv (\mathrm{Re}[h'(z)], -\mathrm{Im}[h'(z)])$. Hence $\mathrm{d}R_z = 0$ if and only if $h'(z) = 0$, and the set of critical points of R in B_1 has no accumulation points in B_1 and is at most countable. By Sard's theorem, almost all $y \in \mathbb{R}$ are regular values of $R|\partial B_1$. If y is a regular value both of R and $R|\partial B_1$ then the level set $R^{-1}(y)$ is a compact one-dimensional C^1 submanifold with boundary in $\overline{B_1}$, $\partial R^{-1}(y) = R^{-1}(y) \cap \partial B_1$ and $R^{-1}(y)$ is not tangent to ∂B_1 at any point. Moreover, $R^{-1}(y) \cap B_1$ is a smooth submanifold of B_1 [12, 11]. By the classification of compact connected one-dimensional manifolds [11], every component of $R^{-1}(y)$ is diffeomorphic either to a circle or to a closed interval. But the first possibility is excluded because $R(z)$ is a harmonic function. In fact, if $\overline{U} \subset B_1$, U is an open set, $\partial U \simeq S^1$ is a smooth submanifold of B_1 and $R(z)$ is constant on ∂U then $R(z)$ is constant on U and so is $h(z)$. Consequently, $h(z)$ is constant on B_1, a contradiction with our assumptions. Thus every component Γ of $R^{-1}(y)$ is diffeomorphic to a closed interval, $\partial\Gamma = \{a, b\} = \Gamma \cap \partial B_1$, and Γ is tangent to ∂B_1 neither at a nor at b.

Let $z \in B_1$ be such that $\mathrm{d}R_z \neq 0$. By the local submersion theorem [11], R is locally equivalent at z to the canonical submersion $\mathbb{R}^2 \to \mathbb{R}$. Hence z possesses an open neighborhood U such that $R(U)$ is an open interval. We know that almost every $y \in R(U)$ is a regular value both of R and $R|\partial B_1$. By the Fubini theorem, for almost every $w \in U$, $R(w)$ is a regular value both of R and $R|\partial B_1$. The same claim is true for almost all $w \in B_1$ because the set of critical points of R in B_1 is at most countable. It follows that for almost all $(\chi_1, J_1) \in \mathbb{R}\times\,]0, +\infty[$, $R(\varrho(J_1)e^{i\chi_1}) \neq R(0)$ is a regular value both of R and $R|\partial B_1$.

Suppose now that an initial condition $(\chi_1(0), J_1(0))$ has been chosen so that

$$y = R\Big(\varrho(J_1(0))e^{i\chi_1(0)}\Big) \neq R(0)$$

is a regular value both of R and $R|\partial B_1$. Let Γ be the component of $R^{-1}(y)$ containing the point $\varrho(J_1(0))e^{i\chi_1(0)}$. Since the Hamiltonian $\mathcal{Z}(\chi_1, J_1)$ is an integral of motion the Hamiltonian trajectory $z(t) = \varrho(J_1(t))e^{i\chi_1(t)}$ is constrained to the submanifold $\Gamma \subset \overline{B_1}$. We have to show that $z(t)$ reaches the boundary ∂B_1 as $t \to +\infty$. The tangent vector to the trajectory at the point $z(t)$ equals

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = i\varrho(J_1(t))\varrho'(J_1(t))\, \overline{h'(z(t))}\, .$$

Since $0 \notin \Gamma$, $\varrho'(J_1) > 0$ for all $J_1 > 0$ and $h'(z)$ has no zeroes on Γ (because y is a regular value) it follows that $z(t)$ leaves any compact subset of B_1 in a finite time. It remains to show that $z(t)$ does not reach ∂B_1 in a finite time. But by equations of motion (1.37), $|J_1'(t)| \leq \max_{z\in\partial B_1} |h'(z)|$ and so $J_1(t)$ cannot grow faster than linearly.

This reasoning shows (1.38). From (1.37) and (1.38) it follows (1.39); one has only to justify the sign of the limit. Obviously, the limit must be nonnegative. Denote $\partial R = R|\partial B_1$. Then ∂R can be regarded as a function of the angle variable, $\partial R(x) = \mathrm{Re}[h(e^{ix})]$, and one has

$$(\partial R)'\Big(\chi_1(\infty)\Big) = -\,\mathrm{Im}\Big[e^{i\chi_1(\infty)}\,h'\Big(e^{i\chi_1(\infty)}\Big)\Big] \neq 0$$

because $y = \partial R(\chi_1(\infty))$ is a regular value of ∂R. $\square$

As a next step, one has to apply the inverted canonical transformation, from (ψ, J) to (φ, I),

$$\psi = \varphi + \varepsilon\,\frac{\partial S_1(\varphi, J)}{\partial J}\,, \qquad I = J + \varepsilon\,\frac{\partial S_1(\varphi, J)}{\partial \varphi}\,.$$

Using (1.33) one can estimate (recalling that $\lambda = \mu/\nu$)

$$\left|\frac{\partial G_k(\varphi_1, J_1)}{\partial J_1}\right| \leq \frac{\lambda}{2}\,\beta'(J_1) \sum_{\substack{n\in\mathbb{Z}\smallsetminus\{0\}\\ n\neq -k\lambda}} \frac{1}{|n+k\lambda|}\,\beta(J_1)^{|n|-1} \leq \frac{\mu\beta'(J_1)}{1-\beta(J_1)}\,.$$

Hence, using (1.30),

$$\left|\frac{\partial G_k(\varphi_1, J_1)}{\partial J_1}\right| \leq C\,\frac{1-\beta(J_1)}{\beta(J_1)} \tag{1.40}$$

where the constant does not depend on k and φ_1, J_1. To estimate $|\partial G_k/\partial\varphi_1|$ we need the following lemma.

Lemma 1.4: For all β, $0 \leq \beta < 1$, and all $a \in \mathbb{R} \smallsetminus \mathbb{Z}$,

$$\sup_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \frac{\beta^{|n+j|}}{|n-a|} \leq \frac{1}{\mathrm{dist}(a,\mathbb{Z})} + 2 + 6|\log(1-\beta)|, \tag{1.41}$$

and for all $a \in \mathbb{Z}$,

$$\sup_{j\in\mathbb{Z}} \sum_{\substack{n\in\mathbb{Z}\\ n\neq a}} \frac{\beta^{|n+j|}}{|n-a|} \leq 1 + 3|\log(1-\beta)|. \tag{1.42}$$

Proof. Notice that inequality (1.41) is invariant if a is replaced either by $-a$ or by $k+a$, $k \in \mathbb{Z}$. Thus we can restrict ourselves to the interval $0 < a \leq 1/2$. Then $|a| = \mathrm{dist}(a,\mathbb{Z})$ and $|n-a| \geq |n|/2$. This observation reduces (1.41) to (1.42) with $a = 0$. Similarly, inequality (1.42) is invariant if a is replaced by $k+a$, $k \in \mathbb{Z}$. It follows that, in both cases, it suffices to show (1.42) for $a = 0$ and with j being restricted to the range $j \geq 0$.

Splitting the range of summation in n into the subranges $n \leq -j-1$, $-j \leq n \leq -1$ and $1 \leq n$, one gets

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{\beta^{|n+j|}}{|n|} \leq 2|\log(1-\beta)| + \sum_{m=1}^{j} \frac{\beta^{j-m}}{m}.$$

Furthermore,

$$\sum_{1\le m\le j/2} \frac{\beta^{j-m}}{m} \le \sum_{1\le m\le j/2} \frac{\beta^m}{m} \le |\log(1-\beta)|$$

and

$$\sum_{j/2<m\le j} \frac{\beta^{j-m}}{m} \le \sum_{j/2<m\le j} \frac{1}{m} \le 1.$$

This shows the lemma. □

Clearly,

$$\left|\frac{\partial G_k(\varphi_1,J_1)}{\partial\varphi_1}\right| \le \frac{\lambda}{2} \sum_{\substack{n\in\mathbb{Z}\setminus\{0\}\\ n\neq -k\lambda}} \frac{1}{|n+k\lambda|}\,\beta(J_1)^{|n|}\,.$$

Writing $k\lambda = -j-a$, with $j\in\mathbb{Z}$ and $a = s/\nu$, $s = 0,1,\ldots,\nu-1$, one can apply Lemma 1.4 to show that

$$\left|\frac{\partial G_k(\varphi_1,J_1)}{\partial\varphi_1}\right| \le c' + c''|\log(1-\beta(J_1))\,| \tag{1.43}$$

where the constants c', c'' do not depend on k and φ_1, J_1.

Let us discuss the resonant case when $\omega_2/\omega_1 = \mu/\nu$, μ and ν are coprime positive integers and ν is such that

$$\operatorname{supp}\mathscr{F}[f]\cap(\mathbb{Z}\nu\setminus\{0\})\neq\emptyset, \tag{1.44}$$

and this happens if and only if $\langle f(\varphi)\rangle_{\mathbb{Z}\nu}$ is not a constant function. Then $h(z)$ defined in (1.36) obeys the assumptions of Proposition 1.3 and so for almost all initial conditions $(\chi_1(0),J_1(0))$, equalities (1.38) and (1.39) hold. In particular, Theorem 1.3 implies that

$$1-\beta(J_1(t)) = O(t^{-1}) \ \text{ as } t\to+\infty. \tag{1.45}$$

Putting $f_\nu(\varphi) = \langle f(\varphi)\rangle_{\mathbb{Z}\nu}$ one also has

$$\lim_{t\to+\infty}\frac{J_1(t)}{t} = \operatorname{Im}\!\left[e^{i\chi_1(\infty)}\,h'\!\left(e^{i\chi_1(\infty)}\right)\right] = -\frac{\varepsilon\omega_2}{2}\,f_\nu'\!\left(-\frac{\chi_1(\infty)}{\nu}-\frac{\pi\lambda}{2}\right) > 0. \tag{1.46}$$

From definitions (1.32), (1.33) and from assumption (1.23) one can readily see that $S(\varphi,J)$ is C^1 in φ_2 and C^∞ in φ_1, J_1 (and does not depend on J_2). Moreover, from (1.40) and (1.45) it follows that

$$\frac{\partial S_1(\varphi(t),J(t))}{\partial J_1} = O(t^{-1}) \ \text{ as } t\to+\infty.$$

Similarly, estimate (1.43) implies

$$\frac{\partial S_1(\varphi(t),J(t))}{\partial\varphi_1} = O(\log(t)) \ \text{ as } t\to+\infty.$$

Now one can deduce the asymptotic behavior of $\varphi_1(t)$ and $I_1(t)$. Observe that $\psi_2' = \omega_2$ and $S_1(\varphi,J)$ does not depend on J_2, hence $\psi_2(t) = \varphi_2(t) = \omega_2 t$. Furthermore, $\psi_1 = (\chi_1+\nu\psi_2)/\mu$ and so

$$\lim_{t\to+\infty}(\psi_1(t)-\omega_1 t) = \frac{1}{\mu}\,\chi_1(\infty).$$

Putting $\phi(\infty) = \chi_1(\infty)/\mu$ and taking into account (1.46) we arrive at the following conclusion.

Corollary 1.5: Suppose (1.29) is true. In the resonant case (1.44) and for almost all initial conditions $(\varphi_1(0), I_1(0))$,

$$\lim_{t\to+\infty} (\varphi_1(t) - \omega_1 t) = \phi(\infty) \in \mathbb{R}, \quad \lim_{t\to+\infty} \frac{I_1(t)}{t} = C > 0, \tag{1.47}$$

and

$$C = -\frac{\varepsilon\omega_2}{2} f'_\nu\Big(-\Big(\phi(\infty) + \frac{\pi}{2}\Big)\lambda\Big), \text{ with } f_\nu(\varphi) = \langle f(\varphi)\rangle_{\mathbb{Z}\nu}. \tag{1.48}$$

Discussion of the non-resonant case

Discussion of the non-resonant case $\operatorname{supp}\mathscr{F}[f]\cap(\mathbb{Z}\nu\setminus\{0\}) = \emptyset$ is simple if one considers only the first order approximation. One readily deduces that the first-order von Zeipel solution $I_1(t)$ is bounded in the non-resonant case.

Let us finish this Section with a short note concerning the higher order approximations of the Hamiltonian. In general, for $\nu > 1$ it is true that $\mathcal{K}_1(\psi, J) = 0$. Also for any $\nu > 1$ it is possible to compute the function S_1:

$$\begin{aligned} S_1(\varphi, J) = \frac{1}{2}\operatorname{Re}\Bigg[& -2i\arctan\frac{\beta\cos\varphi_1}{1+\beta\sin\varphi_1} \\ & + \frac{\omega_1}{\omega_1-\omega_2} i\beta e^{i\varphi_1} {}_2F_1\left(1, 1-\frac{\omega_2}{\omega_1}, 2-\frac{\omega_2}{\omega_1}, i\beta e^{i\varphi_1}\right) \\ & + \frac{\omega_1}{\omega_1+\omega_2} i\beta e^{-i\varphi_1} {}_2F_1\left(1, 1+\frac{\omega_2}{\omega_1}, 2+\frac{\omega_2}{\omega_1}, -i\beta e^{-i\varphi_1}\right)\Bigg]\exp(-i\varphi_2) \end{aligned}$$

The symbol ${}_2F_1$ denotes the hypergeometric function. If $\nu = 2$, then the second term of the approximative Hamiltonian is quite nontrivial

$$\begin{aligned} \mathcal{K}_2(\varphi, J) = & -\frac{\omega_2\mu(-1)^\mu}{2^4 p_\theta(1+\mu/2)}\beta^{\mu+2}\Big(\frac{1}{\beta}-\beta\Big)^2 {}_2F_1\left(1, 1+\frac{\mu}{2}, 2+\frac{\mu}{2}, \beta^2\right)\cos(\mu\psi_1 - 2\psi_2) \\ & - \frac{\mu\omega_2}{2^5 p_\theta}\Big(\frac{1}{\beta}-\beta\Big)^2\beta^2 \\ & \times\left(\frac{1}{1-\mu/2}\,{}_2F_1\left(1, 1-\frac{\mu}{2}, 2-\frac{\mu}{2}, \beta^2\right) + \frac{1}{1+\mu/2}\,{}_2F_1\left(1, 1+\frac{\mu}{2}, 2+\frac{\mu}{2}, \beta^2\right)\right). \end{aligned}$$

Again, we have one integral of motion. Following the same steps as in the end of the last subsection, but with the matrix

$$R = \begin{pmatrix} \mu & -2 \\ \frac{1-\mu}{2} & 1 \end{pmatrix},$$

it follows that $P_2 = 2J_1 + \mu J_2$ is conserved. The transformed Hamiltonian now reads

$$\mathcal{K}(\chi, P) = \frac{\omega_1}{2}P_2 + \varepsilon^2\mathcal{K}_2(\chi, P) + \mathcal{O}(\varepsilon^3). \tag{1.49}$$

For the level curves in the χ_1, P_1-plane see Figure 1.5. It appears that in this case it is not possible to have $P_1 \to \infty$. Let me just present comparison between the level curves obtained by von Zeipel's method and the numerical solution of the original problem, see Figure 1.6.

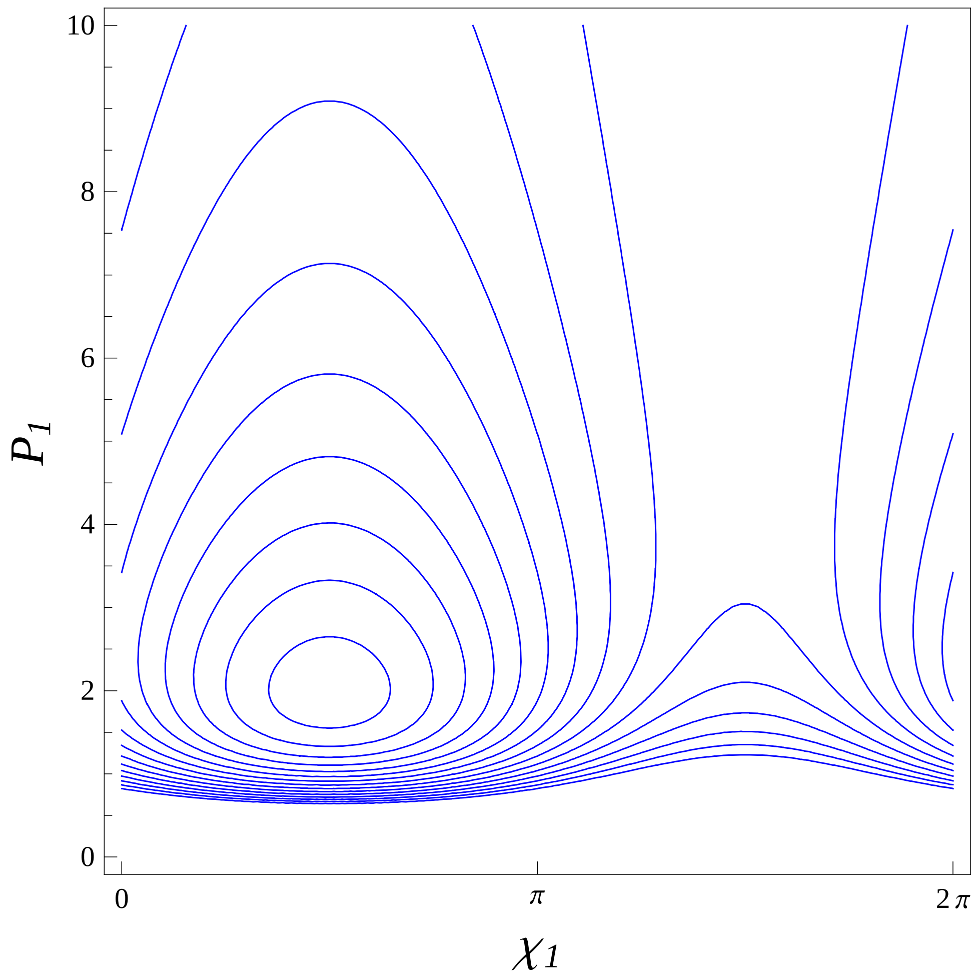

Figure 1.5: Typical level curves of Hamiltonian (1.49).

1.3 Decoupled equations and asymptotic behavior: action-angle variables

Formulation of the problem

Applying the substitution

$$F(t) = 2I(t) + |a(t)|, \;\; \phi(t) = \varphi(t) - bt, \tag{1.50}$$

to the original equations (1.13), (1.14) we obtain the system of differential equations

$$F'(t) = \frac{a(t)\,a'(t)}{F(t) + \sqrt{F(t)^2 - a(t)^2}\,\sin(bt + \phi(t))}, \tag{1.51}$$

$$\phi'(t) = -\frac{a(t)\,a'(t)\cos(bt + \phi(t))}{\sqrt{F(t)^2 - a(t)^2}\left(F(t) + \sqrt{F(t)^2 - a(t)^2}\,\sin(bt + \phi(t))\right)}, \tag{1.52}$$

where $a(t) = p_\theta - \varepsilon f(\Omega t)$ (see (1.5) and (1.15)). The real function $f(t)$ is supposed to be continuously differentiable and 2π-periodic. Originally, $\varepsilon > 0$ was a small parameter

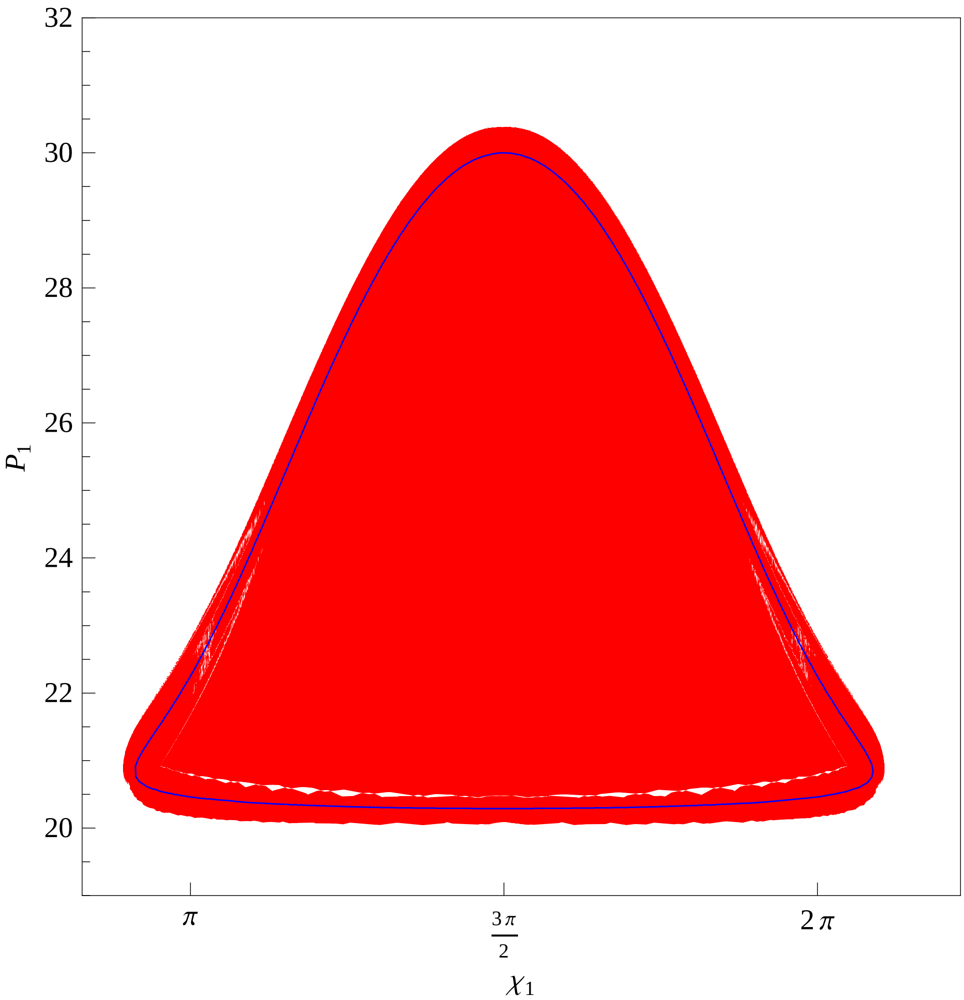

Figure 1.6: Comparison of the orbits of the second order approximation (blue) and the numerical solution of the original system (red) for $q = 2$. Values of various constants are $p_\theta = \Omega = 1$, $b = q = 2$, $\varepsilon = 1/2$, $I_1(0) = 30$ and $\varphi_1(0) = \pi/2$.

but this assumption is not used any more in the current Section at all. The only thing we assume is that ε is small enough so that the function $a(t)$ has no zeroes and hence it is everywhere of the same sign as p_θ. Recall that for definiteness p_θ is supposed to be positive. Clearly, the functions $a(t)$ and $a'(t)$ are bounded on $\mathbb{R}$.

Equations (1.51) and (1.52) are nonlinear and coupled together. To decouple them we replace $\phi(t)$ on the RHS of (1.51) and $F(t)$ on the RHS of (1.52) by the respective leading asymptotic terms, as learned from the averaging method (see Corollary 1.5). This is done under the assumption that the solution has already reached the domain $F(t) \geq F_0 \gg p_\theta$ where $F(t)$ is sufficiently large and starts to grow. Let us point out an essential difference between equations (1.51) and (1.52). Note that for all $F, a, s \in \mathbb{R}$ such that $F \geq |a| > 0$ one has

$$\left| \frac{a\cos(s)}{F + \sqrt{F^2 - a^2}\sin(s)} \right| \leq 1,$$

and, consequently, it follows from (1.52) that

$$|\phi'(t)| \leq \frac{|a'(t)|}{\sqrt{F(t)^2 - a(t)^2}}\,. \tag{1.53}$$

This means that the RHS of (1.52) is inversely proportional to $F(t)$. Anything similar cannot be claimed, however, for equation (1.51).

Thus, to formulate a problem with decoupled equations, we replace $\phi(t)$ in (1.51) by the expected limiting value $\phi \equiv \phi(\infty) \in \mathbb{R}$, i.e. the simplified equation reads

$$F'(t) = \frac{a(t)\,a'(t)}{F(t) + \sqrt{F(t)^2 - a(t)^2}\,\sin(bt + \phi)}\,. \tag{1.54}$$

We first analyze this equation in next Subsection. Equation (1.52) is then analyzed in second Subsection under the assumption that $F(t)$ grows linearly. In that case $\phi(t)$ is actually shown to approach a constant value as t tends to infinity.

A resonance effect for Equation (1.54)

Analyzing equation (1.54) we prefer to work with a rescaled time (or one can choose the units so that $b = 1$) and, simplifying the notation, we consider a differential equation of the form

$$g'(t) = \frac{\varrho(t)}{g(t) + \sqrt{g(t)^2 - a(t)^2}\,\sin(t + \phi)} \tag{1.55}$$

where $\varrho(t)$, $a(t)$ are continuously differentiable real functions, $a(t)$ is strictly positive and ϕ is a real constant. In the resonant case the functions $\varrho(t)$, $a(t)$ are supposed to be 2π-periodic which means for the original data that $\Omega \in \mathbb{N}$.

In the first step we estimate the growth of a solution on an interval of length $\pi/2$. Let $\|f\| = \max |f(t)|$ denote the norm in $C([0, \pi/2])$, and put

$$A = \min_{0 \leq t \leq \pi/2} a(t) > 0. \tag{1.56}$$

Consider for a moment the differential equation

$$h'(t) = \frac{\varrho(t)}{h(t) - \sqrt{h(t)^2 - a(t)^2}\,\cos(t)} \tag{1.57}$$

on the interval $[0, \pi/2]$ with an initial condition $h(0) = h_0 > \|a\|$. The goal is to show that for large values of h_0, an essential contribution to the growth of a solution $h(t)$ on this interval comes from a narrow neighborhood of the point $t = 0$.

Remark 1.6: If $\varrho(t)$ is nonnegative on the interval $[0, \pi/2]$ then a solution $h(t)$ to (1.57) surely exists and is unique. In the general case the existence and uniqueness is guaranteed provided the initial condition h_0 is sufficiently large. From (1.57) one derives that $|h'(t)| \leq 2\|\varrho\| h(t)/A^2$ and so

$$\exp\!\left(-2\|\varrho\|t/A^2\right) h_0 \leq h(t) \leq \exp\!\left(2\|\varrho\|t/A^2\right) h_0 \tag{1.58}$$

as long as $h(t)$ makes sense. Consequently, a sufficient condition for existence of a solution is $h_0 > \exp(\pi\|\varrho\|/A^2)$.

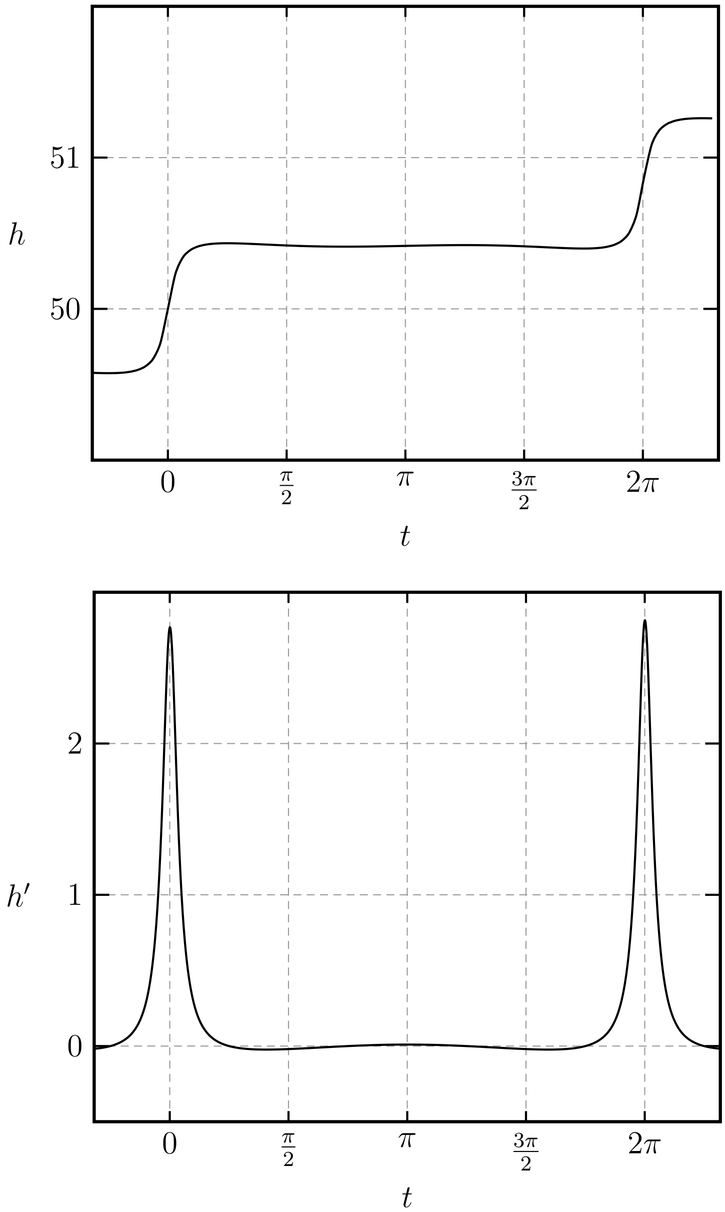

Figure 1.7: Illustration of the kicked behaviour of solutions of equation (1.57). Our particular choice of parameters is $h(0) = 50$, $a(t) = 5 + \cos t + \frac{1}{3}\sin(2t) - \frac{1}{2}\sin(3t)$, and $\rho(t) = \cos(2t)$.

Lemma 1.7: Let $\varrho, a \in C^1([0, \pi/2])$ be real functions, $\varrho(0) \neq 0$ and $a(t) > 0$ on $[0, \pi/2]$. Consider the set of solutions $h(t)$ to the differential equation (1.57) on the interval $[0, \pi/2]$ with a variable initial condition $h(0) = h_0$ for h_0 sufficiently large. Then

$$h\Big(\frac{\pi}{2}\Big) = h_0 + \frac{\pi \varrho(0)}{a(0)} + O\Big(h_0^{-1}\log(h_0)\Big) \quad \text{as } h_0 \to +\infty.$$

Proof. Let us fix η, $0 < \eta \leq \pi/2$, so that $|\varrho(t)| > 0$ on the interval $[0, \eta\,[$, i.e. $\varrho(t)$ does not change its sign on that interval. Thus any solution $h(t)$ to (1.57) is strictly monotone on $[0, \eta]$. For $\eta \leq t \leq \pi/2$ one can estimate $|h'(t)| \leq C/h(t)$ where $C = \|\varrho\|/(1-\cos(\eta))$.

In view of (1.58) it follows that

$$h(\pi/2) - h(\eta) = O\left(h_0^{-1}\right) \text{ as } h_0 \to +\infty. \tag{1.59}$$

Set

$$h_1 = \min\{h(0), h(\eta)\},\ h_2 = \max\{h(0), h(\eta)\},\ \Delta = h(\eta) - h_0.$$

Then $|\Delta| = h_2 - h_1$. Set, for $x \geq a > 0$,

$$\Psi(x, a, t) = \frac{1}{x - \sqrt{x^2 - a^2}\,\cos(t)}\,.$$

One has $h'(t) = \varrho(t)\,\Psi(h(t), a(t), t)$. If $x \geq 2u/\sqrt{3} > 0$ then $\sqrt{x^2 - u^2} \geq x/2$ and

$$\left|\frac{\partial}{\partial x}\Psi(x, u, t)\right| = \frac{\left|\sqrt{x^2 - u^2} - x\cos(t)\right|}{\sqrt{x^2 - u^2}\left(x - \sqrt{x^2 - u^2}\cos(t)\right)^2} \leq \frac{2}{x}\,\Psi(x, u, t), \tag{1.60}$$

$$\left|\frac{\partial}{\partial u}\Psi(x, u, t)\right| = \frac{u\cos(t)}{\sqrt{x^2 - u^2}\left(x - \sqrt{x^2 - u^2}\cos(t)\right)^2} \leq \frac{3}{u}\,\Psi(x, u, t). \tag{1.61}$$

Observe also that, for $x \geq u > 0$,

$$\int_0^{\pi/2} \Psi(x, u, t)\,\mathrm{d}t = \frac{2}{u}\,\arctan\!\left(\frac{x + \sqrt{x^2 - u^2}}{u}\right) \leq \frac{\pi}{u}\,. \tag{1.62}$$

Assuming that h_0 is sufficiently large and using (1.60), (1.62) one can estimate

$$\begin{aligned}
\left|\Delta - \int_0^{\eta} \varrho(t)\Psi(h_0, a(t), t)\,\mathrm{d}t\right| &\leq \int_0^{\eta} |\varrho(t)|\,|\Psi(h(t), a(t), t) - \Psi(h_0, a(t), t)|\,\mathrm{d}t \\
&\leq \frac{2\|\varrho\|}{h_1}\int_0^{\pi/2}\left(\int_{h_1}^{h_2} \Psi(x, A, t)\,\mathrm{d}x\right)\mathrm{d}t \\
&\leq \frac{2\pi\|\varrho\|}{Ah_1}\,|\Delta|.
\end{aligned}$$

In view of (1.58) it follows that

$$\Delta = \left(1 + O(h_0^{-1})\right)\int_0^{\eta} \varrho(t)\Psi(h_0, a(t), t)\,\mathrm{d}t.$$

Furthermore, with the aid of (1.61) one finds that

$$|\varrho(t)\Psi(h_0, a(t), t) - \varrho(0)\Psi(h_0, a(0), t)| \leq C'\Psi(h_0, A, t)t$$

where

$$C' = \left(1 + \frac{9\|\varrho\|^2}{A^2}\right)^{1/2}\sqrt{\|\varrho'\|^2 + \|a'\|^2}\,.$$

Note that

$$\int_0^{\eta} \Psi(h_0, A, t)\,t\,\mathrm{d}t \leq \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin(t)}{h_0 - \sqrt{h_0^2 - A^2}\cos(t)}\,\mathrm{d}t = O\!\left(h_0^{-1}\log(h_0)\right)$$

and

$$\begin{aligned}\int_0^{\eta} \varrho(0)\Psi(h_0, a(0), t)\,\mathrm{d}t &= \varrho(0)\int_0^{\pi/2} \Psi(h_0, a(0), t)\,\mathrm{d}t + O\Big(h_0^{-1}\Big)\\ &= \frac{2\varrho(0)}{a(0)}\arctan\left(\frac{h_0+\sqrt{h_0^2-a(0)^2}}{a(0)}\right) + O\Big(h_0^{-1}\Big)\\ &= \frac{\pi\varrho(0)}{a(0)} + O\Big(h_0^{-1}\Big)\,.\end{aligned}$$

Altogether this means that

$$h(\eta) - h_0 = \Delta = \frac{\pi\varrho(0)}{a(0)} + O\Big(h_0^{-1}\log(h_0)\Big)\,.$$

Recalling (1.59), the lemma follows. □

Consider the mapping $H : h(0) \mapsto h(\pi/2)$, where $h(t)$ runs over solutions to the differential equation (1.57). From the general theory of ordinary differential equations it is known that H is a C^1 mapping well defined on a neighborhood of $+\infty$. Lemma 1.7 claims that $H(x) = x + \pi\varrho(0)/a(0) + O(x^{-1}\log(x))$. On the basis of similar arguments, the inverse mapping $H^{-1} : h(\pi/2) \mapsto h(0)$ is also well defined and C^1 on a neighborhood of $+\infty$. From the asymptotic behavior of $H(x)$ one readily derives that $H^{-1}(y) = y - \pi\varrho(0)/a(0) + O(y^{-1}\log(y))$. These considerations make it possible to reverse the roles of the boundary points 0 and $\pi/2$. Moreover, splitting the interval $[0, 2\pi]$ into four subintervals of length $\pi/2$ one arrives at the following lemma.

Lemma 1.8: Let $\varrho, a \in C^1([0, 2\pi])$ be real functions, $\varrho(\pi) \neq 0$ and $a(t) > 0$ on $[0, 2\pi]$. Consider the set of solutions $h(t)$ to the differential equation

$$h'(t) = \frac{\varrho(t)}{h(t) + \sqrt{h(t)^2 - a(t)^2}\,\cos(t)}$$

on the interval $[0, 2\pi]$ with a variable initial condition $h(0) = h_0$ for h_0 sufficiently large. Then

$$h(2\pi) = h_0 + \frac{2\pi\varrho(\pi)}{a(\pi)} + O\Big(h_0^{-1}\log(h_0)\Big) \quad \text{as } h_0 \to +\infty.$$

In the next step, applying repeatedly Lemma 1.8 one can show that solutions of the simplified differential equation in the resonant case $\Omega \in \mathbb{N}$ (with $b = 1$) are unbounded and grow with time at least linearly provided the initial condition is sufficiently large and the phase ϕ belongs to a certain interval.

Proposition 1.9: Suppose $\varrho(t)$, $a(t)$ are continuously differentiable 2π-periodic real functions, $\phi \in \mathbb{R}$, $a(t)$ is everywhere positive and

$$\varrho\Big(-\phi - \frac{\pi}{2}\Big) > 0. \tag{1.63}$$

Let $g(t)$ be a solution of the differential equation (1.55) on the interval $t \geq 0$ with the initial condition $g(0) = g_0 \geq 1$. If g_0 is sufficiently large then

$$g(t) = \frac{\varrho\big(-\phi - \frac{\pi}{2}\big)}{a\big(-\phi - \frac{\pi}{2}\big)}\,t + O\Big(\log(t)^2\Big) \quad \text{as } t \to +\infty.$$

To return back to the original notation and equation (1.54) one can apply the substitution $F(t) = g(bt)$, $a(t) = \tilde{a}(bt)$, $\varrho(t) = \tilde{a}(t)\tilde{a}'(t)$. Equation (1.54) transforms into (1.55) (with $a(t)$ being replaced by $\tilde{a}(t)$), and Proposition 1.9 is directly applicable.

Corollary 1.10: Suppose $f \in C^2(\mathbb{R})$ is 2π-periodic and

$$\frac{\Omega}{b} \in \mathbb{N},\ f'\left(-\frac{\Omega}{b}\left(\phi + \frac{\pi}{2}\right)\right) < 0. \tag{1.64}$$

Then any solution of (1.54) such that $F(0)$ is sufficiently large fulfills

$$F(t) = \varepsilon\,\Omega\left|f'\left(-\frac{\Omega}{b}\left(\phi + \frac{\pi}{2}\right)\right)\right| t + O\Big(\log(t)^2\Big) \quad \text{as } t \to +\infty.$$

To conclude let us emphasize that the replacement of the phase $\phi(t)$ by a constant ϕ was quite crucial for the estimates. In fact, suppose $F(t)$ is sufficiently large. Then in the case of the original equation (1.51), too, essential contributions to the growth of $F(t)$ are achieved at the moments of time when $\sin(bt + \phi(t)) = -1$. If $\phi(t)$ equals a constant then these moments of time are well defined and the growth of $F(t)$ can be estimated. On the contrary, without a sufficiently precise information about $\phi(t)$ one loses any control on the growth of $F(t)$.

An analysis of equation (1.52)

On the contrary, here we wish to verify that if $F(t)$ grows linearly, possibly with a logarithmic correction, then any solution $\phi(t)$ to (1.52) approaches sufficiently rapidly a constant value as t tends to infinity. At this moment, the periodicity of the functions $a(t)$ is not important. It suffices if one knows that it takes values from a bounded interval separated from zero.

Proposition 1.11: Suppose $a(t) \in C^1(\mathbb{R})$ fulfills

$$0 < A_1 \le a(t) \le A_2,\ |a'(t)| \le A_3,$$

for some positive constants A_1, A_2, A_3. Furthermore, suppose $F(t) \in C(\mathbb{R})$ has the asymptotic behavior

$$F(t) = \alpha t + O\Big(\log(t)^2\Big) \quad \text{as } t \to +\infty, \tag{1.65}$$

with a positive constant α. Under these assumptions, if $\phi(t)$ obeys the differential equation (1.52) on a neighborhood of $+\infty$ then there exists a finite limit $\lim_{t\to+\infty}\phi(t) = \phi(\infty)$ and

$$\phi(t) = \phi(\infty) + O\left(\frac{\log(t)}{t}\right) \quad \text{as } t \to +\infty. \tag{1.66}$$

Proof. Put $\zeta(t) = bt - \phi(t)$. Recall that $\phi'(t)$ obeys the estimate (1.53). Choose $t_* \in \mathbb{R}$ such that

$$|\phi'(t)| \le \frac{b}{2}, \quad \forall t \ge t_*,$$

hence the function $\zeta(t)$ is strictly increasing and $b/2 \le \zeta'(t) \le 3b/2$. Moreover, we choose t_* sufficiently large so that

$$F(t+s) \le \sqrt{2}\, F(t) \text{ for } 0 \le s \le 3\pi b, \text{ and } F(t) \ge \sqrt{2}\, A_2, \quad \forall t \ge t_*. \tag{1.67}$$

Fix $\ell \in \mathbb{N}$ such that $\zeta(2\pi(\ell+1)) \ge t_*$. Put $\tau_k = \zeta(2\pi(\ell+k))$, $k \in \mathbb{N}$. Note that $\pi b \le \tau_{k+1} - \tau_k \le 3\pi b$. For a given $k \in \mathbb{N}$ put

$$F_1 = \min_{t \in [\tau_k, \tau_{k+1}]} F(t), \ F_2 = \max_{t \in [\tau_k, \tau_{k+1}]} F(t).$$

One has

$$\int_{\tau_k}^{\tau_{k+1}} |\phi'(t)|\, \mathrm{d}t \le$$

$$\le \frac{2}{b} \int_{\zeta(2\pi(\ell+k))}^{\zeta(2\pi(\ell+k+1))} |\phi'(t)|\zeta'(t)\, \mathrm{d}t \le \frac{2A_2A_3}{b\sqrt{F_1^2 - A_2^2}} \int_{2\pi(\ell+k)}^{2\pi(\ell+k+1)} \frac{|\cos s|}{\tilde{F}(s) + \sqrt{\tilde{F}(s)^2 - \tilde{a}(s)^2}\, \sin s}\, \mathrm{d}s$$

where $\tilde{F} = F \circ \zeta^{-1}$ and $\tilde{a} = a \circ \zeta^{-1}$. Put $M_+ = 2\pi(\ell+k) + [\,0, \pi\,]$ and $M_- = 2\pi(\ell+k) + [\,\pi, 2\pi\,]$. One has

$$\int_{M_+} \frac{|\cos s|}{\tilde{F}(s) + \sqrt{\tilde{F}(s)^2 - \tilde{a}(s)^2}\, \sin s}\, \mathrm{d}s \le 2 \int_0^{\pi/2} \frac{\cos s}{F_1 + \sqrt{F_1^2 - A_2^2}\, \sin s}\, \mathrm{d}s \le \frac{2\log 2}{\sqrt{F_1^2 - A_2^2}}.$$

For $s \in M_-$ one can estimate

$$\frac{1}{\tilde{F}(s) + \sqrt{\tilde{F}(s)^2 - \tilde{a}(s)^2}\, \sin s} \le \frac{F_2 + \sqrt{F_2^2 - A_1^2}\, |\sin s|}{F_1^2 \cos^2 s + A_1^2 \sin^2 s} < \frac{2}{F_1 + \sqrt{F_1^2 - A_1^2}\, \sin s}$$

where we have used that

$$\frac{F_2}{\sqrt{F_2^2 - A_1^2}} \le \frac{F_1}{\sqrt{F_1^2 - A_1^2}}, \quad F_2^2 - A_1^2 \le 2F_1^2 - A_1^2 \le 4F_1^2 - 5A_1^2 < 4(F_1^2 - A_1^2),$$

as it follows from (1.67). Thus one arrives at the estimates

$$\begin{aligned} \int_{M_-} \frac{|\cos s|}{\tilde{F}(s) + \sqrt{\tilde{F}(s)^2 - \tilde{a}(s)^2}\, \sin s}\, \mathrm{d}s \ &\le \ 4 \int_0^{\pi/2} \frac{\cos s}{F_1 - \sqrt{F_1^2 - A_1^2}\, \sin s}\, \mathrm{d}s \\ &\le \ \frac{4}{\sqrt{F_1^2 - A_1^2}} \log\left(\frac{2F_1^2}{A_1^2}\right) \end{aligned}$$

and

$$\int_{\tau_k}^{\tau_{k+1}} |\phi'(t)|\, \mathrm{d}t \le \frac{32A_2A_3}{bF_1^2} \log\left(\frac{2F_1}{A_1}\right).$$

Referring to the asymptotic behavior (1.65) one concludes that there exists a constant $C_* > 0$ such that

$$\int_{\tau_1}^{\infty} |\phi'(t)|\, \mathrm{d}t = \sum_{k=1}^{\infty} \int_{\tau_k}^{\tau_{k+1}} |\phi'(t)|\, \mathrm{d}t \le C_* \sum_{j=\ell+1}^{\infty} \frac{\log(j)}{j^2} < \infty.$$

Hence the limit $\lim_{t \to +\infty} \phi(t) = \phi(\infty) \in \mathbb{R}$ does exist and (1.66) follows. □

Asymptotic behavior of the original dynamical system

Corollary 1.10 and Proposition 1.11 can also be interpreted in the following way. Let us pass from the differential equations (1.51), (1.52) to the integral equations

$$F(t) - F(0) - \int_0^t \frac{a(s)\, a'(s)}{F(s) + \sqrt{F(s)^2 - a(s)^2}\, \sin(bs + \phi(s))}\, \mathrm{d}s = 0,$$

$$\phi(t) - \phi(\infty) - \int_t^\infty \frac{a(s)\, a'(s) \cos(bs + \phi(s))}{\sqrt{F(s)^2 - a(s)^2} \left(F(s) + \sqrt{F(s)^2 - a(s)^2} \sin(bs + \phi(s))\right)}\, \mathrm{d}s = 0.$$

Suppose $\phi(\infty)$ satisfies $\alpha = -\varepsilon\, \Omega f'(-(\Omega/b)(\phi(\infty) + \pi/2)) > 0$. If $F(0)$ is sufficiently large then the functions

$$F(t) = \alpha t + F(0),\ \phi(t) = \phi(\infty),\ t > 0,$$

can be regraded as an approximate solution of this system of integral equations with errors of order $O(\log(t)^2)$ for the first equation and of order $O(\log(t)/t)$ for the second one.

One has to admit, however, that this argument still does not represent a complete mathematical proof of the asymptotic behavior of the action-angle variables $I(t) = (F(t) - |a(t)|)/2$, $\varphi(t) = bt + \phi(t)$. Moreover, it should be emphasized that these relations were derived under the essential assumption that the dynamical system had already reached the regime characterized by an acceleration with an unlimited energy growth (this is reflected by the assumption that $F(0)$ is sufficiently large). Nevertheless, on the basis of the above analysis as well as on the basis of numerous numerical experiments we formulate the following conjecture.

Conjecture 1.12: If $\Omega/b \in \mathbb{N}$ then the regime of acceleration for the original (true) dynamical system is described by the asymptotic behavior

$$I(t) = Ct + O\big(\log(t)^2\big),\ \ \varphi(t) = bt + \phi(\infty) + O\left(\frac{\log(t)}{t}\right) \quad \textit{as } t \to +\infty, \tag{1.68}$$

where $\phi(\infty)$ is a real constant and

$$C = -\frac{1}{2}\, \varepsilon \Omega f'(-\xi) > 0,\ \textit{with}\ \ \xi = \frac{\Omega}{b} \left(\phi(\infty) + \frac{\pi}{2}\right). \tag{1.69}$$

1.4 Asymptotic behavior: guiding center coordinates

Given the asymptotic relations (1.68), (1.69) it is desirable to describe the accelerated motion in terms of the original Cartesian coordinates q. The description becomes more transparent if the motion is decomposed into a motion of the guiding center X and a relative motion of the particle with respect to this center which is characterized by a gyroradius vector R and a gyrophase ϑ [21]. Let $v = p - A$ be the velocity. Thus we write $q = X + R$ where, by definition,

$$X = q + \frac{1}{b}\, v^{\perp},\ \ R = q - X = -\frac{1}{b}\, v^{\perp}. \tag{1.70}$$

We use the polar decompositions

$$q = r\,(\cos\theta, \sin\theta),\;\; X = |X|(\cos\chi, \sin\chi),\;\; R = |R|(\cos\vartheta, \sin\vartheta). \tag{1.71}$$

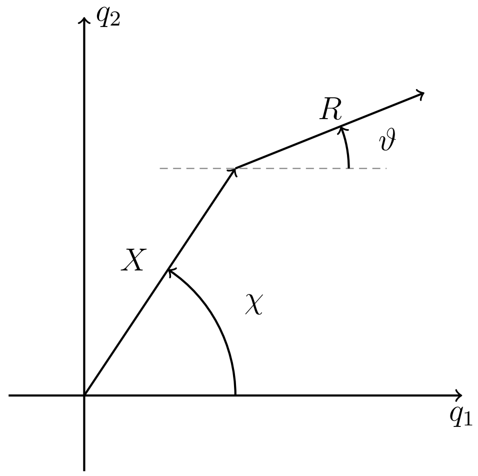

Figure 1.8: The guiding center coordinates.

The quantities X, R were introduced (under different names) and studied in [6] where one can also find several formulas given below, notably those given in (1.72). Note that $p \cdot q^{\perp} = p_\theta$, $|p|^2 = p_r^2 + p_\theta^2/r^2$. A direct computation yields

$$|R|^2 - |X|^2 = \frac{2a}{b}\,,$$
$$|v|^2 = p_r^2 + \frac{b^2 r^2}{4} + \frac{a^2}{r^2} + ba.$$

Using (1.10), (1.11) one derives the equalities

$$|X|^2 = \frac{1}{b}\,(2I + |a| - a),\;\; |R|^2 = \frac{1}{b}\,(2I + |a| + a) \tag{1.72}$$

and

$$r = \Big(|X|^2 + |R|^2 + 2|X||R|\sin(\varphi)\Big)^{1/2},\;\; p_r = \frac{b}{r}\,|X||R|\cos(\varphi). \tag{1.73}$$

On the other hand, one has

$$r^2 = |X|^2 + |R|^2 + 2X\cdot R = \frac{2}{b}\left(2I + |a| + 2\sqrt{I(I+|a|)}\,\cos(\vartheta - \chi)\right). \tag{1.74}$$

By comparison of (1.74) with (1.73) one finds that

$$\vartheta = \varphi + \chi - \frac{\pi}{2}\;\;(\mathrm{mod}\;2\pi). \tag{1.75}$$

Observe from (1.72) that if $a(t)$ is an everywhere positive function then $|R(t)| > |X(t)|$ and so the center of coordinates always stays in the domain encircled by the spiral-like trajectory. On the contrary, if $a(t)$ is an everywhere negative function then the center of coordinates is never encircled by the trajectory. This can be nicely seen in Figures 1.1 and 1.2.

From (1.72) and (1.68) one deduces the asymptotic behavior

$$|X| = \sqrt{\frac{2Ct}{b}} + O\left(\frac{\log(t)^2}{\sqrt{t}}\right),$$
$$|R| = \sqrt{\frac{2Ct}{b}} + O\left(\frac{\log(t)^2}{\sqrt{t}}\right) \quad \text{as } t \to +\infty. \tag{1.76}$$

We still need some information about the asymptotic behavior of the phase $\chi(t)$. To this end, let us compute the derivative $\chi'(t)$. This can be done by differentiating the equality

$$r\,(\cos\theta, \sin\theta) = |X|(\cos\chi, \sin\chi) + |R|(\cos\vartheta, \sin\vartheta)$$

and then taking the scalar product with the vector $(-\sin\theta, \cos\theta)$. One has

$$\theta' = \frac{\partial H}{\partial p_\theta} = \frac{a}{r^2} + \frac{b}{2}$$

where H is the Hamiltonian (1.4) expressed in polar coordinates. Furthermore, in view of (1.75), $\vartheta' = \chi' + \varphi'$. After some straightforward manipulations one finally arrives at the differential equation

$$\chi' = \frac{|R|a'\cos\varphi}{|X|br^2}\,. \tag{1.77}$$

Equation (1.77) admits an asymptotic analysis with the aid of similar methods as those used in Section 1.3. In order to spare some space we omit the details. We still assume that $\Omega/b \in \mathbb{N}$. Recalling (1.68), (1.69) one observes that the main contribution to the growth of $\chi(t)$ over a period $T = 2\pi/b$ equals

$$\chi((n+1)T) - \chi(nT) \quad \sim \quad \frac{1}{4CnT} \lim_{\alpha\to 0} \int_0^T \frac{a'(t)\cos(bt + \phi(\infty))}{1 + \sqrt{1-\alpha^2}\,\sin(bt + \phi(\infty))}\,\mathrm{d}t.$$

Proceeding this way one finally concludes that

$$\chi(t) = D\log(t) + \chi(\infty) + o(1) \quad \text{as } t \to +\infty \tag{1.78}$$

where $\chi(\infty)$ is a real constant and

$$D = \frac{1}{4\pi f'(-\xi)} \int_0^\pi \left(f'\left(\frac{\Omega}{b}t - \xi\right) - f'\left(-\frac{\Omega}{b}t - \xi\right)\right) \frac{\sin(t)}{1-\cos(t)}\,\mathrm{d}t.$$

Given the Fourier series $f'(t) = \sum_{k=1}^\infty (a_k\cos(kt) + b_k\sin(kt))$ one can also express

$$D = \frac{1}{2}\sum_{k=1}^\infty (a_k\sin(k\xi) + b_k\cos(k\xi)) \Big/ \sum_{k=1}^\infty (a_k\cos(k\xi) - b_k\sin(k\xi))\,.$$

Thus (1.68), (1.78) and (1.75) imply that

$$\vartheta(t) = bt + D\log(t) + \phi(\infty) + \chi(\infty) - \frac{\pi}{2} + o(1) \quad \text{as } t \to +\infty. \tag{1.79}$$

Relations (1.76), (1.78) and (1.79) give a complete information about the asymptotic behavior of the trajectory $q(t) = X(t) + R(t)$. The length of the guiding center position vector and that of the gyroradius vector are almost equal and grow with the square root of t, and $|X(t)|^2 - |R(t)|^2 = -2a(t)/b$. The gyrophase $\vartheta(t)$ grows linearly with the frequency b while the growth of the phase of the guiding center, $\chi(t)$, is only logarithmic and so comparatively very slow.

Finally let us state a formula for the acceleration rate using now all physical constants (including m and e). One has

$$\gamma_{\rm acc} = -\frac{e}{2\pi} \lim_{T\to\infty} \frac{1}{T} \int_0^T \Theta'(t)\Phi'(t)dt, \text{ with } \Theta'(t) = \frac{\omega_c |a|}{2\Big(2I + |a| + 2\sqrt{I(I+|a|)} \sin\varphi\Big)}.$$

Applying Conjecture 1.12 we find (to lowest order in the flux amplitude) the positive acceleration rate

$$\gamma_{\rm acc} = -\frac{e\omega_c}{4\pi} \Phi'\Big(-\frac{1}{\omega_c}\Big(\varphi(\infty) + \frac{\pi}{2}\Big)\Big) > 0. \tag{1.80}$$

Moreover, by equation (1.76) one has for the guiding center $|X(t)|^2 \sim 2\gamma_{\rm acc} t/(m\omega_c^2)$.

Chapter 2

Quantum Mechanics

2.1 Introduction

Now we consider a quantum point particle of mass m and charge e moving on the plane in the presence of a homogeneous magnetic field of magnitude b; here all constants m, e, b are supposed to be positive. Assume further that the particle is driven by an Aharonov-Bohm magnetic flux concentrated along a line intersecting the plane in the origin and whose strength $\Phi(t)$ is oscillating with frequency Ω. Let us study this model in the framework of the non-relativistic quantum mechanics.

In the time-independent case, the Hamiltonian corresponding to a homogeneous magnetic field and a constant Aharonov-Bohm flux of magnitude Φ_0 has the form

$$\frac{\hbar^2}{2m}\left(-\frac{1}{r}\,\partial_r r\partial_r+\frac{1}{r^2}\left(-i\partial_\theta-\frac{e\Phi_0}{2\pi\hbar c}+\frac{eBr^2}{2\hbar c}\right)^2\right)$$

where (r,θ) are polar coordinates on the plane. So the Hilbert space in question is $L^2(\mathbb{R}_+\times\mathbb{R},r\,\mathrm{d}r\,\mathrm{d}\theta)$.

It is convenient to set $\hbar=m=e=1$.

Making use of the rotational symmetry of the model we restrict ourselves to a fixed eigenspace of the angular momentum $J_3=-i\partial_\theta$ with an eigenvalue j_3, $j_3\in\mathbb{Z}$. Put $p:=j_3-\Phi_0/(2\pi)$. Then this restriction leads to the radial Hamiltonian

$$H(p)=\frac{1}{2}\left(-\frac{1}{r}\,\partial_r r\partial_r+\frac{1}{r^2}\left(p+\frac{br^2}{2}\right)^2\right)$$

in the Hilbert space $\mathscr{H}=L^2(\mathbb{R}_+,r\,\mathrm{d}r)$. Without loss of generality, we can assume that $p>0$ (note that $H(-p)-H(p)$ is a constant). The boundary conditions at the origin are chosen to be the regular ones (then $H(p)$ is the so called Friedrichs self-adjoint extension of the symmetric operator defined on smooth functions with compact support, see Appendix B). If $0<p<1$, then more general boundary conditions are admissible [9], but here we confine ourselves to the above standard choice.

Note that the cyclotron frequency ω_c is equal to b. The operator $H(p)$ has a simple discrete spectrum, the eigenvalues are

$$E_n(p)=b(n+p+1/2),\quad n=0,1,2,\dots, \tag{2.1}$$

with the corresponding normalized eigenfunctions

$$\psi_n(p;r)=c_n(p)\,r^p\,L_n^{(p)}\left(\frac{br^2}{2}\right)\exp\left(-\frac{br^2}{4}\right),\quad n=0,1,2,\dots,$$

where

$$c_n(p)=\left(\frac{b}{2}\right)^{(p+1)/2}\left(\frac{2\,n!}{\Gamma(n+p+1)}\right)^{1/2},\quad n=0,1,2,\dots,$$

are the normalization constants and $L_n^{(p)}$ are the generalized Laguerre polynomials. The set $\{\psi_n(p)\}_{n=0}^\infty$ forms an orthonormal basis of $L^2(\mathbb{R},r\,\mathrm{d}r)$. This information determines the operator $H(p)$ unambiguously.

Next we consider the periodically time-dependent Hamiltonian $H(a(t))$ where $a(t) = p+\varepsilon f(\Omega t)$, $f(t)$ is a 2π-periodic continuously differentiable function, $\Omega > 0$ is a frequency and ε is a small parameter. Thus the Aharonov-Bohm flux depends on time as

$$\Phi(t) = \Phi_0 - 2\pi\varepsilon\, f(\Omega t). \tag{2.2}$$

Without loss of generality one can assume that $\int_0^{2\pi} f(t)\mathrm{d}t = 0$. As discussed in [4], the domain of $H(a(t))$ in principle depends on t which makes the discussion from the mathematical point of view more delicate, particularly in the case $0 < a(t) < 1$. But here we ignore these mathematical subtleties.

This model has already been studied in the framework of classical mechanics in [5], and Chapter 1 of this thesis. It turns out that in the resonant case, when Ω is an integer multiple of ω_c, the classical trajectory eventually reaches an asymptotic domain where it resembles a spiral whose circles expand, as t approaches infinity, with the rate $t^{1/2}$. As the particle moves along the circles of the spiral-like orbit, approximately with frequency ω_c, the extremal distances to the origin converge to zero and to infinity, respectively. At the same time, the energy of the particle grows linearly with time. If $\mathcal{E}(t)$ is the energy of the particle depending on time, then the acceleration rate, as computed in Chapter 1 is given by the formula

$$\gamma_{\mathrm{acc}} := \lim_{t\to\infty} \frac{\mathcal{E}(t)}{t} = \frac{\omega_c}{4\pi}\,|\Phi'(\tau)|, \tag{2.3}$$

where τ is a real number depending on asymptotic parameters of the trajectory.

The purpose of this Chapter is to show that in the framework of quantum mechanics one can derive a formula analogous to (2.3). To this end and because of complexity of the problem, we restrict ourselves to the case when $f(t)$ is a sinusoidal function. Moreover, we study in fact an approximate time evolution which we derive with the aid of the quantum averaging method.

2.2 The Floquet operator and the quasienergy

Let $U(t, t_0)$ be the propagator (evolution operator) associated with $H(a(t))$. Without going into details, we take its existence and natural properties for granted [4]. An important characteristic of the dynamical properties of the system is the time evolution over a period which is described by the Floquet (monodromy) operator $U(T, 0)$, with $T = 2\pi/\Omega$. Our goal is to study the asymptotic behavior of the mean value of the energy

$$\Big\langle U(T,0)^N\psi, H(p)U(T,0)^N\psi\Big\rangle$$

for an initial condition ψ as N tends to infinity. We focus on the resonant case when $\Omega = \mu\omega_c$ for some natural number μ.

A basic tool in the study of time-dependent quantum systems is the quasienergy operator K. It is nothing but the full Schrödinger operator (including the time derivative); thus we put $K = -i\partial_t + H(a(t))$. It acts in the so called extended Hilbert space which is in our case $\mathscr{K} = L^2\Big((0,T)\times\mathbb{R}_+, r\,\mathrm{d}t\,\mathrm{d}r\Big)$, and the time derivative is taken with the periodic boundary conditions. In general, this is a way, very similar to the approach usually applied in classical mechanics, how to pass from a time-dependent system to an autonomous one[1]. The price to be paid for it is that one has to work with more complex operators in the extended Hilbert space.

An important property of the quasienergy consists in its close relationship to the Floquet operator [13, 25]. In more detail, if $\psi(t,r) \in \mathscr{K}$ is an eigenfunction or a generalized eigenfunction of K, $K\psi = \eta\psi$, which also implies that $\psi(t+T,r) = \psi(t,r)$, then the wave function $e^{-i\eta t}\psi(t,r)$ solves the Schrödinger equation with the initial condition $\psi_0(r) = \psi(0,r)$. It follows that $U(T,0)\psi_0 = e^{-i\eta T}\psi_0$. Thus from the spectral decomposition of the quasienergy one can deduce the spectral decomposition of the Floquet operator.

Let $K_0 = -i\partial_t + H(p)$ be the unperturbed quasienergy operator. Its complete set of normalized eigenfunctions is

$$\left\{T^{-1/2}e^{im\Omega t}\psi_n(p;r);\ m \in \mathbb{Z}, n \in \mathbb{N}_0\right\}$$

with the corresponding eigenvalues $m\Omega + E_n(p)$. Thus K_0 has a pure point spectrum which is in the resonant case ($\Omega = \mu\omega_c$) infinitely degenerated. To take into account these degeneracies we perform the following transformation of indices. Denote by $[x]$ and $\{x\}$ the integer and the fractional part of a real number x, respectively, i.e. $x = [x] + \{x\}$, $[x] \in \mathbb{Z}$ and $0 \leq \{x\} < 1$. Furthermore, let $\rho(\mu,k) = \mu\{k/\mu\}$ be the remainder in division of an integer k by μ. The transformation of indices we wish to apply is a one-to-one map of $\mathbb{Z}\times\mathbb{N}_0$ onto itself sending (m,n) to (k,ℓ), with $k = \mu m + n$ and $\ell = [n/\mu]$, and conversely,

$$\begin{aligned} m &= m(k,\ell) := [k/\mu] - \ell, \\ n &= n(k,\ell) := \mu\ell + \rho(\mu,k). \end{aligned}$$

Using the new indices (k,ℓ) we put

$$\Psi_{k,\ell}(p;t,r) = T^{-1/2}\, e^{im(k,\ell)\Omega t}\, \psi_{n(k,\ell)}(p;r).$$

Then the vectors $\Psi_{k,\ell}$, $(k,\ell) \in \mathbb{Z}\times\mathbb{N}_0$, form an orthonormal basis in the extended Hilbert space $\mathscr{K}$. For a fixed integer $k \in \mathbb{Z}$ let P_k be the orthogonal projection onto the subspace in $\mathscr{K}$ spanned by the vectors $\Psi_{k,\ell}$, $\ell \in \mathbb{N}_0$. Then

$$K_0 = \sum_{k\in\mathbb{Z}} \lambda_k P_k \quad \text{where } \lambda_k = \omega_c(k + p + 1/2).$$

Furthermore, using the basis $\{\Psi_{k,\ell}\}$ one can identify $\mathscr{K}$ with the Hilbert space $\ell^2(\mathbb{Z}\times\mathbb{N}_0)$. In particular, partial differential operators in the variables t and r like the quasienergy are identified in this way with matrix operators. In the sequel we denote matrix operators by bold uppercase letters.

2.3 The quantum averaging method

The full quasienergy operator $K = K(\varepsilon)$ depends on the small parameter ε. Let us write $K(\varepsilon)$ as a formal power series, $K(\varepsilon) = K_0 + \varepsilon K_1 + \varepsilon^2 K_2 + \ldots$. In our case,

$$K_1 = f(\Omega t)\,\omega_c\left(\frac{p}{r^2\omega_c} + \frac{1}{2}\right),\ K_2 = \frac{f(\Omega t)^2}{2r^2}\,,$$

[1] We have already mentiond this classical approach in Chapter 1, Section 1.2.

and $K_3 = K_4 = \ldots = 0$. The ultimate goal of the quantum averaging method in the case of resonances is a unitary transformation resulting in a partial (block-wise) diagonalization of $K(\varepsilon)$. Thus one seeks a skew-Hermitian operator $W(\varepsilon)$ so that $e^{W(\varepsilon)}K(\varepsilon)e^{-W(\varepsilon)}$ commutes with K_0 which is the same as saying that it commutes with all projections P_k. This goal is achievable in principle through an infinite recurrence which in reality should be interrupted at some step. Here we shall be content with the first order approximation.

Let us introduce the (block-wise) diagonal part of an operator A in $\mathscr{K}$ as $\operatorname{diag} A := \sum_{k\in\mathbb{Z}} P_k A P_k$. Thus $\operatorname{diag} A$ surely commutes with K_0. The off-diagonal part is then defined as $\operatorname{offdiag} A := A - \operatorname{diag} A$. Developing formally in ε one has $W(\varepsilon) = \varepsilon W_1 + \mathcal{O}(\varepsilon^2)$ and

$$e^{W(\varepsilon)}K(\epsilon)e^{-W(\varepsilon)} = K_0 + \varepsilon K_1 + \varepsilon\,[W_1, K_0] + \mathcal{O}(\varepsilon^2).$$

Choosing W_1 as

$$W_1 = \sum_{\substack{k_1,\,k_2\\ k_1\neq k_2}} (\lambda_{k_1} - \lambda_{k_2})^{-1} P_{k_1} K_1 P_{k_2}$$

one has $[W_1, K_0] = -\operatorname{offdiag} K_1$ and

$$e^{W(\varepsilon)}K(\varepsilon)e^{-W(\varepsilon)} = K_0 + \varepsilon\operatorname{diag} K_1 + \mathcal{O}(\varepsilon^2).$$

The solution W_1 is also expressible in terms of an averaging integral, and this explains the name of the method [23, 16].

After switching on the perturbation, any unperturbed eigenvalue λ_k gives rise to a perturbed spectrum which, in the first order approximation, equals the spectrum of the operator $\lambda_k + \varepsilon P_k K_1 P_k$ in $\operatorname{Ran} P_k \subset \mathscr{K}$. If the degeneracy of λ_k is infinite then the character of the perturbed spectrum may be arbitrary, depending on the properties of $P_k K_1 P_k$. The corresponding perturbed (generalized) eigenvectors span a subspace which is the range of the orthogonal projection

$$\begin{aligned} P_k(\varepsilon) := e^{-W(\varepsilon)} P_k e^{W(\varepsilon)} &= P_k - \varepsilon\,[W_1, P_k] + \mathcal{O}(\varepsilon^2) \\ &= P_k - \varepsilon\,(S_k K_1 P_k + P_k K_1 S_k) + \mathcal{O}(\varepsilon^2) \end{aligned}$$

where $S_k = \sum_{\ell,\ell\neq k} (\lambda_\ell - \lambda_k)^{-1} P_\ell$ is the reduced resolvent of K_0 taken at the isolated eigenvalue λ_k. Thus the first order averaging method is in fact nothing but the standard quantum perturbation method in the first order but accomplished on the extended Hilbert space simultaneously for all eigenvalues of K_0 (compare to [18, Chp. II§2]).

Our strategy in the remainder of the chapter is based on replacing the true quasi-energy $K(\varepsilon)$ by its first order approximation $K_{(1)} := K_0 + \varepsilon\operatorname{diag} K_1$ and, consequently, $U(T, 0)$ will be replaced by an approximate Floquet operator $U_{(1)}$ associated with $K_{(1)}$.

2.4 The approximate Floquet operator

To determine the approximate Floquet operator $U_{(1)}$ one has to solve the spectral problem for $K_{(1)}$. To this end, as already pointed out above, one can employ the orthonormal basis $\{\Psi_{k\ell}\}$ in order to identify operators in $\mathscr{K}$ with infinite matrix operators in $\ell^2(\mathbb{Z} \times \mathbb{N}_0)$. Let $\{e^1_k; k \in \mathbb{Z}\}$ denote the standard basis in $\ell^2(\mathbb{Z})$, and $\{e^2_\ell; \ell \in \mathbb{N}_0\}$ denote the standard basis in $\ell^2(\mathbb{N}_0)$. It is convenient to write $\ell^2(\mathbb{Z} \times \mathbb{N}_0)$ as the tensor product of Hilbert spaces $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}_0)$ which also means identification of the standard basis in $\ell^2(\mathbb{Z} \times \mathbb{N}_0)$ with the set of vectors $\{e^1_k \otimes e^2_\ell; (k,\ell) \in \mathbb{Z} \times \mathbb{N}_0\}$.

Let $\boldsymbol{P}_k$ be the orthogonal projector onto the one-dimensional subspace $\mathbb{C}e^1_k \subset \ell^2(\mathbb{Z})$. Then the matrix operator $\boldsymbol{K}_{(1)}$ corresponding to $K_{(1)}$ takes the form

$$\boldsymbol{K}_{(1)} = \sum_{k \in \mathbb{Z}} \boldsymbol{P}_k \otimes (\lambda_k + \varepsilon \boldsymbol{A}_k)$$

where $\boldsymbol{A}_k$ is a matrix operator in $\ell^2(\mathbb{N}_0)$ with the entries

$$(\boldsymbol{A}_k)_{\ell_1,\ell_2} = \langle \Psi_{k,\ell_1}, K_1 \Psi_{k,\ell_2} \rangle \,.$$

To compute the matrix entries one observes that

$$\begin{aligned}\langle \Psi_{k,\ell_1}, K_1 \Psi_{k,\ell_2} \rangle &= \mathscr{F}[f](\ell_2 - \ell_1) \\ &\qquad \times \Big\langle \psi_{n(k,\ell_1)}(p), (\partial H(p)/\partial p)\, \psi_{n(k,\ell_2)}(p) \Big\rangle\end{aligned}$$

where $\mathscr{F}[f](r) = (2\pi)^{-1} \int_0^{2\pi} e^{-ir\cdot s} f(s)\,\mathrm{d}s$ stands for the rth Fourier coefficient of f. Recall that, by our assumptions, $\mathscr{F}[f](0) = 0$. Moreover, for $\ell_1 \neq \ell_2$ one has

$$\Big\langle \psi_{n(k,\ell_1)}(p), H(p) \psi_{n(k,\ell_2)}(p) \Big\rangle = 0.$$

Differentiating this identity with respect to p and using the explicit formulas for the scalar products $\langle \psi_{n_1}(p), \partial \psi_{n_2}(p)/\partial p \rangle$ derived in [4] one finally obtains the relation

$$(\boldsymbol{A}_k)_{\ell_1,\ell_2} = \frac{\omega_c}{2} \mathscr{F}[f](\ell_2 - \ell_1) \min\left\{ \frac{\gamma(p; n(k,\ell_2))}{\gamma(p; n(k,\ell_1))}, \frac{\gamma(p; n(k,\ell_1))}{\gamma(p; n(k,\ell_2))} \right\}$$

where $\gamma(p;n) := \Big(\Gamma(n+p+1)/n!\Big)^{1/2}$.

Note that $n(k,\ell)$ is μ–periodic in the integer variable k and so is the matrix $\boldsymbol{A}_k$, i.e. $\boldsymbol{A}_{k+\mu} = \boldsymbol{A}_k$. Moreover, since $\mu\omega_c = 2\pi/T$ one also has $e^{-i\lambda_{k+\mu}T/\hbar} = e^{-i\lambda_k T/\hbar}$. For an integer s, $0 \leq s < \mu$, let $\mathscr{H}_s$ be the closed subspace in the original Hilbert space $\mathscr{H} = L^2(\mathbb{R}_+, r\,\mathrm{d}r)$ spanned by the vectors $\psi_{s+j\mu}(r)$, $j = 0, 1, 2, \ldots$. Then $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \ldots \oplus \mathscr{H}_{\mu-1}$ and from the relationship between $K_{(1)}$ and $U_{(1)}$, as recalled above, it follows that every subspace $\mathscr{H}_s$ is invariant with respect to $U_{(1)}$.

In the example which we study below in more detail (for a particular choice of $f(t)$), the matrix operators $\boldsymbol{A}_s$ have purely absolutely continuous spectra. For the sake of simplicity of the notation let us confine ourselves to this case. For a fixed index s as above, suppose that all generalized eigenvectors and eigenvalues of $\boldsymbol{A}_s$ are parametrized by a parameter $\theta \in (a_s, b_s)$. Let us call them $\boldsymbol{x}_s(\theta)$ and $\eta_s(\theta)$, respectively, i.e. $\boldsymbol{A}_s \boldsymbol{x}_s(\theta) = \eta_s(\theta) \boldsymbol{x}_s(\theta)$, and write $\boldsymbol{x}_s(\theta) = (\xi_{s;0}(\theta), \xi_{s;1}(\theta), \xi_{s;2}(\theta), \ldots)$. The generalized eigenvectors $\boldsymbol{x}_s(\theta)$ are supposed to be normalized to the δ function, i.e. $\langle \boldsymbol{x}_s(\theta_1), \boldsymbol{x}_s(\theta_2) \rangle = \delta(\theta_1 - \theta_2)$, which in fact means that $\xi_{s;\ell}(\theta)$ as a function in the variables $\ell \in \mathbb{N}_0$ and $\theta \in (a_s, b_s)$ is a kernel of a unitary mapping between the Hilbert spaces $\ell^2(\mathbb{N}_0)$ and $L^2((a_s, b_s), \mathrm{d}\theta)$. Thus the spectral decomposition of $\boldsymbol{A}_s$ reads: $\forall \boldsymbol{v} \in \ell^2(\mathbb{N}_0)$,

$$\boldsymbol{A}_s \boldsymbol{v} = \int_{a_s}^{b_s} \eta_s(\theta) \langle \boldsymbol{x}_s(\theta), \boldsymbol{v} \rangle \, \boldsymbol{x}_s(\theta)\,\mathrm{d}\theta.$$

Put

$$\varphi_s(\theta, r) = \sum_{j=0}^{\infty} \xi_{s;j}(\theta)\, \psi_{s+j\mu}(p; r). \tag{2.4}$$

Then again,

$$\int_0^\infty \overline{\varphi_s(\theta_1, r)}\varphi_s(\theta_2, r)\, r\mathrm{d}r = \delta(\theta_1 - \theta_2)$$

and for all $\psi(r) \in \mathscr{H}_s$

$$\begin{aligned} U_{(1)}\psi(r) &= e^{-2\pi i(s+p+1/2)/\mu} \\ &\quad \times \int_{a_s}^{b_s} e^{-i\epsilon\,\eta_s(\theta)T/\hbar} \langle \varphi_s(\theta, \cdot), \psi(\cdot)\rangle \varphi_s(\theta, r)\, \mathrm{d}\theta. \end{aligned} \tag{2.5}$$

2.5 Example: $f(t) = \cos(t)$

We still assume that $s \in \{0, 1, \ldots, \mu - 1\}$ is fixed. In the remainder of the paper we discuss the example when $f(t) = \cos(t)$. In that case an immediate computation gives

$$(\boldsymbol{A}_s)_{j_1,j_2} = \frac{\omega_c}{4}\, \delta_{|j_1-j_2|,1} \left(\prod_{\nu=1}^{\mu} \frac{\mu j_< + s + \nu}{\mu j_< + s + p + \nu} \right)^{1/2}.$$

where $j_< = \min\{j_1, j_2\}$.

Thus $\boldsymbol{A}_s$ is a Jacobi (tridiagonal) matrix with zero diagonal, i.e. a matrix of the type $(\omega_c/4)\boldsymbol{J}$ where

$$\boldsymbol{J} = \begin{pmatrix} 0 & \alpha_0 & 0 & 0 & \ldots \\ \alpha_0 & 0 & \alpha_1 & 0 & \ldots \\ 0 & \alpha_1 & 0 & \alpha_2 & \ldots \\ 0 & 0 & \alpha_2 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2.6}$$

with the matrix entries α_j being positive for all j. This is an elementary fact that the spectrum of $\boldsymbol{J}$ is simple since any eigenvector or generalized eigenvector is unambiguously determined by its first entry. Moreover, one observes, while applying the unitary transformation determined by the diagonal matrix with the diagonal $\{1, -1, 1, -1, \ldots\}$, that the matrices $\boldsymbol{J}$ and $-\boldsymbol{J}$ are unitarily equivalent, and so the spectrum of $\boldsymbol{J}$ is symmetric with respect to the origin.

In our case, $\alpha_j = 1 - p/(2j) + O(j^{-2})$ as $j \to \infty$. Hence $\boldsymbol{J}$ is rather close to the "free" Jacobi matrix $\boldsymbol{J}_0$ for which $\alpha_j = 1$ for all j. The spectral problem for $\boldsymbol{J}_0$ is readily solvable explicitly (see below). It turns out that the spectral properties of $\boldsymbol{J}$ are close to those of $\boldsymbol{J}_0$ as well [14], see also [24]. In particular, it is known that the singular continuous spectrum of $\boldsymbol{J}$ is empty, the essential spectrum coincides with the absolutely continuous spectrum and equals the interval $[-2, 2]$. Furthermore, there are no embedded eigenvalues, i.e. if η is an eigenvalue of $\boldsymbol{J}$ then $|\eta| \geq 2$.

Splitting $\boldsymbol{J}$ into the sum of the upper triangular and the lower triangular part, one notes that $\|\boldsymbol{J}\| \leq 2\sup\{\alpha_0, \alpha_1, \alpha_2, \ldots\}$. In our example, $\alpha_j \leq 1$ for all j and so $\|\boldsymbol{J}\| \leq 2$ and, consequently, the spectrum of $\boldsymbol{J}$ is contained in the interval $[-2, 2]$. This means that the only possible eigenvalues of $\boldsymbol{J}$ are ± 2. But one can exclude even this possibility. In fact, suppose that $\boldsymbol{J}\boldsymbol{u} = 2\boldsymbol{u}$, with $\boldsymbol{u} = (u_0, u_1, u_2, \ldots)$ and $u_0 = 1$. Then

$\alpha_{j-1}u_{j-1} + \alpha_j u_{j+1} = 2u_j$ for $j = 0, 1, 2, \ldots$ (while putting $u_{-1} = 0$). Summing this equality for $j = 0, 1, \ldots, n$, and using that $\alpha_j \leq 1$, one finds that $u_{n+1} \geq u_n + 1$ for $n = 0, 1, 2, \ldots$. Hence $u_j \geq j + 1$ for all j, and so $\boldsymbol{u}$ is not square summable. Let us summarize that the spectrum of $\boldsymbol{J}$ is simple and purely absolutely continuous and equals $[-2, 2]$.

Let us parametrize the spectrum of $\boldsymbol{A}_s = \boldsymbol{A}_s(p)$ by a continuous parameter θ, $0 < \theta < \pi$, so that $\eta(\theta) := (\omega_c/2)\cos(\theta)$ is a point from the spectrum and $\boldsymbol{x}(p;\theta)$ is the corresponding normalized generalized eigenvector with components $\xi_j(p;\theta)$, $j = 0, 1, 2, \ldots$ (here we drop the index s at $\boldsymbol{x}$ and ξ in order to simplify the notation). The asymptotic behavior of the components ξ_j is known [15, 7]; one has

$$\xi_j(p;\theta) \sim A(p;\theta)\cos\Big(j\theta - (p/2)\cot(\theta)\log(j+1) + \phi(p;\theta)\Big) \tag{2.7}$$

for $j \gg 0$. Here $A(p;\theta)$ is a normalization constant and $\phi(p;\theta)$ is a phase which depends on the initial conditions imposed on the sequence $\{\xi_j\}$ (i.e. $\xi_{-1} = 0$) but the asymptotic methods employed in the cited articles do not provide an explicit value for it. In the limit case $p = 0$ the generalized eigenvectors are known explicitly, namely $\xi_j(0;\theta) = \sqrt{2/\pi}\,\sin((j+1)\theta)$ for all j. Hence $\phi(0;\theta) = \theta - \pi/2$.

The generalized eigenvectors are supposed to be normalized so that

$$\Big\langle \boldsymbol{x}(p;\theta_1), \boldsymbol{x}(p;\theta_2)\Big\rangle = \delta(\theta_1 - \theta_2).$$

For $p = 0$, one can use the equality

$$\sum_{n=1}^{\infty} e^{inx} = \pi\delta(x) - \mathscr{P}\frac{1}{1 - e^{-ix}} \tag{2.8}$$

which is valid for $x = \theta_1 - \theta_2 \in (-\pi, \pi)$ and where the symbol $\mathcal{P}$ indicates the regularization of a nonintegrable singularity in the sense of the principal value. It follows that $\sum_{n=-\infty}^{\infty} e^{inx} = 2\pi\delta(x)$, and the normalization is an immediate consequence.

Proof of formula (2.8). Let $\varphi \in \mathscr{D}(-\pi, \pi)$ be a test function, that is a smooth function with compact support in the interval $(-\pi, \pi)$. The limit

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} e^{ikx},\, \varphi\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\int_{-\pi}^{\pi}\frac{-1}{k^2}e^{ikx}\varphi''(x)\,\mathrm{d}x$$

exists for any test function a defines a generalized function which is denoted by $\sum_{k=1}^{\infty} e^{ikx}$. Splitting the domain of integration we get

$$\begin{aligned}\left(\sum_{k=1}^{n} e^{ikx},\, \varphi\right) &= \lim_{\delta\to 0_+}\left(\int_{-\delta}^{\delta}\sum_{k=1}^{n} e^{ikx}\varphi(x)\,\mathrm{d}x + \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right) e^{ix}\frac{1 - e^{inx}}{1 - e^{ix}}\varphi(x)\,\mathrm{d}x\right) = \\ &= \left(\mathscr{P}\frac{1}{e^{-ix} - 1},\, \varphi\right) - \lim_{\delta\to 0_+}\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right)\frac{e^{inx}}{e^{-ix} - 1}\varphi(x)\,\mathrm{d}x.\end{aligned}$$

In order to simplify the second term on the right hand side we use the substitution and obtain

$$\lim_{\delta\to 0_+}\int_{\delta}^{\pi}\frac{e^{-inx}}{e^{ix} - 1}\varphi(-x) + \frac{e^{inx}}{e^{-ix} - 1}\varphi(x)\,\mathrm{d}x = \int_{0}^{\pi}\frac{e^{-inx}}{e^{ix} - 1}\varphi(-x) + \frac{e^{inx}}{e^{-ix} - 1}\varphi(x)\,\mathrm{d}x. \tag{2.9}$$

The last equality is correct since

$$\lim_{x\to 0}\left(\frac{e^{-inx}}{e^{ix}-1}\varphi(-x)+\frac{e^{inx}}{e^{-ix}-1}\varphi(x)\right)=2i\varphi'(0)-(2n+1)\varphi(0).$$

The Equation (2.9) thus becomes

$$\begin{aligned}\varphi(0)\cdot\int_0^\pi\left(\frac{e^{-inx}}{ix}+\frac{e^{inx}}{-ix}\right)\mathrm{d}x+\int_0^\pi e^{-inx}\left(\frac{\varphi(-x)}{e^{ix}-1}-\frac{\varphi(0)}{ix}\right)+\\ +e^{inx}\left(\frac{\varphi(x)}{e^{-ix}-1}-\frac{\varphi(0)}{-ix}\right)\mathrm{d}x=\\ =-2\varphi(0)\int_0^\pi\frac{\sin nx}{x}\mathrm{d}x+\int_{-\pi}^\pi e^{inx}\left(\frac{\varphi(x)}{e^{-ix}-1}-\frac{\varphi(0)}{-ix}\right)\mathrm{d}x=\\ =-2\varphi(0)\int_0^{n\pi}\frac{\sin y}{y}\,\mathrm{d}y+\sqrt{2\pi}\mathscr{F}\left[\frac{\varphi(x)}{e^{-ix}-1}-\frac{\varphi(0)}{-ix}\right](-n)\end{aligned}$$

Consequently

$$\lim_{n\to\infty}\left(\sum_{k=1}^n e^{ikx},\,\varphi\right)=\pi\varphi(0)-\left(\mathscr{P}\frac{1}{1-e^{-ix}},\,\varphi\right).$$

This proves (2.8). □

For general p, the contribution to the δ function should come from the most singular and, at the same time, the leading term in the asymptotic expansion of $\xi_j(p;\theta)$, as given in (2.7). This time, when investigating the singularity near the diagonal $\theta_1=\theta_2$ in the scalar product of two generalized eigenvectors, one is lead to considering the sum $\sum_{n=1}^\infty n^{iax}e^{inx}$ where $a=p/(2\sin^2\theta_1)$ is a real constant. Using the Lerch function[2] $\Phi(z,s,v)$ one has for $|z|<1$,

$$\sum_{n=1}^\infty n^s z^n=z\,\Phi(z,s,1)=\Gamma(1-s)\sum_{n=-\infty}^\infty(-\log(z)+2\pi ni)^{-1+s}. \tag{2.10}$$

From here one deduces that, for any real a,

$$\sum_{n=1}^\infty n^{iax}e^{inx}=\pi\delta(x)+i\mathcal{P}\frac{1}{x}+g_a(x) \tag{2.11}$$

where $g_a(x)$ is a regular distribution, i.e. a locally integrable function. Thus in the general case, too, the normalization constant equals $A(p;\theta)=\sqrt{2/\pi}$.

[2] Lerch function is defined by

$$\Phi(z,s,v)=\sum_{n=0}^\infty(v+n)^{-s}z^n,\quad |z|<1,\ v\neq 0,-1,\dots,$$

and has a series expansion

$$\Phi(z,s,v)=z^{-v}\Gamma(1-s)\sum_{n=-\infty}^\infty(-\log(z)+2\pi ni)^{s-1}e^{2\pi nvi},$$

where $0<v\le 1$, $\operatorname{Re}s<0$, $\left|\arg(-\log(z)2\pi ni)\right|\le\pi$, see [10, § 9.55].

Proof of formula (2.11). Let $\varphi \in \mathscr{D}(-\pi,\pi)$ be a test function and note that

$$\left(n^{\varepsilon+iax}e^{inx-\varepsilon n},\,\varphi\right) = \int_{-\pi}^{\pi} n^{\varepsilon}e^{-\varepsilon n}\frac{e^{ix(n+a\ln n)}}{\left(i(n+a\ln n)\right)^2}\varphi''(x)\,\mathrm{d}x$$

holds for any sufficiently large integer n and $\varepsilon \geq 0$. Hence

$$\left|\left(n^{iax+\varepsilon}e^{inx-\varepsilon n},\,\varphi\right)\right| \leq \frac{2\pi\|\varphi''\|_\infty}{(n+a\ln n)^2}. \tag{2.12}$$

and

$$\begin{aligned}\lim_{N\to\infty}\left(\sum_{n=1}^{N} n^{iax}e^{inx},\,\varphi\right) &= \sum_{n=1}^{\infty}\left(n^{iax}e^{inx},\,\varphi\right) = \lim_{\varepsilon\to 0_+}\sum_{n=1}^{\infty}\left(n^{\varepsilon+iax}e^{n(ix-\varepsilon)},\,\varphi\right) = \\ &= \lim_{\varepsilon\to 0_+}\left(\sum_{n=1}^{\infty} n^{\varepsilon+iax}e^{n(ix-\varepsilon)},\,\varphi\right) = \\ &= \lim_{\varepsilon\to 0_+}\left(e^{ix-\varepsilon}\Phi\left(e^{ix-\varepsilon},-\varepsilon-iax,1\right),\,\varphi\right).\end{aligned}$$

Using the series expansion of Lerch function the last expression is equal to

$$\lim_{\varepsilon\to 0_+}\left(\Gamma(1+\varepsilon+iax)\sum_{n=-\infty}^{\infty}\left(\varepsilon-ix+2n\pi i\right)^{-\varepsilon-iax-1},\,\varphi\right) =$$

$$= \lim_{\varepsilon\to 0_+}\left(\frac{i}{x+i\varepsilon},\,\Gamma(1+\varepsilon+iax)(-ix+\varepsilon)^{-\varepsilon-iax},\,\varphi\right) + \tag{2.13}$$

$$+ \lim_{\varepsilon\to 0_+}\left(\Gamma(1+\varepsilon+iax)\sum_{n\neq 0}\left(-ix+\varepsilon+2n\pi i\right)^{-\varepsilon-iax-1},\,\varphi\right) \tag{2.14}$$

Recalling the well known Sokhotski formula,

$$\lim_{\varepsilon\to 0_+}\frac{1}{x+i\varepsilon} = -i\pi\delta(x) + \mathscr{P}\frac{1}{x}, \quad \text{in } \mathscr{D}'(-\pi,\pi),$$

the limit in (2.13) is

$$(\pi\delta,\,\varphi) + \left(i\mathscr{P}\frac{1}{x},\,\varphi\right) + \left(i\left(\Gamma(1+iax)(-ix)^{-iax}-1\right)\frac{1}{x},\,\varphi\right).$$

Third term comes from a regular distribution as does the contribution from (2.14). Altogether we have derived the equality

$$\lim_{N\to\infty}\left(\sum_{n=1}^{\infty} n^{iax}e^{inx},\,\varphi\right) = \left(\pi\delta + i\mathscr{P}\frac{1}{x} + g_a(x),\,\varphi\right), \quad \varphi \in \mathscr{D}(-\pi,\pi),$$

where $g_a(x)$ is a regular generalized function. This completes the proof of (2.11) □

2.6 The phase $\phi(p;\theta)$ near the spectral point 0

As already mentioned, the phase $\phi(p;\theta)$ in the asymptotic solution (2.7) remains undetermined. Though in the sequel we shall not need this information, let us remark that a bit more can be said about the behavior of the phase near the spectral point 0 (the center of the spectrum) which corresponds to the value of the parameter $\theta = \pi/2$.

First of all, 0 always belongs to the spectrum of the Jacobi matrix $\boldsymbol{J}$. Putting $\boldsymbol{u} = (u_0, u_1, u_2, \ldots)$, with $u_{2j+1} = 0$ and

$$u_{2j} = (-1)^j \prod_{k=0}^{j-1} \frac{\alpha_{2k}}{\alpha_{2k+1}} \tag{2.15}$$

for $j = 0, 1, 2, \ldots$, one has $\boldsymbol{J}\boldsymbol{u} = \boldsymbol{0}$ and $u_0 = 1$. Recalling that, in our example, $\alpha_j = 1 - p/(2j) + O(j^{-2})$ one derives that

$$u_{2j} = (-1)^j u_\infty \Big(1 + p/(8j) + O(j^{-2})\Big) \quad \text{as } j \to \infty,$$

where $u_\infty = \lim_{j\to\infty} (-1)^j u_{2j}$ is a finite constant (depending on p, however). Comparing to (2.7), with $A(p;\theta) = \sqrt{2/\pi}$ and $\theta = \pi/2$, one finds that $\boldsymbol{x}(p;\pi/2) = (\sqrt{2/\pi}/u_\infty)\boldsymbol{u}$. Moreover, $\phi(p;\pi/2) = 0$.

Differentiating the equality $\boldsymbol{J}\boldsymbol{x}(p;\theta) = 2\cos(\theta)\boldsymbol{x}(p;\theta)$ with respect to θ at the point $\pi/2$ and using the substitution

$$\partial \boldsymbol{x}(p;\pi/2)/\partial\theta = -\left(2\sqrt{2/\pi}\big/u_\infty\right)\boldsymbol{v},$$

with $\boldsymbol{v} = (v_0, v_1, v_2, \ldots)$, one arrives at the equation $\boldsymbol{J}\boldsymbol{v} = \boldsymbol{u}$. From (2.7) one deduces that

$$v_j \sim \frac{1}{2}\, u_\infty \sin\!\left(j\,\frac{\pi}{2}\right)\left(j + \frac{p}{2}\,\log(j+1) + \frac{\partial\phi(p;\pi/2)}{\partial\theta}\right) \tag{2.16}$$

for $j \gg 0$. This suggests that one can seek a solution $\boldsymbol{v}$ such that $v_{2j} = 0$ for all j. This assumption on $\boldsymbol{v}$ is in fact necessary and makes the solution unambiguous since otherwise one could add to $\boldsymbol{v}$ any nonzero multiple of $\boldsymbol{u}$ which would violate the asymptotic behavior (2.16). Given that all odd elements of the vector $\boldsymbol{u}$ and all even elements of $\boldsymbol{v}$ vanish the equation $\boldsymbol{J}\boldsymbol{v} = \boldsymbol{u}$ effectively reduces to a linear system with a lower triangular matrix which is explicitly solvable. Using (2.15) one can express the solution as

$$v_{2j+1} = \frac{1}{\alpha_{2j}u_{2j}} \sum_{k=0}^{j} (u_{2k})^2, \quad j = 0, 1, 2, \ldots. \tag{2.17}$$

Noting that

$$\sum_{k=0}^{j}\left(1 + \frac{p}{4k+2}\right) = j + 1 + \frac{p}{4}\Big(\log(4j+4) + \gamma\Big) + O(j^{-1}),$$

where γ is the Euler constant, and that $(u_{2k})^2 = u_\infty^2(1 + p/(4k) + O(k^{-2}))$ one derives

$$\sum_{k=0}^{j}\left(\frac{u_{2k}}{u_\infty}\right)^2 = j + 1 + \frac{p}{4}\left(\log(4j+4) + \gamma\right) + \sum_{k=0}^{\infty}\left(\left(\frac{u_{2k}}{u_\infty}\right)^2 - 1 - \frac{p}{4k+2}\right) + O(j^{-1}).$$

Using (2.17) and comparing to (2.16) one finally arrives at the relation

$$\frac{\partial\phi(p;\pi/2)}{\partial\theta} = 1 + \frac{p}{2}\Big(\log(2)+\gamma\Big) + 2\sum_{k=0}^{\infty}\left(\left(\frac{u_{2k}}{u_\infty}\right)^2 - 1 - \frac{p}{4k+2}\right).$$

2.7 Acceleration

We again fix an integer s, $0 \le s < \mu$. Suppose one is given a test function $\varrho(\theta) \in C_0^\infty((0,\pi))$. Recalling (2.4) we put

$$\psi(r) = \int_0^\pi \varphi_s(\theta,r)\varrho(\theta)\,\mathrm{d}\theta.$$

In what follows, we drop the index s and, whenever convenient, write simply H instead of $H(p)$. Using (2.5), one has, for $N \in \mathbb{N}$,

$$\begin{aligned}\left\langle U_{(1)}^N\psi, HU_{(1)}^N\psi\right\rangle &= \int_0^\pi\int_0^\pi e^{i\varepsilon(\cos\theta_1-\cos\theta_2)\omega_c TN/2} \\ &\qquad\times\langle\varphi(\theta_1,r),H\varphi(\theta_2,r)\rangle\,\overline{\varrho(\theta_1)}\varrho(\theta_2)\,\mathrm{d}\theta_1\mathrm{d}\theta_2 \\ &= \sum_{j=0}^{\infty}E_{s+j\mu}(p)\left|\int_0^\pi e^{-i\varepsilon\cos(\theta)\omega_c TN/2}\xi_j(p;\theta)\varrho(\theta)\,\mathrm{d}\theta\right|^2.\end{aligned}$$

Recall (2.1) and note that $\{\xi_j(p;\theta)\}_{j=0}^\infty$ is an orthonormal basis in $L^2((0,\pi),\mathrm{d}\theta)$ and so

$$\sum_{j=0}^{\infty}\left|\int_0^\pi e^{-i\varepsilon\cos(\theta)\omega_c TN/2}\xi_j(p;\theta)\varrho(\theta)\,\mathrm{d}\theta\right|^2 = \int_0^\pi|\varrho(\theta)|^2\,\mathrm{d}\theta.$$

Hence the leading contribution to the acceleration rate comes from the expression

$$\mu\omega_c\sum_{j=0}^{\infty}(j+1)\left|\int_0^\pi e^{-i\varepsilon\cos(\theta)\omega_c TN/2}\xi_j(p;\theta)\varrho(\theta)\,\mathrm{d}\theta\right|^2.$$

Furthermore, restricting this sum to an arbitrarily large but finite number of summands results in an expression which is uniformly bounded in N. This means that one can replace $\xi_j(p;\theta)$ by the leading asymptotic term, as given in (2.7) (with $A(p;\theta) = \sqrt{2/\pi}$). Hence the leading contribution to the acceleration rate is expressible as

$$\frac{2\Omega}{\pi}\int_0^\pi\int_0^\pi h(\theta_1,\theta_2)e^{i\varepsilon(\cos\theta_1-\cos\theta_2)\omega_c TN/2}\,\overline{\varrho(\theta_1)}\varrho(\theta_2)\,\mathrm{d}\theta_1\mathrm{d}\theta_2$$

where

$$\begin{aligned}h(\theta_1,\theta_2) = \sum_{j=0}^{\infty}(j+1)\cos&\Big(j\theta_1 - \frac{p}{2}\cot(\theta_1)\log(j+1)+\phi(p;\theta_1)\Big) \\ &\times\cos\Big(j\theta_2 - \frac{p}{2}\cot(\theta_2)\log(j+1)+\phi(p;\theta_2)\Big).\end{aligned} \tag{2.18}$$

The singular part of the distribution $h(\theta_1,\theta_2)$ is supported on the diagonal $\theta_1=\theta_2$. The sum in (2.18) can be evaluated analogously as in (2.11) with the result

$$h(\theta_1,\theta_2) = -\frac{1}{2}\frac{\partial}{\partial\theta_2}\mathcal{P}\frac{1}{\theta_1-\theta_2} - \frac{\pi}{2}\left(\frac{\partial\phi(p;\theta_1)}{\partial\theta}-1\right)\delta(\theta_1-\theta_2) \\ + \text{ a regular distribution.}$$

Estimating the acceleration rate we can restrict ourselves to a sufficiently small but fixed neighbourhood of the diagonal with a radius $d>0$. Thus we arrive at the expression

$$\frac{\Omega}{\pi}\mathcal{P}\underset{|\theta_1-\theta_2|<d}{\int_0^\pi\int_0^\pi}\frac{1}{\theta_1-\theta_2}\frac{\partial}{\partial\theta_2}\Big(e^{-i\varepsilon\sin(\theta_1)(\theta_1-\theta_2)\omega_c TN/2}\,\overline{\varrho(\theta_1)}\varrho(\theta_2)\Big)\mathrm{d}\theta_1\mathrm{d}\theta_2.$$

Further we carry out the differentiation, as indicated in the integrand, and get rid of the terms which are not proportional to N or which are non singular. Moreover, we use the substitution $\theta_2=\theta_1+u$. Thus we obtain

$$-\frac{i\varepsilon\Omega\omega_c TN}{2\pi}\int_0^\pi\sin(\theta_1)|\varrho(\theta_1)|^2\left(\mathcal{P}\int_{-d}^{d}\frac{1}{u}e^{i\varepsilon\sin(\theta_1)\omega_c TNu/2}\,\mathrm{d}u\right)\mathrm{d}\theta_1$$
$$=\frac{\varepsilon\Omega\omega_c TN}{\pi}\int_0^\pi\sin(\theta)|\varrho(\theta)|^2\left(\int_0^d\frac{1}{u}\sin\!\Big(\frac{\varepsilon}{2}\sin(\theta)\omega_c TNu\Big)\mathrm{d}u\right)\mathrm{d}\theta.$$

Finally note that, for any a real,

$$\lim_{N\to\infty}\int_0^d\frac{1}{u}\,\sin(aNu)\,\mathrm{d}u=\frac{\pi}{2}\,\operatorname{sign}a.$$

We conclude that the formula for the acceleration rate in the first-order averaging approximation reads

$$\gamma_{\mathrm{acc}} := \lim_{N\to\infty}\left\langle U_{(1)}^N\psi, H(p)U_{(1)}^N\psi\right\rangle\Big/(NT\|\psi\|^2)$$
$$=\frac{|\epsilon|\omega_c\Omega}{2}\int_0^\pi\sin(\theta)|\varrho(\theta)|^2\,\mathrm{d}\theta\Big/\int_0^\pi|\varrho(\theta)|^2\,\mathrm{d}\theta. \tag{2.19}$$

Here we have used that $\|\psi\|^2=\int_0^\pi|\varrho(\theta)|^2\,\mathrm{d}\theta$. Formula (2.19) can be compared to formula (2.3), as derived for a classical particle, in the case when $\Phi(t)$ is given by (2.2) and $f(t)=\cos(t)$. Then (2.3) gives the acceleration rate $\gamma_{\mathrm{acc}}=|\varepsilon|\omega_c\Omega\sin(\xi)/2$ where $\xi\in(0,\pi)$ depends on some data which can be learned from the asymptotic behavior of the classical trajectory. In addition, we note to this comparison that the classical case suggests, as discussed in [5], that the first-order averaging approximation yields in fact the correct acceleration rate (valid for the original system), and this is so even if the parameter ε is not assumed to be very small.

2.8 Numerical analysis

Purpose of this Section is to present our numerical results that support the predicted acceleration rate (2.19). For the sake of simplicity we take $\mu=1$ and $s=0$. Let us fix a function $\rho(\theta)$,

$$\rho(\theta) = \left(\frac{20}{\pi}\right)^{1/4} \exp\Big(-10(2-\theta)^2 + 8i\theta\Big). \tag{2.20}$$

For our numerical purposes it can be assumed to be in $C_0^\infty((0,\pi))$. For this choice of $\rho(\theta)$ we put

$$\eta_0(r) = \int_0^\pi \varphi_0(\theta, r)\rho(\theta)\mathrm{d}\theta.$$

In order to solve the time-dependent Schrödinger equation

$$i\partial_t \eta(t) = H(a(t))\eta(t), \quad \eta(0) = \eta_0,$$

we truncate the Fourier expansion of $\eta(t)$,

$$\eta(t) = \sum_{n=0}^{\infty} \eta_n(t)\psi_n(a(t)),\ \eta_n(t) = \langle \psi_n(a(t)), \eta(t)\rangle,\ n = 0, 1, \ldots,$$

at some fixed order $n_{\max}$. In this way we obtain a system of ordinary differential equations for the Fourier coefficients

$$\begin{aligned} i\eta_n'(t) &= E_n(a(t))\eta_n(t) - ia'(t) \sum_{j=0}^{n_{\max}} \langle \psi_n(a(t)), \psi_j'(a(t))\rangle \eta_j(t), \\ \eta_n(0) &= \langle \psi_n(a(0)), \eta_0\rangle,\ n = 0, 1, \ldots, n_{\max}, \end{aligned}$$

where explicit formulas for the scalar products are known, see [4]. In order to approximately solve this system we employ explicit Runge-Kutta method of order 4 (RK4) with fixed time stepsize $h = 0.001$. The mean value of energy at time t is then approximated by

$$\Big\langle \eta(t), H(a(t))\eta(t) \Big\rangle \approx \sum_{n=0}^{n_{\max}} E_n(a(t))|\eta_n(t)|^2.$$

We present the results of our numerical experiments in Figure 2.1 and Figure 2.2.

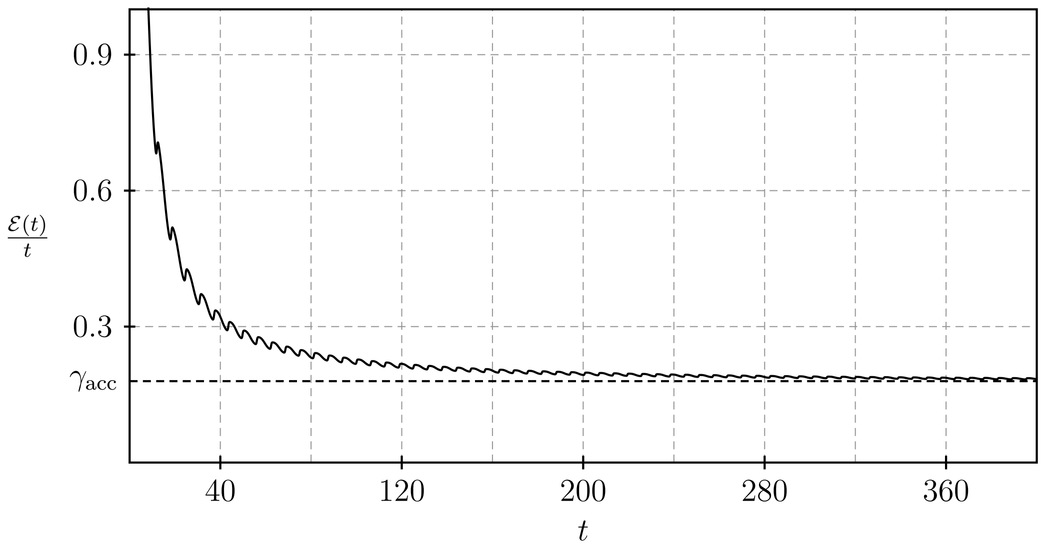

Figure 2.1: The numerical computation of the ratio of the energy mean value $\mathcal{E}(t) = \sum_{n=0}^{n_{\max}} E_n(t)|\eta_n(t)|^2$ and time t. Various parameters are set to be $p = 2.5$, $b = 1$, $\varepsilon = 0.4$, $\Omega = 1$, $f(t) = \varepsilon \sin(\Omega t)$. The initial value is computed using (2.20), in this case the predicted rate of acceleration is $\gamma_{\text{acc}} \approx 0.1796$.

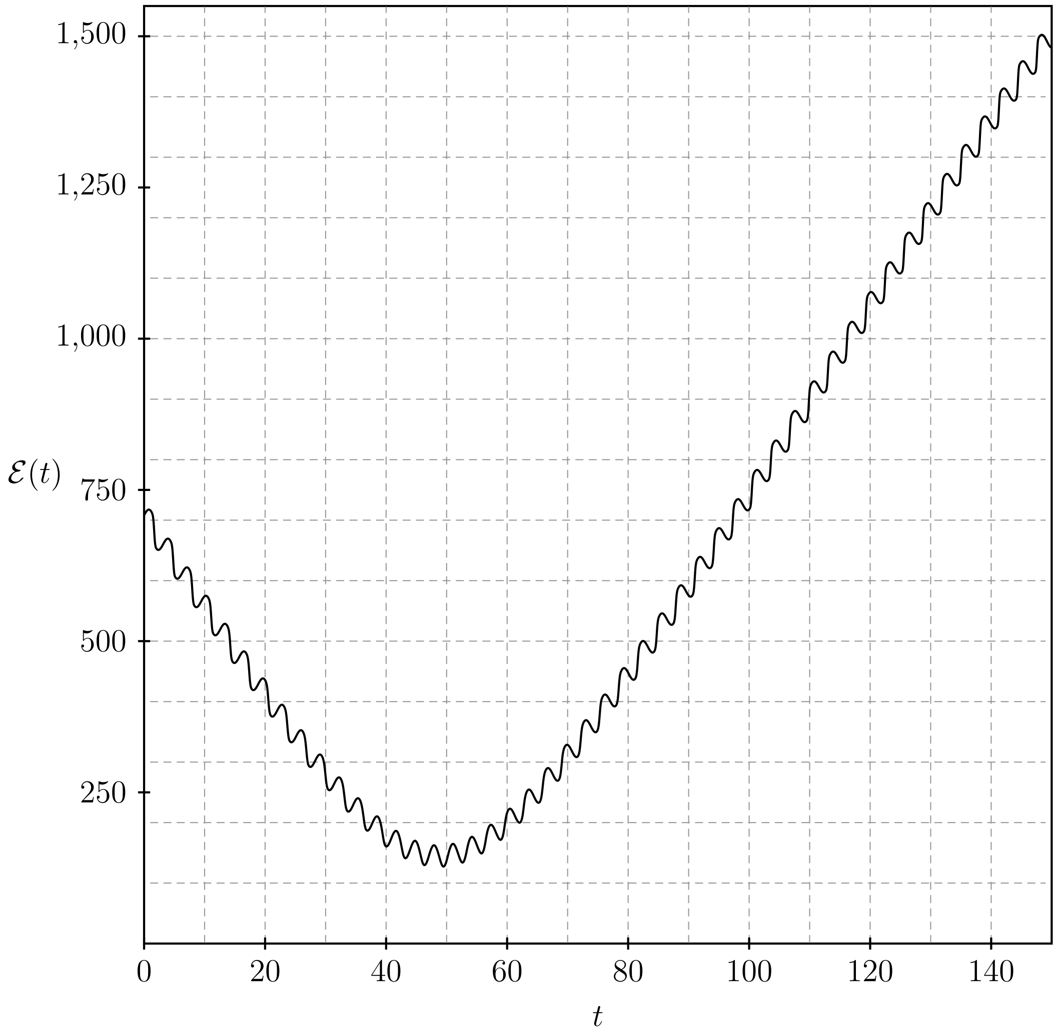

Figure 2.2: The mean value of energy, $\mathcal{E}(t) = \sum_{n=0}^{n_{\max}} E_n(t)|\eta_n(t)|^2$, computed numerically according to the description in the present Section. Particular values of those constants are $p = 0.5$, $b = 2 = \Omega$, $\varepsilon = 0.7$, $f(t) = \varepsilon \sin(\Omega t)$, and initial condition $\eta_0(r) = \exp\Big(-(10-r)^2 + ir\Big)$.

Appendix A

Evaluation of Auxiliary Integrals

This Section contains the proof and corollaries of the following

Lemma A.1: For $n \in \mathbb{N}_0$ and $|\beta| < 1$, $\beta \in \mathbb{R}$ it is true that

$$\int_0^{2\pi} \frac{\cos nt}{1+\beta \sin t}\,\mathrm{d}t = 2\pi \frac{\beta^n}{\sqrt{1-\beta^2}\left(1+\sqrt{1-\beta^2}\right)^n} \cos\frac{\pi n}{2}, \tag{A.1}$$

$$\int_0^{2\pi} \frac{\sin nt}{1+\beta \sin t}\,\mathrm{d}t = -2\pi \frac{\beta^n}{\sqrt{1-\beta^2}\left(1+\sqrt{1-\beta^2}\right)^n} \sin\frac{\pi n}{2}, \tag{A.2}$$

$$\int_0^{2\pi} \frac{\cos nt \cos t}{1+\beta \sin t}\,\mathrm{d}t = 2\pi \frac{\beta^{n-1}}{\left(1+\sqrt{1-\beta^2}\right)^n} \sin\frac{\pi n}{2}, \tag{A.3}$$

and for[1] $n \in \mathbb{N}$ one has

$$\int_0^{2\pi} \frac{\sin nt \cos t}{1+\beta \sin t}\,\mathrm{d}t = 2\pi \frac{\beta^{n-1}}{\left(1+\sqrt{1-\beta^2}\right)^n} \cos\frac{\pi n}{2}. \tag{A.4}$$

Proof. We will prove the equality (A.1). The proof of the rest is analogous. For the sake of brevity denote the LHS of (A.1) by the symbol L. Note that under our assumptions it holds

$$\sin nt = \sum_{k=0}^{n} \binom{n}{k} \cos^k t \sin^{n-k} t \sin\frac{\pi}{2}(n-k),$$

$$\cos nt = \sum_{k=0}^{n} \binom{n}{k} \cos^k t \sin^{n-k} t \cos\frac{\pi}{2}(n-k),$$

$$\frac{1}{1+\beta\sin t} = \sum_{m=0}^{\infty} (-\beta)^m \sin^m t.$$

Plugging these relations into the left hand side of (A.1) and using the well known Beta function

$$\int_0^{2\pi} \cos^k t \sin^n t\mathrm{d}t = \frac{\left(1+(-1)^k\right)\left(1+(-1)^n\right)}{2} B\left(\frac{1+k}{2}, \frac{1+n}{k}\right), \quad n,k \in \mathbb{N}_0,$$

one obtains, after some minor adjustments,

$$L = \sum_{k=0}^{n} \cos\frac{\pi}{2}(n-k) \sum_{m=0}^{\infty} (-\beta)^m (-1)^k \frac{(1+(-1)^k)(1+(-1)^{m+n})}{2} \times B\left(\frac{1+k}{2}, \frac{1}{2}(1+m+n-k)\right).$$

Summands with k or $m+n$ odd are zero. Hence we can assume that k and $m+n$ are even. Furthermore if n is also odd then $\cos(\pi(n-k)/2) = 0$ and therefore $J = 0$. We must investigate the case of $n = 2N$ where $N \in \mathbb{N}_0$. After re-notation of indices we clearly have

$$J = 2\sum_{k=0}^{N} \binom{2N}{2k} (-1)^{N-k} \sum_{m=0}^{\infty} \beta^{2m} B\left(\frac{1}{2}k, \frac{1}{2}+N+m-k\right).$$

[1] For $n = 0$ this is obviously zero.

Rewriting the Beta function in terms of Gamma function and using the Gauss hypergeometric series

$$F(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{m=0}^{\infty} \frac{\Gamma(a+m)\Gamma(b+m)}{\Gamma(c+m)} \frac{z^m}{m!},$$

we arrive at

$$J = (-1)^N \frac{2\pi(2N)!}{2^N} \sum_{k=0}^{N} \frac{(-1)^k}{(2N-2k)!!(2k)!!} F^{\mathrm{reg}}(1, N-k+1/2, N+1, \beta^2),$$

where $F^{\mathrm{reg}}(a,b,c,z) = F(a,b,c,z)/\Gamma(c)$ is regularised hypergeometric function. Next step is to take advantage of the symmetry in interchange of a, b and of the integral representation of hypergeometric function

$$F(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,\mathrm{d}t.$$

It turns out that

$$J = \pi \frac{(-1)^{N+1}}{2^{2N}} \beta^{2N+1} F\Big(N+1, N+3/2, 2N+2, \beta^2\Big),$$

where the binomial theorem was used. Final step is to look in [1] and find relation 15.1.14:

$$F(a, a+1/2, 2a, z) = 2^{2a-1}(1-z)^{-1/2}\Big(1+\sqrt{1-z}\Big)^{1-2a}.$$

Hence

$$J = 2\pi(-1)^N \frac{\beta^{2N}}{\sqrt{1-\beta^2}\Big(1+\sqrt{1-\beta^2}\Big)^{2N}}.$$

Combining results for odd and even n one obtains the formula which was to be proved.

For the sake of completeness note, that in the computation of the two last integrals one needs [1], 15.1.13

$$F(a, a+1/2, 2a+1, z) = 2^{2a}(1+\sqrt{1-z})^{-2a}. \qquad \square$$

The Proposition just proved immediately implies

Corollary A.2: For $n \in \mathbb{Z} \setminus \{0\}$ and $\beta \in \mathbb{R}$, $|\beta| < 1$ it is true that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(\varphi)\exp(-in\varphi)}{1+\beta\sin(\varphi)} \mathrm{d}\varphi = \frac{1}{\beta} \left(\frac{\beta}{1+\sqrt{1-\beta^2}} \right)^{|n|} \exp\left(i\Big(n - \mathrm{sign}(n)\Big)\frac{\pi}{2} \right).$$

Obviously, if $n = 0$ the integral is zero. Further, assuming that $n \in \mathbb{Z}$ and β as above, the following equality

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\exp(-in\varphi)}{1+\beta\sin(\varphi)} \mathrm{d}\varphi = \frac{1}{\sqrt{1-\beta^2}} \left(\frac{\beta}{1+\sqrt{1-\beta^2}} \right)^{|n|} \exp\left(in\frac{\pi}{2} \right).$$

holds.

And finally

Corollary A.3: Suppose that $\alpha > 1$ and $n \in \mathbb{Z} \setminus \{0\}$ then the following formula

$$\frac{1}{2\pi}\int_0^{2\pi} \arctan\left(\frac{\cos\varphi}{\alpha+\sin\varphi}\right)\exp(-in\varphi)\mathrm{d}\varphi = \frac{\exp(\frac{in\pi}{2})}{2in}\left(\frac{1}{\alpha}\right)^{|n|}.$$

holds. The integral is zero if $n = 0$.

Proof. For $n \neq 0$ this is straightforward consequence of integration by parts and the preceding Corollary. If $n = 0$, then denote

$$F(\alpha) := \frac{1}{2\pi}\int_0^{2\pi} \arctan\left(\frac{\cos\varphi}{\alpha+\sin\varphi}\right)\mathrm{d}\varphi$$

for $\alpha > 1$ and observe that

$$F'(\alpha) = -\frac{1}{2\pi}\int_0^{2\pi} \frac{\cos\varphi}{\alpha^2+2\alpha\sin\varphi+1}\mathrm{d}\varphi = 0.$$

Since $\lim\limits_{\alpha\to+\infty} F(\alpha) = 0$ we have $F(\alpha) = 0$ for all $\alpha > 1$. $\square$

Let us note, that one can arrive at similar conclusion in an alternative way. For $\alpha > 0$ consider

$$f(\varphi) := \arctan\frac{\cos\varphi}{\alpha+\sin\varphi}, \quad \varphi \in [0, 2\pi].$$

For any $|z| < 1$ we have the identity

$$\begin{aligned}\arctan z &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}z^{2n+1} = \frac{i}{2}\sum_{n=1}^{\infty}\left(\frac{(-i)^n}{n}z^n - \frac{i^n}{n}z^n\right) = \\ &= \frac{i}{2}\Big(-\ln(1+iz) + \ln(1-iz)\Big) = \frac{i}{2}\ln\frac{1-iz}{1+iz}.\end{aligned} \tag{A.5}$$

Employing the identity

$$\arctan z + \arctan\frac{1}{z} = \frac{\pi}{2}, \quad z \neq 0$$

we conclude, that (A.5) is valid for any $z \in \mathbb{R}$. Consequently

$$\begin{aligned}f(\varphi) &= \frac{i}{2}\ln\frac{1-i\frac{\cos\varphi}{\alpha+\sin\varphi}}{1+i\frac{\cos\varphi}{\alpha+\sin\varphi}} = \frac{i}{2}\ln\frac{1-\frac{i}{\alpha}e^{i\varphi}}{1+\frac{i}{\alpha}e^{-i\varphi}} = -\frac{i}{2}\left(\sum_{n=1}\frac{i^n}{n\alpha^n}e^{in\varphi} + \sum_{n=1}^{\infty}\frac{i^{-n}}{-n\alpha^n}e^{-in\varphi}\right) = \\ &= -\frac{i}{2}\sum_{n\neq 0}\frac{i^n}{n\alpha^{|n|}}e^{in\varphi}.\end{aligned}$$

From there it is easy to read out the Fourier coefficients of f.

Appendix B

Friedrichs Extension

In this Appendix we would like to point out some of the properties of the Friedrichs extension of the operator

$$H(p) = \frac{1}{2}\left(-\frac{1}{r}\partial_r r \partial_r + v(r)\right), \quad \text{where } v(r) = \frac{1}{r^2}\left(p + \frac{br^2}{2}\right)^2, \tag{B.1}$$

with the domain given by $\operatorname{dom} H(p) = C_0^\infty(\mathbb{R}_+)$ acting in the Hilbert space $\mathscr{H} = L^2(\mathbb{R}_+, r\,\mathrm{d}r)$. It is sufficient to assume that $p \in \mathbb{R}$ is nonzero and $b \in \mathbb{R}_+$. Let us consider the corresponding quadratic form

$$\mathfrak{t}(p)(\varphi, \psi) = \frac{1}{2}\int_0^\infty \left(\overline{\varphi'}\psi' + \frac{1}{r^2}\left(p + \frac{br^2}{2}\right)^2 \overline{\varphi}\,\psi \right) r\,\mathrm{d}r,$$

where $\varphi, \psi \in \operatorname{dom} \mathfrak{t} = C_0^\infty(\mathbb{R}_+)$. We will further suppress the dependence on p because it does not play any significant role in the following discussion. Our only assumption is that p is nonzero.

The form $\mathfrak{t}$ is a densely defined sesquilinear and bounded from below with lower bound 0. According to [18, Corollary VI-1.28] the form $\mathfrak{t}$ is closable and its closure $\bar{\mathfrak{t}}$ has the same lower bound as $\mathfrak{t}$.

By virtue of [18, Theorem VI-2.6] there is an operator T associated with $\bar{\mathfrak{t}}$. This operator is self-adjoint and bounded from below by the same bound as $\mathfrak{t}$. In particular, T enjoys the following properties

a) $\operatorname{dom} T \subset \operatorname{dom} \bar{\mathfrak{t}}$ and

$$\bar{\mathfrak{t}}(\varphi, \psi) = \langle \varphi, T\psi \rangle$$

for every $\varphi \in \operatorname{dom} \bar{\mathfrak{t}}$ and $\psi \in \operatorname{dom} T$.

b) $\operatorname{dom} T$ is a core of $\bar{\mathfrak{t}}$.

c) If $\varphi \in \operatorname{dom} \bar{\mathfrak{t}}$, $\psi \in \mathscr{H}$ and

$$\bar{\mathfrak{t}}(\eta, \varphi) = \langle \eta, \psi \rangle$$

holds for any η belonging to a core of $\bar{\mathfrak{t}}$, then $\varphi \in \operatorname{dom} T$ and $T\varphi = \psi$.

The operator T is uniquely determined by the first property and is called the Friedrichs extension of (B.1).

Observe, that $\mathfrak{t} = \mathfrak{s}_1 + \mathfrak{s}_2$, where

$$\mathfrak{s}_1(\varphi, \psi) = \frac{1}{2}\int_0^\infty \overline{\varphi'}\psi'\, r\,\mathrm{d}r,$$
$$\mathfrak{s}_2(\varphi, \psi) = \frac{1}{2}\int_0^\infty v(r)\overline{\varphi}\psi\, r\,\mathrm{d}r,$$

where $\varphi, \psi \in \operatorname{dom} \mathfrak{s}_1 = \operatorname{dom} \mathfrak{s}_2 = C_0^\infty(\mathbb{R}_+)$. Both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are of the form

$$s_i(\psi, \varphi) = \langle S_i\psi,\, S_i\varphi \rangle, \quad i = 1, 2,$$

and the operators S_i, $i = 1, 2$ act in $\mathscr{H}$ by

$$S_1\varphi = \frac{1}{\sqrt{2}}\varphi' \quad \text{and} \quad S_2\varphi = \frac{1}{\sqrt{2}}v^{1/2}\varphi, \quad \text{for any } \varphi \in \operatorname{dom} S_i = C_0^\infty(\mathbb{R}_+),\ i = 1, 2.$$

It follows that $\overline{\mathfrak{s}_i}(\psi,\varphi) = \left\langle \overline{S_i}\psi,\, \overline{S_i}\varphi \right\rangle$ for any $\varphi, \psi \in \operatorname{dom} \overset{\mathfrak{s}_i}{=} \operatorname{dom} \overline{S_i}$, $i = 1, 2$. In our case one can show that[1]

$$\operatorname{dom} \overline{S_1} = \Big\{ \psi \in \mathscr{H} \,;\, \psi \text{ is a.c. in } \mathbb{R}_+,\ \psi' \in \mathscr{H},\ \lim_{r\to 0_+} \overline{\psi}\eta r = 0 \text{ for any } \eta \in \operatorname{dom} S_1^* \Big\},$$
$$\operatorname{dom} \overline{S_2} = \Big\{ \psi \in \mathscr{H} \,;\, v^{1/2}\psi \in \mathscr{H} \Big\},$$

where

$$\operatorname{dom} S_1^* = \Big\{ \psi \in \mathscr{H} \,;\, \psi \text{ is a.c. in } \mathbb{R}_+,\ \psi' + \frac{1}{r}\psi \in \mathscr{H} \Big\}.$$

According to the [18, Theorem 1.31] we know that $\overline{\mathfrak{t}} \subset \overline{\mathfrak{s}_1} + \overline{\mathfrak{s}_2}$, in particular we have the inclusion

$$\operatorname{dom} \overline{\mathfrak{t}} \subset D := \operatorname{dom} \overline{\mathfrak{s}_1} \cap \operatorname{dom} \overline{\mathfrak{s}_2}.$$

Note that if $\psi \in D$ then $\psi' + \frac{\psi}{r} \in L^2\big(]0,1[,\, r\,\mathrm{d}r\big)$ and

$$\big(|\psi|^2\big)' = \Big|\psi' + \frac{\psi}{r}\Big|^2 r - |\psi'|^2 r - \frac{1}{r}|\psi|^2 \in L^1\big(]0,1[, \mathrm{d}r\big).$$

Therefore there exist a finite limit $\lim_{r\to 0_+} |\psi(r)| \in \mathbb{R}$. Since $\psi/r \in L^2\big(]0,1[, r\,\mathrm{d}r\big)$ we conclude that $\lim_{r\to 0_+} \psi(r) = 0$. Similarly, for any $\eta \in \operatorname{dom} S_1^*$ we have

$$\big(|\eta|^2 r^2\big)' = \Big(\overline{\eta}\big(\eta' + r^{-1}\eta\big) + \big(\overline{\eta'} + r^{-1}\overline{\eta}\big)\eta\Big) r^2 \in L^1\big((0,1), \mathrm{d}r\big)$$

and so there is a finite limit $\lim_{r\to 0_+} r|\eta| \in \mathbb{R}$. This implies that the last condition of $\operatorname{dom} \overline{S_1}$ can be dropped in D.

The operator T indeed is an extension of the operator $H(p)$ defined in (B.1). Suppose that we have $\psi \in \operatorname{dom} T$ and set $\varphi := T\psi \in \mathscr{H}$. For every $\eta \in \operatorname{dom} \overline{\mathfrak{t}}$ it is true that

$$\overline{\mathfrak{t}}(\eta, \psi) = \langle \eta, \varphi \rangle. \tag{B.2}$$

In particular,

$$\int_0^\infty \overline{\eta'} \left(\frac{r}{2}\psi' + z \right) \mathrm{d}r = 0, \quad \text{for all } \eta \in C_0^\infty(\mathbb{R}_+),$$

where z is an absolutely continuous function satisfying $z' = \eta r - \frac{r}{2} v\psi \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathrm{d}r)$. Consequently $\frac{r}{2}\psi' + z = \text{const}$ and ψ' is an absolutely continuous function. It follows that

$$T\psi = \varphi = \frac{1}{2}\Big(-\psi'' - \frac{1}{r}\psi' + v(r)\psi \Big) = \mathscr{H}.$$

Since $\psi \in \operatorname{dom} T$ belongs also to $\operatorname{dom} \overline{\mathfrak{t}} \subset D$ it follows that

$$\lim_{r\to 0_+} \psi(r) = 0.$$

For a detailed discussion of a characterization of the Friedrichs extension one can consult the reference [22].

[1] "a.c." is a shorthand for "absolutely continuous."

Appendix C

Jacobi Matrix Operators and Absolutely Continuous Spectrum

In this Appendix we will review certain class of Jacobi operators and their spectral properties. In the following it will be assumed that $\{\lambda_n\}_{n=1}^{\infty}$ is a real positive sequence such that there is a positive α, $0 < \alpha < 1$, for which the condition

$$\frac{1}{\alpha} < \lambda_n \leq 1 \tag{C.1}$$

holds for all $n \in \mathbb{N}$. Moreover, we assume that the sequence satisfy

$$\lim_{n\to\infty} \lambda_n = 1.$$

In order to simplify some expressions it is convenient to set $\lambda_0 := -1$. Keep in mind that the condition (C.1) holds only for positive n, not for $n = 0$.

Let $\{e_n\}_{n=1}^{\infty}$ denote the standard basis in $\ell^2 := \ell^2(\mathbb{N}, \mathbb{C})$. We are interested in the analysis of the Jacobi matrix operator with matrix (with respect to the standard basis of ℓ^2)

$$\boldsymbol{J} = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 & \cdots \\ \lambda_1 & 0 & \lambda_2 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \lambda_3 & \\ 0 & 0 & \lambda_3 & 0 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}$$

Under our assumptions $\boldsymbol{J}$ is a well defined bounded self-adjoint operator in ℓ^2. In fact, the operator

$$\boldsymbol{A} = \sum_{n=1}^{\infty} \lambda_n \langle e_n, \cdot \rangle$$

is a bounded diagonal self-adjoint operator. Let $\boldsymbol{S}$ be the right shift operator, $\boldsymbol{S}e_n := e_{n+1}$ for all $n \in \mathbb{N}$. The adjoint $\boldsymbol{S}^*$ is then the left shift operator. Our $\boldsymbol{J}$ can be expressed in the form $\boldsymbol{J} = \boldsymbol{SA} + \boldsymbol{AS}^*$. Since $\|\boldsymbol{S}\| = \|\boldsymbol{S}^*\| = 1$ and $\|\boldsymbol{A}\| = \sup_{n\in\mathbb{N}} |\lambda_n|$ we see that

$$\|\boldsymbol{J}\| \leq 2 \sup_{n\in\mathbb{N}} |\lambda_n|.$$

As an elementary consequence of this estimate we conclude that the spectrum of $\boldsymbol{J}$ belongs to the interval $[-2, 2]$, $\sigma(\boldsymbol{J}) \subset [-2, 2]$.

The article by Janas and Naboko, [15] is concerned with a similar problem. Their approach relies on the theory due to Gilbert and Pearson [26]. The idea is that if one shows that all solutions of the (generalized) eigenfunction equation, with spectral parameter $\lambda \in (a, b)$ are bounded, then $(a, b) \in \sigma(J)$ and the spectrum of J has absolutely continuous component filling (a, b). There is also an article by Simon, [30], which contains a short proof of this interesting result (in his own words "All eigenfunctions bounded implies purely a.c. spectrum."). This result is reproduced in Section C.2.

The material of this Appendix is organized into several parts. In Section C.1 we summarize results concerning Weyl m-function and its relation to the spectrum of $\boldsymbol{J}$. Section C.2 reproduces the result of Simon, in particular [30]. Finally, in Section C.3 we show that the spectrum if $\boldsymbol{J}$ is absolutely continuous in $(-2, 2)$.

C.1 Spectral measure and Weyl m-function

Note that e_1 is a cyclic vector for $\boldsymbol{J}$, this means that the set $\{\boldsymbol{J}^n e_1\,;\, n = 0, 1, 2, \ldots\}$ is dense in ℓ^2. Consequently (see [29], first volume, page 226), if one denotes

$$\mathrm{d}\rho(\lambda) = \mathrm{d}\langle e_1, P_\lambda e_1\rangle,$$

where $\boldsymbol{J} = \int_{\sigma(J)} \lambda \,\mathrm{d}P_\lambda$, then $\boldsymbol{J}$ is unitarily equivalent to the operator of multiplication by λ in the Hilbert space $L^2\big(\sigma(\boldsymbol{J}), \mathrm{d}\rho(\lambda)\big)$.

The measure $\mathrm{d}\rho$, also known as **spectral measure** of $\boldsymbol{J}$, is a finite measure. More precisely,

$$\rho(\mathbb{R}) = \int_{\mathbb{R}} 1\,\mathrm{d}\rho(\lambda) = \|e_1\|^2 = 1.$$

Thus it makes sense to consider its Borel transform. The Borel transform of $\mathrm{d}\rho$ is a complex function m defined by the formula

$$m(z) = \int_{\sigma(\boldsymbol{J})} \frac{\mathrm{d}\rho(\lambda)}{\lambda - z} = \Big\langle e_1, (\boldsymbol{J} - z)^{-1} e_1 \Big\rangle \tag{C.2}$$

for any $z \in \mathbb{C} \setminus \mathbb{R}$. This function is also called **Weyl m-function** of $\boldsymbol{J}$.

It is important to note that due to our choice of $\lambda_0 = -1$ the vector $\xi(z) = (\boldsymbol{J} - z)^{-1} e_1 \in \ell^2$ solves the recurrence relation

$$\lambda_{n-1}\xi_{n-1}(z) + \lambda_n \xi_{n+1}(z) = z\xi_n(z), \quad n \in \mathbb{N} \tag{C.3}$$

with initial condition $\xi_0(z) = 1$. Thus from the very definition (C.2) it follows that in fact $m(z) = \xi_1(z)$.

From the recurrence relation (C.3) we conclude that the equality

$$\begin{aligned}(\bar{z} - z)|\xi_n(z)|^2 &= \begin{vmatrix} \xi_n(z) & \overline{\xi_n(z)} \\ \lambda_{n-1}\xi_{n-1}(z) + \lambda_n(z)\xi_{n+1}(z) & \overline{\lambda_{n-1}\xi_{n-1}(z) + \lambda_n \xi_{n+1}(z)} \end{vmatrix} = \\ &= \lambda_{n-1}\Big(\xi_n(z)\overline{\xi_{n-1}(z)} - \overline{\xi_n(z)}\xi_{n-1}(z)\Big) + \lambda_n\Big(\xi_n(z)\overline{\xi_{n+1}(z)} - \overline{\xi_n(z)}\xi_{n+1}(z)\Big)\end{aligned}$$

holds for any positive integer n. Summing over $n = 1, 2, 3, \ldots$ and manipulating with the sum on right hand side one obtains the equality

$$(\bar{z} - z)\sum_{n=1}^{\infty} |\xi_n(z)|^2 = \lambda_0\Big(\xi_1(z)\overline{\xi_0(z)} - \overline{\xi_1(z)}\xi_0(z)\Big).$$

Recalling that $\lambda_0 = -1$, $\xi_0(z) = 1$, and $\xi_1(z) = m(z)$ we arrive at an important formula[1]

$$\operatorname{Im} z \cdot \sum_{n=1}^{\infty} |\xi_n(z)|^2 = \operatorname{Im} m(z). \tag{C.4}$$

Since $m(z)$ is the Borel transform of $\mathrm{d}\rho$, one can obtain $\mathrm{d}\rho$ via the Stieltjes inversion formula from $m(z)$ (see [24], in particular page 301). More precisely, the following proposition holds true.

[1]The symbol $\operatorname{Im} z$ denotes the imaginary part of a complex number z.

Proposition C.1: The measure $\frac{1}{\pi}\operatorname{Im} m(\lambda+i\varepsilon)\mathrm{d}\lambda$ (here $\mathrm{d}\lambda$ denotes the Lebesgue measure) converges weakly to $\mathrm{d}\rho(\lambda)$ in the sense that the equality

$$\lim_{\varepsilon\to 0+}\frac{1}{\pi}\int_{-\infty}^{\infty} f(\lambda)\operatorname{Im}\Big(m(\lambda+i\varepsilon)\Big)\mathrm{d}\lambda=\int_{-\infty}^{\infty} f(\lambda)\mathrm{d}\rho(\lambda)$$

holds for any continuous f bounded by const$/(1+\lambda^2)$.

A set $M\subset\mathbb{R}$ is called an **essential support** for $\mathrm{d}\rho$ if M is a support (i.e. $\rho(\mathbb{R}\setminus M)=0$) and any subset $M_0\subset M$ which does not support ρ (i.e. $\rho(M_0)=0$) has Lebesgue measure zero. Let us denote (there is no danger of confusion, since we defined $m(z)$ only for $z\in\mathbb{C}\setminus\mathbb{R}$)

$$\operatorname{Im} m(\lambda):=\limsup_{\varepsilon\to 0+}\operatorname{Im} m(\lambda+i\varepsilon)$$

and let $L(\rho)$ be the set of all $\lambda\in\mathbb{R}$ for which

$$\lim_{\varepsilon\to 0+}\operatorname{Im} m(\lambda+i\varepsilon)$$

exists, be it finite or infinite.

Recall the unique decomposition of the measure $\mathrm{d}\rho$ with respect to the Lebesgue measure,

$$\mathrm{d}\rho=\mathrm{d}\rho_{\mathrm{ac}}+\mathrm{d}\rho_{\mathrm{s}},$$

where ρ_{ac} is absolutely continuous with respect to Lebesgue measure (i.e. every set $A\subset\mathbb{R}$ of zero Lebesgue measure satisfies $\rho_{\mathrm{ac}}(A)=0$) and ρ_{s} is singular with respect Lebesgue measure (i.e. ρ_{s} is supported on a set of zero Lebesgue measure). The singular part can be further decomposed into a singular continuous and pure point part,

$$\mathrm{d}\rho_s=\mathrm{d}\rho_{\mathrm{sc}}+\mathrm{d}\rho_{\mathrm{pp}},$$

where ρ_{sc} is continuous on $\mathbb{R}$ and ρ_{pp} is a step function.

The following Lemma shows the relation between the Weyl m-function and essential supports of various measures (see [24] page 303).

Lemma C.2: Essential supports M, M_{ac}, M_{s} for ρ, ρ_{ac}, ρ_{s} respectively are given by

$$\begin{aligned}M&=\Big\{\lambda\in L(\rho)\,;\,0<\operatorname{Im} m(\lambda)\le\infty\Big\},\\ M_{\mathrm{ac}}&=\Big\{\lambda\in L(\rho)\,;\,0<\operatorname{Im} m(\lambda)<\infty\Big\},\\ M_{\mathrm{s}}&=\Big\{\lambda\in L(\rho)\,;\,\operatorname{Im} m(\lambda)=\infty\Big\}.\end{aligned}$$

An obvious corollary of this Lemma is a criterion for the absolute continuity. More precisely, if $\operatorname{Im} m(\lambda)<\infty$ for all $\lambda\in(a,b)$, then ρ is absolutely continuous in (a,b). In the next Section we will show how to find a bound for the imaginary part of a Weyl function.

C.2 A criterion for the absolute continuity

The main result of this Section is Theorem C.5 and the proof follows [30]. For each $z\in\mathbb{C}$ let us denote by $\{u_1(n,z)\}_n$ and $\{u_2(n,z)\}_n$ solutions of

$$\lambda_{n-1}u_j(n-1,z)+\lambda_n u_j(n+1,z)=zu_j(n,z),\quad n\in\mathbb{N},\ j=1,2,\tag{C.5}$$

with initial conditions

$$u_1(0,z) = 0,\ u_1(1,z) = 1,$$
$$u_2(0,z) = 1,\ u_2(1,z) = 0.$$

We are interested in the set

$$S = \Big\{ x \in \mathbb{R}\,;\, u_1(\cdot, x) \text{ and } u_2(\cdot, x) \text{ are bounded on } \mathbb{N} \Big\}.$$

For a given $z \in \mathbb{C}$ and $n \in \mathbb{N}_0$ set

$$T(z,n,0) := \begin{pmatrix} u_1(n+1,z) & u_2(n+1,z) \\ u_1(n,z) & u_2(n,z) \end{pmatrix}.$$

This matrix is invertible. In fact, for $n = 0$ the very definition of sequences $u_j(z)$ implies the equality $\det T(\lambda, 0, 0) = 1$. On the other hand, taking $n \in \mathbb{N}$ and using the recurrence relation (C.5) n times one obtains $\det T(\lambda, n, 0) = -1/\lambda_n$. Taking into account our convention for λ_0 we see that this relation holds also for $n = 0$.

Thus it makes sense to define

$$T(z,n,m) := T(z,n,0)T(z,m,0)^{-1}, \quad n, m \in \mathbb{N}_0,\ z \in \mathbb{C}.$$

Let us note some properties of these matrices.

Lemma C.3: The matrix $T(z,n,m)$ defined above for $n, m \in \mathbb{N}_0$ and $z \in \mathbb{C}$ is invertible,

$$\det T(z,n,m) = \frac{\lambda_m}{\lambda_n}.$$

If one denotes **the transfer matrix**

$$A(z,n) = \begin{pmatrix} \frac{z}{\lambda_n} & -\frac{\lambda_{n-1}}{\lambda_n} \\ 1 & 0 \end{pmatrix}, \quad n \in \mathbb{N}_0,$$

then $T(z,n,0) = A(z,n)T(z,n-1,0)$ for any $n \in \mathbb{N}$ and

$$T(z,n,m-1) = A(z,n)A(z,n-1)\cdots A(z,m+1)A(z,m).$$

Proof. The first part of the Lemma should be clear from the discussion preceding this Lemma. The rest follows again from the recurrence relation and $T(z,0,0) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ $\square$

Suppose now that $\{\eta_n\}$ solves (C.3) (where ξ is replaced by η) with prescribed initial conditions η_0 and η_1. Then using the uniqueness and linearity we get $\eta_n = \eta_0 u_2(n,z) + \eta_1 u_1(n,z)$ for any $n \in \mathbb{N}_0$. Thus, if we denote

$$\Phi(n) = \begin{pmatrix} \eta_{n+1} \\ \eta_n \end{pmatrix}$$

then

$$\Phi(n) = T(z,n,0) \begin{pmatrix} \eta_1 \\ \eta_0 \end{pmatrix}.$$

Consequently

$$\Phi(n) = T(z, n, m)\Phi(m).$$

For any real x set[2]

$$c(x) := \sup_{n,m\in\mathbb{N}_0} \left\|T(x, n, m)\right\|_2$$

Now we can estimate

$$\begin{aligned}\left\|T(x,n,m)\right\|_2 = \left\|T(x,n,0)T(x,m,0)^{-1}\right\|_2 \le \left\|T(x,n,0)\right\|_2 \left\|T(x,m,0)^{-1}\right\|_2 = \\ = \frac{1}{\lambda_m}\left\|T(x,n,0)\right\|_2 \left\|T(x,m,0)\right\|_2 \le \frac{\alpha}{2}\left(\left\|T(x,n,0)\right\|_2^2 + \left\|T(x,m,0)\right\|_2^2\right)\end{aligned}$$

Therefore $c(x)$ can be estimated as follows

$$c(x) \le \alpha \sup_{n\in\mathbb{N}_0} \|T(x,n,0)\|_2^2.$$

The right hand side of the last inequality is finite if and only if $\lambda \in S$. We are ready to prove the following

Lemma C.4: If x belongs to S, then

$$\liminf_{\varepsilon\to 0_+} \operatorname{Im} m(x + i\varepsilon) \ge \frac{1}{\left(\alpha c(x)\right)^2} \tag{C.6}$$

$$\limsup_{\varepsilon\to 0_+} |m(x + i\varepsilon)| \le \left(\alpha c(x)\right)^2. \tag{C.7}$$

Proof. Suppose that $x \in \mathbb{R}$ and $\varepsilon > 0$. Using Lemma C.3 and observing that

$$A(x + i\varepsilon, n) = \frac{i\varepsilon}{\lambda_n}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + A(x, n)$$

we arrive at the relation

$$T(x + i\varepsilon, n, 0) = T(x, n, 0) + \sum_{j=0}^{n-1} \frac{i\varepsilon}{\lambda_j} T(x, n, j+1)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} T(x + i\varepsilon, j, 0).$$

Consequently

$$\left\|T(x + i\varepsilon, n, 0)\right\|_2 \le c(x) + \varepsilon\alpha c(x)\sum_{j=0}^{n-1}\left\|T(x + i\varepsilon, j, 0)\right\|_2$$

and then it follows that

$$\|T(x + i\varepsilon, n, 0)\|_2 \le \sum_{k=0}^{n}\binom{n}{k} c(x)^{k+1}(\varepsilon\alpha)^k = c(x)\left(1 + \varepsilon\alpha c(x)\right)^n \le c(x)e^{\varepsilon n\alpha c(x)}. \tag{C.8}$$

[2]Here we use the Hilbert-Schmidt norm, for $A \in \mathrm{Mat}(\mathbb{C}, 2)$, $A = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ this is defined by

$$\|A\|_2 = \sqrt{|a^2| + |b|^2 + |c|^2 + |d|^2}.$$

Recall that the operator norm ($\|\cdot\|$, the norm in $\mathscr{B}(\mathbb{C}^2)$ induced by the usual Euclidean norm of $\mathbb{C}^2$) satisfies $\|A\| \le \|A\|_2$. If the matrix A is invertible, with $\alpha = \det A \ne 0$, then it immediately follows from the definition and the explicit form of A^{-1} that $\|A^{-1}\|_2 = \|A\|_2/|\alpha|$.

Indeed, suppose that we are given $\beta, \gamma \geq 0$ and a sequence $a_n \geq 0$, $n \in \mathbb{N}_0$ such that it satisfies

$$a_n \leq \beta + \beta\gamma \sum_{j=0}^{n-1} a_j$$

for any $n \in \mathbb{N}$ and $a_0 \leq \beta$. Then it holds that

$$a_n \leq \sum_{j=0}^{n} \binom{n}{j} \beta^{j+1}\gamma^j.$$

Now recall the definition of $\xi(z)$, in Equation (C.3). We have the equality

$$\begin{pmatrix} \xi_{n+1}(x+i\varepsilon) \\ \xi_n(x+i\varepsilon) \end{pmatrix} = T(x+i\varepsilon, n, 0) \begin{pmatrix} m(x+i\varepsilon) \\ 1 \end{pmatrix}.$$

And so

$$\left\| \begin{pmatrix} m(x+i\varepsilon) \\ 1 \end{pmatrix} \right\| \leq \left\| T(x+i\varepsilon, n, 0)^{-1} \right\|_2 \cdot \left\| \begin{pmatrix} \xi_{n+1}(x+i\varepsilon) \\ \xi_n(x+i\varepsilon) \end{pmatrix} \right\| \leq$$
$$\leq \alpha c(x) e^{\varepsilon n \alpha c(x)} \left\| \begin{pmatrix} \xi_{n+1}(x+i\varepsilon) \\ \xi_n(x+i\varepsilon) \end{pmatrix} \right\|.$$

where we have again used the relation between Hilbert-Schmidt norm of the matrix and its inverse, and the estimate (C.8). We have arrived at

$$\left\| \begin{pmatrix} \xi_{n+1}(x+i\varepsilon) \\ \xi_n(x+i\varepsilon) \end{pmatrix} \right\| \geq \frac{e^{-\varepsilon n \alpha c(x)}}{\alpha c(x)} \sqrt{1 + |m(x+i\varepsilon)|^2}.$$

Squaring and then summing over $n = 1, 3, 5, \ldots$ we finally get

$$\sum_{n=1}^{\infty} |\xi(x+i\varepsilon)|^2 \geq \frac{1}{\alpha c(x)} \cdot \frac{1}{e^{\varepsilon \alpha c(x)} - 1} \big(1 + |m(x+i\varepsilon)|^2\big).$$

And using the formula (C.4)

$$\operatorname{Im} m(x+i\varepsilon) \geq \frac{1}{\big(\alpha c(x)\big)^2} \cdot \frac{\varepsilon \alpha c(x)}{e^{\varepsilon \alpha c(x)} - 1} \big(1 + |m(x+i\varepsilon)|^2\big).$$

Consequently

$$\liminf_{\varepsilon \to 0+} \frac{\operatorname{Im} m(x+i\varepsilon)}{1 + \big|m(x+i\varepsilon)\big|^2} \geq \frac{1}{\big(\alpha c(x)\big)^2}.$$

Since $1 + |m(x+i\varepsilon)|^2 \geq 1$ this immediately implies (C.6). Because[3]

$$\frac{1 + \big|m(x+i\varepsilon)\big|^2}{\operatorname{Im} m(x+i\varepsilon)} \geq |m(x+i\varepsilon)|$$

we also get (C.7). This completes the proof of the Theorem. □

[3]For $z \in \mathbb{C}_+$ we have $|z| \leq |z| \cdot \frac{|z|}{\operatorname{Im} z} \leq \frac{1+|z|^2}{\operatorname{Im} z}$. Since we assume $\varepsilon > 0$ and $\lambda \in \mathbb{R}$ it indeed holds $\operatorname{Im} m(\lambda + i\varepsilon) > 0$, cf. (C.4).

Finally, we can state the main theorem of the present Section.

Theorem C.5 ([30])**:** On S, the spectral measure ρ for $\boldsymbol{J}$ is purely absolutely continuous in the sense that

(i) $\rho_{\mathrm{ac}}(T) > 0$ for any $T \subset S$ with $|T| > 0$ (where $|\cdot|$ is the Lebesgue measure),

(ii) $\rho_{\mathrm{s}}(S) = 0$.

Proof. Since for any $\lambda \in S$ we have proved the estimate (C.7) the Lemma C.2 implies that $\rho_{\mathrm{s}}(S) = 0$.

Let $T \subset S$ with nonzero Lebesgue measure. Take any sequence $\{f_n\}_{n=1}^{\infty}$ of continuous functions with compact support such that $|f_n(x)| \leq 1$ and $\lim_{n\to\infty} f_n(x) = \chi_T(x)$ for almost all $x \in \mathbb{R}$. Then using the Lebesgue theorem and Proposition C.1 and Equation (C.6) we obtain

$$\begin{aligned}\rho(T) &= \int_{\mathbb{R}} \chi_T(x)\mathrm{d}\rho(x) = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\mathrm{d}\rho(x) = \\ &= \frac{1}{\pi} \lim_{n\to\infty} \lim_{\epsilon\to 0+} \int_{\mathbb{R}} f_n(x)\,\mathrm{Im}\, m(x+i\epsilon)\mathrm{d}x \geq \frac{1}{\alpha\pi} \cdot \sup_{x\in T} \frac{1}{c(x)^2} \cdot |T|. \qquad \square\end{aligned}$$

C.3 Application to our example

Let us now turn our attention to our particular example. We assume that the sequence $\{\lambda_n\}_{n=1}^{\infty}$ satisfies all conditions mentioned at the beginning of this Appendix.

Theorem C.6: Let $\lambda \in (-2, 2)$ be fixed and denote by u_n the solution of recurrence relation

$$\lambda_{n-1}u_{n-1} + \lambda_n u_{n+1} = \lambda u_n, \quad n \in \mathbb{N}$$

with prescribed initial conditions $u_0, u_1 \in \mathbb{C}$. If the series

$$\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n|$$

converges, then $\{u_n\}_{n=1}^{\infty}$ is bounded.

Proof. We have already seen how to express the solution using the transfer matrix $A(\lambda, n)$, in particular

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A(\lambda, n)A(\lambda, n-1)\cdots A(\lambda, 2)A(\lambda, 1)\begin{pmatrix} u_1 \\ u_0 \end{pmatrix}, \quad n \in \mathbb{N}. \tag{C.9}$$

Recall (Lemma C.3) that

$$A(\lambda, n) = \begin{pmatrix} \frac{\lambda}{\lambda_n} & -\frac{\lambda_{n-1}}{\lambda_n} \\ 1 & 0 \end{pmatrix}.$$

In the rest of the proof I will suppress the dependence on λ. Looking at the discriminant of the characteristic polynomial,

$$D_n = \frac{\lambda^2}{\lambda_n^2} - 4\frac{\lambda_{n-1}}{\lambda_n} \to \lambda^2 - 4 < 0, \quad \text{as } n \to \infty,$$

we see that there exists $n_0 \in \mathbb{N}$ (which depends only on the sequence $\{\lambda_n\}$ and $\lambda \in (-2, 2)$) such that for all $n > n_0$ we have $D_n < 0$, hence the transfer matrix $A(\lambda, n)$ is diagonalizable. Eigenvalues of $A(\lambda, n)$, $n > n_0$ are

$$\eta_\pm(n) = \frac{1}{2}\left(\frac{\lambda}{\lambda_n} \pm i\sqrt{4\frac{\lambda_{n-1}}{\lambda_n} - \frac{\lambda^2}{\lambda_n^2}}\right).$$

Note that since $\eta_-(n) = \overline{\eta_+(n)}$ both eigenvalues have the same modulus, in particular

$$|\eta_\pm(n)| = \frac{1}{2}\sqrt{\frac{\lambda^2}{\lambda_n^2} + 4\frac{\lambda_{n-1}}{\lambda_n} - \frac{\lambda^2}{\lambda_n^2}} = \sqrt{\frac{\lambda_{n-1}}{\lambda_n}}.$$

Corresponding eigenvectors are

$$v_+(n) = \begin{pmatrix} \eta_+(n) \\ 1 \end{pmatrix} \quad \text{and} \quad v_-(n) = \begin{pmatrix} \eta_-(n) \\ 1 \end{pmatrix}.$$

Thus $A(\lambda, n) = T_n B_n T_n^{-1}$, where $B_n = \operatorname{diag}\big(\eta_+(n), \eta_-(n)\big)$ and $T_n = \big(v_+(n), v_-(n)\big)$.

Equation (C.9) implies

$$\begin{aligned} \left\| \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} \right\| &\le \|T_n B_n T_n^{-1} T_{n-1} B_{n-1} T_{n-1}^{-1} \cdots T_{n_0+1} B_{n_0+1} T_{n_0+1}^{-1}\| \\ &\qquad \times \|A(\lambda, n_0) \cdots A(\lambda, 1)\| \sqrt{u_1^2 + u_0^2} \qquad \text{(C.10)} \\ &\le \|T_n\| \left(\prod_{k=n_0+2}^{n} \|T_k^{-1} T_{k-1}\| \right) \left(\prod_{k=n_0+1}^{n} \|B_k\| \right) \operatorname{const}(\lambda, n_0, u_0, u_1) \qquad \text{(C.11)} \end{aligned}$$

The second product behaves nicely,

$$\prod_{k=n_0+1}^{n} \|B_n\| = \prod_{k=n_0+1}^{n} \sqrt{\frac{\lambda_{k-1}}{\lambda_k}} = \sqrt{\frac{\lambda_{n_0}}{\lambda_n}} = O(1), \quad n \to \infty.$$

The first one is a little bit more involved. First of all, observe that (recall $\det T_n = \eta_+(n) - \eta_-(n) = i\sqrt{|D_n|}$)

$$T_{n+1}^{-1} T_n = \frac{1}{i\sqrt{|D_{n+1}|}} \begin{pmatrix} \eta_+(n) - \eta_-(n+1) & \eta_-(n) - \eta_-(n+1) \\ -\eta_+(n) + \eta_+(n+1) & -\eta_-(n) + \eta_+(n+1) \end{pmatrix}.$$

This matrix converges to the identity matrix. We need to estimate the rate of this convergence. So

$$\left\|T_{n+1}^{-1} T_n\right\| \le 1 + \left\|T_{n+1}^{-1} T_n - I\right\|$$

Note that the matrix $T_{n+1}^{-1} T_n - I$ is of the form $\left(\begin{smallmatrix} a_n & b_n \\ \bar{b}_n & \bar{a}_n \end{smallmatrix}\right)$. Its norm can be estimated by its Hilbert-Schmidt norm $\sqrt{2}\sqrt{|a_n|^2 + |b_n|^2}$. In our case we have

$$\begin{aligned} a_n &= \frac{1}{i\sqrt{|D_n|}}\big(\eta_+(n) - \eta_-(n+1)\big) - 1, \\ b_n &= \frac{1}{i\sqrt{|D_n|}}\big(\eta_-(n) - \eta_-(n+1)\big). \end{aligned}$$

It is straight forward to show that

$$a_n = O\Big(|\lambda_{n+1} - \lambda_n| + |\lambda_n - \lambda_{n-1}|\Big),$$
$$b_n = O\Big(|\lambda_{n+1} - \lambda_n| + |\lambda_n - \lambda_{n-1}|\Big).$$

Hence

$$\left\|T_{n+1}^{-1} T_n\right\| = 1 + O\Big(|\lambda_{n+1} - \lambda_n| + |\lambda_n - \lambda_{n-1}|\Big).$$

The boundedness of (C.11) now follows from the convergence of (it is our assumption!) the series $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n|$. In fact[4],

$$\prod_{n=1}^{\infty} \Big(1 + \gamma|\lambda_{n+1} - \lambda_n|\Big) = \exp \sum_{n=1}^{\infty} \ln \Big(1 + \gamma|\lambda_{n+1} - \lambda_n|\Big) \leq \exp \left(\gamma \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n|\right),$$

with $\gamma > 0$. □

Remark C.7: Theorem C.6 is in particular applicable to the case of $\lambda_n = \sqrt{\frac{n}{n+\alpha}}$, $\alpha > 0$. In fact

$$\lambda_{n+1} - \lambda_n = O(1/n^2),$$

as $n \to \infty$. This example occurs in the Chapter 2.

Remark C.8: Also note that in our case we have $0 < \lambda_n \leq 1$ for all $n \in \mathbb{N}$. This implies that $\pm 2 \notin \sigma_{\mathrm{pp}}(J)$ and hence we have complete information about the spectrum.

Indeed. Let u be a solution of the eigenvalue equation $Ju = 2u$ with $u_1 = 1$ (this is no restriction, any other solution of this equation is a scalar multiple of this one). We will prove by induction that $u_{n+1} \geq u_n$ holds for any $n \in \mathbb{N}$ and thus $u \notin l^2$.

- In fact, since $\lambda_1 u_2 = 2u_1$ we have $u_2 \geq u_1$.
- Suppose that $u_{n+1} \geq u_n$ holds for $n \in \mathbb{N}$. Then

$$u_{n+2} = \frac{1}{\lambda_{n+1}}\Big(2u_{n+1} - \lambda_n u_n\Big) \geq u_{n+1} + \left(\frac{2}{\lambda_{n+1}} - 1\right)u_{n+1} - \frac{\lambda_n}{\lambda_{n+1}} u_n.$$

  Since $2/\lambda_{n+1} - 1$ is positive we can use our induction assumption to conclude that

$$u_{n+2} \geq u_{n+1} + \left(\frac{2}{\lambda_{n+1}} - 1 - \frac{\lambda_n}{\lambda_{n+1}}\right)u_n = u_{n+1} + \frac{2 - \lambda_n - \lambda_{n+1}}{\lambda_{n+2}} u_n \geq u_{n+1},$$

  since we know that this u_n and the fraction are positive.

The case of $\lambda = -2$ can be dealt with similarly.

[4] $\ln(1 + x) < x$ for $x > 0$.

C.4 Asymptotic analysis of the generalized eigenvectors

According to previous Sections it is necessary to study the boundedness of generalized eigenvectors of $\boldsymbol{J}$. This Section is thus devoted to the asymptotic analysis of those eigenvectors. This problem is of course interesting in its own right and the results of the Chapter 2 are based on the information obtained in the current Section. Our exposition roughly follows [27].

In accordance with [27] we introduce a number of useful notions. Suppose that $\xi = \{\xi_n\}_{n=1}^{\infty}$ is a sequence in a Banach space $(\mathscr{X}, \|\cdot\|)$. We shall say that $\xi \in l^1$ if and only if $\sum_{n=1}^{\infty} \|\xi_n\| < \infty$. Let Δ be the forward difference operator, $(\Delta\xi)_n := \xi_{n+1} - \xi_n$ for each $n \geq 1$. We shall say that sequence ξ belongs to the class D^1 if and only if it is bounded and $\Delta\xi \in l^1$.

Our interest in this section is the asymptotic behavior of solutions of the recurrence equation

$$x_{n+1} = A(n)x_n, \quad n \geq 1,$$

where $A(n) = V(n) + R(n)$ is a complex 2×2 matrix, $n \geq 1$, $V \in D^1$, and $R \in l^1$. Furthemore, we assume that $V(n)$ is a real matrix and that[5]

$$\limsup_{n\to\infty} \operatorname{discr} V(n) < 0 \quad \text{and} \quad \lim_{n\to\infty} V(n) = V_\infty.$$

Under these assumptions it follows that there is $n_0 \geq 1$ such that $V(n)$ has two distinct (complex conjugate) eigenvalues $\eta_\pm(n)$, $\eta_-(n) = \overline{\eta_+(n)}$. Thus $|\eta_+(n)| = |\eta_-(n)|$. In the rest of these notes we will need the following Lemma (when we are dealing with our particular example from the previous Section, we can check conclusions of this Lemma directly, see the last Remark of this section).

Lemma C.9 ([27], Lemma 1.7)**:** Let $B = \{B(n)\}_{n\geq n_0}$ be a sequence of $d \times d$ complex matrices, $\lim_{n\to\infty} B(n) = B_\infty$. If $B \in D^1$ and B_∞ has d different eigenvalues $\mu_1, \dots, \mu_d$, then there exist $n_1 \geq n_0$, a sequence of diagonal matrices $\Lambda = \{\Lambda(n)\}_{n\geq n_1} \in D_1$, and of invertible matrices $T = \{T(n)\}_{n\geq n_1} \in D_1$ such that

$$B(n) = T(n)\Lambda(n)T(n)^{-1}, \quad n \geq n_1,$$

$\lim_{n\to\infty} \Lambda(n) = \Lambda_\infty := \operatorname{diag}(\mu_1, \dots, \mu_d)$ and $\lim_{n\to\infty} T(n) = T_\infty$, where T_∞ is invertible and $B_\infty = T_\infty \Lambda_\infty T_\infty^{-1}$.

As a first step we will make a "change of variables" so that the new matrix is partially diagonalized. According to the Lemma above there is some $n_1 \geq n_0$ such that $V(n) = T(n)\Lambda(n)T(n)^{-1}$, $n \geq n_1$, where $T(n)$ are invertible $\{T(n)\}_{n\geq n_1} \in D^1$ and

$$\Lambda(n) = \operatorname{diag}(\eta_+(n), \eta_-(n)).$$

Moreover $\lim_{n\to\infty} \Lambda(n) = \Lambda_\infty$, $\lim_{n\to\infty} T(n) = T_\infty$ and $V_\infty = T_\infty \Lambda_\infty T_\infty^{-1}$. We set

$$y_n := T(n)^{-1}x_n, \quad n \geq n_1.$$

[5] If A is a 2×2 complex matrix then, by $\operatorname{discr} A$ we denote the discriminant of the characteristic polynomial of A.

Thus the sequence y satisfies a new recurrence equation

$$y_{n+1} = \Big(\Lambda(n) + R'(n)\Big)y_n,\ n \ge n_1,$$

where

$$R'(n) = \Big(T(n+1)^{-1} - T(n)^{-1}\Big)A(n)T(n) + T(n+1)^{-1}R(n)T(n),\ n \ge n_1,$$

is again of class l^1, that is $R' \in l^1$.

Let us now denote

$$\varphi_\pm(n,m) := \prod_{j=m}^{n} \eta_\pm(j), \quad \Phi(n,m) := \prod_{j=m}^{n} \Lambda(j) = \operatorname{diag}\Big(\varphi_+(n,m), \varphi_-(n,m)\Big), \quad n \ge m,$$

and for $n < m$ we set $\varphi_\pm(n,m) = 1$ and $\Phi(n,m) = \operatorname{diag}(1,1)$. Note that the order of multiplication in the matrix product is irrelevant because Λ's commute. Let us fix a sequence $\gamma = \{\gamma(n)\}_{n\ge n_1} \subset \mathbb{C}$, $\gamma(n) \neq 0$, $n \ge n_1$, and set (N_0 will be specified below)

$$\mathscr{X}_\gamma = \Big\{\xi = \{\xi_n\}_{n\ge N_0} \subset \mathbb{C}^2\,;\ \|\xi\|_\gamma := \sup_{n\ge N_0} \frac{\|\xi_n\|}{|\gamma(n)|} < \infty\Big\}.$$

$\mathscr{X}_\gamma$ is a linear space and together with the norm $\|\cdot\|_\gamma$ forms a Banach space. Our next step is essentially contained in the following Lemma.

Lemma C.10: Denote[6]

$$\beta(n) := \frac{1}{|\gamma(n)|}\sum_{j=n}^{\infty}\left|\frac{\gamma(j)}{\varphi_+(n,j)}\right|\|R'(j)\| \tag{C.12}$$

and suppose that there is $N_0 \ge n_1$ such that

$$\sup_{n\ge N_0} \beta(n) = \beta_* < 1. \tag{C.13}$$

Then the linear operator $L : \mathscr{X}_\gamma \to \mathscr{X}_\gamma$,

$$(L\xi)(n) := \sum_{j=n}^{\infty} \Phi(n-1,N_0)\Phi(j,N_0)^{-1}R'(j)\xi(j),\ \xi \in \mathscr{X}_\gamma,\ n \ge N_0, \tag{C.14}$$

is well defined and

$$\|L\|_\gamma \le \beta_* < 1.$$

Proof. Let us first check the convergence of the series in (C.14). The norm ($\|\cdot\|$ to be precise) of the summand can be estimated by (recall that our eigenfunctions have identical modulus)

$$\le \left|\frac{\varphi_+(n-1,N_0)}{\varphi_+(j,N_0)}\right| \|R'(j)\|\,\|\xi(j)\| \le \left|\frac{\gamma(j)}{\varphi_+(j,n)}\right|\|R'(j)\|\,\|\xi\|_\gamma.$$

[6]This definition is correct, $\beta(n)$ is either finite or infinite.

Using the assumption (C.13) one see that indeed the series in (C.14) converges. Consequently

$$\frac{\|(L\xi)(n)\|}{|\gamma(n)|} \le \beta(n)\|\xi\|_\gamma \le \beta_*\|\xi\|_\gamma,\ n \ge N_0$$

and so $L\xi \in \mathscr{X}_\gamma$ with

$$\|L\xi\|_\gamma \le \beta_*\|\xi\|_\gamma.$$

Hence for the induced operator norm we infer that $\|L\|_\gamma \le \beta_* < 1$. □

Suppose that the assumptions of the preceding Lemma are satisfied and moreover that

$$\sup_{n\ge N_0}\left|\frac{\varphi_+(n-1,N_0)}{\gamma(n)}\right| = \alpha < \infty. \tag{C.15}$$

Then the vectors

$$b_+(n) = \varphi_+(n-1,N_0)e_1, \quad b_-(n) = \varphi_-(n-1,N_0)e_2, \quad n \ge N_0$$

belong to $\mathscr{X}_\gamma$. According to the Lemma C.10 the equation

$$(L+I)y_\pm = b_\pm$$

has a unique solution in $\mathscr{X}_\gamma$. In fact, $-1 \notin \sigma(L)$. Moreover $y_\pm \ne 0$ and

$$\|y_\pm\|_\gamma = \left\|(L+I)^{-1}b_\pm\right\|_\gamma \le \left\|(L+I)^{-1}\right\|_\gamma \|b_\pm\|_\gamma \le \frac{\alpha}{1-\beta_*}.$$

In other words, there are unique sequences $y_\pm \in \mathscr{X}_\gamma$ such that

$$y_\pm(n) = b_\pm(n) - (Ly_\pm)(n) = b_\pm(n) - \sum_{j=n}^{\infty}\Phi(n-1,N_0)\Phi(j,N_0)^{-1}R'(j)y_\pm(j),\ n \ge N_0 \tag{C.16}$$

and $y_\pm = O(\gamma)$, where O is with respect to the original norm $\|\cdot\|$.

Let us now show, that these sequences $y_\pm$ solve the equation

$$y_\pm(n+1) = \Big(\Lambda(n) + R'(n)\Big)y_\pm(n), \quad n \ge N_0.$$

This can be verified by a direct computation. In fact,

$$\begin{aligned}
&\Big(\Lambda(n) + R'(n)\Big)y_\pm(n) = \\
&= \Big(\Lambda(n) + R'(n)\Big)b_\pm(n) - \sum_{j=n}^{\infty}\Phi(n,N_0)\Phi(j,N_0)^{-1}R'(j)y_\pm(j) - R'(n)\Big(Ly_\pm\Big)(n) \\
&= \Lambda(n)b_\pm(n) + R'(n)\Big(b_\pm(n) - y_\pm(n) - (Ly_\pm)(n)\Big) - (Ly_\pm)(n+1) \\
&= b_\pm(n+1) - (Ly_\pm)(n+1) = y_\pm(n+1).
\end{aligned}$$

Let us apply this result to our particular example of the previous Section. In this case we have $|\eta_\pm(n)| = 1$ for $n \ge n_1$. Since $R' \in D^1$ we can choose $\gamma \equiv 1$ (then both conditions (C.15) and (C.13) are satisfied) and from (C.16) we conclude that

$$y_\pm(n) - b_\pm(n) = O\Big(\sum_{j=n}^{\infty}\|R'(j)\|\Big), \quad \text{as } n \to \infty.$$

Passing back to the original "coordinates" we get

$$x_\pm(n) = \varphi_\pm(n-1, N_0)T(n)e_{\frac{1}{2}} + O\Big(\sum_{j=n}^{\infty} \|R'(j)\|\Big), \quad \text{as } n \to \infty.$$

This can be rewritten to the following form

$$x_\pm(n) = \begin{pmatrix} \varphi_\pm(n, N_0) \\ \varphi_\pm(n-1, N_0) \end{pmatrix} + O\Big(\sum_{j=n}^{\infty} \|R'(j)\|\Big).$$

Remark C.11: In our particular case of

$$A(n) = \begin{pmatrix} \frac{\lambda}{\lambda_n} & -\frac{\lambda_{n-1}}{\lambda_n} \\ 1 & 0 \end{pmatrix},$$

where $\lambda \in (-2, 2)$ and $0 < \lambda_n < 1$ for each $n \geq 1$ and

$$\lim_{n\to\infty} \lambda_n = 1, \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

we take

$$V(n) := \begin{pmatrix} \frac{\lambda}{\lambda_n} & -1 \\ 1 & 0 \end{pmatrix} \qquad\qquad R(n) := \begin{pmatrix} 0 & 1 - \frac{\lambda_{n-1}}{\lambda_n} \\ 0 & 0 \end{pmatrix}$$

Surely we have $V \in D^1$ and $R \in l^1$. Moreover,

$$\operatorname{discr} V(n) = \frac{\lambda^2}{\lambda_n^2} - 4 \to \lambda^2 - 4 < 0, \quad V_\infty = \lim_{n\to\infty} V(n) = \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}.$$

There exists $n_0 \geq 1$ such that the $\operatorname{discr} V(n) < 0$ for any $n \geq n_0$ and $V(n)$ has two complex conjugate unimodular eigenvalues

$$\eta_\pm(n) = \frac{1}{2}\left(\frac{\lambda}{\lambda_n} \pm i\sqrt{4 - \frac{\lambda^2}{\lambda_n^2}}\right).$$

Corresponding eigenvectors are

$$v_\pm(n) = \begin{pmatrix} \eta_\pm(n) \\ 1 \end{pmatrix}.$$

We conclude that there are two linearly independent solutions of the generalized eigenfunction equation with asymptotics

$$u_\pm(n) = \varphi_\pm(n-1, N_0) + o(1).$$

Taking

$$\lambda_n = \sqrt{\frac{n}{n+\alpha}}, \;\; \alpha > 0,$$

we obtain

$$\eta_{\pm}(n) = \frac{1}{2}\Big(\lambda \pm i\sqrt{4-\lambda^2}\Big) + \frac{\alpha\lambda}{4n}\Big(1 \mp \frac{i\lambda}{\sqrt{4-\lambda^2}}\Big) + O(1/n^2)$$

as $n \to \infty$.

In order to somewhat simplify our equations let us use $\theta \in (0, \pi)$ to denote rescale the parameter $\lambda = 2\cos\theta \in (-2, 2)$. With this choice we obtain

$$\begin{aligned}\eta_{\pm}(n) &= \cos\theta \pm i\sin\theta + \frac{\alpha\cos\theta}{2n}\Big(1 \mp \frac{i\cos\theta}{\sin\theta}\Big) + O(1/n^2) = \\ &= e^{\pm i\theta}\Big(1 \mp \frac{i\alpha}{2n}\operatorname{ctg}\theta\Big) + O(1/n^2), \quad n \to \infty.\end{aligned}$$

Consequently,

$$\varphi_{\pm}(n, 1) = \prod_{k=1}^{n} \eta_{\pm}(k) = \exp\Bigg(\pm in\theta \mp \frac{i\alpha}{2}\operatorname{ctg}\theta \ln n + f(\theta)\Bigg) + o(1),$$

as $n \to \infty$

Acknowledgement

I would like to thank my supervisor professor Šťovíček for numerous helpful discussions and patience through the whole time of my PhD study. I also wish to thank professor Asch for his support and discussions during my stays in CPT Marseille. Next, I would like to acknowledge support from the grant No. 202/08/H072 of the Czech Science Foundation and grant No. LC06002 of the Ministry of Education of the Czech Republic. Last but not least, I thank my family and friends for their encouragement during my studies.

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