Czech Technical University in Prague
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Applications of the WKB method in scattering theory for quantum 2D models

Research Project

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Declaration

I declare that I wrote my research project independently and exclusively with use of the cited bibliography.

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Matej Hazala

Title: **Scattering on a magnetic Aharonov-Bohm flux in a plane**

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Obor: Mathemaical Physics
Druh práce: Research Project

Abstract: We present a general formulation of the stationary scattering on finite cylinder with magnetic flux in two dimensions. By using partial wave decomposition we break up the task into songle modes, then derive a solution of the corresponding Schrödinger. Threating inside potential area and neighbour hood of infinity separately, we use WKB method to find approximate solution inside cylinder potential, and in neoghbourhood of infinity WKB method is used to find phase shift approximation, followed by taking a look at large energy behaviour of the phase shift and numerical comparison with exact expression. In the end, scattering amplitude is determined and consequently given differential cross section.

Key words: scattering theory, WKB approximation

Contents

Introduction

Main goal of this project is to investigate 2D quantum scattergin on the orthogonal finite cylinder potential with magnetic flux contained inside and to apply WKB method in order to conveniently solve this task.

First chapter contains most important relations mainly of special functions which are extensively used thorugout the rest of the document.

In second chapter is posed the task we are dealing with mathemathicly, giving us equation we are interested in.

Chapter number three is dedicated to solving the equation inside the cylinder potential, especially in the neoghbourhood of zero were comparison of exact and WKB based solution is given.

Fourth chapter deals with the asymptotic region in the neogbourhood of the infinity. Brief application of general asymptotic scattering approach leads to exact expression of phase shift and realtions with scattering amplitude, which will serve the determining of differential cross section later on.

In the fifth chapter WKB approximation of phase shift is made, followed by determining leading terms for large energy behaviour of error term and phase shift itself. At the and of the chapter is also a numerical comparison of approximation with exact expression.

Sisth chapter gives us desired approximate expressions of scattering amplitude based on previous results followed by determining croos section.

Chapter 1

Important relations

This chapter contans well known realtions and properties, mainly of Bessel functions, which are extensively used throughout the whole document. Theorems will be given gradually in chapters where used.

C_ν, D_ν denote any of Bessel Functions $J_\nu, Y_\nu, \nu \in \mathbb{R}$, for which following relations are taken from ([9], Chp. 10).

Asymptotics

$z \to 0$:

$$\begin{aligned} &J_0(z) \to 1, && J_\nu(z) \sim \left(\frac{z}{2}\right)^\nu / \Gamma(\nu+1), && \nu \neq -1, -2, -3, ... \\ &Y_0(z) \sim \frac{2}{\pi} \ln z, && Y_\nu(z) \sim -\left(\frac{z}{2}\right)^{-\nu} \frac{\Gamma(\nu)}{\pi}, && \nu > 0 \end{aligned} \tag{1.0.1}$$

$z \to \infty$:

$$\begin{aligned} J_\nu(z) &= \sqrt{\frac{2}{\pi z}} \left[\sin\left(z - \frac{\pi}{2}(\nu - 1/2)\right) + e^{|\mathrm{Im} z|} O(1)\right] \\ Y_\nu(z) &= \sqrt{\frac{2}{\pi z}} \left[\cos\left(z - \frac{\pi}{2}(\nu - 1/2)\right) + e^{|\mathrm{Im} z|} O(1)\right] \end{aligned} \tag{1.0.2}$$

Recurencies

$$C_{-n}(z) = (-1)^n C_n, \qquad n \in \mathbb{N}, \qquad C_{\nu-1}(z) + C_{\nu+1}(z) = \frac{2\nu}{z} C_\nu(z) \tag{1.0.3}$$

Derivatives

$$\begin{aligned} C'_\nu(z) &= -C_{\nu+1}(z) + \frac{\nu}{z} C_\nu(z) \\ &= +C_{\nu-1}(z) - \frac{\nu}{z} C_\nu(z) \end{aligned} \tag{1.0.4}$$

$$\begin{aligned} \frac{2}{\pi z} &= J_{\nu+1}(z) Y_\nu(z) - J_\nu(z) Y_{\nu+1}(z) \\ &= J_\nu(z) Y'_\nu(z) - J'_\nu(z) Y_\nu(z) \end{aligned} \tag{1.0.5}$$

Integrals

$$\int z^{\nu+1}C_\nu(z)dz = z^{\nu+1}C_{\nu+1}(z) \tag{1.0.6}$$

$$\int zC_\mu(az)D_\mu(az)dz = \frac{z^2}{4}\left[2C_\mu(az)D_\mu(az) - C_{\mu-1}(az)D_{\mu+1}(az) - C_{\mu+1}(az)D_{\mu-1}(az)\right] \tag{1.0.7}$$

$$\int z^{\mu+\nu+1}C_\mu(az)D_\nu(az)dz = \frac{z^{\mu+\nu+2}}{2(\mu+\nu+1)}\left[C_\mu(az)D_\nu(az) + C_{\mu+1}(az)D_{\nu+1}(az)\right], \qquad \mu+\nu \neq -1 \tag{1.0.8}$$

$$\int\limits_x^{+\infty} \frac{J_0(t)}{t}dt = -\gamma - \ln\left(\frac{x}{2}\right) + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k(k!)^2}\left(\frac{x}{2}\right)^{2k} \tag{1.0.9}$$

Series

$$\begin{aligned}\cos(z\cos\theta) &= J_0(z) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(z)\cos(2n\theta)\\ \sin(z\cos\theta) &= 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(z)\cos(2(n+1)\theta)\end{aligned} \tag{1.0.10}$$

Graf's addition theorem

$$C_\nu(w)\genfrac{}{}{0pt}{}{\cos}{\sin}(\nu\chi) = \sum_{n=-\infty}^{\infty} C_{n+\nu}(u)J_n(v)\genfrac{}{}{0pt}{}{\cos}{\sin}(n\theta)\ , \qquad \left|ve^{\pm i\theta}\right| < |u|\ , \tag{1.0.11}$$

$$\text{where} \quad w = \sqrt{u^2+v^2-2uv\cos\theta}\ , \quad u - v\cos\theta = w\cos\chi\ , \quad v\sin\theta = w\sin\chi$$

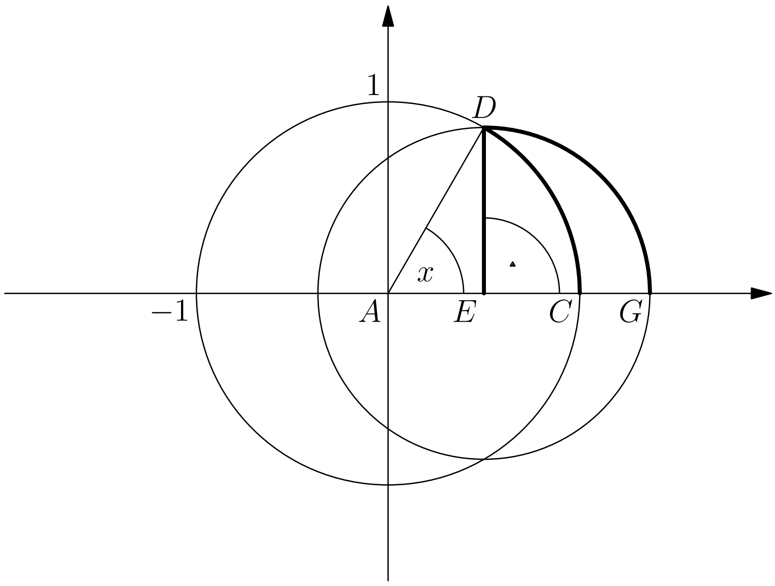
in special case $u = v$:

$$C_\nu\left(2u\sin\frac{\theta}{2}\right)e^{i\nu(\pi-\theta)/2} = \sum_{n=-\infty}^{\infty} C_{n+\nu}(u)J_n(u)e^{in\theta} \tag{1.0.12}$$

Jordan's inequality

$$\frac{2}{\pi}x \leq \sin x \leq x, \qquad x \in \left[0, \frac{\pi}{2}\right] \tag{1.0.13}$$

Proof. Simply prooven by the picture of unit circle, angle x and anoter circle of radius $\sin x$.



$$|DE| \leq |\widehat{DC}| \leq |\widehat{DG}| \qquad \Leftrightarrow \qquad \sin x \leq x \leq \frac{\pi}{2}x \qquad \Rightarrow \qquad \frac{2}{\pi}x \leq \sin x \leq x$$

□

Chapter 2

Scattering setup

General backgroung to scattering theory in this and following chapters is given in [2, 4, 6].

2.1 Equation

Problem of our interest is a stationary formulation of the scattering problem for a charged mass m particle with a positive energy $E = k^2$ on a finite cylindric potential potential V with embeded magnetix flux μ, perpendicular to the trajectory of the particle. In order to simplify notation, $m = \frac{1}{2}$, $\hbar = 1$. This setup means solving Schrödinger equation, which in cylindrical coordinates (r, θ, z), where z-axis coincide with axis of cylinder and vector of magnetic induction $\vec{B} = \nabla \times \vec{A}$, has a following form:

$$-\left[r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \left(\frac{\partial}{\partial \theta} - iA_\theta(r)\right)^2 - V(r)\right] \psi(r, \theta) = k^2 \psi(r, \theta), \tag{2.1.1}$$

where

$$\begin{aligned} A_\theta(r) &= \mu \frac{r^2}{X^2}, \quad r < X, & V(r) &= V, \quad r < X, \\ &= \mu, \qquad r > X, & &= 0, \quad r > X. \\ A_r &= 0 & \vec{B} &= \left(0, 0, \frac{\partial A_\theta(r)}{\partial r}\right) \\ A_z &= 0 \end{aligned} \tag{2.1.2}$$

Introducing factorization and wave decomposition

$$\psi(r, \theta) = \sum_{n \in \mathbb{Z}} R(r) e^{in\theta}, \tag{2.1.3}$$

(2.1.1) becomes

$$r^2 R''(r) + rR'(r) + \left[r^2 \left(k^2 - V(r)\right) - \left(n - A_\theta(r)\right)^2\right] R(r) = 0, \qquad r \in (0, +\infty), n \in \mathbb{Z}. \tag{2.1.4}$$

From this expression should be clear that $R(r)$ depends on n despite not being explicitely denoted in (2.1.3) due to later convenience.

In analogy with classical scattering, we are looking for solution consisting of incident wave and scattered outgoing wave

$$\psi(r, \theta) = \sum_{n \in \mathbb{Z}} R(r) e^{in\theta} \qquad \stackrel{!}{=} \qquad e^{ikr\cos\theta} + f(\theta) \frac{e^{ikr}}{\sqrt{r}}, \tag{2.1.5}$$

so we can determine scattering amplitude $f(\theta)$ and afterwards derive formula for differential cross section

$$\frac{d\sigma(\theta)}{d\theta} = |f(\theta)|^2. \tag{2.1.6}$$

Chapter 3

Zero neigbourhood

3.1 Theorems and proofs

Theorem 1. *([3], Chp. 5 §4, 5) Differential equation*

$$\frac{d^2w}{dz^2} + f(z)\frac{dw}{dz} + g(z)w = 0 \tag{3.1.1}$$

where for $|z| < r$ exist convergent expansions

$$f(z) = \frac{1}{z}\sum_{s=0}^{\infty} f_s z^s, \qquad g(z) = \frac{1}{z^2}\sum_{s=0}^{\infty} g_s z^s, \tag{3.1.2}$$

in which at least one of the coefficients f_0, g_0, g_1 is nonzero.

Let α be a root of the equation

$$Q(\alpha) := \alpha(\alpha-1) + f_0\alpha + g_0 = 0, \tag{3.1.3}$$

given by solving (3.1.1) while restricting ourselves to the leading terms

$$\frac{d^2w}{dz^2} + \frac{f_0}{z}\frac{dw}{dz} + \frac{g_0}{z^2}w = 0 \qquad \Rightarrow \qquad w = z^\alpha,$$

and substituing the solution back into (3.1.1).

Then series

$$w(z) = z^\alpha \sum_{s=0}^{\infty} a_s z^s, \tag{3.1.4}$$

converges and solve equation (3.1.1) for $|z| < r$, if the roots of (3.1.3) differs by a noninteger value.

Proof. Substituing (3.1.4) and (3.1.2) into (3.1.1) gives following:

$$\sum_{s=0}^{\infty} a_s(s+\alpha)(s+\alpha-1)z^{s+\alpha-2} + \sum_{j=0}^{\infty} a_j(j+\alpha)z^{j+\alpha-1}\sum_{s=0}^{\infty} f_s z^{s-1} + \sum_{j=0}^{\infty} a_j z^{j+\alpha}\sum_{s=0}^{\infty} g_s z^{s-2} = 0$$

$$\sum_{s=0}^{\infty} a_s(s+\alpha)(s+\alpha-1)z^{s+\alpha-2} + \sum_{s=0}^{\infty}\sum_{j=0}^{\infty} f_s a_j(j+\alpha)z^{j+s+\alpha-2} + \sum_{s=0}^{\infty}\sum_{j=0}^{\infty} g_s a_j z^{j+s+\alpha-2} = 0$$

$$\sum_{s=0}^{\infty} a_s(s+\alpha)(s+\alpha-1)z^{s+\alpha-2} + \sum_{s=0}^{\infty}\sum_{j=0}^{\infty}\left[f_s(j+\alpha) + g_s\right] a_j z^{j+s+\alpha-2} = 0$$

$$\sum_{s=0}^{\infty} a_s(s+\alpha)(s+\alpha-1)z^{s+\alpha-2} + \sum_{s=0}^{\infty}\left[\left[f_0(s+\alpha)+g_0\right]a_s + \sum_{j=0}^{s-1}\left[f_{s-j}(j+\alpha)+g_{s-j}\right]a_j\right] z^{s+\alpha-2} = 0$$

Whence for coefficients of power $z^{s+\alpha-2}$ we have

$$Q(\alpha+s)a_s = -\sum_{j=0}^{s-1}\left[f_{s-j}(j+\alpha)+g_{s-j}\right]a_j, \qquad s \geq 1. \tag{3.1.5}$$

This equation recursively determines coefficients $a_s, s \geq 1$ for given a_0. Formula clearly doesn't work if $Q(\alpha+s)=0$ for some positive integer s, giving condition on difference of roots of indicial equation.

To prove convergence we take arbitrary positive $\rho < r$ and denote

$$K = \max_{|z|=\rho}\left\{|zf(z)|, |z^2 g(z)|\right\}.$$

Cauchy's integral formula then yealds

$$|f_s| = \left|\frac{f^{(s-1)}(0)}{(s-1)!}\right| = \left|\frac{1}{2\pi}\int_0^{2\pi}\frac{f(\rho e^{i\varphi})}{\rho^s e^{is\varphi}}\rho e^{i\varphi}d\varphi\right| \leq K\rho^{-s}, \qquad |g_s| \leq K\rho^{-s}.$$

Next, let β denotes the second root of (3.1.3), $n := \lfloor|\alpha-\beta|\rfloor$, and define

$$b_s = |a_s|, \qquad s = 0,1,\ldots,\lfloor|\alpha-\beta|\rfloor, \tag{3.1.6}$$

$$s\left(s-|\alpha-\beta|\right)b_s = K\sum_{j=0}^{s-1}\left(|\alpha|+j+1\right)b_j\rho^{j-s}, \qquad s > \lfloor|\alpha-\beta|\rfloor. \tag{3.1.7}$$

Using (3.1.5) and identity given by factorization by roots

$$Q(\alpha+s) = \left[(\alpha+s)-\alpha\right]\left[(\alpha+s)-\beta\right] = s(s+\alpha-\beta)$$

we show, using induction, that $|a_s| \leq |b_s|$:

$s \leq \lfloor|\alpha-\beta|\rfloor$: $b_s = |a_s|$

$s-1 \to s$:

$$\begin{aligned}|s(s+\alpha-\beta)a_s| &= \left|\sum_{j=0}^{s-1}\left[f_{s-j}(j+\alpha)+g_{s-j}\right]a_j\right| \\ &\leq K\sum_{j=0}^{s-1}\left[j+|\alpha|+1\right]|a_j|\rho^{j-s} \\ &\leq K\sum_{j=0}^{s-1}\left[j+|\alpha|+1\right]b_j\rho^{j-s} = s(s-|\alpha-\beta|)b_s\end{aligned}$$

$$\underbrace{\frac{s+(\alpha-\beta)}{s-|\alpha-\beta|}}_{\geq 1}|a_s| \leq b_s \qquad \Rightarrow \qquad |a_s| \leq b_s$$

Substracting (3.1.7) for $s-1$ from the one for s multiplied by $|\rho|$ gives simple recurent relation for b_s.

$$\rho s(s-|\alpha-\beta|)b_s - (s-1)(s-1-|\alpha-\beta|)b_{s-1} = K(|\alpha|+s)b_{s-1} \qquad \Big/ \frac{1}{s^2 b_s}$$

$$\rho\left(1-\frac{|\alpha-\beta|}{s}\right)-\left(1-\frac{1}{s}\right)\left(1-\frac{1+|\alpha-\beta|}{s}\right)\frac{b_{s-1}}{b_s}=K\frac{(|\alpha|+s)}{s^2}\frac{b_{s-1}}{b_s}\quad\xrightarrow{s\to+\infty}\quad\rho-\frac{b_{s-1}}{b_s}=0$$

This means that radius of covergence of the series $\sum_{s=0}^{\infty} b_s z^s$ is ρ, and since this series is majorant to (3.1.4), radius of convergence of (3.1.4) is at least ρ, and due to possibility to choose ρ arbitrary close to r, radius of convergence is at least r. Thus, we verified that (3.1.4) solve (3.1.1) within the region $|z| < r$. □

Remark 1. *Second linearly independent solution $w_2(z)$, when exponents differ by a nonnegative integer $n = \alpha - \beta$, is given by*

$$w_2(z) = w_1(z)v(z), \qquad\qquad w_1(z) = z^\alpha \sum_{s=0}^{\infty} a_s z^s,$$

where $w_1(z)$ is solution from theorem 1. Substituing it into (3.1.1) leads to equation for $v(z)$:

$$v''(z) + \left[2\frac{w_1'(z)}{w_1(z)} + f(z)\right] v'(z) = 0,$$

which can be asily solved.

$$v'(z) = \exp\left[-\int\left[2\frac{w_1'(z)}{w_1(z)} + f(z)\right]dz\right] = \exp\left[-2\ln w_1(z) - \int f(z)dz\right]$$
$$v(z) = \int \frac{1}{[w_1(z)]^2}\exp\left[-\int f(z)dz\right]$$

Let's see how does $w_2(z)$ behave in the neighbourhood of zero.

$$\frac{1}{[w_1(z)]^2}\exp\left[-\int f(z)dz\right] = \frac{1}{z^{2\alpha}\left(a_0 + a_1 z + O(z^2)\right)^2}\exp\left[-f_0\ln z - f_1 z + O(z^2)\right]$$

From (3.1.3), when compared with factorization by roots, follows $f_0 = 1 - \alpha - \beta = 1 + n - 2\alpha$, whence

$$\frac{1}{[w_1(z)]^2}\exp\left[-\int f(z)dz\right] = \frac{\phi(z)}{z^n + 1},$$

where $\phi(z)$ is analytic at $z = 0$, thus it can be expanded into series

$$\phi(z) = \sum_{s=0}^{\infty} \phi_s z^s,$$

and ϕ_s can be expressed i terms of a_s, f_s, particulary $\phi_0 = a_0^{-2}$. Consequently,

$$w_2(z) = w_1(z)\left[-\sum_{s=0}^{n-1}\frac{\phi_s}{(n-s)z^{n-s}} + \phi_n \ln z + \sum_{s=n+1}^{\infty}\frac{\phi_s z^{s-n}}{s-n}\right].$$

When $n = 0$,

$$\begin{aligned} w_2(z) &= \phi_0 w_1(z)\ln(z) + z^{\alpha+1}\sum_{s=0}^{\infty} b_s z^s, \\ &\sim \frac{z^\alpha \ln z}{a_0}, \qquad z \to 0. \end{aligned} \tag{3.1.8}$$

When $n > 0$,

$$\begin{aligned} w_2(z) &= \phi_n w_1(z)\ln(z) + z^{\beta}\sum_{s=0}^{\infty} c_s z^s, \\ &\sim -\frac{z^\beta}{n a_0}, \qquad z \to 0. \end{aligned} \tag{3.1.9}$$

Theorem 2. *([3], Chp. 6 §2, 5) In a given interval (a_1, a_2), let $f(x)$ be a positive, real, twice continuously differentiable function, $g(x)$ a continuous real or complex function, u large positive parameter, and*

$$F(x) = \int \left[f^{-1/4} \frac{d^2}{dx^2} f^{-1/4} - g f^{1/2} \right] dx.$$

Then in this interval the differential equation

$$\frac{d^2 w}{dx^2} = \left[u^2 f(x) + g(x) \right] w \tag{3.1.10}$$

has twice continuously differentiable solutions

$$w_j(u,x) = u^{-1/2} f^{-1/4}(x) \exp\left[(-1)^{j+1} u \int f^{1/2}(x) dx \right] \left[1 + \varepsilon_j(u,x) \right], \qquad j = 1, 2, \tag{3.1.11}$$

such that

$$|\varepsilon_j(u,x)|, \frac{|\varepsilon_j'(u,x)|}{2u f^{1/2}(x)} \le \exp\left[\frac{1}{2u} \int_{a_j}^{x} |F'(t)| dt \right] - 1, \qquad j = 1, 2,$$

provided that $\int_{a_j}^{x} |F'(t)| dt < \infty$. If $g(x)$ is real, then the solutions are real.

It suffices to establish the theorem for the case $j = 1$, corresponding result for $j = 2$ then follows on replacing x in (3.1.10) by $-x$.

Proof. Due to observation above we proove only the case $j = 1$. Substitution

$$w(x) = f^{-1/4}(x) W(\xi(x)), \qquad \xi(x) = u \int_{a_1}^{x} f^{1/2}(t) dt,$$

modifies (3.1.10) as follows.

$$\frac{dW}{dx} = u f^{1/2} \frac{dW}{d\xi}, \qquad \frac{d^2 W}{dx^2} = \frac{u}{2} f^{-1/2} \frac{df}{dx} \frac{dW}{d\xi} + u^2 f \frac{d^2 W}{d\xi^2},$$

$$\begin{aligned} \left[u^2 f + g \right] f^{-1/4} W &= W \frac{d^2}{dx^2} f^{-1/4} + 2 \left(\frac{dW}{dx} \right) \left(\frac{d}{dx} f^{-1/4} \right) + f^{-1/4} \frac{d^2 W}{dx^2}, \\ &= W \frac{d^2}{dx^2} f^{-1/4} + \cancel{2 \left(u f^{1/2} \frac{dW}{d\xi} \right) \left(-\frac{1}{4} f^{-5/4} \frac{df}{dx} \right) + \frac{u}{2} f^{-3/4} \frac{df}{dx} \frac{dW}{d\xi}} + u^2 f^{3/4} \frac{d^2 W}{d\xi^2}. \end{aligned}$$

$$\frac{d^2 W}{d\xi^2} = \left[1 + u^{-2} \left(\frac{g}{f} - f^{-3/4} \frac{d^2}{dx^2} f^{-1/4} \right) \right] W = \left[1 + \psi(\xi(x)) \right] W,$$

where we denoted

$$\psi(\xi(x)) := u^{-2} \left(\frac{g(x)}{f(x)} - f^{-3/4}(x) \frac{d^2}{dx^2} f^{-1/4}(x) \right).$$

Introduction of another substitution

$$W(\xi) = e^{\xi} \left[1 + h(\xi) \right]$$

leads to equation

$$h''(\xi) + 2h'(\xi) = \psi(\xi) \left[1 + h(\xi) \right]$$

which can be solved by variation of constants.

$$h''(\xi) + 2h'(\xi) = 0 \qquad \Rightarrow \qquad h(\xi) = c_1 e^{-2\xi} + c_2$$
$$h'(\xi) = -2c_1 e^{-2\xi} + c_1' e^{-2\xi} + c_2', \qquad c_1' e^{-2\xi} + c_2' \stackrel{!}{=} 0$$
$$h''(\xi) + 2h'(\xi) = e^{-2\xi}(4c_1 - 2c_1') + 2\left(-2c_1 e^{-2\xi}\right) = -2c_1' e^{-2\xi} = \psi(\xi)\left[1 + h(\xi)\right]$$

$$c_1' = -\frac{1}{2} e^{2\xi}\psi(\xi)\left[1 + h(\xi)\right], \qquad c_2' = \frac{1}{2}\psi(\xi)\left[1 + h(\xi)\right]$$
$$c_1 = -\frac{1}{2}\int_{\alpha_1}^{\xi} e^{2t}\psi(t)\left[1 + h(t)\right]dt, \qquad c_2 = \frac{1}{2}\int_{\alpha_1}^{\xi} \psi(t)\left[1 + h(t)\right]dt$$

$$h(\xi) = \frac{1}{2}\int_{\alpha_1}^{\xi} \left[1 - e^{2(t-\xi)}\right]\psi(t)\left[1 + h(t)\right]dt \tag{3.1.12}$$

Assuming $|\alpha_1| < \infty, \psi(\xi)$ continuous at α_1, later equation can be solved by successive aproximations.

$$h_0(\xi) := 0, \qquad h_s(\xi) := \frac{1}{2}\int_{\alpha_1}^{\xi} \left[1 - e^{2(t-\xi)}\right]\psi(t)\left[1 + h_{s-1}(t)\right]dt, \qquad s \geq 1$$

To proove convergence of the series, we show by induction following inequality.

$$|h_s(\xi) - h_{s-1}(\xi)| \leq \frac{\Psi^s(\xi)}{s!2^s}, \qquad \Psi(\xi) = \int_{\alpha_1}^{\xi} |\psi(t)|dt.$$

$s = 1:$

$$|h_1(\xi)| = \left|\frac{1}{2}\int_{\alpha_1}^{\xi} \left[1 - e^{2(t-\xi)}\right]\psi(t)dt\right| \leq \frac{1}{2}\int_{\alpha_1}^{\xi} |\psi(t)|dt$$

$s > 1:$

$$\begin{aligned}
|h_{s+1}(\xi) - h_s(\xi)| &= \left|\frac{1}{2}\int_{\alpha_1}^{\xi} \left[1 - e^{2(t-\xi)}\right]\psi(t)\left[h_s(t) - h_{s-1}(t)\right]dt\right| \\
&\leq \frac{1}{2}\int_{\alpha_1}^{\xi} |\psi(t)|\left|h_s(t) - h_{s-1}(t)\right|dt \\
&\leq \frac{1}{2}\int_{\alpha_1}^{\xi} |\psi(t)|\frac{\Psi^s(t)}{s!2^s}dt = \frac{\Psi^{s+1}(\xi)}{(s+1)!2^{s+1}}
\end{aligned}$$

$$|h(\xi)| \leq \sum_{s=0}^{\infty} |h_{s+1}(\xi) - h_s(\xi)| \leq \sum_{s=0}^{\infty} \frac{\Psi^{s+1}(\xi)}{(s+1)!2^{s+1}} = e^{\Psi(\xi)/2} - 1$$

$$h_1'(\xi) = \int\limits_{\alpha_1}^{\xi} \left[1 - e^{2(t-\xi)}\right] \psi(t)dt = 2h_1(\xi),$$

$$h_{s+1}'(\xi) - h_s'(\xi) = \int\limits_{\alpha_1}^{\xi} \left[1 - e^{2(t-\xi)}\right] \psi(t) \left[h_s(t) - h_{s-1}(t)\right] dt = 2\left[h_{s+1}(\xi) - h_s(\xi)\right]$$

$$|h'(\xi)| = \left|\sum_{s=0}^{\infty} \left[h_s'(t) - h_{s-1}'(t)\right]\right| = \left|2\sum_{s=0}^{\infty} \left[h_{s+1}(\xi) - h_s(\xi)\right]\right| \leq 2\left(e^{\Psi(\xi)/2} - 1\right)$$

$$|h(\xi)|, \frac{1}{2}|h'(\xi)| \leq e^{\Psi(\xi)} - 1$$

$$w(x) = u^{-1/2} f^{-1/4}(x) W(\xi(x)) = u^{-1/2} f^{-1/4}(x) e^{\xi}(x) \left[1 + h(\xi(x))\right] =$$
$$= u^{-1/2} f^{-1/4}(x) \exp\left[u \int\limits_{a_1}^{x} f^{1/2}(t)dt\right] \left[1 + h(\xi)\right]$$

Comparing with (3.1.11) it is clear that

$$\varepsilon_1(u, x) = h(\xi(x)),$$

and using defining relations

$$\Psi(\xi(x)) = \int\limits_{\alpha_1}^{\xi(x)} u^{-2} \left|\frac{g}{f} - f^{-3/4}\frac{d^2}{dx^2}f^{-1/4}\right| dt = u^{-1} \int\limits_{\alpha_1}^{\xi(x)} \underbrace{\left|u^{-1}f^{-1/2}\right|}_{\left(\frac{d\xi}{dt}\right)^{-1}} |F'(x)|\, dt =$$
$$= u^{-1} \int_{a_1}^{x} |F'(x)|\, dt, \qquad \xi(a_1) = \alpha_1,$$

$$\varepsilon_1'(u, x) = \frac{dh}{dx} = \frac{dh}{d\xi}(\xi(x)) u f^{1/2}(x),$$

what gives desired error bounds

$$|\varepsilon_1(u, x)|, \frac{|\varepsilon_1'(u, x)|}{2uf^{1/2}(x)} \leq \exp\left[\frac{1}{2u}\int\limits_{a_1}^{x} |F'(t)|dt\right] - 1.$$

In case of infinit α_1 or discontinuity of $\psi(\xi)$ at α_1, convergence of $\int_{\alpha_1}^{\xi} |\psi(t)|dt$ is sufficient to ensure convergence of all integrals appearing in previous calculations.

It is also worthy to mention that if $g(x)$ has a finite step discontinuity, it is carried into $\psi(\xi)$ and consequently also into $\Psi(\xi)$ and $h(\xi)$. Thus the solution has also finite step discontinuity at the same point. However, except of that, validity of the theorem is not violated. □

3.2 Exact solution and its asymptotic

In this case, we are dealing with (2.1.4) in region $r < X$:

$$r^2 R''(r) + rR'(r) + \left[r^2\left(k^2 - V\right) - \left(n - \mu\frac{r^2}{X^2}\right)^2\right] R(r) = 0, \qquad n \in \mathbb{Z}, \tag{3.2.1}$$

which can be, for convenience, treated under restriction of $n \in \mathbb{N}_0$ as follows:

$$r^2 R''(r) + rR'(r) + \left[r^2 \left(k^2 - V\right) - \left(n - \frac{r^2(\pm\mu)}{X^2}\right)^2 \right] R(r) = 0, \qquad n \in \mathbb{N}_0, \tag{3.2.2}$$

where upper sign corresponds to originally positive, and lower sign to negative n. Introducing substitution

$$R(r) = 2^{\frac{n+1}{2}} \mathrm{e}^{-\frac{r^2(\pm\mu)}{2X^2}} r^n G(z), \qquad z = r^2 \frac{(\pm\mu)}{X^2},$$

(3.2.2) is transformed into confluent hypergeometrical equation ([9], Chp. 13 §2(i), 2(v))

$$zG''(z) + (n + 1 - z)G'(z) - aG(z) = 0, \qquad n \in \mathbb{N}_0,$$

where

$$a = -\frac{(k^2 - V)X^2 - 2(\pm\mu)}{4(\pm\mu)}.$$

Pair of indipendent solutions is

$$\begin{aligned} G_1(z) &= {}_1F_1(a, n + 1, z) = 1 + O(z), \\ G_2(z) &= U(a, n + 1, z) = \frac{\Gamma(n)}{\Gamma(a)} z^{-n} + O(z^{-n+1}). \end{aligned}$$

Whence pair of corresponding solutions in terms of R(r) is

$$\begin{aligned} R_1(r) = 2^{\frac{n+1}{2}} \mathrm{e}^{-\frac{r^2(\pm\mu)}{2X^2}} r^n \, {}_1F_1\left(a, n + 1, r^2 \frac{(\pm\mu)}{X^2}\right) &= 2^{\frac{n+1}{2}} r^n + O\left(r^{n+2}\right), \\ R_2(r) = 2^{\frac{n+1}{2}} \mathrm{e}^{-\frac{r^2(\pm\mu)}{2X^2}} r^n U\left(a, n + 1, r^2 \frac{(\pm\mu)}{X^2}\right) &= 2^{\frac{n+1}{2}} \frac{\Gamma(n)}{\Gamma(a)} \frac{X^{2n}}{(\pm\mu)^n} r^{-n} + O(r^{-n+2}). \end{aligned}$$

It is clear that only the first one of two is convergent, thus inside the cylinder of potential V we will be interested only in solution of the form

$$R(r) = C 2^{\frac{n+1}{2}} \mathrm{e}^{-\frac{r^2(\pm\mu)}{2X^2}} r^n \, {}_1F_1\left(a, n + 1, r^2 \frac{(\pm\mu)}{X^2}\right) = C 2^{\frac{n+1}{2}} r^n + O\left(r^{n+2}\right). \tag{3.2.3}$$

3.3 Approximate solution

Introducing transformation

$$\widetilde{R}(r) = \sqrt{r} R(r),$$

(3.2.1) become

$$\widetilde{R}''(r) = k^2 \underbrace{\left[-1 + \frac{V}{k^2} + \frac{\left(n - \mu \frac{r^2}{X^2}\right)^2 - \frac{1}{4}}{k^2 r^2} \right]}_{f(r) :=} \widetilde{R}(r). \tag{3.3.1}$$

In order to find small argument asymptotics of the solutions corresponding to $n \neq 0$, resp. $n = 0$, it is favourable to treat these two cases separately in different ways.

3.3.1 $n \neq 0$

For convenience we denote

$$f(r) = ar^{-2} + b + cr^2, \qquad a = \frac{n^2 - 1/4}{k^2}, \quad b = (-1 + \frac{V}{k^2} - \frac{2\mu n}{k^2 X^2}), \quad c = \frac{\mu^2}{k^2 X^4}.$$

Furthermore, by setting

$$u = \sqrt{k^2 + \frac{1}{4a}},$$

(3.3.1) gains following form.

$$\widetilde{R}''(r) = \left[u^2 f(r) + g(r)\right] \widetilde{R}(r), \qquad g(r) = -\frac{f(r)}{4a}.$$

This equation has according to Theorem 2 convergent solution at vicinity of zero of the form

$$\widetilde{R}(r) = u^{-1/2} f^{-1.4} \exp\left[u \int\limits_0^r f^{1/2}(x)dx\right] (1 + \varepsilon(k,r)),$$

where

$$|\varepsilon(k,r)| \leq \exp\left[\frac{1}{2u}\int\limits_0^r |F'(x)|dx\right] - 1, \qquad F'(x) = f^{-1/4}(x)\frac{d}{dx}f^{-1/4}(x) - g(x)f^{-1/2}(x).$$

After biref look at our $f(r)$ can be seen that for sufficiently small r will be $f(r)$ indeed positive, as needed to use Theorem 2.

Performing straightforward calculations one obtains

$$\begin{aligned} F'(r) &= \frac{-3a^2 r\left(b + 5cr^2\right) + ar^3\left(3b^2 + 4bcr^2 + 6c^2 r^4\right) + r^5\left(b + cr^2\right)^3}{4a\left(a + br^2 + cr^4\right)^{5/2}} \\ &= -\frac{3b}{4a^{3/2}} r + O(r^2). \end{aligned}$$

Consequently

$$|\varepsilon(k,r)| \leq \left[1 + \frac{3|b|}{8ua^{3/2}} r^2 + O(r^3)\right] - 1 = \frac{3|b|}{8ua^{3/2}} r^2 + O(r^3),$$

$$\begin{aligned} \widetilde{R}(r) &= u^{-1/2}\left(ar^{-2} + b + cr^2\right)^{-1.4} \exp\left[u\int\limits_0^r \left(ar^{-2} + b + cr^2\right)^{1/2}(x)dx\right](1 + \varepsilon(k,r)) \\ &= u^{-1/2} a^{-1/4} r^{1/2}\left(1 + O(r^2)\right) \exp\left[u\int\limits_0^r a^{1/2} r^{-1}\left(1 + O(r^2)\right)(x)dx\right](1 + \varepsilon(k,r)) \\ &= \frac{r^{1/2}}{\sqrt{n}}\left(1 + O(r^2)\right)\exp\left[n\ln r + O(r^2)\right](1 + \varepsilon(k,r)) \\ &= \frac{r^{n+1/2}}{\sqrt{n}}\left(1 + O(r^2)\right) \end{aligned}$$

and finally

$$R(r) = \frac{r^n}{\sqrt{n}}\left(1 + O(r^2)\right). \tag{3.3.2}$$

Comparing this resul with (3.2.3) is obvious that both expressions agrees.

3.3.2 $n = 0$

Previous procedure cannot be used in this case due to nonpositivity of previously chosen $f(r)$. Instead we will examine original equation (3.2.1), which can be now rewritten as

$$R''(r) + \underbrace{\frac{1}{r}}_{f(r):=} R'(r) + \underbrace{\left[\left(k^2 - V\right) - \frac{\mu^2}{X^4} r^2\right]}_{g(r):=} R(r) = 0, \qquad r \in (0, +\infty), \tag{3.3.3}$$

by applying Theorem 1. Whence denoting

$$g(r) = g_0 r^{-2} + g_2 + g_4 r^2, \qquad g_0 = 0 \qquad g_2 = k^2 - V, \qquad g_4 = -\frac{\mu^2}{X^4}$$

and considering only problematic leading terms of $f(r), g(r)$ as $r \to 0$ leads to

$$R''(r) + \frac{1}{r} R'(r) = 0.$$

This equation has an exact solution of the form $R(r) = z^\alpha$, where α satisfies

$$\alpha(\alpha - 1) + \alpha = 0 \qquad \Rightarrow \qquad \alpha = 0.$$

Thus (3.3.3) has a analytic solution

$$R(r) = r^\alpha \sum_{s=0}^{+\infty} a_s r^s = \sum_{s=0}^{+\infty} a_s r^s,$$

where for given a_0

$$a_s = -\frac{1}{s^2} \sum_{j=0}^{s-1} g_{s-j} a_j, \qquad s \in \mathbb{Z}_+.$$

This already quite simple form, compared to one of full generality of referenced theorem, can be due to form of $g(r)$ simplified even more

$$R(r) = \sum_{s=0}^{+\infty} a_{2s} t^{2s}, \qquad a_2 = -g_2 a_0, \qquad a_{2s} = -\frac{1}{(2s)^2} \left(g_4 a_{2s-4} + g_2 a_{2s-2}\right), \quad s \geq 2. \tag{3.3.4}$$

Whence

$$R(r) = a_0 + O(r^2), \tag{3.3.5}$$

what, as well as in case $n \neq 0$, agrees with (3.2.3).

Chapter 4

Neighbourhood of infinity

Analysis in this chapter is based on procedures given in [1, 5, 7, 8].

4.1 Exact Solution

In this case we are dealing with (2.1.4) in region $r > X$

$$r^2R''(r) + rR'(r) + \left[k^2r^2 - \left(n - (\pm\mu)\right)^2\right] R(r) = 0, \qquad r \in (0, +\infty), n \in \mathbb{N}_0, \tag{4.1.1}$$

where for later convenience as in previous chapter, restriction is put on n and plus, resp. minus sign before μ coresponds to positive, resp. negative n. This is Bessel equation, therefore solution can be written in terms of Bessel functions as

$$R(r) = B_1J_{n\mp\mu}(kr) + B_2Y_{n\mp\mu}(kr), \qquad r > X, n \in \mathbb{N}_0, \tag{4.1.2}$$

It is worth reminding that $R(r)$ as well as constants B_1, B_2 depends on n, however in most of the following calculations we are dealing with expressions with arbtrary fixed n, it is convenient not to write it explicitely.

In order to determine B_1, B_2 we need to connect solutions at the boundary of the cylinder potential $V(r = X)$. Recalling (3.2.3) and (4.1.2)

$$\begin{aligned} R(r) &= C2^{\frac{n+1}{2}} e^{-\frac{r^2(\pm\mu)}{2X^2}} r^n \, {}_1F_1\left(\frac{-k^2X^2 + VX^2 + 2(\pm\mu)}{4(\pm\mu)}, n+1, \frac{r^2(\pm\mu)}{X^2}\right), \quad r < X, \\ &= B_1J_{n\mp\mu}(kr) + B_2Y_{n\mp\mu}(kr), \qquad r > X, \end{aligned} \qquad n \in \mathbb{Z}_0, \tag{4.1.3}$$

and substituing into bundary conditons

$$R(X+) = R(X-), \qquad R'(X+) = R'(X-),$$

gives after some straightforward algebra

$$\begin{aligned} B_1 = &-\frac{\pi}{n+1} C e^{\mp\mu/2} 2^{\frac{n-3}{2}} X^n \times \\ &\times \Bigg[k(n+1)X \, {}_1F_1\left(\frac{-k^2X^2 + VX^2 + 2(\pm\mu)}{4(\pm\mu)}; n+1; \pm\mu\right) \left[Y_{n\mp\mu+1}(kX) - Y_{n\mp\mu-1}(kX)\right] + \\ &+ Y_{n\mp\mu}(kX) \Bigg[2(n+1)(n\mp\mu) \, {}_1F_1\left(\frac{-k^2X^2 + VX^2 + 2(\pm\mu)}{4(\pm\mu)}; n+1; \pm\mu\right) + \\ &+ \left(-k^2X^2 + VX^2 + 2(\pm\mu)\right) \, {}_1F_1\left(\frac{-k^2X^2 + VX^2 + 6(\pm\mu)}{4(\pm\mu)}; n+2; \pm\mu\right)\Bigg]\Bigg] \end{aligned}$$

$$B_2 = \frac{\pi}{n+1} C e^{\mp\mu/2} 2^{\frac{n-3}{2}} X^n \times$$
$$\times \left[k(n+1) X \, {}_1F_1 \left(\frac{-k^2X^2 + VX^2 + 2(\pm\mu)}{4(\pm\mu)}; n+1; \pm\mu \right) [J_{n\mp\mu+1}(kX) - J_{n\mp\mu-1}(kX)] + \right.$$
$$+ J_{n\mp\mu}(kX) \left[2(n+1)(n\mp\mu) \, {}_1F_1 \left(\frac{-k^2X^2 + VX^2 + 2(\pm\mu)}{4(\pm\mu)}; n+1; \pm\mu \right) + \right.$$
$$\left. \left. - \left(k^2X^2 - VX^2 - 2(\pm\mu) \right) {}_1F_1 \left(\frac{-k^2X^2 + VX^2 + 6(\pm\mu)}{4(\pm\mu)}; n+2; \pm\mu \right) \right] \right]$$

4.2 Asymptotic region

Being intrested in asymptotic behavior for large r allow us to use large argument asymptotics of Bessel functions (1.0.2) in (4.1.2) and some trigonometry afterwards gives

$$\begin{aligned} R(r) &= \sqrt{\frac{2}{\pi k r}} \left[B_1 \sin\left(kr - \frac{\pi}{2}(\nu - 1/2) \right) - B_2 \cos\left(kr - \frac{\pi}{2}(\nu - 1/2) \right) \right] \\ &= \Big/ B_1 = B\cos\delta_n, \qquad B_2 = -B\sin\delta_n \qquad \Rightarrow \qquad \tan\delta_n = -\frac{B_2}{B_1} \Big/ \qquad (4.2.1) \\ &= \sqrt{\frac{2}{\pi k r}} B \sin\left(kr - \frac{\pi}{2}(\nu - 1/2) + \delta_n \right) \\ &= \sqrt{\frac{2}{\pi k r}} B \frac{1}{2i} \left[e^{i\left(kr - \frac{\pi}{2}(\nu-1/2) + \delta_n\right)} - e^{-i\left(kr - \frac{\pi}{2}(\nu-1/2)+\delta_n\right)} \right]. \end{aligned}$$

At this point it is good to remind ourselves, that we are interested in solution consisting of incident and outgoing wave and rewrite incident part in terms of Bessel functions using (1.0.10).

$$e^{ikr\cos\theta} = \sum_{n\in\mathbb{Z}} i^n J_n(kr) e^{in\theta} \sim \frac{1}{\sqrt{2\pi k r}} \sum_{n\in\mathbb{Z}} \left[e^{i(kr-\pi/4)} + e^{-i(kr - n\pi - \pi/4)} \right] e^{in\theta}$$

Comparing two previous expressions one can see, that setting

$$B := e^{i\left(\frac{n\pi}{2} + \frac{\mu\pi}{2} + \delta_n\right)} \qquad (4.2.2)$$

leads to desired form

$$\begin{aligned} \psi(r,\theta) &\sim \sum_{n\in Z} \frac{1}{\sqrt{2\pi k r}} \left[e^{i\left(kr + \mu\pi - \frac{\pi}{4} + 2\delta_n\right)} + e^{-i\left(kr - n\pi - \frac{\pi}{4}\right)} \right] e^{in\theta} \\ &\sim e^{ikr\cos\theta} + \sum_{n\in Z} \frac{1}{\sqrt{2\pi k r}} \left[e^{i\left(kr + \mu\pi - \frac{\pi}{4} + 2\delta_n\right)} - e^{i\left(kr - \frac{\pi}{4}\right)} \right] e^{in\theta} \\ &\sim e^{ikr\cos\theta} + \frac{e^{ikr}}{\sqrt{r}} \underbrace{\left(\frac{e^{-i\pi/4}}{\sqrt{2\pi k}} \sum_{n\in Z} \left[e^{i(\mu\pi + 2\delta_n)} - 1 \right] e^{in\theta} \right)}_{f(\theta)}. \qquad (4.2.3) \end{aligned}$$

Clearly to to determine scattering amplitude $f(\theta)$, we need to know phase shifts δ_n. HOwever those can be easily derived from already found expressions of B_1, B_2 according to (4.2.1):

$$\delta_n = \text{arctg}\left(-\frac{B_2}{B_1} \right) \qquad (4.2.4)$$

Chapter 5

Phase shift approximation

5.1 Theorems and proofs

Theorem 3. *([3], Chp. 6 §2, 5) In a given interval (a_1, a_2), let $f(x)$ be a positive, real, twice continuously differentiable function, $g(x)$ a continuous real or complex function, u large positive parameter, and*

$$F(x) = \int \left[f^{-1/4} \frac{d^2}{dx^2} f^{-1/4} - g f^{1/2} \right] dx.$$

Then in this interval the differential equation

$$\frac{d^2 w}{dx^2} = \left[-u^2 f(x) + g(x) \right] w \tag{5.1.1}$$

has twice continuously differentiable solutions

$$w_j(u,x) = u^{-1/2} f^{-1/4}(x) \exp\left[(-1)^{j+1} iu \int f^{1/2}(x) dx \right] \left[1 + \varepsilon_j(u,x) \right], \qquad j = 1, 2, \tag{5.1.2}$$

such that

$$|\varepsilon_j(u,x)|, \frac{|\varepsilon_j'(u,x)|}{2u f^{1/2}(x)} \leq \exp\left[\frac{1}{u} \int\limits_{a_j}^{x} |F'(t)| dt \right] - 1, \qquad j = 1, 2,$$

when $\int_{a_j}^{x} |F'(t)| dt < \infty$. If $g(x)$ is real, then the solutions are real.

Proof. Only difference compared to previous proof is absence of the coefficient $\frac{1}{2}$ in bound of $\varepsilon(u,x)$ arising from from (3.1.12) due to bound given by $|1 - e^{2i(t-\xi)}| \leq 2$. □

Remark 2. *([3], Chp. 6 §5.4) Theorems 2 and 3 can be generalized to the case*

$$\frac{d^2 w}{dx^2} = \left[+u^2 f(u,x) + g(u,x) \right] w, \qquad resp. \qquad \frac{d^2 w}{dx^2} = \left[-u^2 f(u,x) + g(u,x) \right] w, \tag{5.1.3}$$

when folowing conditions are fulfilled:

1. $f(u,x) > 0$,
2. $\frac{\partial^2 f(u,x)}{\partial x^2}$ *and* $g(u,x)$ *are continuous functions of* x,
3. $\int\limits_{a_1}^{a_2} |F'(t)| dt = O(u)$ *as* $u \to \infty$.

Theorem 4. *([3], Chp. 12 §6.2) If $k > 0, n \le 0, g(r)$ continuous and for finite r satisfying*

$$G_n(k,r) = \pi \int\limits_0^{X_n/k} t|Y_n(kt)|J_n(kt)|g(t)|dt + \frac{\pi}{2} \int\limits_{X_n/k}^{r} t\left[J_n^2(kt) + Y_n^2(kt)\right]|g(t)|dt < +\infty \tag{5.1.4}$$

where $X_n > 0$ is smallest root of $J_n(r) + Y_n(r) = 0$, then equation

$$\frac{d^2\widetilde{R}}{dr^2} = \left[-k^2 + \frac{n^2 - \frac{1}{4}}{r^2} + g(r)\right]\widetilde{R}\,, \qquad 0 < r < +\infty \tag{5.1.5}$$

has a solution

$$\widetilde{R}(k,r) = r^{1/2}\left(J_n(kr) + \varepsilon_n(k,r)\right), \tag{5.1.6}$$

where

$$\begin{aligned} |\varepsilon_n(k,r)| &\le \sqrt{2}|J_n(kr)|\left[\exp\left(G_n(k,r)\right) - 1\right] && kr \le X_n, \\ &\le \left(J_n^2(kr) + Y_n^2(kr)\right)^{1/2}\left[\exp\left(G_n(k,r)\right) - 1\right] && kr \ge X_n. \end{aligned} \tag{5.1.7}$$

Proof. Substituing (5.1.6) into (5.1.5) leads to differential equation for error term

$$r^2\varepsilon_n''(k,r) + r\varepsilon_n'(k,r) = \left[-k^2r^2 + n^2\right]\varepsilon_n(k,r) + g(r)r^2\left[J_n(kr) + \varepsilon_n(k,r)\right], \qquad 0 < r < +\infty, \tag{5.1.8}$$

which is equivalent to

$$\varepsilon_n(k,r) = \frac{\pi}{2}\int\limits_0^r K(r,t)\left[J_n(kt) + \varepsilon_n(k,t)\right]tg(t)dt, \qquad 0 < r < +\infty, \tag{5.1.9}$$

where

$$K(r,t) = Y_n(kr)J_n(kt) - J_n(kr)Y_n(kt).$$

Equivalence can be verified by differentiating integral expression and using basic relatons of Bessel functions:

$$\varepsilon_n'(k,r) = \frac{\pi}{2}\int\limits_0^r \frac{\partial K(r,t)}{\partial r}\left[J_n(kt) + \varepsilon_n(k,t)\right]tg(t)dt + \frac{\pi}{2}\underbrace{K(r,r)}_{=0}\left[J_n(kr) + \varepsilon_n(k,r)\right]rg(r),$$

$$\varepsilon_n''(k,r) = \frac{\pi}{2}\int\limits_0^r \frac{\partial^2 K(r,t)}{\partial r^2}\left[J_n(kt) + \varepsilon_n(k,t)\right]tg(t)dt + \frac{\pi}{2}\underbrace{\left.\frac{\partial K(r,t)}{\partial r}\right|_{t=r}}_{=\frac{2}{\pi r}}\left[J_n(kr) + \varepsilon_n(k,r)\right]rg(r).$$

Altogether,

$$\begin{aligned} r^2\varepsilon_n''(k,r) + r\varepsilon_n'(k,r) = \frac{\pi}{2}\int\limits_0^r &\left[r^2\frac{\partial^2 K(r,t)}{\partial r^2} + r\frac{\partial K(r,t)}{\partial r}\right]\left[J_n(kt) + \varepsilon_n(k,t)\right]tg(t)dt \\ &+ \left[J_n(kr) + \varepsilon_n(k,r)\right]r^2g(r), \end{aligned}$$

where

$$\begin{aligned} r^2\frac{\partial^2 K(r,t)}{\partial r^2} + r\frac{\partial K(r,t)}{\partial r} &= \underbrace{\left[r^2k^2Y_n''(kr) + krY_n'(kr)\right]}_{[-k^2r^2+n^2]Y_n(kr)}J_n(kt) - \underbrace{\left[r^2k^2J_n''(kr) + krJ_n'(kr)\right]}_{[-k^2r^2+n^2]J_n(kr)}Y_n(kt) \\ &= \left[-k^2r^2 + n^2\right]K(r,t), \end{aligned}$$

gives (5.1.8). Having integral equation (5.1.9), we use method of succesive approximations to solve it and write its solution as a series:

$$\varepsilon_n(k,r) = \sum_{s=0}^{+\infty} \big[h_{s+1}(r) - h_s(r)\big],$$

$$h_0(r) = 0, \qquad h_1(r) = \frac{\pi}{2}\int_0^r K(r,t)J_n(kt)tg(t)dt,$$

$$h_s(r) = \frac{\pi}{2}\int_0^r K(r,t)\big[J_n(kt) + h_{s-1}(t)\big]tg(t)dt, \qquad s \geq 1. \tag{5.1.10}$$

In order to show convergence of the series let us introduce auxiliary function

$$\begin{aligned} E_n(x) &= \sqrt{\frac{|Y_n(x)|}{J_n(x)}}, && 0 < x \leq X_n, \\ &= 1, && x \geq X_n, \end{aligned}$$

and take a closer look at it. Let us denote j_n, resp. y_n smallest positive zero of $J_n(x)$, resp. $Y_n(x)$, then for every $n \in \mathbb{N}$ holds $y_n < j_n$. Thus, using small argument behavior of Bessel functions (1.0.1), it is clear that $J_n(x) + Y_n(x)$ is negative as $x \to 0_+$ and positive as $x \to y_n$, what leaves no other option than $Y_n(x) < 0, J_n(x) > 0$ for $0 < x \leq X_n$. Furhermore

$$\big(E_n^2(x)\big)' = \left(\frac{|Y_n(x)|}{J_n(x)}\right)' = \left(-\frac{Y_n(x)}{J_n(x)}\right)' \overset{(1.0.5)}{=} -\frac{2}{\pi x J_n^2(x)},$$

what means that $E_n(x)$ is continuous, positive nonincreasing function of x. Having found key properties of $E_n(x)$, we introduve two more auxiliary functions $M_n(x), \theta_n(x)$ by setting

$$J_n(x) = E_n^{-1}(x)M_n(x)\cos\theta_n(x), \qquad Y_n(x) = E_n(x)M_n(x)\sin\theta_n(x),$$

wha gives us

$$\begin{aligned} M_n(x) &= \sqrt{2|Y_n(x)|J_n(x)}, && 0 < x \leq X_n, & \theta_n(x) &= -\frac{\pi}{4}, && 0 < x \leq X_n, \\ &= \sqrt{J_n^2(x) + Y_n^2(x)}, && x \geq X_n, & &= \tan^{-1}\frac{Y_n(x)}{J_n(x)}, && x \geq X_n. \end{aligned}$$

Having intorduced all auxiliary functions we need, getting desired bounds of $J_n, Y_n, K(r,t)$ follows.

$$\begin{aligned} E_n^{-1}(kt)M_n(kt) &= \sqrt{2}|J_n(kt)| && \geq |J_n(kt)|, && 0 < x \leq X_n, \\ &= \sqrt{J_n^2(x) + Y_n^2(x)} && \geq |J_n(k)t|, && x \geq X_n, \end{aligned}$$

$$\begin{aligned} E_n(kt)M_n(kt) &= \sqrt{2}|Y_n(kt)| && \geq |Y_n(kt)|, && 0 < x \leq X_n, \\ &= \sqrt{J_n^2(x) + Y_n^2(x)} && \geq |Y_n(k)t|, && x \geq X_n. \end{aligned}$$

$$\begin{aligned} |K(r,t)| &= \big|Y_n(kr)J_n(kt) - J_n(kr)Y_n(kt)\big| \\ &= \big|E_n(kr)E_n^{-1}(kt)\sin\theta_n(kr)\cos\theta_n(kt) - E_n^{-1}(kr)E_n(kt)\cos\theta_n(kr)\sin\theta_n(kt)\big|\, M_n(kr)M_n(kt) \\ &= \big|\underbrace{E_n(kr)^2E_n^{-2}(kt)}_{<1}\sin\theta_n(kr)\cos\theta_n(kt) - \cos\theta_n(kr)\sin\theta_n(kt)\big|E_n^{-1}(kr)E_n(kt)M_n(kr)M_n(kt) \\ &\leq E_n^{-1}(kr)E_n(kt)M_n(kr)M_n(kt) \end{aligned}$$

Now, in order to prove convergence of the series, we first use induction to show following inequality:

$$|h_s(kr) - h_{s-1}(kr)| \le E_n^{-1}(kr)M_n(kr)\frac{G_n(k,r)^s}{s!}, \qquad G_n(k,r) = \frac{\pi}{2}\int_0^r M_n^2(kt)tg(t)dt$$

$s = 1$:

$$\begin{aligned} h_1(r) &= \frac{\pi}{2}\int_0^r K(r,t)J_n(kt)tg(t)dt \\ &\le E_n^{-1}(kr)M_n(kr)\frac{\pi}{2}\int_0^r E_n(kt)M_n(kt)E_n^{-1}(kt)M_n(kt)tg(t)dt \\ &= E_n^{-1}(kr)M_n(kr)\frac{\pi}{2}\int_0^r M_n^2(kt)tg(t)dt \\ &= E_n^{-1}(kr)M_n(kr)G_n(k,r) \end{aligned}$$

$s - 1 \to s$:

$$\begin{aligned} h_s(r) - h_{s-1}(r) &= \frac{\pi}{2}\int_0^r K(r,t)\big[h_s(t) - h_{s-1}(t)\big]tg(t)dt \\ &\le E_n^{-1}(kr)M_n(kr)\frac{\pi}{2}\int_0^r E_n(kt)M_n(kt)\left[2E_n^{-1}(kt)M_n(kt)\frac{G_n(k,t)^s}{s!}\right]tg(t)dt \\ &= E_n^{-1}(kr)M_n(kr)\frac{\pi}{2}\int_0^r M_n(kt)^2\frac{G_n(k,t)^s}{s!}tg(t)dt \\ &= E_n^{-1}(kr)M_n(kr)\frac{G_n(k,r)^{s+1}}{(s+1)!} \end{aligned}$$

Whence,

$$|\varepsilon_n(k,r)| \le \sum_{s=0}^{+\infty}\big|h_{s+1}(r) - h_s(r)\big| \le E_n^{-1}(kr)M_n(kr)\sum_{s=0}^{+\infty}\frac{G_n(k,r)^{s+1}}{(s+1)!} = E_n^{-1}(kr)M_n(kr)\left[\exp\big(G_n(k,r)\big) - 1\right].$$

□

5.2 Large energy approximation

Original equation (2.1.4) can be, as before, transformed using substitution $\widetilde{R} = \sqrt{r}R(r)$ into

$$\frac{d^2\widetilde{R}}{dr^2} = \left[-k^2 + \frac{n^2 - 1/4}{r^2} + g(r)\right]\widetilde{R} \tag{5.2.1}$$

where

$$\begin{aligned} g(r) &= \mu^2\frac{r^2}{X^4} - \frac{2n\mu}{X^2} + V, && r < X, \\ &= \frac{\mu^2 - 2n\mu}{r^2}, && r > X. \end{aligned} \tag{5.2.2}$$

It is clear that only discontinuity is a step at $r = X$ and

$$\int\limits_0^{+\infty} |g(r)|dr < +\infty.$$

Appllyinng (3) to (5.2.1) on interval $(X, +\infty)$ leads to pair of independent solutions

$$\widetilde{R}_j(k,r) = k^{-1/2} e^{(-1)^{j+1}ikr}\left[1 + \widetilde{\varepsilon}_j(k,r)\right], \qquad \widetilde{\varepsilon}_j(k,r) \le \exp\left[\frac{1}{k}\int_{a_j}^{r} |g(t)|dt\right] - 1, \qquad j = 1,2,$$

where $a_1 = X, a_2 = +\infty$. There is nothing preventing us from choosing another pair of independent solutions taken as linear combinations of previous ones

$$e^{ikr}, e^{-ikr} \qquad \rightarrow \qquad \sin(kr), \cos(kr).$$

Consequently any solution of (5.2.1) can be written as

$$\widetilde{R}(k,r) = \left(\frac{1}{k\pi}\right)^{1/2} (1+\alpha_n)\sin\left(kr - \frac{\pi}{2}\left(n - \mu - \frac{1}{2}\right) + \delta_n\right) + O(1), \qquad 1 + \alpha_n > 0, \delta_n \in (-\pi, \pi].$$

Using Theorem 4 and large argument asymptotics of $J_n(x)$ we get another similar equality

$$\widetilde{R}(k,r) = \left(\frac{2}{\pi k}\right)^{1/2}\left[\sin\left(kr - \frac{\pi}{2}\left(n - \frac{1}{2}\right)\right) + O\left(\frac{1}{kr}\right)\right] + r^{1/2}\varepsilon_n(k,r).$$

Substraction of last two expresions leads to

$$(1+\alpha_n)\sin\left(kr - \frac{\pi}{2}\left(n - \mu - \frac{1}{2}\right) + \delta_n\right) - \sin\left(kr - \frac{\pi}{2}\left(n - \frac{1}{2}\right)\right) = \left(\frac{\pi k r}{2}\right)^{1/2} \varepsilon_n(k,r) + O(1). \tag{5.2.3}$$

Here, instead of $\varepsilon_n(k,r)$ we take only first term of its Liouville-Neumann Expansion (5.1.10) from Theorem 4:

$$\varepsilon_n(k,r) \simeq \frac{\pi}{2} Y_n(kr) \int\limits_0^r J_n^2(kt)tg(t)dt - \frac{\pi}{2} J_n(kr)\int\limits_0^r J_n(kt)Y_n(kt)tg(t)dt$$

and using large argument asymptotics as $r \to +\infty$ we get

$$\varepsilon_n(k,r) \simeq -\left(\frac{\pi}{2kr}\right)^{1/2} \cos\left(kr - \frac{\pi}{2}\left(n - \frac{1}{2}\right)\right)\int\limits_0^{+\infty} J_n^2(kt)tg(t)dt +$$
$$-\left(\frac{\pi}{2kr}\right)^{1/2} \sin\left(kr - \frac{\pi}{2}\left(n - \frac{1}{2}\right)\right)\int\limits_0^{+\infty} J_n(kt)Y_n(kt)tg(t)dt.$$

Now comparing with (5.2.3) after convenient cosmetic adjustments

$$\sin\left(\dots + \delta_n + \frac{\mu\pi}{2}\right) = \sin(\dots)\cos\left(\delta_n + \frac{\mu\pi}{2}\right) + \cos(\dots)\sin\left(\delta_n + \frac{\mu\pi}{2}\right),$$

one has

$$\left(\frac{\pi k r}{2}\right)^{1/2}\varepsilon_n(k,r) + O(1) = \sin\left(kr - \frac{\pi}{2}\left(n - \frac{1}{2}\right)\right)\left[(1+\alpha_n)\cos\left(\delta_n + \frac{\mu\pi}{2}\right) - 1\right] +$$
$$+ \cos\left(kr - \frac{\pi}{2}\left(n - \frac{1}{2}\right)\right)(1+\alpha_n)\sin\left(\delta_n + \frac{\mu\pi}{2}\right).$$

From this expression is apparent, that

$$(1+\alpha_n)\sin\left(\delta_n+\frac{\mu\pi}{2}\right)=-\frac{\pi}{2}\int\limits_0^{+\infty}J_n^2(kt)tg(t)dt, \tag{5.2.4}$$

thus we have expression of phase shift, however, we still need to ged rid of α_n to obtain phase shift explicitely. In order to deal wit this obsticle, let us start with bound of $\varepsilon_n(k,r)$ (5.1.7) from Theorem 4, where using asymptotic behavior of Bessel functions as $r\to+\infty$ (1.0.2) gives

$$|\varepsilon_n(k,+\infty)|\le\left(\frac{2}{\pi kr}\right)^{1/2}\big(\exp(G_n(k,+\infty))-1\big)\le\exp(G_n(k,+\infty))-1$$

Thus from (5.2.3) follows

$$\left|(1+\alpha_n)\sin\left(kr-\frac{\pi}{2}\left(n-\mu-\frac{1}{2}\right)+\delta_n\right)-\sin\left(kr-\frac{\pi}{2}\left(n-\frac{1}{2}\right)\right)\right|\le\exp(G_n(k,+\infty))-1+O(1).$$

Using convenient substiution

$$\sigma\cos\eta=(1+\alpha_n)\cos\left(\delta_n+\frac{\mu\pi}{2}\right),\qquad\qquad \sigma\sin\eta=(1+\alpha_n)\sin\left(\delta_n+\frac{\mu\pi}{2}\right)$$

leads to

$$(1+\alpha_n)\sin\left(\dots+\delta_n+\frac{\mu\pi}{2}\right)-\sin\left(\dots\right)=\sigma\sin(\dots+\eta),\qquad (1+\alpha_n)e^{i\left(\delta_n+\frac{\mu\pi}{2}\right)}=\sigma e^{i\eta},$$

what combined with previous inequality immediately gives

$$|\sigma|\le\exp(G_n(k,+\infty))-1.$$

If $\sigma\le1$, using Jordan's inequality (1.0.13) we derive

$$|\alpha_n|\le\sigma,\qquad\qquad \left|\delta_n+\frac{\mu\pi}{2}\right|\le\sin^{-1}\sigma\le\frac{\pi\sigma}{2},$$

what finally leads to

$$|\alpha_n|,\frac{2}{\pi}\left|\delta_n+\frac{\mu\pi}{2}\right|\le\exp(G_n(k,+\infty))-1.$$

Thus

$$G_n(k,+\infty)\xrightarrow{k\to+\infty}0\quad\Rightarrow\quad(1+\alpha_n)\sin\left(\delta_n+\frac{\mu\pi}{2}\right)\sim\delta_n+\frac{\mu\pi}{2}+O(G_n^2(k,+\infty))\qquad\text{for large }k$$

and comparing with (5.2.4) one obtains desired approximation of phase shift

$$\delta_n+\frac{\mu\pi}{2}=-\frac{\pi}{2}\int\limits_0^{+\infty}J_n^2(kt)tg(t)dt+O(G_n^2(k,+\infty))\qquad\text{for large }k. \tag{5.2.5}$$

5.3 Error term behaviour for large energies

Let's take a closer look at the error term of phase shift expression (5.2.5) that we just found. For suficently large k we can suppose that $X_n/k < X$, thus

$$G_n(k,+\infty) = \pi \int_0^{X_n/k} t|Y_n(kt)|J_n(kt)|g(t)|dt + \frac{\pi}{2} \int_{X_n/k}^{+\infty} t\left[J_n^2(kt)+Y_n^2(kt)\right]|g(t)|dt$$

$$\overset{(5.2.2)}{=} \pi \int_0^{X_n/k} t|Y_n(kt)|J_n(kt)\left|\frac{\mu^2 t^2}{X^4} - \frac{2\mu n}{X^2} + V\right| dt +$$

$$+ \frac{\pi}{2} \int_{X_n/k}^{X} t\left[J_n^2(kt)+Y_n^2(kt)\right]\left|\frac{\mu^2 t^2}{X^4} - \frac{2\mu n}{X^2} + V\right| dt +$$

$$+ \frac{\pi}{2} \int_{X}^{+\infty} t\left[J_n^2(kt)+Y_n^2(kt)\right]\left|\frac{\mu^2 - 2\mu n}{t^2}\right| dt$$

$$= \Big/ y = kt \Big/ = \frac{\pi}{k^2} \int_0^{X_n} y|Y_n(y)|J_n(y)\left|\frac{\mu^2 y^2}{X^4 k^2} - \frac{2\mu n}{X^2} + V\right| dy +$$

$$+ \frac{\pi}{2k^2} \int_{X_n}^{kX} y\left[J_n^2(y)+Y_n^2(y)\right]\left|\frac{\mu^2 y^2}{X^4 k^2} - \frac{2\mu n}{X^2} + V\right| dy +$$

$$+ \frac{\pi}{2}\left|\mu^2 - 2\mu n\right| \int_{kX}^{+\infty} \left[J_n^2(y)+Y_n^2(y)\right] \frac{dy}{y} \tag{5.3.1}$$

First term is clearly of order $\frac{1}{k^2}$. Second term can be conveniently integrated, and using large argumetnt asymptotics of Bessel functions (1.0.2) stripped of irelevant factors will lead to result we are looking for (C_n stands for J_n, Y_n):

$$C_n(kt) \sim \frac{1}{\sqrt{kt}}$$

$$\frac{1}{k^2}\int_{X_n}^{kX} yC_n^2(y)dy \overset{(1.0.7)}{=} \frac{y^2}{2k^2}\Big[C_n(y)^2 - C_{n-1}(y)C_{n+1}(y)\Big]_{X_n}^{kX} \sim \frac{1}{k} \tag{5.3.2}$$

$$\frac{1}{k^4}\int_{X_n}^{kX} y^3 C_n^2(y)dy = \frac{1}{k^4}\int_{X_n}^{kX} y^2\left[yC_n^2(y)\right]dy \overset{p.p.}{=} \left[\frac{y^4}{2k^4}\left[C_n^2(y) - C_{n-1}(y)C_{n+1}(y)\right]\right]_{X_n}^{kX} +$$

$$-\frac{1}{k^4}\int_{X_n}^{kX} y^3\left[C_n^2(y) - C_{n-1}(y)C_{n+1}(y)\right]dy \tag{5.3.3}$$

$$\frac{1}{k^4}\int\limits_{X_n}^{kX} y^3 C_{n-1}(y)C_{n+1}(y)dy \overset{(1.0.3)}{=} \frac{(-1)^{n-1}}{k^4}\int\limits_{X_n}^{kX} y^3 C_{-n+1}(y)C_{n+1}(y)dy =$$
$$\overset{(1.0.8)}{=} (-1)^{n-1}\frac{y^4}{6k^4}\Big[C_{-n+1}(y)C_{n+1}(y) + C_{-n+2}(y)C_{n+2}(y)\Big]_{X_n}^{kX} =$$
$$\overset{(1.0.3)}{=} \frac{y^4}{6k^4}\Big[C_{n-1}(y)C_{n+1}(y) - C_{n-2}(y)C_{n+2}(y)\Big]_{X_n}^{kX} \tag{5.3.4}$$

$$\Rightarrow \qquad \frac{1}{k^4}\int\limits_{X_n}^{kX} y^3 C_n^2(y)dy = \left[\frac{y^4}{4k^4}\left[C_n^2(y) - C_{n-1}(y)C_{n+1}(y)\right]\right]_{X_n}^{kX} +$$
$$+\frac{1}{2k^4}\int\limits_{X_n}^{kX} y^3 C_{n-1}(y)C_{n+1}(y)dy$$
$$= \left[\frac{y^4}{4k^4}\left[C_n^2(y) - C_{n-1}(y)C_{n+1}(y)\right]\right]_{X_n}^{kX} +$$
$$+\left[\frac{y^4}{12k^4}\left[C_{n-1}(y)C_{n+1}(y) - C_{n-2}(y)C_{n+2}(y)\right]\right]_{X_n}^{kX}$$
$$= \left[\frac{y^4}{4k^4}\left[C_n^2(y) - \frac{2}{3}C_{n-1}(y)C_{n+1}(y) - \frac{1}{3}C_{n-2}(y)C_{n+2}(y)\right]\right]_{X_n}^{kX}$$
$$\sim \frac{1}{k} \tag{5.3.5}$$

In the third term of $G_n(k, +\infty)$ exppression (5.3.1) is due to large lower bound of integration sufficient to substitute asymptotics of Bessel functions (1.0.2) straight into integral:

$$\int\limits_{kX}^{+\infty}\left[J_n^2(y) + Y_n^2(y)\right]\frac{dy}{y} \sim \int\limits_{kX}^{+\infty}\frac{dy}{y^2} \sim \frac{1}{k} \tag{5.3.6}$$

All in all

$$G_n(k, +\infty) \sim \frac{1}{k}. \tag{5.3.7}$$

5.4 Large energy behavior of phase shift

δ_n is also of order $\frac{1}{k}$, as it consists of almost the same integrals, hence can be also integrated in terms of Bessel fuctions up to first order in $\frac{1}{k}$. Using previous calculations we have

$$\delta_n + \frac{\mu\pi}{2} = -\frac{\pi}{2}\int_0^{+\infty} J_n^2(kt)tg(t)dt + O(G_n^2(k,+\infty))$$

$$\overset{(5.2.2)}{\underset{(5.3.7)}{=}} -\frac{\pi}{2}\int_0^X J_n^2(kt)t\left[\frac{\mu^2t^2}{X^4} - \frac{2\mu n}{X^2} + V\right]dt - \frac{\pi}{2}\int_X^{+\infty} J_n^2(kt)\frac{\mu^2 - 2\mu n}{t}dt + O\left(\frac{1}{k^2}\right)$$

$$= \Big/ y = kt \Big/ = -\frac{\pi\mu^2}{2X^4k^4}\int_0^{kX} y^3 J_n^2(y)dy - \frac{\pi}{2k^2}\left[-\frac{2\mu n}{X^2} + V\right]\int_0^{kX} yJ_n^2(y)dy +$$

$$-\frac{\pi}{2}\left[\mu^2 - 2\mu n\right]\int_{kX}^{+\infty} J_n^2(y)\frac{dy}{y} + O\left(\frac{1}{k^2}\right)$$

$$\overset{(5.3.2)(5.3.3)(5.3.4)(1.0.3)}{=} -\frac{\pi\mu^2}{8}\left[J_n^2(kX) - \frac{2}{3}J_{n-1}(kX)J_{n+1}(kX) - \frac{1}{3}J_{n-2}(kX)J_{n+2}(kX)\right] +$$

$$-\frac{\pi X^2}{4}\left[-\frac{2\mu n}{X^2} + V\right]\left[J_n(kX)^2 - J_{n-1}(kX)J_{n+1}(kX)\right] +$$

$$-\frac{\pi}{2}\left[\mu^2 - 2\mu n\right]\int_{kX}^{+\infty} J_n(y)\left[-J_{n+2}(y) + \frac{2(n+1)}{y}J_{n+1}(y)\right]\frac{dy}{y} + O\left(\frac{1}{k^2}\right)$$

$$\overset{(1.0.3)}{=} -\frac{\pi\mu^2}{8}\left[J_n^2(kX) + \frac{2}{3}J_{n+1}^2(kX) - \frac{1}{3}J_n^2(kX) + O\left(\frac{1}{k^2}\right)\right] +$$

$$-\frac{\pi X^2}{4}\left[-\frac{2\mu n}{X^2} + V\right]\left[J_n(kX)^2 + J_{n+1}^2(kX) + O\left(\frac{1}{k^2}\right)\right] +$$

$$-\frac{\pi}{2}\left[\mu^2 - 2\mu n\right]\int_{kX}^{+\infty}\left[(-1)^{n+1}J_n(y)J_{-n-2}(y) + O\left(\frac{1}{k^2}\right)\right]\frac{dy}{y} + O\left(\frac{1}{k^2}\right)$$

$$\overset{(1.0.8)}{=} -\frac{\pi\mu^2}{12}\left[J_n^2(kX) + J_{n+1}^2(kX)\right] - \frac{\pi}{4}\left[-2\mu n + VX^2\right]\left[J_n(kX)^2 + J_{n+1}^2(kX)\right] +$$

$$-\frac{\pi}{2}\left[\mu^2 - 2\mu n\right]\frac{(-1)^{n+1}}{2}\left[J_n(kX)J_{-n-2}(kX) + J_{n+1}(kX)J_{-n-1}(kX)\right] + O\left(\frac{1}{k^2}\right)$$

$$\overset{(1.0.3)}{=} -\frac{\pi}{4}\left[\frac{\mu^2}{3} - 2\mu n + VX^2\right]\left[J_n^2(kX) + J_{n+1}^2(kX)\right] +$$

$$-\frac{\pi}{4}\left[\mu^2 - 2\mu n\right]\left[J_n^2(kX) + J_{n+1}^2(kX) + O\left(\frac{1}{k^2}\right)\right] + O\left(\frac{1}{k^2}\right)$$

$$= -\frac{\pi}{4}\left[VX^2 + \frac{4}{3}\mu^2 - 4\mu n\right]\left[J_n^2(kX) + J_{n+1}^2(kX)\right] + O\left(\frac{1}{k^2}\right). \qquad (5.4.1)$$

5.5 Numerical comparison

Following figures show graphical comparison of exact expression of phase shift (4.2.4) with approximation (5.2.5) as a function of energy k, for a few values of n, when $V = 2, X = 1, \mu = 0.1$, representing week magnetic field.

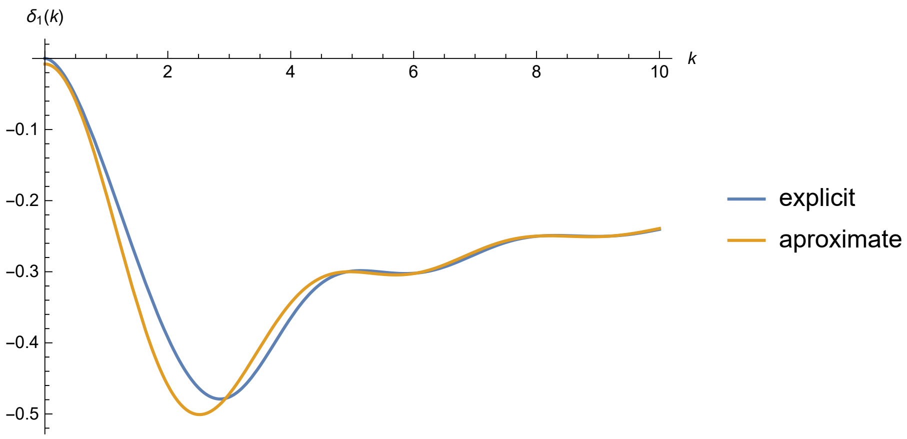


Figure 5.1: $\delta_1(k)$

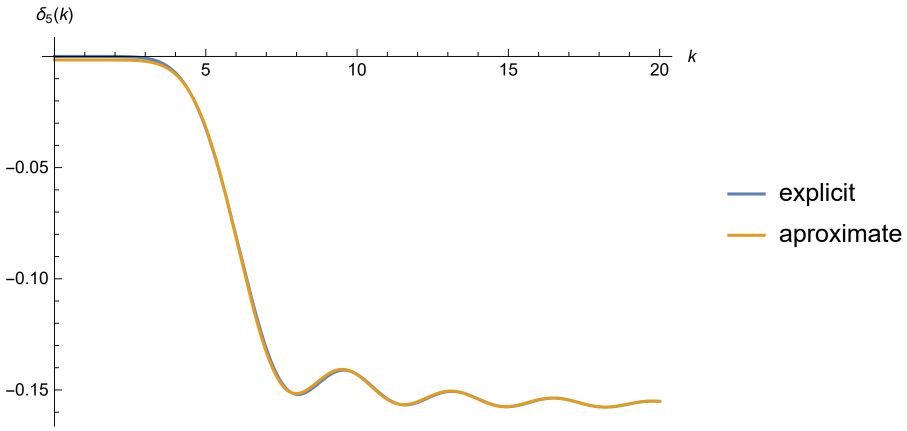


Figure 5.2: $\delta_5(k)$

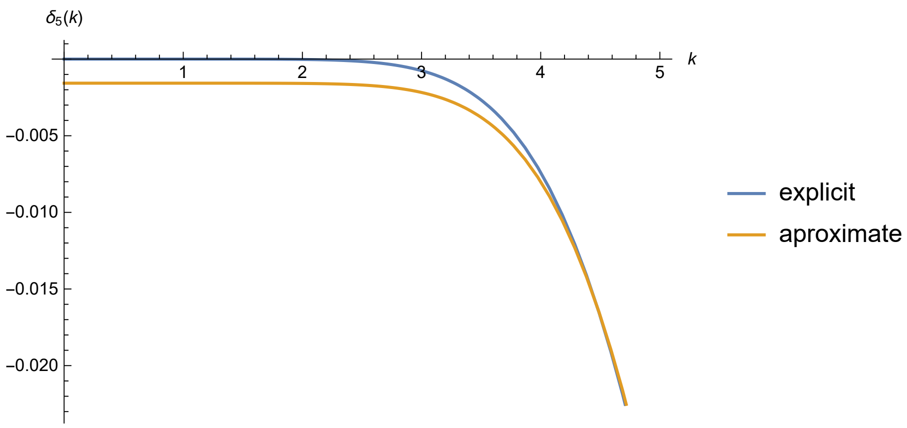


Figure 5.3: $\delta_5(k)$ - detail

In case of stronger magnetic field $\mu = 1$ accuracy for low energies is worse, however quickly improoves as energy increases.

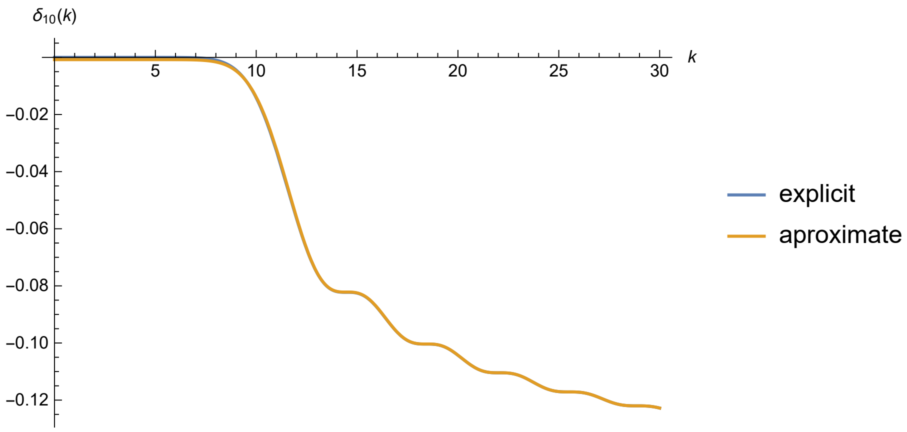


Figure 5.4: $\delta_{10}(k)$

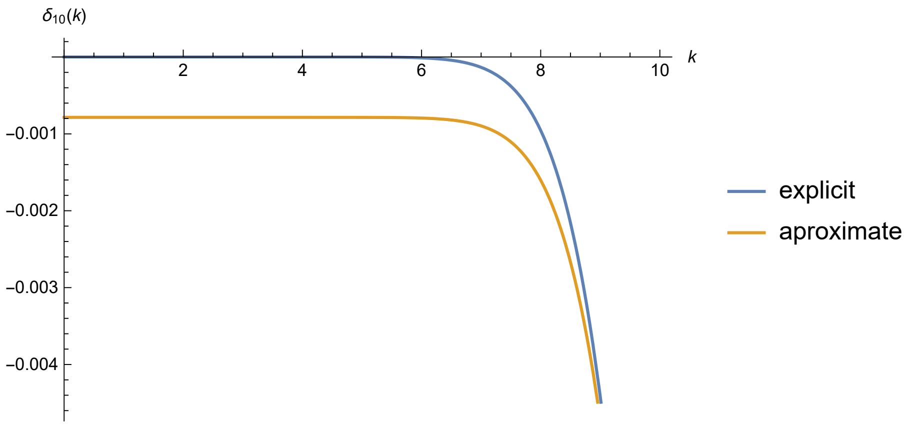


Figure 5.5: $\delta_{10}(k)$ - detail

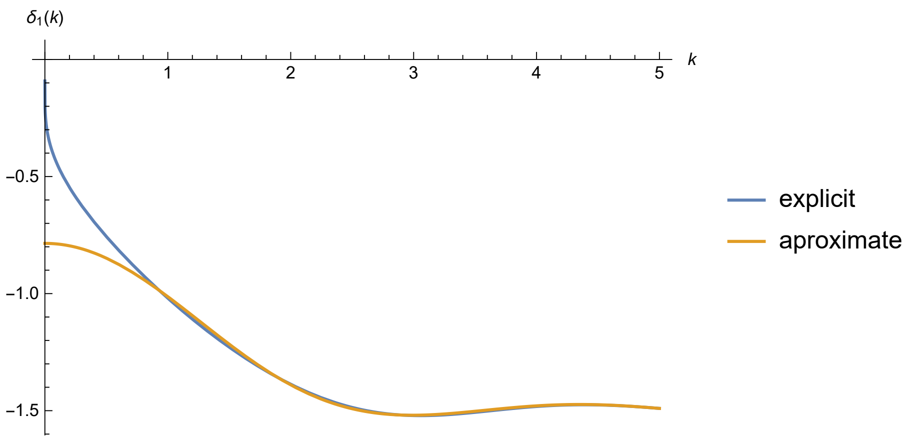


Figure 5.6: $\delta_1(k)$ - detail

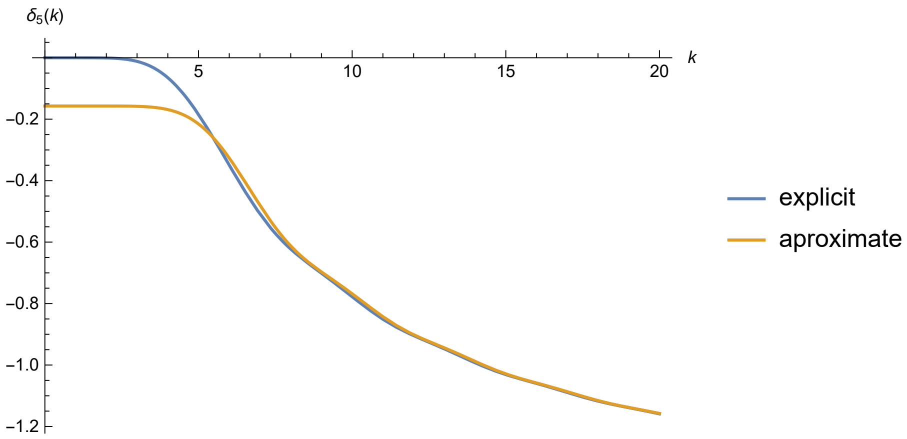


Figure 5.7: $\delta_5(k)$

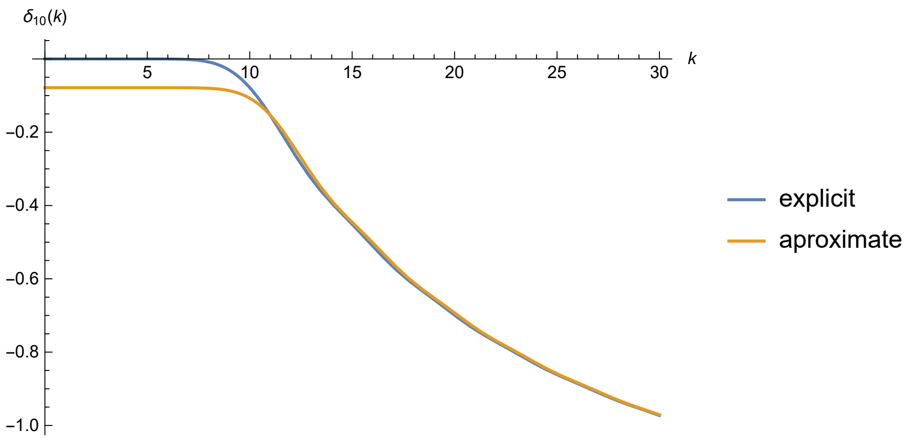


Figure 5.8: $\delta_{10}(k)$

Chapter 6

Cross section

6.1 Scattering amplitude

Having phase shift expression we can calculate scattering amplitude from (4.2.3):

$$
\begin{aligned}
f(\theta) &= \left(\frac{\mathrm{e}^{-i\pi/4}}{\sqrt{2\pi k}} \sum_{n\in\mathrm{Z}} \left[\mathrm{e}^{i(\mu\pi+2\delta_n)} - 1 \right] \mathrm{e}^{in\theta} \right) \\
&\sim \frac{e^{-i\pi/4}}{\sqrt{2\pi k}} \sum_{n\in\mathrm{Z}} i\left(\mu\pi + 2\delta_n\right) e^{in\theta} \\
&\overset{(5.2.5)}{\underset{(5.3.7)}{\sim}} \sqrt{\frac{2}{\pi k}} e^{i\pi/4} \sum_{n\in\mathrm{Z}} \left[-\frac{\pi}{2} \int_0^{+\infty} J_n^2(kt) t g(t) dt \right] e^{in\theta} \\
&\sim -\sqrt{\frac{\pi}{2k}} e^{i\pi/4} \int_0^{+\infty} \left[\sum_{n\in\mathrm{Z}} J_n^2(kt) t g(t) e^{in\theta} \right] dt,
\end{aligned}
$$

where

$$
\begin{aligned}
\int_0^{+\infty} \left[\sum_{n\in\mathrm{Z}} J_n^2(kt) t g(t) e^{in\theta} \right] dt = \int_0^{X} \left[\sum_{n\in\mathrm{Z}} J_n^2(kt) t \left[\frac{\mu^2 t^2}{X^4} - \frac{2\mu n}{X^2} + V \right] e^{in\theta} \right] dt + \\
+ \int_X^{+\infty} \left[\sum_{n\in\mathrm{Z}} J_n^2(kt) \frac{\mu^2 - 2\mu n}{t} e^{in\theta} \right] dt. \qquad (6.1.1)
\end{aligned}
$$

In order to perform sumation it is convenient to write down equalities given by Graf's theorem:

$$
\sum_{n\in\mathrm{Z}} J_n^2(kt) e^{in\theta} \overset{(1.0.12)}{=} J_0\left(2kt \sin\left(\frac{\theta}{2} \right) \right), \qquad (6.1.2)
$$

$$
\sum_{n\in\mathrm{Z}} n J_n^2(kt) e^{in\theta} = -i \frac{\partial}{\partial\theta} \sum_{n\in\mathrm{Z}} J_n^2(kt) e^{in\theta} = -i \frac{\partial}{\partial\theta} J_0\left(2kt \sin\left(\frac{\theta}{2} \right) \right), \qquad (6.1.3)
$$

which come handy in calculating (6.1.1) one term after another:

$$\frac{\mu^2}{X^4}\int_0^X\left[t^3\sum_{n\in\mathbb{Z}}J_n^2(kt)e^{in\theta}\right]dt \overset{(6.1.2)}{=} \frac{\mu^2}{X^4}\int_0^X t^2\left[tJ_0\left(2kt\sin\left(\frac{\theta}{2}\right)\right)\right]dt$$

$$\overset{p.p.}{=} \frac{\mu^2}{X^4}\left[\frac{t^3}{2k\sin\left(\frac{\theta}{2}\right)}J_1\left(2kt\sin\left(\frac{\theta}{2}\right)\right)\right]_0^X - \frac{\mu^2}{X^4k\sin\left(\frac{\theta}{2}\right)}\int_0^X t^2J_1\left(2kt\sin\left(\frac{\theta}{2}\right)\right)dt$$

$$\overset{(1.0.6)}{=} \frac{\mu^2}{2kX\sin\left(\frac{\theta}{2}\right)}J_1\left(2kX\sin\left(\frac{\theta}{2}\right)\right) - \frac{2\mu^2}{X^4}\left[\frac{t^2}{\left(2k\sin\left(\frac{\theta}{2}\right)\right)^2}J_2\left(2kt\sin\left(\frac{\theta}{2}\right)\right)\right]_0^X$$

$$= \frac{\mu^2}{2kX\sin\left(\frac{\theta}{2}\right)}J_1\left(2kX\sin\left(\frac{\theta}{2}\right)\right) - \frac{2\mu^2}{\left(2kX\sin\left(\frac{\theta}{2}\right)\right)^2}J_2\left(2kX\sin\left(\frac{\theta}{2}\right)\right)$$

$$= \frac{\mu^2}{2kX\sin\left(\frac{\theta}{2}\right)}J_1\left(2kX\sin\left(\frac{\theta}{2}\right)\right) + O\left(\frac{1}{k^{5/2}}\right),$$

$$-\frac{2\mu}{X^2}\int_0^X\left[t\sum_{n\in\mathbb{Z}}nJ_n^2(kt)e^{in\theta}\right]dt \overset{(6.1.3)}{=} \frac{2i\mu}{X^2}\frac{\partial}{\partial\theta}\int_0^X tJ_0\left(2kt\sin\left(\frac{\theta}{2}\right)\right)dt$$

$$\overset{(1.0.6)}{=} \frac{2i\mu}{X^2}\frac{\partial}{\partial\theta}\left[\frac{X}{2k\sin\left(\frac{\theta}{2}\right)}J_1\left(2kX\sin\left(\frac{\theta}{2}\right)\right)\right]$$

$$= -\frac{i\mu\cos\left(\frac{\theta}{2}\right)}{2kX\sin^2\left(\frac{\theta}{2}\right)}J_1\left(2kX\sin\left(\frac{\theta}{2}\right)\right) +$$

$$+ \frac{i\mu}{kX\sin\left(\frac{\theta}{2}\right)}J_1'\left(2kX\sin\left(\frac{\theta}{2}\right)\right)kX\cos\left(\frac{\theta}{2}\right)$$

$$\overset{(1.0.4)}{=} -\frac{i\mu\cos\left(\frac{\theta}{2}\right)}{kX\sin^2\left(\frac{\theta}{2}\right)}J_1\left(2kX\sin\left(\frac{\theta}{2}\right)\right) + \frac{i\mu\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}J_0\left(2kX\sin\left(\frac{\theta}{2}\right)\right),$$

$$V\int_0^X\left[t\sum_{n\in\mathbb{Z}}J_n^2(kt)e^{in\theta}\right]dt \overset{(6.1.2)}{=} V\int_0^X tJ_0\left(2kt\sin\left(\frac{\theta}{2}\right)\right)dt$$

$$\overset{(1.0.6)}{=} \frac{VX}{2k\sin\left(\frac{\theta}{2}\right)}J_1\left(2kX\sin\left(\frac{\theta}{2}\right)\right),$$

$$\mu^2\int_X^{+\infty}\left[\frac{1}{t}\sum_{n\in\mathbb{Z}}J_n^2(kt)e^{in\theta}\right]dt \overset{(6.1.2)}{=} \mu^2\int_X^{+\infty}\frac{1}{t}J_0\left(2kt\sin\left(\frac{\theta}{2}\right)\right)dt$$

$$\overset{(1.0.3)}{=} \mu^2\int_X^{+\infty}\frac{1}{t}\left[-J_{-2}\left(2kt\sin\left(\frac{\theta}{2}\right)\right)+O\left(\frac{1}{k^{3/2}}\right)\right]dt$$

$$\overset{(1.0.6)}{=} -\frac{\mu^2}{2kX\sin\left(\frac{\theta}{2}\right)}J_1\left(2kX\sin\left(\frac{\theta}{2}\right)\right)+O\left(\frac{1}{k^{5/2}}\right),$$

$$-2\mu \int\limits_X^{+\infty} \left[\frac{1}{t} \sum_{n \in Z} n J_n^2(kt) e^{in\theta} \right] dt \overset{(6.1.3)}{=} 2i\mu \int\limits_X^{+\infty} \frac{1}{t} \frac{\partial}{\partial \theta} J_0 \left(2kt \sin\left(\frac{\theta}{2}\right)\right) dt$$

$$= 2i\mu \int\limits_X^{+\infty} \frac{1}{t} J_0' \left(2kt \sin\left(\frac{\theta}{2}\right)\right) kt \cos\left(\frac{\theta}{2}\right) dt$$

$$= i\mu \cos\left(\frac{\theta}{2}\right) \int\limits_X^{+\infty} 2k J_0' \left(2kt \sin\left(\frac{\theta}{2}\right)\right) dt$$

$$= -\frac{i\mu \cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} J_0 \left(2kX \sin\left(\frac{\theta}{2}\right)\right).$$

Putting all together results in

$$f(\theta) \sim -\sqrt{\frac{\pi}{2k}} e^{i\pi/4} \Bigg[\cancel{\frac{\mu^2}{2kX \sin\left(\frac{\theta}{2}\right)} J_1 \left(2kX \sin\left(\frac{\theta}{2}\right)\right)} +$$

$$-\frac{i\mu \cos\left(\frac{\theta}{2}\right)}{kX \sin^2\left(\frac{\theta}{2}\right)} J_1 \left(2kX \sin\left(\frac{\theta}{2}\right)\right) + \cancel{\frac{i\mu \cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} J_0 \left(2kX \sin\left(\frac{\theta}{2}\right)\right)} +$$

$$+\frac{VX}{2k \sin\left(\frac{\theta}{2}\right)} J_1 \left(2kX \sin\left(\frac{\theta}{2}\right)\right) - \cancel{\frac{\mu^2}{2kX \sin\left(\frac{\theta}{2}\right)} J_1 \left(2kX \sin\left(\frac{\theta}{2}\right)\right)} +$$

$$\cancel{-\frac{i\mu \cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} J_0 \left(2kX \sin\left(\frac{\theta}{2}\right)\right)} \Bigg]$$

$$\sim -\sqrt{\frac{\pi}{8}} \frac{VXe^{i\pi/4}}{k^{3/2} \sin\left(\frac{\theta}{2}\right)} J_1 \left(2kX \sin\left(\frac{\theta}{2}\right)\right) \left[1 - \frac{4i\mu \cos\left(\frac{\theta}{2}\right)}{VX^2 \sin\left(\frac{\theta}{2}\right)} \right]. \tag{6.1.4}$$

6.2 Differential cross section

Scattering analysis would not be complete without determining cross section. It's relation to the scattering amplitude

$$\frac{d\sigma(\theta)}{d\theta} = |f(\theta)|^2 \qquad \sigma = \int\limits_0^{2\pi} \frac{d\sigma(\theta)}{d\theta} d\theta, \tag{6.2.1}$$

gives using (6.1.4) immediately

$$\frac{d\sigma(\theta)}{d\theta} = |f(\theta)|^2 \sim \frac{\pi V^2 X^2}{8k^3 \sin^2\left(\frac{\theta}{2}\right)} J_1^2 \left(2kX \sin\left(\frac{\theta}{2}\right)\right) \left[1 + \frac{16\mu^2 \cos^2\left(\frac{\theta}{2}\right)}{V^2 X^4 \sin^2\left(\frac{\theta}{2}\right)} \right]. \tag{6.2.2}$$

Clearly there is singularity at $\theta = 0$. Hwever, this result shuld not surprise us, using (1.0.1) gives

$$\frac{d\sigma(\theta)}{d\theta} \sim \frac{\pi V^2 X^4}{8k} \left[1 + \frac{16\mu^2 \cos^2\left(\frac{\theta}{2}\right)}{V^2 X^4 \sin^2\left(\frac{\theta}{2}\right)} \right] \sim \frac{2\pi\mu^2}{k \sin^2\left(\frac{\theta}{2}\right)}, \qquad \theta \to 0. \tag{6.2.3}$$

This result is in agreement with formula for differential cross section of scattering on a magnetic vortex of infinitely small radius (see (21) in [1]).

Complete cross section is then given as

$$\sigma \sim \int\limits_0^{2\pi} \frac{\pi V^2 X^2}{8k^3 \sin^2\left(\frac{\theta}{2}\right)} J_1^2 \left(2kX \sin\left(\frac{\theta}{2}\right)\right) \left[1 + \frac{16\mu^2 \cos^2\left(\frac{\theta}{2}\right)}{V^2 X^4 \sin^2\left(\frac{\theta}{2}\right)} \right] d\theta. \tag{6.2.4}$$

Chapter 7

Conclusion

We dealt with 2D quantum scattering of a charged particle by finite cylinder potential containing magnetic flux perpendicular to the plane where particle's movement is considered. Using partial wave decomposition, we broke up the problem into solving Schrödinger equations for individual modes, for which exact solutions were found for both nighbourhood of zero, inside the cylinder potential, and infinity, far outside the potential area.

Inside the cylinder we used WKB approach to find convenient approimations of exact results, following by their comparison with leading terms of exact ones in vicinity of zero.

In case of asymptotic region of infinity we again used WKB approach to determine convenient approximation of phase shift, compared to quite obscure exact expressions, folowed by determinig qualitative large energy behaviour and numerical comparison which showed great fit. Consequently we calculated differential cross section which was found to be in great agreement with result of well known scattering on magnetic vortex of infinitely small radius.

Approximation of total cross section can be considered little inconvenient and further investigation may be apropriate as well as generalising problem to potentials of different shapes or multiple cylinders. However, goal of this project was to investigate scattering on finite cylinder, which was fullfiled.

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