Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Scattering on a magnetic Aharonov-Bohm flux in a plane

Bachelor's Thesis

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Matej Hazala

Název práce: **Rozptyl na magnetickém toku Aharonova-Bohma v rovině**

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Abstrakt: Uvedeme obecnou formulaci stacionárního rozptylu ve dvou dimenzích a užitím parciálního vlnového rozvoje odvodíme řešení příslušné Schrödingerovy rovnice a následně odvodíme diferenciální účinný průřez. Pro konečný válcový potenciál odvodíme také přibližné výrazy užitím asymptotiky velkých energií a WKB přiblížení a pak numericky porovnáme získané výsledky. Též se budeme zabývat rozptylem na δ-potenciálu, jakožto limitního případu konečného válcového potenciálu.

Klíčová slova: teorie rozptylu, WKB metoda

Title: **Scattering on a magnetic Aharonov-Bohm flux in a plane**

Author: Matej Hazala

Abstract: We present a general formulation of the stationary scattering theory in two dimensions. By using partial wave decomposition we derive a solution of the corresponding Schrödinger equation and consequently we derive the differential cross section. For finite cylindrical potential we determinate also approximate expressions by using the large energy asymptotic and the WKB approximation and subsequently compare numerically the obtained results. We are dealing also with the scattering on a δ-potential as a limiting case of the finite cylindrical potential.

Key words: scattering theory, WKB approximation

Contents

Notation

$\mathbb{N}$	set of all positive integers
$\mathbb{N}_0$	set of all non-negative integers
$\mathbb{Z}$	set of all integers
$\mathbb{C}$	complex plane (excluding infinity)
x, y	real variable
z	complex variable
ph	phase of complex number
Re	real part
Δ	Laplace operator

Introduction

As it is evident from the title, this thesis should have dealt with scattering on Aharonov-Bohm flux, based on idea of a magnetic flux inside hard cylinder orthogonal to the plane, where we were looking for solutions. However, this intention was not fulfilled. In the paper we are dealing only with the cylindrical potential without magnetic flux, as it has provided enough interesting situations, on which we have decided to focus.

The first chapter introduces special functions, namely the Bessel functions and the Airy functions, used throughout the thesis, and summarizes their properties and relations, which are important for calculations in the rest of the paper.

In the second chapter there is described classical scattering to outline the basic ideas of scattering in two dimensions and to give results, which are serving later for comparison with the classical limit of high energies, as we are expecting their agreement.

In the third chapter the WKB method is used to find an approximate solutions of the Schrödinger equation. This task should be solved in two separate regions. After finding solutions in the both regions, the Airy functions will be used to connect them.

In the fourth chapter the problem of the stationary quantum scattering is firstly introduced by using the partial wave decomposition to solve the corresponding Schrödinger equation for spherically symmetric potential, and subsequently to derive differential and total cross section expressions. Then it follows a transformation of the differential Schrödinger equation to an integral equation, which is used in the next chapter, by applying the Green function.

The fifth chapter deals with scattering on a finite cylindrical potential by using integral and differential approach to derive some exact expressions and then there are used large energy asymptotic and the WKB method to derive simplified expressions including differential and total cross section. Subsequently we make some numerical comparisons of those simplified results and the exact ones. In the end of the chapter we solve the scattering on a δ-potential realized as a limit of finite cylinder.

Chapter 1

Special functions

This chapter is meant to be a summary of known properties of the Bessel functions and the Airy functions used in this paper. If it is not stated otherwise, the listed relations and properties are taken from a monograph dedicated to properties of special functions [1].

1.1 Bessel functions

1.1.1 Definition

The Bessel functions are defined as solutions of the Bessel's equation:

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+\left(z^2-\nu^2\right)w=0\ . \tag{1.1}$$

The Bessel function of the first kind

$$J_\nu(z)=\left(\frac{z}{2}\right)^\nu\sum_{k=0}^{\infty}(-1)^k\frac{\left(\frac{z}{2}\right)^{2k}}{k!\Gamma(\nu+k+1)} \tag{1.2}$$

It is an analytic function of $z\in\mathbb{C}$, except for a branch point $z=0$ when ν is not an integer.

The Bessel function of the second kind

$$Y_\nu(z)=\frac{J_\nu(z)cos(\nu\pi)-J_{-\nu}(z)}{sin(\nu\pi)} \tag{1.3}$$

This function has a branch point at $z=0$ whether or not is ν an integer. When ν is an integer it can be written as

$$Y_n(z)=\frac{1}{\pi}\left.\frac{\partial J_\nu(z)}{\partial\nu}\right|_{\nu=n}+\frac{(-1)^n}{\pi}\left.\frac{\partial J_{\nu(z)}}{\partial\nu}\right|_{\nu=-n}\ ,\qquad n\in\mathbb{Z}\ , \tag{1.4}$$

and when $n\in\mathbb{N}_0$, it can be written in power series representation in the following way:

$$Y_n(z)=-\frac{\left(\frac{z}{2}\right)^{-n}}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k}+\frac{2}{\pi}\ln\left(\frac{z}{2}\right)J_n(z)+$$

$$-\frac{\left(\frac{z}{2}\right)^{n}}{\pi}\sum_{k=0}^{\infty}\left(\psi(k+1)+\psi(n+k+1)\right)\frac{(-1)^k\left(\frac{z}{2}\right)^{2k}}{k!(n+k)!}\ ,\qquad n\in\mathbb{N}_0\ ,\ \psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}\ . \tag{1.5}$$

The Bessel functions of the third kind

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z) \tag{1.6}$$

$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z) \tag{1.7}$$

They are also called the Hankel functions.

1.1.2 Properties

Cylinder functions $\mathscr{C}_\nu$ denote any linear combination of $J_\nu(z)$, $Y_\nu(z)$, $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$, in which coefficients are independent of z and ν.

$$\mathscr{C}_{-n}(z) = (-1)^n \mathscr{C}_n(z) \tag{1.8}$$

Wronskians

$$\mathscr{W}\left\{J_\nu(z), Y_\nu(z)\right\} = J_{\nu+1}(z)Y_\nu(z) - J_\nu(z)Y_{\nu+1}(z) = \frac{2}{\pi z} \tag{1.9}$$

$$\mathscr{W}\left\{J_\nu(z), H_\nu^{(1)}(z)\right\} = J_{\nu+1}(z)H_\nu^{(1)}(z) - J_\nu(z)H_{\nu+1}^{(1)}(z) = \frac{2i}{\pi z} \tag{1.10}$$

$$\mathscr{W}\left\{J_\nu(z), H_\nu^{(2)}(z)\right\} = J_{\nu+1}(z)H_\nu^{(2)}(z) - J_\nu(z)H_{\nu+1}^{(2)}(z) = -\frac{2i}{\pi z} \tag{1.11}$$

Recurrence relations

$$\mathscr{C}_{\nu-1}(z) + \mathscr{C}_{\nu+1}(z) = \frac{2\nu}{z}\mathscr{C}_\nu(z) \tag{1.12}$$

$$\mathscr{C}_{\nu-1}(z) - \mathscr{C}_{\nu+1}(z) = 2\mathscr{C}'_\nu(z) \tag{1.13}$$

$$\mathscr{C}'_\nu(z) = \mathscr{C}_{\nu-1}(z) - \frac{\nu}{z}\mathscr{C}_\nu(z) \tag{1.14}$$

$$\mathscr{C}'_\nu(z) = -\mathscr{C}_{\nu+1}(z) + \frac{\nu}{z}\mathscr{C}_\nu(z) \tag{1.15}$$

Asymptotic properties

$\mathbf{z \to 0}$:

$$J_0(z) \to 1\ , \qquad Y_0(z) \sim \frac{2}{\pi}\ln z\ , \qquad H_0^{(1)}(z) \sim -H_0^{(2)}(z) \sim \frac{2i}{\pi}\ln z \tag{1.16}$$

$$J_\nu(z) \sim \left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)}\ , \qquad \nu \neq -1, -2, -3, ... \tag{1.17}$$

$$Y_\nu \sim -\frac{1}{\pi}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}\ , \qquad \operatorname{Re}\nu > 0 \text{ or } \nu = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, ... \tag{1.18}$$

$$Y_{-\nu} \sim -\frac{1}{\pi}\cos(\nu\pi)\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}\ , \qquad \operatorname{Re}\nu > 0,\ \nu \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ... \tag{1.19}$$

$$H_\nu^{(1)}(z) \sim -H_\nu^{(2)}(z) \sim -\frac{i}{\pi}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}\ , \qquad \operatorname{Re}\nu > 0 \tag{1.20}$$

$\mathbf{z \to \infty}$:

$$J_\nu(z) \to \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\ , \qquad -\pi + \delta \le \operatorname{ph} z \le 2\pi - \delta,\ 0 < \delta \tag{1.21}$$

$$Y_\nu(z) \to \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\ , \qquad -\pi + \delta \le \operatorname{ph} z \le 2\pi - \delta,\ 0 < \delta \tag{1.22}$$

$$H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}e^{i\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)}\ , \qquad -\pi + \delta \le \operatorname{ph} z \le 2\pi - \delta,\ 0 < \delta \tag{1.23}$$

$$H_\nu^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}e^{-i\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)}\ , \qquad -2\pi + \delta \le \operatorname{ph} z \le \pi - \delta,\ 0 < \delta \tag{1.24}$$

Graf's addition theorem

$$\mathscr{C}_\nu(w)\frac{\cos}{\sin}(\nu\chi) = \sum_{k=-\infty}^{\infty} \mathscr{C}_{\nu+k}(u) J_k(v) \frac{\cos}{\sin}(k\alpha) \ , \qquad \left|ve^{\pm i\alpha}\right| < |u| \ , \tag{1.25}$$

$$\text{where} \quad w = \sqrt{u^2+v^2-2uv\cos\alpha} \ , \quad u - v\cos\alpha = w\cos\chi \ , \quad v\sin\alpha = w\sin\chi$$

Integrals

The following integral relations are from [2], and they are written here in a form modified to our purposes.

$$\int_0^a xJ_n(\lambda x)J_n(\mu x) = \left(\lambda^2-\mu^2\right)^{-1}\left[\lambda a J_{n+1}(\lambda a)J_n(\mu a) - \mu a J_n(\lambda a) J_{n+1}(\mu a)\right] \ , \qquad \lambda \neq \mu, \ n \in \mathbb{N}_0 \tag{1.26}$$

$$\int_0^a xJ_n(\lambda x)H_n^{(1)}(\mu x) = \left(\lambda^2-\mu^2\right)^{-1}\left[\frac{-2i}{\pi}\left(\frac{\lambda}{\mu}\right)^n + \lambda a J_{n+1}(\lambda a)H_n^{(1)}(\mu a) - \mu a J_n(\lambda a)H_{n+1}^{(1)}(\mu a)\right] \ , \qquad \lambda \neq \mu, \ n \in \mathbb{N}_0 \tag{1.27}$$

$$\int_0^\infty \frac{xJ_0(ax)}{x^2+k^2}dx = \frac{\pi i}{2}H_0^{(1)}(iak) \ , \qquad a > 0, \ \operatorname{Re} k > 0, \ -\pi \leq \operatorname{ph} k \leq \frac{\pi}{2} \tag{1.28}$$

1.2 Airy functions

1.2.1 Definition

The Airy functions are defined as solutions of the Airy's equation:

$$\frac{d^2w}{dz^2} = zw \ . \tag{1.29}$$

The standard solutions $Ai(z)$, $Bi(z)$ can be written in the integral representations for a real argument in the following way:

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt \ , \tag{1.30}$$

$$Bi(x) = \frac{1}{\pi}\int_0^\infty \exp\left(-\frac{t^3}{3} + xt\right) dt + \frac{1}{\pi}\int_0^\infty \sin\left(\frac{t^3}{3} + xt\right) dt \ . \tag{1.31}$$

1.2.2 Asymptotic properties

For simplicity of expressions, let us denote:

$$\zeta = \frac{2}{3}z^{\frac{3}{2}} \ , \qquad u_0 = 1 \ , \qquad u_k = \frac{(2k+1)(2k+3)(2k+5)\dots(6k-1)}{(216)^k(k)!} \ . \tag{1.32}$$

$\mathbf{z \to \infty}$:

$$Ai(z) \sim \frac{e^{-\zeta}}{2\sqrt{\pi}z^{\frac{1}{4}}}\sum_{k=0}^{\infty}(-1)^k\frac{u_k}{\zeta^k} \ , \qquad |\operatorname{ph} z| \leq \pi - \delta \tag{1.33}$$

$$Ai(-z) \sim \frac{1}{\sqrt{\pi}z^{\frac{1}{4}}}\left(\cos\left(\zeta - \frac{\pi}{4}\right)\sum_{k=0}^{\infty}(-1)^k\frac{u_{2k}}{\zeta^{2k}} + \sin\left(\zeta - \frac{\pi}{4}\right)\sum_{k=0}^{\infty}(-1)^k\frac{u_{2k+1}}{\zeta^{2k+1}}\right) \ , \qquad |\operatorname{ph} z| \leq \frac{2}{3}\pi - \delta \tag{1.34}$$

$$Bi(z) \sim \frac{e^{\zeta}}{\sqrt{\pi} z^{\frac{1}{4}}} \sum_{k=0}^{\infty} \frac{u_k}{\zeta^k} \ , \qquad |\mathrm{ph}\, z| \leq \frac{1}{3}\pi - \delta \tag{1.35}$$

$$Bi(-z) \sim \frac{1}{\sqrt{\pi} z^{\frac{1}{4}}} \left(-\sin\left(\zeta - \frac{\pi}{4}\right) \sum_{k=0}^{\infty} (-1)^k \frac{u_{2k}}{\zeta^{2k}} + \cos\left(\zeta - \frac{\pi}{4}\right) \sum_{k=0}^{\infty} (-1)^k \frac{u_{2k+1}}{\zeta^{2k+1}} \right) \ , \qquad |\mathrm{ph}\, z| \leq \frac{2}{3}\pi - \delta \tag{1.36}$$

Chapter 2

Classical scattering

Scattering of two particles of masses m_1, m_2 interacting via potential $V(\mathbf{r})$ depending on the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, by analogy with the classical scattering, can be reduced to the case of scattering of a particle with the reduced mass $m = \frac{m_1 m_2}{m_1 + m_2}$ on the potential $V(\mathbf{r})$, simply by changing the coordinate system to the one, which is fixed in the center of mass of the system consisting of those two particles. Whilst brief summary of the classical scattering can be found in [9], here we present it in a more detail way.

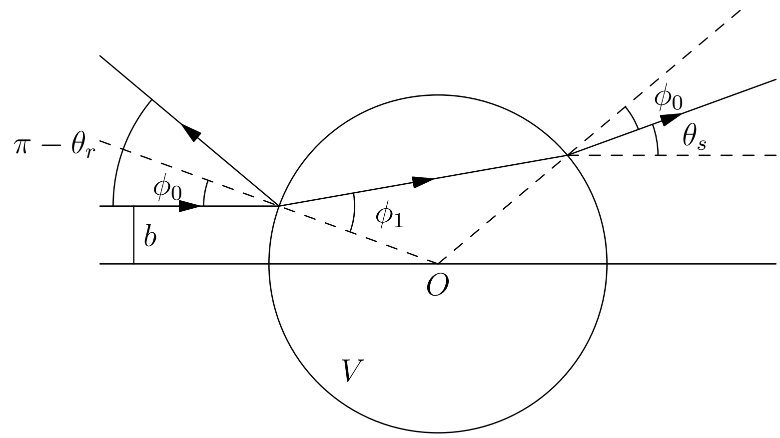

Figure 2.1: A classical scattering scheme.

Hence, we consider scattering of the particle with a mass m on the cylindrical potential V with a radius R in a 2D plane as follows:

$$V(r) = \begin{cases} V_0 \,, & r < R \,, \\ 0 \,, & r > R \,. \end{cases} \tag{2.1}$$

Due to symmetry of the problem it is convenient to restrict the discussion to the upper half-plane. To stress difference between the reflection at the boundary of the cylinder and the deflection caused by passing through the cylinder, the scattering angle is used with the corresponding indices. According to Fig. 2.1, b denotes the scattering parameter, θ_r denotes the scattering angle in the case of reflection, θ_s denotes the scattering angle in case of deflection. It is evident that, there is no scattering for $b > R$, and if $E < V_0$, only the reflection at the boundary is realized. Thus in further calculations we suppose $b < R$ and $E > V_0$, if is not said otherwise. From Fig. 2.1 it is also palpable, that the following relations are satisfied.

$$\sin \phi_0 = \frac{b}{R} \tag{2.2}$$

$$\theta_r = \pi - 2\phi_0 = \pi - 2\sin^{-1}\left(\frac{b}{R}\right) \tag{2.3}$$

We can immediately see that the scattering angle in the case of the reflection at the boundary of the cylinder is

$$\theta_r = 2\cos^{-1}\left(\frac{b}{R}\right) \ . \tag{2.4}$$

For the further calculation of the total cross section it is convenient to derive the maximum scattering angle. In a such situation, we use the equation for energy:

$$E_n = E\cos^2\phi_0 = V_0 \quad \Rightarrow \quad \sin\phi_0 = \sqrt{\frac{E - V_0}{E}} \ , \tag{2.5}$$

where E_n is the energy corresponding to the radial part of the initial velocity $\vec{v}_0$ due to surface of the potential cylinder. Later it will be evident that the maximum angle is the same for both reflection at the boundary and deflection inside the cylinder, as follows from the course of scattering, when the particle enters cylinder while $E_n > V_0$ and reflects from cylinder when $E_n < V_0$, depending on the scattering parameter b. To simplify following calculations let us denote

$$n = \sqrt{\frac{E - V_0}{E}} \ . \tag{2.6}$$

Now from (2.3) it is easy to derive

$$\theta_{r\ max} = 2\cos^{-1} n \ . \tag{2.7}$$

Hence, using (2.4) one obtains θ_r in terms of b.

$$\theta_r = 2\cos^{-1}\frac{b}{R} \ , \qquad b \in (nR, R) \tag{2.8}$$

In the case of the deflection in the potential area, using the energy conservation law:

$$E = \frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 + V_0 \ , \tag{2.9}$$

where $\vec{v}_1$ is the velocity inside cylinder, and the conservation law of tangential inertia:

$$v_1 \sin\phi_1 = v_0 \sin\phi_0 \ , \tag{2.10}$$

one obtains

$$\sin\phi_1 = \frac{b}{Rn} \ , \tag{2.11}$$

and using the relation between angles, as it is palpable from Fig. 2.1

$$\theta_s = 2\left(\phi_1 - \phi_0\right) \ , \tag{2.12}$$

is derived

$$\theta_s = 2\left(\sin^{-1}\left(\frac{b}{Rn}\right) - \sin^{-1}\left(\frac{b}{R}\right)\right) \ , \qquad b \in (0, nR) \ . \tag{2.13}$$

In order to verify the derived maximum scattering angle, using the energy conservation law (2.9), one obtains

$$\sin\phi_1 = 1 \ . \tag{2.14}$$

Hence, recalling the equalities (2.5) and (2.12) we obtain what we have expected - the same result as in the case of reflection.

$$\theta_{s\ max} = 2\left(\frac{\pi}{2} - \sin^{-1} n\right) = 2\cos^{-1} n \tag{2.15}$$

To find the terms contributing to the differential cross section:

$$\frac{d\sigma(\theta)}{d\theta} = \left|\frac{db(\theta)}{d\theta}\right| = \left|\frac{db_r(\theta_r)}{d\theta_r}\right|_{b_r\in(nR,R)} + \left|\frac{db_s(\theta_s)}{d\theta_s}\right|_{b_s\in(0,nR)} , \tag{2.16}$$

it is needed to write b in terms of θ. In the case of reflection it is apparent from (2.4).

$$b_r = R\cos\frac{\theta_r}{2} \quad \Rightarrow \quad \frac{db_r}{d\theta_r} = -\frac{R}{2}\sin\frac{\theta_r}{2} \tag{2.17}$$

In the second case by using the previous equalities we have:

$$\frac{b}{Rn} = \sin\phi_1 = \sin\left(\frac{\theta_s}{2}+\phi_0\right) = \cos\phi_0\sin\frac{\theta_s}{2} + \sin\phi_0\cos\frac{\theta_s}{2} = \sqrt{1-\frac{b^2}{R^2}}\sin\frac{\theta_s}{2} + \frac{b}{R}\cos\frac{\theta_s}{2} , \tag{2.18}$$

$$b_s = \frac{Rn\sin\frac{\theta_s}{2}}{\sqrt{1-2n\cos\frac{\theta_s}{2}+n^2}} , \tag{2.19}$$

$$\frac{db_s}{d\theta_s} = \frac{\frac{Rn}{2}\cos\frac{\theta_s}{2}}{\left(1-2n\cos\frac{\theta_s}{2}+n^2\right)^{\frac{1}{2}}} + \frac{-\frac{Rn^2}{2}\sin^2\frac{\theta_s}{2}}{\left(1-2n\cos\frac{\theta_s}{2}+n^2\right)^{\frac{3}{2}}} . \tag{2.20}$$

Hence, differential cross section is

$$\frac{d\sigma(\theta)}{d\theta} = \frac{R}{2}\sin\frac{\theta}{2} + \frac{\frac{Rn}{2}\left(\cos\frac{\theta}{2}-n\right)\left(1-n\cos\frac{\theta}{2}\right)}{\left(1-2n\cos\frac{\theta}{2}+n^2\right)^{\frac{3}{2}}} , \qquad \theta\in(0,2\cos^{-1}n) . \tag{2.21}$$

The total scattering cross section, considering both - the upper and the lower half-plane, is thus

$$\sigma = 2\int_0^{\pi}\left|\frac{d\sigma}{d\theta}\right|d\theta = 2\int_0^{2\cos^{-1}n} d\theta\frac{R}{2}\sin\frac{\theta}{2} + \frac{\frac{Rn}{2}\cos\frac{\theta}{2}}{\left(1-2n\cos\frac{\theta}{2}+n^2\right)^{\frac{1}{2}}} + \frac{-\frac{Rn^2}{2}\sin^2\frac{\theta}{2}}{\left(1-2n\cos\frac{\theta}{2}+n^2\right)^{\frac{3}{2}}} =$$

$$\stackrel{p.p.}{=} 2R(1-n) + 2\left[\frac{Rn\sin\frac{\theta_s}{2}}{\left(1-2n\cos\frac{\theta_s}{2}+n^2\right)^{\frac{1}{2}}}\right]_0^{2\cos^{-1}n} = 2R . \tag{2.22}$$

Chapter 3

WKB approximation

This method is used to find an approximate solution of the partial differential equations. In our case it would be the Schrödinger equation:

$$H\psi(y) = -\frac{\hbar^2}{2m}\frac{d^2}{dy^2}\psi(y) + U(y)\psi(y) = E\psi(y) \ . \tag{3.1}$$

This method is briefly described in [6]. In order to transform this equation into a dimensionless form the following substitution is used:

$$\begin{aligned} y &= ax \ , \qquad [a] = m \ , \\ U(y) &= uf(x) = uf\left(\frac{y}{a}\right) \ , \\ \varphi(x) &= \psi(y) = \psi(ax) \ . \end{aligned} \tag{3.2}$$

The introduced constants are appropriately chosen. How to choose them will be stated later.

$$-\frac{\hbar^2}{2ma^2u}\frac{d^2}{dy^2}\varphi(x) + \left(f(x) - \frac{E}{u}\right)\varphi(x) = 0 \tag{3.3}$$

To simplify this expression we denote:

$$\xi^2 = \frac{\hbar^2}{2ma^2u} \ , \qquad r(x) = \frac{E}{u} - f(x) \ . \tag{3.4}$$

Thus (3.1) transforms to

$$\xi^2\frac{d^2}{dx^2}\varphi(x) + r(x)\varphi(x) = 0 \ . \tag{3.5}$$

By using another substitution

$$\varphi(x) = \exp\left(\frac{i}{\xi}\int_{x_0}^{x} q(\tilde{x})d\tilde{x}\right) \ , \tag{3.6}$$

one obtains

$$i\xi q'(x) - q^2(x) + r(x) = 0 \ . \tag{3.7}$$

It is Riccati's equation with a solution, which can be found in the form of a series.

$$q(x) = \sum_{n=0}^{\infty}(-i\xi)^n q_n(x) \tag{3.8}$$

From this expression it can be seen that the previously introduced constants are needed to be chosen in such a way, in which ξ is sufficiently small. Hence, we get the equation:

$$\sum_{n=0}^{\infty}(-i\xi)^{n+1}q_n'(x) + \sum_{n=0}^{\infty}\sum_{m=0}^{n}(-i\xi)^n q_m(x)q_{n-m}(x) - r(x) = 0 \ , \tag{3.9}$$

and by comparing terms corresponding to the same power we find the q_n expressions of an arbitrary order depending on lower order terms. We are interested in the approximation of the first order:

$$q(x) \sim q_0(x) - i\xi q_1(x) \ . \tag{3.10}$$

For classically allowed region corresponding to $r(x) > 0$ one obtains:

$$q_0(x) = \pm\sqrt{r(x)} \ , \qquad r(x) > 0 \ , \tag{3.11}$$

$$q_1(x) = -\frac{1}{2}\frac{q_0'(x)}{q_0(x)} = -\frac{1}{4}\frac{d}{dx}\ln r(x) \ , \qquad r(x) > 0 \ , \tag{3.12}$$

which used in (3.6) leads to:

$$\varphi^{\pm}(x) = \exp\left(\frac{i}{\xi}\int\limits_{x_0}^{x}\left(\pm\sqrt{r(\tilde{x})} + \frac{i\xi}{4}\frac{d}{d\tilde{x}}\ln r(\tilde{x})\right)d\tilde{x}\right) = \frac{r^{\frac{1}{4}}(x_0)}{r^{\frac{1}{4}}(x)}\exp\left(\pm\frac{i}{\xi}\int\limits_{x_0}^{x} r^{\frac{1}{2}}(\tilde{x})d\tilde{x}\right) \ , \qquad r(x) > 0 \ . \tag{3.13}$$

For the classically forbidden region corresponding to $r(x) < 0$, we get:

$$q_0(x) = \pm i\sqrt{|r(x)|} \ , \qquad r(x) < 0 \ , \tag{3.14}$$

$$q_1(x) = -\frac{1}{4}\frac{d}{dx}\ln|r(x)| \ , \qquad r(x) < 0 \ , \tag{3.15}$$

$$\varphi^{\pm}(x) = \frac{|r(x_0)|^{\frac{1}{4}}}{|r(x)|^{\frac{1}{4}}}\exp\left(\pm\frac{1}{\xi}\int\limits_{x_0}^{x}|r(\tilde{x})|^{\frac{1}{2}}d\tilde{x}\right) \ , \qquad r(x) < 0 \ . \tag{3.16}$$

3.1 Connection of solutions

To connect solutions from both the classically allowed and the forbidden region in vicinity of the turning point x_t, which satisfies

$$r(x_t) = 0 \ , \tag{3.17}$$

we expand $r(x)$ in a power series up to the first order, obtaining the linear approximation.

$$r(x) \sim r'(x_t)(x - x_t) \tag{3.18}$$

Hence, solutions (3.13), (3.16) can be rewritten as:

$$\begin{aligned}\varphi(x) &= \frac{C_+}{r(x)^{\frac{1}{4}}}\exp\left(\frac{i}{\xi}\int\limits_{x_t}^{x} r^{\frac{1}{2}}(\tilde{x})d\tilde{x}\right) + \frac{C_-}{r(x)^{\frac{1}{4}}}\exp\left(-\frac{i}{\xi}\int\limits_{x_t}^{x} r^{\frac{1}{2}}(\tilde{x})d\tilde{x}\right) \sim \\ &\sim \frac{C_+}{(r'(x_t)(x-x_t))^{\frac{1}{4}}}\exp\left(\frac{i}{\xi}\int\limits_{x_t}^{x}(r'(x_t)(\tilde{x}-x_t))^{\frac{1}{2}}\,d\tilde{x}\right) + \\ &+\frac{C_-}{(r'(x_t)(x-x_t))^{\frac{1}{4}}}\exp\left(-\frac{i}{\xi}\int\limits_{x_t}^{x}(r'(x_t)(\tilde{x}-x_t))^{\frac{1}{2}}\,d\tilde{x}\right) \ , \qquad r(x) > 0 \ ,\end{aligned} \tag{3.19}$$

$$\varphi(x) = \frac{D_+}{|r(x)|^{\frac{1}{4}}}\exp\left(\frac{1}{\xi}\int\limits_{x_t}^{x}|r(\tilde{x})|^{\frac{1}{2}}d\tilde{x}\right) + \frac{D_-}{|r(x)|^{\frac{1}{4}}}\exp\left(-\frac{1}{\xi}\int\limits_{x_t}^{x}|r(\tilde{x})|^{\frac{1}{2}}d\tilde{x}\right) \sim$$

$$\sim \frac{D_+}{|r'(x_t)(x-x_t)|^{\frac{1}{4}}} \exp\left(\frac{1}{\xi}\int_{x_t}^{x} |r'(x_t)(\tilde{x}-x_t)|^{\frac{1}{2}} d\tilde{x}\right) +$$

$$\frac{D_-}{(r'(x_t)(x-x_t))^{\frac{1}{4}}} \exp\left(-\frac{1}{\xi}\int_{x_t}^{x} |r'(x_t)(\tilde{x}-x_t)|^{\frac{1}{2}} d\tilde{x}\right) , \qquad r(x) < 0 , \tag{3.20}$$

and (3.5) become

$$\xi^2 \frac{d^2}{dx^2}\varphi(x) + r'(x_t)(x-x_t)\varphi(x) = 0 . \tag{3.21}$$

For clarity we distinguish two cases depending on $r'(x)$. Let us begin with the case $r'(x) < 0$, representing ascending potential. Applying the substitution:

$$z = \left(\frac{-r'(x_t)}{\xi^2}\right)^{\frac{1}{3}} (x - x_t) ,$$

$$\phi(z) = \varphi(x) , \tag{3.22}$$

one obtains the Airy's equation:

$$\left(\frac{d^2}{dz^2} - z\right)\phi(z) = 0 \tag{3.23}$$

with solutions $Ai(z)$, $Bi(z)$, and using asymptotic expansions (1.33) - (1.36) only up to zero order, one finds following relations.

$\mathbf{z \gg 0}$:

$$Ai(z) \sim \frac{1}{2\sqrt{\pi} z^{\frac{1}{4}}} \exp\left(-\frac{2}{3} z^{\frac{3}{2}}\right)$$

$$Bi(z) \sim \frac{1}{\sqrt{\pi} z^{\frac{1}{4}}} \exp\left(\frac{2}{3} z^{\frac{3}{2}}\right)$$

$\mathbf{z \ll 0}$:

$$Ai(z) \sim \frac{1}{\sqrt{\pi}(-z)^{\frac{1}{4}}} \cos\left(\frac{2}{3}(-z)^{\frac{3}{2}} - \frac{\pi}{4}\right)$$

$$Bi(z) \sim \frac{-1}{\sqrt{\pi}(-z)^{\frac{1}{4}}} \sin\left(\frac{2}{3}(-z)^{\frac{3}{2}} - \frac{\pi}{4}\right)$$

At this point, it is good to realize that:

$$r(x) > 0 \qquad \Leftrightarrow \qquad (x - x_t) < 0 \qquad \Leftrightarrow \qquad z < 0 . \tag{3.24}$$

In order to find connection formulas between the classically allowed and the forbidden region, we rewrite solutions (3.19), (3.20) in terms of the Airy functions asymptotics.

$$\phi(z) \sim \frac{C_+}{(-z)^{\frac{1}{4}}} \exp\left(-\frac{2i}{3}(-z)^{\frac{3}{2}}\right) + \frac{C_-}{(-z)^{\frac{1}{4}}} \exp\left(\frac{2i}{3}(-z)^{\frac{3}{2}}\right) \sim$$

$$\sim C_+\sqrt{\pi}e^{-i\frac{\pi}{4}}\left(Ai(z) + iBi(z)\right) + C_-\sqrt{\pi}e^{i\frac{\pi}{4}}\left(Ai(z) - iBi(z)\right) , \qquad z < 0 \tag{3.25}$$

$$\phi(z) \sim \frac{D_+}{z^{\frac{1}{4}}} \exp\left(\frac{2}{3}(-z)^{\frac{3}{2}}\right) + \frac{D_-}{z^{\frac{1}{4}}} \exp\left(-\frac{2}{3}z^{\frac{3}{2}}\right) \sim D_+\sqrt{\pi}Bi(z) + 2D_-\sqrt{\pi}Ai(z) , \qquad z > 0 \tag{3.26}$$

Whence, relations between coefficients are easy to derive:

$$C_+ = \frac{1}{2}D_+e^{-i\frac{\pi}{4}} + D_-e^{i\frac{\pi}{4}} , \qquad C_- = \frac{1}{2}D_+e^{i\frac{\pi}{4}} + D_-e^{-i\frac{\pi}{4}} ,$$

$$D_+ = C_+e^{i\frac{\pi}{4}} + C_-e^{-i\frac{\pi}{4}} , \qquad D_- = \frac{1}{2}\left(C_+e^{-i\frac{\pi}{4}} + C_-e^{i\frac{\pi}{4}}\right) , \tag{3.27}$$

and the connection formulas are evident as well, due to expressions of the Airy functions for $z \gg 0$ and $z \ll 0$.

$$\begin{array}{ccc} r(x) > 0 & & r(x) < 0 \\ \hline \frac{C}{r(x)^{\frac{1}{4}}} \cos\left(\frac{1}{\xi}\int\limits_{x_t}^{x} r(\tilde{x})^{\frac{1}{2}} d\tilde{x} - \frac{\pi}{4}\right) & \leftrightarrow & \frac{C}{2|-r(x)|^{\frac{1}{4}}} \exp\left(-\frac{1}{\xi}\int\limits_{x_t}^{x} |r(\tilde{x})|^{\frac{1}{2}} d\tilde{x}\right) \\ \frac{-C}{r(x)^{\frac{1}{4}}} \sin\left(\frac{1}{\xi}\int\limits_{x_t}^{x} r(\tilde{x})^{\frac{1}{2}} d\tilde{x} - \frac{\pi}{4}\right) & \leftrightarrow & \frac{C}{|-r(x)|^{\frac{1}{4}}} \exp\left(\frac{1}{\xi}\int\limits_{x_t}^{x} |-r(\tilde{x}|)^{\frac{1}{2}} d\tilde{x}\right) \end{array} \qquad (3.28)$$

From the applied procedure should be apparent that the case $r'(x) > 0$ differs from the previous one only in a few details. Instead of using substitution (3.22) we use the following one:

$$z = \left(\frac{r'(x_t)}{\xi^2}\right)^{\frac{1}{3}} (x_t - x) \ ,$$

$$\phi(z) = \varphi(x) \ . \qquad (3.29)$$

It leads exactly to the same Airy's equation and the same relation between r and z.

$$r(x) > 0 \qquad \Leftrightarrow \qquad (x - x_t) > 0 \qquad \Leftrightarrow \qquad z < 0 \qquad (3.30)$$

The difference in sign of x in this relation causes only change of signs in the arguments of the solutions' exponentials (3.19), (3.20). For relations of coefficients it means just interchanging C_+, D_+ for C_-, D_- as follows:

$$C_- = \frac{1}{2} D_- e^{-i\frac{\pi}{4}} + D_+ e^{i\frac{\pi}{4}} \ , \qquad C_+ = \frac{1}{2} D_- e^{i\frac{\pi}{4}} + D_+ e^{-i\frac{\pi}{4}} \ ,$$

$$D_- = C_- e^{i\frac{\pi}{4}} + C_+ e^{-i\frac{\pi}{4}} \ , \qquad D_+ = \frac{1}{2}\left(C_- e^{-i\frac{\pi}{4}} + C_+ e^{i\frac{\pi}{4}}\right) \ . \qquad (3.31)$$

$$\begin{array}{ccc} r(x) > 0 & & r(x) < 0 \\ \hline \frac{C}{r(x)^{\frac{1}{4}}} \cos\left(-\frac{1}{\xi}\int\limits_{x_t}^{x} r(\tilde{x})^{\frac{1}{2}} d\tilde{x} - \frac{\pi}{4}\right) & \leftrightarrow & \frac{C}{2|-r(x)|^{\frac{1}{4}}} \exp\left(\frac{1}{\xi}\int\limits_{x_t}^{x} |r(\tilde{x})|^{\frac{1}{2}} d\tilde{x}\right) \\ \frac{-C}{r(x)^{\frac{1}{4}}} \sin\left(-\frac{1}{\xi}\int\limits_{x_t}^{x} r(\tilde{x})^{\frac{1}{2}} d\tilde{x} - \frac{\pi}{4}\right) & \leftrightarrow & \frac{C}{|-r(x)|^{\frac{1}{4}}} \exp\left(-\frac{1}{\xi}\int\limits_{x_t}^{x} |-r(\tilde{x}|)^{\frac{1}{2}} d\tilde{x}\right) \end{array} \qquad (3.32)$$

Chapter 4

Stationary scattering

Considering the stationary formulation of the scattering problem for a mass m particle with a positive energy $E = k^2$ in a potential field V, we let in order to simplify notation, $m = \frac{1}{2}$ and $\hbar = 1$. The problem means solving the Schrödinger equation:

$$(-\Delta + V)\,\psi = k^2\psi \ , \tag{4.1}$$

where V is supposed to be localized and vanishing as $r \to +\infty$, and the thought solution has an asymptotic form of a wave incoming in a direction corresponding to an angle θ_0, written in polar coordinates:

$$\psi(r,\theta,\theta_0) \sim e^{ikr\cos(\theta-\theta_0)} + f(\theta,\theta_0)\frac{e^{ikr}}{\sqrt{r}} \ , \qquad r \to +\infty \ . \tag{4.2}$$

Theory behind the quantum scattering and solving the Schrödinger equation is given in [4], [5]. If the potential V depends only on the relative distance and is independent of the angle variable $V = V(r)$, then we can set $f(\theta,\theta_0) = f(\theta-\theta_0)$. Due to symmetry we can also restrict discussion to the case $\theta_0 = 0$.

$$\psi(r,\theta) \sim e^{ikr\cos(\theta)} + f(\theta)\frac{e^{ikr}}{\sqrt{r}} \ , \qquad r \to +\infty \tag{4.3}$$

Differential cross section, in the case of two dimensions also called the differential scattering length, is given by the scattering amplitude $f(\theta)$

$$\frac{d\sigma(\theta)}{d\theta} = |f(\theta)|^2 \ , \tag{4.4}$$

and the total cross section is

$$\sigma = \int_0^{2\pi} \frac{d\sigma(\theta)}{d\theta} d\theta \ . \tag{4.5}$$

4.1 Partial wave decomposition

This approach is discussed in [8]. Suppose the potential is independent of the angular variable $V = V(r)$. We are interested in a solution of the form (4.3). In order to find this kind of solution we start by rewriting the Schrödinger equation (4.1) in the polar coordinates as

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + k^2 - V(r)\right)\psi(r,\theta) = 0 \ . \tag{4.6}$$

We assume a solution in the factorized form $\psi(r,\theta) = R(r)T(\theta)$. Thus we get

$$\frac{1}{R(r)}\left(r^2\frac{\partial^2 R(r)}{\partial r^2} + r\frac{\partial R(r)}{\partial r} + \left((kr)^2 - V(r)r^2\right)R(r)\right) = \frac{1}{T(\theta)}\frac{\partial^2 T(\theta)}{\partial \theta^2} = \text{const.} \ , \tag{4.7}$$

and demanding unambiguity of the solution

$$T(0) = T(2\pi) \ , \tag{4.8}$$

$$T'(0) = T'(2\pi) \tag{4.9}$$

leads to

$$T_n(\theta) = e^{in\theta} \ , \quad n \in \mathbb{Z} \ . \tag{4.10}$$

Hence, (4.6) becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + k^2 - V(r) - \frac{n^2}{r^2} \right) R_n(r) = 0 \ . \tag{4.11}$$

Due to assumed behavior of the potential field $V(r)$ for a large argument, the radial equation reduces to

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + k^2 - \frac{n^2}{r^2} \right) R_n(r) = 0 \ , \qquad r \to +\infty \ . \tag{4.12}$$

This is a well known equation with two independent solutions, the Bessel functions $J_n(kr)$, $Y_n(kr)$. Thus

$$R_n(r) = \tilde{A}_n J_n(kr) + \tilde{B}_n Y_n(kr) \ , \qquad r \to +\infty \ , \tag{4.13}$$

and the general solution of (4.6) can be therefore written as

$$\psi(r,\theta) = \sum_{n=-\infty}^{\infty} \left(\tilde{A}_n J_n(kr) + \tilde{B}_n Y_n(kr) \right) e^{in\theta} \ , \qquad r \to +\infty \ . \tag{4.14}$$

Because we are interested in the asymptotic behavior for large argument, the Bessel functions can be rewritten as (1.21), (1.22), in order to obtain

$$\psi(r,\theta) \sim \sqrt{\frac{2}{\pi kr}} \sum_{n=-\infty}^{\infty} \tilde{A}_n \left(\cos\left(kr - \frac{n\pi}{2} - \frac{\pi}{4}\right) + \frac{\tilde{B}_n}{\tilde{A}_n} \sin\left(kr - \frac{n\pi}{2} - \frac{\pi}{4}\right) \right) e^{in\theta} \ . \tag{4.15}$$

To simplify this expression the following substitution is applied:

$$\begin{aligned} \tilde{A}_n &= \tilde{C}_n \cos\delta_n \ , \\ \tilde{B}_n &= -\tilde{C}_n \sin\delta_n \ , \end{aligned} \tag{4.16}$$

$$\psi(r,\theta) \sim \sqrt{\frac{2}{\pi kr}} \sum_{n=-\infty}^{\infty} \tilde{C}_n \cos\left(kr - \frac{n\pi}{2} - \frac{\pi}{4} + \delta_n\right) e^{in\theta} =$$

$$= \frac{1}{\sqrt{2\pi kr}} \sum_{n=-\infty}^{\infty} \tilde{C}_n \left(e^{i\left(kr - \frac{n\pi}{2} - \frac{\pi}{4} + \delta_n\right)} + e^{-i\left(kr - \frac{n\pi}{2} - \frac{\pi}{4} + \delta_n\right)} \right) e^{in\theta} \ . \tag{4.17}$$

To be able to compare this expression with (4.3), it is useful to expand the incident wave from (4.3) in terms of the Bessel functions:

$$e^{ikr\cos\theta} = \sum_{n=-\infty}^{\infty} i^n J_n(kr) e^{in\theta} \sim \frac{1}{\sqrt{2\pi kr}} \sum_{n=-\infty}^{\infty} \left(e^{i\left(kr - \frac{\pi}{4}\right)} + e^{-i\left(kr - n\pi - \frac{\pi}{4}\right)} \right) e^{in\theta} \ , \qquad r \to +\infty \ . \tag{4.18}$$

Comparing those two expressions it can be seen that, if we set

$$\tilde{C}_n = e^{i\delta_n + i\frac{n\pi}{2}} \ , \tag{4.19}$$

we obtain the solution in the desired form.

$$\psi(r,\theta) \sim \frac{1}{\sqrt{2\pi kr}} \sum_{n=-\infty}^{\infty} \left(e^{i\left(kr - \frac{\pi}{4}\right) + 2i\delta_n} + e^{-i\left(kr - n\pi - \frac{\pi}{4}\right)} \right) e^{in\theta} =$$

$$= e^{ikr\cos\theta} + \frac{1}{\sqrt{2\pi kr}} \sum_{n=-\infty}^{\infty} \left(e^{i\left(kr-\frac{\pi}{4}\right)+2i\delta_n} - e^{i\left(kr-\frac{\pi}{4}\right)} \right) e^{in\theta} =$$

$$= e^{ikr\cos\theta} + \frac{1}{\sqrt{2\pi kr}} e^{i\left(kr-\frac{\pi}{4}\right)} \sum_{n=-\infty}^{\infty} \left(e^{2i\delta_n} - 1\right) e^{in\theta} \tag{4.20}$$

According to (4.3), the scattering amplitude is

$$f(\theta) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi k}} \sum_{n=-\infty}^{\infty} \left(e^{2i\delta_n} - 1\right) e^{in\theta} \ . \tag{4.21}$$

Hence, the form of the differential cross section is clear from (4.4)

$$\frac{d\sigma(\theta)}{d\theta} = \frac{1}{2\pi k} \left| \sum_{n=-\infty}^{\infty} \left(e^{2i\delta_n} - 1\right) e^{in\theta} \right|^2 \ , \tag{4.22}$$

and total cross section is, according to (4.5), given by

$$\sigma = \frac{1}{2\pi k} \int_0^{2\pi} \left| \sum_{n=-\infty}^{\infty} \left(e^{2i\delta_n} - 1\right) e^{in\theta} \right|^2 d\theta \ . \tag{4.23}$$

It is not hard to see that after integration only the diagonal terms will contribute. Thus, we get

$$\sigma = \frac{1}{k} \sum_{n=-\infty}^{\infty} \left|e^{2i\delta_n} - 1\right|^2 = \frac{4}{k} \sum_{n=-\infty}^{\infty} \sin^2\delta_n \ . \tag{4.24}$$

4.2 Green function in 2D

Solution of the differential equation (4.1) can be also obtained by solving the integral equation:

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + (\Delta_{\mathbf{r}} + k^2)^{-1} V(\mathbf{r})\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \int_{\mathbb{R}^2} G(\mathbf{r}, \tilde{\mathbf{r}}) V(\tilde{\mathbf{r}})\psi(\tilde{\mathbf{r}}) d^2\tilde{\mathbf{r}} \ , \tag{4.25}$$

where $\psi_0(\mathbf{r})$ satisfies (4.1) with $V(\mathbf{r}) \equiv 0$ and $G(\mathbf{r}, \tilde{\mathbf{r}})$ is the Green function of the operator corresponding to the motion of a free particle, which satisfies

$$\left(\Delta_{\mathbf{r}} + k^2\right) G(\mathbf{r}, \tilde{\mathbf{r}}) = \delta(\mathbf{r} - \tilde{\mathbf{r}}) \ . \tag{4.26}$$

Using the Fourier transformation we derive

$$G(\mathbf{r}, \tilde{\mathbf{r}}) = \left(\Delta_{\mathbf{r}} + k^2\right)^{-1} \delta(\mathbf{r} - \tilde{\mathbf{r}}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{i(\mathbf{r}-\tilde{\mathbf{r}})\mathbf{p}}}{k^2 - \mathbf{p}^2} d^2\mathbf{p} \ . \tag{4.27}$$

In order to calculate the explicit form of the Green function we evaluate the expression with $(k^2 + i\varepsilon)$ instead of a purely real k^2, where $\varepsilon > 0$ and afterwards let $\varepsilon \to 0_+$. By using the polar coordinates and (1.28) we get

$$\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{i(\mathbf{r}-\tilde{\mathbf{r}})\mathbf{p}}}{k^2 + i\varepsilon - \mathbf{p}^2} d^2\mathbf{p} = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\theta \int_0^{\infty} dp \frac{e^{i|\mathbf{r}-\tilde{\mathbf{r}}|p\cos\theta}}{k^2 + i\varepsilon - p^2} = \frac{1}{2\pi} \int_0^{\infty} dp \frac{pJ_0(|\mathbf{r} - \tilde{\mathbf{r}}|p)}{k^2 + i\varepsilon - p^2} =$$

$$= \frac{-i}{4} H_0^{(1)} \left(i\sqrt{-(k^2 - i\varepsilon)}|\mathbf{r} - \tilde{\mathbf{r}}| \right) \ . \tag{4.28}$$

Hence, the integral equation (4.25) can be written in the form:

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{i}{4} \int_{\mathbb{R}^2} H_0^{(1)}(k|\mathbf{r} - \tilde{\mathbf{r}}|) V(\tilde{\mathbf{r}})\psi(\tilde{\mathbf{r}}) d^2\tilde{\mathbf{r}} \ . \tag{4.29}$$

Chapter 5

Finite cylindrical potential

We assume the following initial settings:

$$V = \begin{cases} V_0 \ , & r < R \ , \\ 0 \ , & r > R \ , \end{cases} \tag{5.1}$$

$$E = k^2 > V_0 \ . \tag{5.2}$$

The goal is to find a solution of the corresponding Schrödinger equation (4.1):

$$(-\Delta + V)\,\psi = k^2\psi \ .$$

There are two different approaches to solve this equation as follows from the previous chapters. One can either solve directly the differential Schrödinger equation or transform it to the integral equation (4.29). In this case we demonstrate both ways.

5.1 Integral approach

To find the solution of (4.6), corresponding to this potential, we start by recalling (4.29) and using the polar coordinates, due to cylindrical symmetry of the considered potential, one obtains

$$\psi(r,\theta) = e^{ikr\cos\theta} - \frac{iV_0}{4}\int_0^{2\pi}\int_0^R H_0^{(1)}(k|re^{i(\theta-\tilde{\theta})} - \tilde{r}|)\psi(\tilde{r},\tilde{\theta})\tilde{r}d\tilde{r}d\tilde{\theta} \ . \tag{5.3}$$

According to (4.13) and (4.14) we denote

$$R_n(r) = \frac{1}{2\pi}\int_0^{2\pi}\psi(r,\theta)e^{-in\theta}d\theta \ . \tag{5.4}$$

Thus, multiplying (5.3) by an appropriate term, integrating over θ, using the fact

$$\frac{1}{2\pi}\int_0^{2\pi} H_0^{(1)}(|re^{i\theta} - \tilde{r}|)e^{in\theta}d\theta = H_n^{(1)}(\max\{r,\tilde{r}\})J_n(\min\{r,\tilde{r}\}) \ , \tag{5.5}$$

and subsequently integrating over $\tilde{\theta}$, one obtains

$$R_n(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{ikr\cos\theta}e^{-in\theta}d\theta - \frac{iV_0}{4}\int_0^{2\pi}\int_0^R\left(\frac{1}{2\pi}\int_0^{2\pi} H_0^{(1)}(k|re^{i(\theta-\tilde{\theta})} - \tilde{r}|)e^{in\theta}d\theta\right)\psi(\tilde{r},\tilde{\theta})\tilde{r}d\tilde{r}d\tilde{\theta} =$$

$$= i^n J_n(kr) - \frac{iV_0}{4} \int_0^{2\pi} \int_0^R H_n^{(1)}(k \max\{r,\tilde{r}\}) J_n(k \min\{r,\tilde{r}\}) \psi(\tilde{r},\tilde{\theta}) \tilde{r} d\tilde{r} d\tilde{\theta} =$$

$$= i^n J_n(kr) - \frac{i\pi V_0}{2} \int_0^R H_n^{(1)}(k \max\{r,\tilde{r}\}) J_n(k \min\{r,\tilde{r}\}) R_n(\tilde{r}) \tilde{r} d\tilde{r} \ . \tag{5.6}$$

As we can see, this equation splits up into two cases:

$$R_n(r) = i^n J_n(kr) - \frac{i\pi V_0}{2} H_n^{(1)}(kr) \int_0^R J_n(k\tilde{r}) R_n(\tilde{r}) \tilde{r} d\tilde{r} \ , \qquad r > R \tag{5.7}$$

$$R_n(r) = i^n J_n(kr) - \frac{i\pi V_0}{2} \left(H_n^{(1)}(kr) \int_0^r J_n(k\tilde{r}) R_n(\tilde{r}) \tilde{r} d\tilde{r} + J_n(kr) \int_r^R H_n^{(1)}(k\tilde{r}) R_n(\tilde{r}) \tilde{r} d\tilde{r} \right) \ , \qquad 0 < r < R \ . \tag{5.8}$$

For $r > R$, previously derived general solution (4.13) is valid, however we rewrite it in the more suitable form according to the expression (5.7).

$$R_n(r) = \tilde{A}_n J_n(kr) + \tilde{B}_n Y_n(kr) = i^n J_n(kr) + C_n H_n^{(1)}(kr) \ , \qquad r > R \tag{5.9}$$

The relation between C_n and δ_n from the previous chapter can be easily derived comparing those two expressions, while bearing in mind the definition of $H_n^{(1)}$.

$$i^n + C_n = \tilde{A}_n = \tilde{C}_n \cos\delta_n = e^{i\delta_n + i\frac{n\pi}{2}} \cos\delta_n \qquad \Rightarrow \qquad C_n = i^n \frac{e^{2i\delta_n} - 1}{2} \tag{5.10}$$

$$iC_n = \tilde{B}_n = -\tilde{B}_n \sin\delta_n = -e^{i\delta_n + i\frac{n\pi}{2}} \sin\delta_n \qquad \Rightarrow \qquad C_n = i^n \frac{e^{2i\delta_n} - 1}{2}$$

For $r < R$, (4.11) is to be solved. Solutions of this equation are again the Bessel functions, $J_n(\sqrt{k^2 - V_0} r)$, $Y_n(\sqrt{k^2 - V_0} r)$. However, due to divergent behavior of the functions Y_n at zero, it is necessary to assume a solution only of the form

$$R_n(r) = D_n J_n(\sqrt{k^2 - V_0} r) \ , \qquad r < R \ . \tag{5.11}$$

Inserting this form into (5.8) while looking at the vicinity of $r = 0$, using the appropriate asymptotic

$$\frac{D_n}{n!} \left(\frac{\sqrt{k^2 - V_0} R}{2} \right)^n = \frac{i^n}{n!} \left(\frac{kR}{2} \right)^n - \frac{i\pi V_0}{2} \frac{D_n}{n!} \left(\frac{kR}{2} \right)^n \int_r^R H_n^{(1)}(k\tilde{r}) J_n(\sqrt{k^2 - V_0} \tilde{r}) \tilde{r} d\tilde{r} \ , \qquad r \to 0 \ , \tag{5.12}$$

and eliminating the common constants one obtains

$$D_n \sqrt{k^2 - V_0}^n = i^n k^n - \frac{i\pi V_0}{2} D_n k^n \int_0^R H_n^{(1)}(k\tilde{r}) J_n(\sqrt{k^2 - V_0} \tilde{r}) \tilde{r} d\tilde{r} \ . \tag{5.13}$$

The first term of (5.8) disappeared using the properties of the involved functions as $r \to 0$.

$$D_n = \frac{i^n k^n}{\sqrt{k^2 - V_0}^n + \frac{i\pi V_0}{2} k^n \int_0^R H_n^{(1)}(k\tilde{r}) J_n(\sqrt{k^2 - V_0} \tilde{r}) \tilde{r} d\tilde{r}} \tag{5.14}$$

From (5.7) and (5.11) it is evident that

$$C_n = -\frac{i\pi V_0}{2} D_n \int_0^R J_n(k\tilde{r}) J_n(\sqrt{k^2 - V_0}) \tilde{r} d\tilde{r} \ , \tag{5.15}$$

and using (1.26) and (1.27) one obtains

$$D_n = \frac{\frac{2i^{n+1}}{\pi R}}{\sqrt{k^2-V_0}J_{n+1}(\sqrt{k^2-V_0}R)H_n^{(1)}(kR) - kJ_n(\sqrt{k^2-V_0}R)H_{n+1}^{(1)}(kR)} \ , \tag{5.16}$$

$$C_n = -i^n \frac{\sqrt{k^2-V_0}J_{n+1}(\sqrt{k^2-V_0}R)J_n(kR) - kJ_n(\sqrt{k^2-V_0}R)J_{n+1}(kR)}{\sqrt{k^2-V_0}J_{n+1}(\sqrt{k^2-V_0}R)H_n^{(1)}(kR) - kJ_n(\sqrt{k^2-V_0}R)H_{n+1}^{(1)}(kR)} \ . \tag{5.17}$$

5.2 Differential approach

For simplicity we start with already derived form of the solution for $r > R$ from (5.9) and for $r < R$ from (5.11). Thus

$$R_n(r) = \begin{cases} R_n(r) = i^n J_n(kr) + C_n H_n^{(1)}(kr) \ , & r > R \ , \\ D_n J_n(\sqrt{k^2-V_0}r) \ , & r < R \ , \end{cases} \tag{5.18}$$

and using the boundary conditions at $r = R$

$$R_n(R) = D_n J_n(\sqrt{k^2-V_0}R) = i^n J_n(kR) + C_n H_n^{(1)}(kR) \ , \tag{5.19}$$

$$R_n'(R) = \sqrt{k^2-V_0} D_n J_n'(\sqrt{k^2-V_0}R) = k i^n J_n'(kR) + k C_n H_n'^{(1)}(kR) \ , \tag{5.20}$$

deriving D_n from both equations and comparing these expressions

$$\frac{i^n J_n(kR) + C_n H_n^{(1)}(kR)}{J_n(\sqrt{k^2-V_0}R)} = \frac{k i^n J_n'(kR) + k C_n H_n'^{(1)}(kR)}{\sqrt{k^2-V_0} J_n'(\sqrt{k^2-V_0}R)} \ ,$$

one obtains

$$C_n = -i^n \frac{\sqrt{k^2-V_0}J_n'(\sqrt{k^2-V_0}R)J_n(kR) - kJ_n(\sqrt{k^2-V_0}R)J_n'(kR)}{\sqrt{k^2-V_0}J_n'(\sqrt{k^2-V_0}R)H_n^{(1)}(kR) - kJ_n(\sqrt{k^2-V_0}R)H_n'^{(1)}(kR)} \ . \tag{5.21}$$

This can be rewritten by using recurrent relations for the Bessel functions given in (1.12) - (1.15).

$$C_n = -i^n \frac{\sqrt{k^2-V_0}J_{n+1}(\sqrt{k^2-V_0}R)J_n(kR) - kJ_n(\sqrt{k^2-V_0}R)J_{n+1}(kR)}{\sqrt{k^2-V_0}J_{n+1}(\sqrt{k^2-V_0}R)H_n^{(1)}(kR) - kJ_n(\sqrt{k^2-V_0}R)H_{n+1}^{(1)}(kR)}$$

D_n in the form (5.16) is obtained straightforwardly by using (1.10).

$$D_n J_n(\sqrt{k^2-V_0}R) = i^n J_n(kR) + C_n H_n^{(1)}(kR) =$$

$$= i^n \left(J_n(kR) - H_n^{(1)}(kR) \frac{\sqrt{k^2-V_0}J_{n+1}(\sqrt{k^2-V_0}R)J_n(kR) - kJ_n(\sqrt{k^2-V_0}R)J_{n+1}(kR)}{\sqrt{k^2-V_0}J_{n+1}(\sqrt{k^2-V_0}R)H_n^{(1)}(kR) - kJ_n(\sqrt{k^2-V_0}R)H_{n+1}^{(1)}(kR)} \right) =$$

$$= \frac{i^n k J_n(\sqrt{k^2-V_0}R) \left(J_{n+1}(kR)H_n^{(1)}(kR) - J_n(kR)H_{n+1}^{(1)}(kR) \right)}{\sqrt{k^2-V_0}J_{n+1}(\sqrt{k^2-V_0}R)H_n^{(1)}(kR) - kJ_n(\sqrt{k^2-V_0}R)H_{n+1}^{(1)}(kR)}$$

$$D_n = \frac{\frac{2i^{n+1}}{\pi R}}{\sqrt{k^2-V_0}J_{n+1}(\sqrt{k^2-V_0}R)H_n^{(1)}(kR) - kJ_n(\sqrt{k^2-V_0}R)H_{n+1}^{(1)}(kR)}$$

5.3 Asymptotic behavior for large energies

In order to compare the previously derived solution of the scattering problem with the classical results and the results obtained by the WKB approximation method as calculated in the next paragraph, it is desirable to find a suitable form of the expression $e^{2i\delta_n(k)}$ for $k^2 \gg V_0$, corresponding to the classical limit. Therefore, from the previously derived relation (5.10) between δ_n and C_n, it is not complicated to see that

$$e^{2i\delta_n(k)} = -\frac{\sqrt{k^2-V_0}J_{n+1}(\sqrt{k^2-V_0}R)H_n^{(2)}(kR) - kJ_n(\sqrt{k^2-V_0}R)H_{n+1}^{(2)}(kR)}{\sqrt{k^2-V_0}J_{n+1}(\sqrt{k^2-V_0}R)H_n^{(1)}(kR) - kJ_n(\sqrt{k^2-V_0}R)H_{n+1}^{(1)}(kR)} , \tag{5.22}$$

which due to (1.8), satisfies

$$\delta_{-n}(k) = \delta_n(k) . \tag{5.23}$$

In this expression we use an approximation by the Taylor series expansion up to the first order:

$$\sqrt{k^2-V_0} \sim k - \frac{V_0}{2k} \tag{5.24}$$

$$J_n(\sqrt{k^2-V_0}R) \sim J_n(kR) - J_n'(kR)\frac{V_0R}{2k} = J_n(kR) - \left(J_{n-1}(kR) - J_{n+1}(kR)\right)\frac{RV_0}{4k} \tag{5.25}$$

$$J_{n+1}(\sqrt{k^2-V_0}R) \sim J_{n+1}(kR) - J_{n+1}'(kR)\frac{V_0R}{2k} = J_{n+1}(kR) - \left(J_n(kR) - J_{n+2}(kR)\right)\frac{RV_0}{4k} \tag{5.26}$$

For simplicity we will not write the arguments of the Bessel functions because they are all the same.

$$e^{2i\delta_n} \sim -\frac{\left(k-\frac{V_0}{2k}\right)H_n^{(2)}\left(J_{n+1} - (J_n - J_{n+2})\frac{RV_0}{4k}\right) - kH_{n+1}^{(2)}\left(J_n - (J_{n-1} - J_{n+1})\frac{RV_0}{4k}\right)}{\left(k-\frac{V_0}{2k}\right)H_n^{(1)}\left(J_{n+1} - (J_n - J_{n+2})\frac{RV_0}{4k}\right) - kH_{n+1}^{(1)}\left(J_n - (J_{n-1} - J_{n+1})\frac{RV_0}{4k}\right)} \tag{5.27}$$

Neglecting the terms

$$\frac{RV_0^2}{8k^2}H_n^{(j)}J_n,\ \frac{RV_0^2}{8k^2}H_n^{(j)}J_{n+2} , \qquad j = 1, 2 , \tag{5.28}$$

and using (1.10), (1.11), one obtains

$$e^{2i\delta_n} \sim -\frac{-\frac{2i}{\pi R} + \frac{RV_0}{4}\left(H_{n+1}^{(2)}J_{n-1} - H_{n+1}^{(2)}J_{n+1} - H_n^{(2)}J_n + H_n^{(2)}J_{n+2}\right) - \frac{V_0}{2k}H_n^{(2)}J_{n+1}}{\frac{2i}{\pi R} + \frac{RV_0}{4}\left(H_{n+1}^{(1)}J_{n-1} - H_{n+1}^{(1)}J_{n+1} - H_n^{(1)}J_n + H_n^{(1)}J_{n+2}\right) - \frac{V_0}{2k}H_n^{(1)}J_{n+1}} . \tag{5.29}$$

It is evident that, the numerator is just the complex conjugate of the denominator. Hence we can write

$$e^{2i\delta_n} \sim \frac{1+X-iY}{1+X+iY} , \tag{5.30}$$

where

$$X = -\frac{1}{8}\pi R^2V_0\Big(J_nY_n - J_{n+2}Y_n - J_{n-1}Y_{n+1} + J_{n+1}Y_{n+1}\Big) - \frac{\pi RV_0}{4k}J_{n+1}Y_n ,$$

$$Y = \frac{1}{8}\pi R^2V_0\Big({J_n}^2 - J_{n+2}J_n + {J_{n+1}}^2 - J_{n-1}J_{n+1}\Big) + \frac{\pi RV_0}{4k}J_nJ_{n+1} . \tag{5.31}$$

Dividing the numerator and the denominator by $1+X$, and using $Y(1+X)^{-1} \sim Y$, we have

$$2\delta_n = \arg\left(e^{2i\delta_n}\right) \sim \arg\left(\frac{1-iY}{1+iY}\right) \sim -2Y . \tag{5.32}$$

Using the recurrence relations:

$$J_{n-1}(kR) = -J_{n+1}(kR) + \frac{2n}{kR}J_n(kR) , \tag{5.33}$$

$$J_{n+2}(kR) = -J_n(kR) + \frac{2(n+1)}{kR}\, J_{n+1}(kR) \ , \tag{5.34}$$

leads to

$$\delta_n(k) \sim -\frac{1}{4}\pi R^2 V_0 \left(J_n(kR)^2 + J_{n+1}(kR)^2 - \frac{2n}{kR}\, J_n(kR) J_{n+1}(kR) \right) \ . \tag{5.35}$$

With this δ_n we can take a look at the differential cross section. Recalling the Graf's addition theorem

$$\sum_{n=-\infty}^{\infty} \mathcal{C}_{n+\nu}(z) J_n(z) e^{in\theta} = \mathcal{C}_\nu \Big(2z \sin\Big(\frac{\theta}{2}\Big)\Big)\, e^{i\nu(\pi-\theta)/2} \ , \tag{5.36}$$

where $\mathcal{C}_\nu$ can stand for any of the functions J_ν, Y_ν, $H_\nu^{(1)}$ or $H_\nu^{(2)}$, immediately follows:

$$\sum_{n=-\infty}^{\infty} J_n(z)^2 e^{in\theta} = J_0\Big(2z \sin\Big(\frac{\theta}{2}\Big)\Big), \tag{5.37}$$

$$\sum_{n=-\infty}^{\infty} J_{n+1}(z)^2 e^{in\theta} = e^{-i\theta} \sum_{n=-\infty}^{\infty} J_n(z)^2 e^{in\theta} = J_0\Big(2z \sin\Big(\frac{\theta}{2}\Big)\Big) e^{-i\theta} \ , \tag{5.38}$$

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} n J_{n+1}(z) J_n(z) e^{in\theta} &= -i\frac{\partial}{\partial\theta} \sum_{n=-\infty}^{\infty} J_{n+1}(z) J_n(z) e^{in\theta} = \frac{\partial}{\partial\theta} \left(J_1\Big(2z \sin\Big(\frac{\theta}{2}\Big)\Big) e^{-i\theta/2} \right) = \\
= \Bigg(z\cos\Big(\frac{\theta}{2}\Big) \left(J_0\Big(2z \sin\Big(\frac{\theta}{2}\Big)\Big) - \frac{1}{2z\sin(\theta/2)} J_1\Big(2z \sin\Big(\frac{\theta}{2}\Big)\Big) \right) &- \frac{i}{2} J_1\Big(2z \sin\Big(\frac{\theta}{2}\Big)\Big) \Bigg) e^{-i\theta/2} = \\
= z\cos\Big(\frac{\theta}{2}\Big) J_0\Big(2z \sin\Big(\frac{\theta}{2}\Big)\Big) e^{-i\theta/2} &- \frac{1}{2\sin(\theta/2)} J_1\Big(2z \sin\Big(\frac{\theta}{2}\Big)\Big) \ .
\end{aligned} \tag{5.39}$$

Altogether,

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \left(J_n(z)^2 + J_{n+1}(z)^2 - \frac{2n}{z} J_n(z) J_{n+1}(z) \right) e^{in\theta} = \\
&= J_0\Big(2z \sin\Big(\frac{\theta}{2}\Big)\Big) \left(1 + e^{-i\theta}\right) - \frac{2}{z} \left(z\cos\left(\frac{\theta}{2}\right) J_0\Big(2z \sin\Big(\frac{\theta}{2}\Big)\Big) e^{-i\theta/2} - \frac{1}{2\sin(\theta/2)} J_1\Big(2z \sin\Big(\frac{\theta}{2}\Big)\Big) \right) = \\
&= \frac{1}{z\sin(\theta/2)} J_1\Big(2z \sin\Big(\frac{\theta}{2}\Big)\Big).
\end{aligned} \tag{5.40}$$

Hence the scattering amplitude is

$$\begin{aligned}
f(\theta) &= \frac{e^{-i\pi/4}}{\sqrt{2\pi k}} \sum_{n=-\infty}^{\infty} \left(e^{2i\delta_n(k)} - 1 \right) e^{in\theta} \sim \sqrt{\frac{2}{\pi k}} e^{i\pi/4} \sum_{n=-\infty}^{\infty} \delta_n(k) e^{in\theta} \sim \\
&\sim -\frac{1}{2} R^2 V_0 \sqrt{\frac{\pi}{2k}} e^{i\pi/4} \sum_{n=-\infty}^{\infty} \left(J_n(kR)^2 + J_{n+1}(kR)^2 - \frac{2n}{kR} J_n(kR) J_{n+1}(kR) \right) e^{in\theta} = \\
&= e^{i\pi/4} \sqrt{\frac{\pi}{8}} \frac{RV_0}{k^{3/2}\sin(\theta/2)} J_1\Big(2kR \sin\Big(\frac{\theta}{2}\Big)\Big) \ ,
\end{aligned} \tag{5.41}$$

and consequently the differential cross section is

$$\frac{\mathrm{d}\sigma(\theta)}{\mathrm{d}\theta} \sim \frac{\pi R^2 {V_0}^2}{8k^3} \left(\frac{J_1(2kR\sin(\theta/2))}{\sin(\theta/2)} \right)^2 \ . \tag{5.42}$$

For k large the differential cross section is concentrated near the value $\theta = 0$, and one has

$$\frac{\mathrm{d}\sigma(\theta)}{\mathrm{d}\theta} \sim \frac{\pi R^2 {V_0}^2}{2k^3} \left(\frac{J_1(kR\theta)}{\theta} \right)^2 \ . \tag{5.43}$$

Finally, beacuse of

$$\int_{-\pi}^{\pi} \frac{J_1(2x\sin(\theta/2))^2}{\sin^2(\theta/2)}\,\mathrm{d}\theta \sim 8\int_0^{\pi} \frac{J_1(x\theta)^2}{\theta^2}\,\mathrm{d}\theta \sim 8\int_0^{\infty} \frac{J_1(x\theta)^2}{\theta^2}\,\mathrm{d}\theta = \frac{32x}{3\pi}\ , \qquad x \gg 0\ , \tag{5.44}$$

the total cross section approximately equals

$$\sigma \sim \frac{\pi R^2 {V_0}^2}{2k^3}\frac{8kR}{3\pi} = \frac{4R^3 {V_0}^2}{3k^2}\ . \tag{5.45}$$

It is interesting and worthy to point out that the total cross section has not the form of the classical one even in the classical limit $k \to +\infty$.

5.4 WKB approximation

In this chapter it is assumed the following relation for energies:

$$k^2 \gg V(r)\ . \tag{5.46}$$

Let us start with (4.11). Using the substitution

$$\tilde{R}_n(r) = R_n(r) r^{\frac{1}{2}}\ , \tag{5.47}$$

it is transformed into

$$\frac{d^2\tilde{R}_n(r)}{dr^2} + F_n(r)\tilde{R}_n(r) = 0\ , \tag{5.48}$$

where we have denoted

$$F_n(r) = k^2 - V(r) - \frac{n^2 - \frac{1}{4}}{r^2}\ . \tag{5.49}$$

From this expression it is evident that the classical turning points r_n satisfy

$$F_n(r_n) = 0\ . \tag{5.50}$$

Using the WKB approximation with the ansatz

$$\tilde{R}_n(r) = \frac{1}{(-F_n(r))^{\frac{1}{4}}} \exp\left(\int_{r_n}^{r} (-F(\tilde{r}))^{\frac{1}{2}}\, d\tilde{r}\right)\ , \qquad r < r_0\ , \tag{5.51}$$

one obtains for the asymptotic region

$$\tilde{R}_n(r) = \frac{1}{F_n^{\frac{1}{4}}(r)} \cos\left(\int_{r_n}^{r} F_n^{\frac{1}{2}}(\tilde{r})d\tilde{r} - \frac{\pi}{4}\right)\ , \qquad r < r_0\ , \tag{5.52}$$

$$\tilde{R}_n(r) \sim \frac{1}{F_n^{\frac{1}{4}}(r)} \cos\left(kr - kr_0 + \int_{r_n}^{\infty} \left(F_n^{\frac{1}{2}}(\tilde{r}) - k\right) d\tilde{r} - \frac{\pi}{4}\right)\ , \qquad r \to \infty\ . \tag{5.53}$$

Comparing this asymptotic form with expression (4.17), the expression of δ_n is obtained.

$$\delta_{n,\mathrm{WKB}} = \frac{n\pi}{2} - kr_0 + \int_{r_n}^{\infty} \left(F_n^{\frac{1}{2}}(\tilde{r}) - k\right) d\tilde{r} \tag{5.54}$$

This can be simplified by expanding $F_n^{1/2}$ up to the first order with an approximate expression of r_n according to the assumption (5.46).

$$r_n^2 = \frac{n^2 - \frac{1}{4}}{k^2 - V(r)} \sim \frac{n^2}{k^2} \tag{5.55}$$

$$\left(k^2-\frac{n^2-\frac{1}{4}}{r^2}-V(r)\right)^{\frac{1}{2}}\sim\frac{k}{r}\sqrt{r^2-r_n^2}\left(1-\frac{V(r)r^2}{k^2\left(r^2-r_n^2\right)}\right)^{\frac{1}{2}}\sim\frac{k}{r}\sqrt{r^2-r_n^2}-\frac{V(r)r}{2k\sqrt{r^2-r_n^2}}$$

$$\delta_{n,\mathrm{WKB}}\sim\frac{n\pi}{2}-kr_n+\left(kr_n-\frac{kr_n\pi}{2}\right)-\frac{1}{2k}\int\limits_{r_n}^{\infty}\frac{V(r)r}{\sqrt{r^2-r_n^2}}dr\sim-\frac{1}{2k}\int\limits_{r_n}^{\infty}\frac{V(r)r}{\sqrt{r^2-r_n^2}}dr \tag{5.56}$$

By using potential in the form (5.1) one obtains

$$\delta_{n,\mathrm{WKB}}\sim-\frac{1}{2k}\int\limits_{r_n}^{R}\frac{V_0r}{\sqrt{r^2-r_n^2}}dr=-\frac{V_0\sqrt{R^2-r_n^2}}{2k}=-\frac{V_0\sqrt{(Rk)^2-n^2}}{2k^2}\,,\qquad R>r_n\Leftrightarrow|n|<Rk\,. \tag{5.57}$$

Thus, with the phase shift already derived above, we can calculate the scattering amplitude, differential and total cross section according to (4.21), (4.4), (4.24).

$$\delta_{n,\mathrm{WKB}}\sim\begin{cases}-\frac{V_0\sqrt{(Rk)^2-n^2}}{2k^2} & |n|<Rk\\ 0 & |n|\geq Rk\end{cases} \tag{5.58}$$

$$f_{\mathrm{WKB}}(\theta)=\frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi k}}\sum_{|n|<Rk}\left(e^{-i\frac{V_0\sqrt{(Rk)^2-n^2}}{k^2}}-1\right)e^{in\theta} \tag{5.59}$$

$$\frac{d\sigma_{\mathrm{WKB}}(\theta)}{d\theta}=\frac{1}{2\pi k}\left|\sum_{|n|<Rk}\left(e^{-i\frac{V_0\sqrt{(Rk)^2-n^2}}{k^2}}-1\right)e^{in\theta}\right|^2 \tag{5.60}$$

$$\sigma\sim\frac{4}{k}\sum_{|n|<Rk}\sin^2\left(\frac{V_0\sqrt{(Rk)^2-n^2}}{2k^2}\right)\sim\frac{4}{k}\sum_{|n|<Rk}\left(\frac{V_0}{2k^2}\right)^2\left((Rk)^2-n^2\right)\sim\frac{V_0^2}{k^5}\sum_{|n|<Rk}\left((Rk)^2-n^2\right)\sim$$

$$\sim\frac{2V_0^2R^3}{k^2}-\frac{V_0^2}{k^5}\frac{Rk(Rk+1)(2Rk+1)}{3}=\frac{2V_0^2R^3}{k^2}-\frac{2V_0^2R^3}{3k^2}-\frac{V_0^2R^2}{k^3}-\frac{V_0^2R}{3k^4} \tag{5.61}$$

Neglecting terms of higher order than $\left(\frac{V_0}{k}\right)^2$ we get the result.

$$\sigma_{\mathrm{WKB}}=\frac{4V_0^2R^3}{3k^2} \tag{5.62}$$

We see that the expression of the total cross section is the same as the one derived by large energy asymptotic. This is not so much surprising as both approaches use classical behavior in some way.

5.5 Numerical comparison

After deriving the cross section using different approaches we should verify the correctness of our results. To this end, we will make some numerical calculations by using the exact, the large energy and the WKB expressions, in order to compare them. For this purpose we assume the settings $R=\frac{1}{2}$, $V_0=1$. Here are some graphical comparisons of the exact expression for $\delta_n(k)$ given in (5.22), the large energy asymptotic expression (5.35) and the WKB expression (5.58) for $n=0,\ldots,4$. Note that for $n>4$, $\delta_{n\,WKB}=0$ for our choice of the initial settings. To compare the differential cross section we set $k=10$ and use the exact expression (4.22), but summation goes only over indices satisfying $n\leq 10$, due to negligible contribution of the rest of terms, the large energy asymptotic expressions (5.42) and (5.43) and the WKB expression (5.60).

From the figures 5.1 - 5.7 it is evident that large energy asymptotic expressions (5.35) and (5.42) are in very good agreement with their exact analogs and asymptotic differential cross section for small θ (5.43) works also well within the area of its validity. The WKB approximation provides not perfect but possibly usable results.

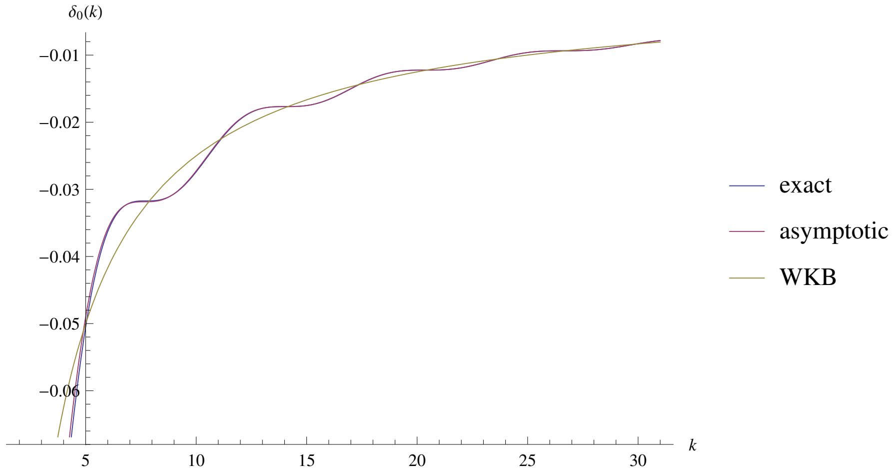

Figure 5.1: Comparison of $\delta_0(k)$.

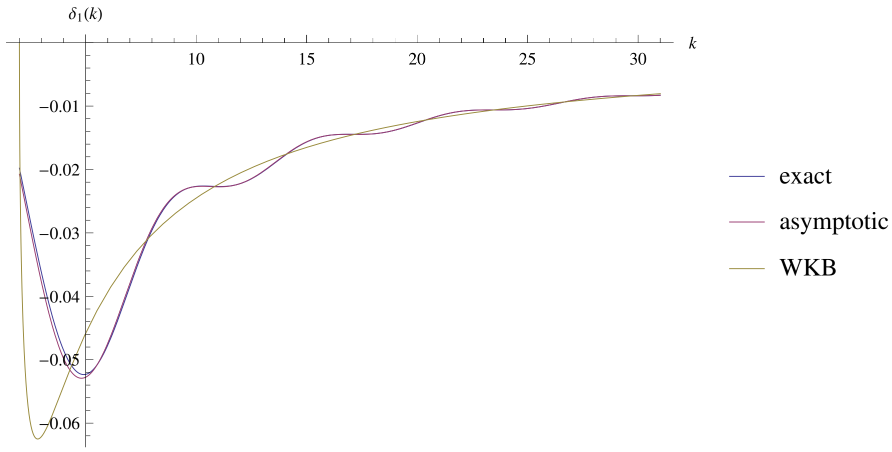

Figure 5.2: Comparison of $\delta_1(k)$.

5.6 Scattering on a δ-function potential as the limiting case

This topic is discussed in [3]. The δ-function potential can be realized as the limit of the finite cylindrical potential

$$V = \begin{cases} V_0 = \frac{2}{R^2 \ln \frac{R}{a}} \ , & r < R \ , \\ 0 \ , & r > R \ , \end{cases} \tag{5.63}$$

such that $R \to 0$. Reason for this particular form in case $r < R$ is that we wish to obtain the non-zero scattering in the limit case. This goal is actually achieved by regularization of the δ-function potential described in [7]. Due to assumption $R \to 0$ and $V_0 \xrightarrow{R \to 0} -\infty$, there are two possible situations, which we will discuss. Before that we should mention the fact that according to (1.8) we can restrict our discussion to coefficients C_n, D_n with indices $n \in \mathbb{N}_0$. Denoting

$$X = -i^n \left[\sqrt{k^2 - V_0} J_{n+1}(\sqrt{k^2 - V_0} R) J_n(kR) - k J_n(\sqrt{k^2 - V_0} R) J_{n+1}(kR) \right] \ , \tag{5.64}$$

$$Y = \left[\sqrt{k^2 - V_0} J_{n+1}(\sqrt{k^2 - V_0} R) H_n^{(1)}(kR) - k J_n(\sqrt{k^2 - V_0} R) H_{n+1}^{(1)}(kR) \right] \ , \tag{5.65}$$

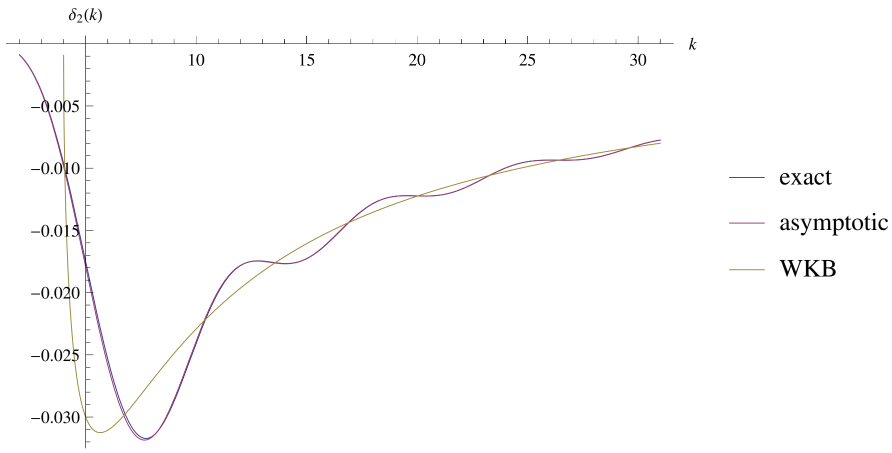

Figure 5.3: Comparison of $\delta_2(k)$.

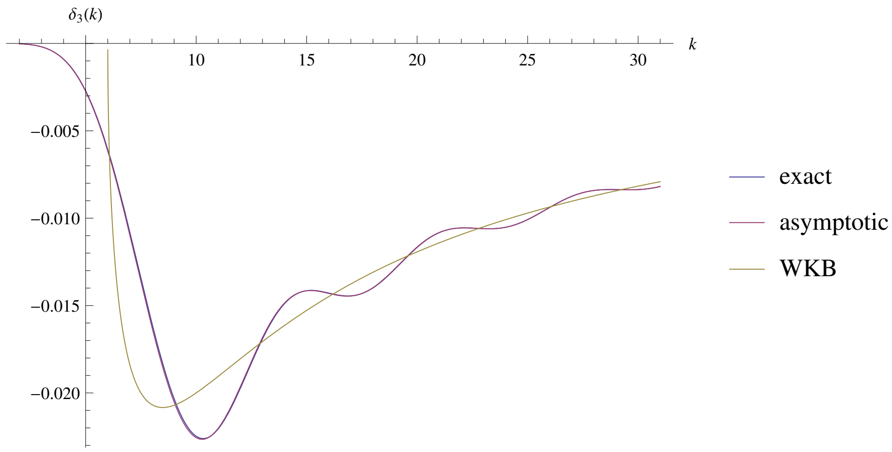

Figure 5.4: Comparison of $\delta_3(k)$.

we have

$$C_n = \frac{X}{Y} , \qquad D_n = \frac{\frac{2i^{n+1}}{\pi R}}{Y} . \tag{5.66}$$

Thus, by using asymptotic properties of the Bessel functions (1.16) - (1.20) we derive the following relations:

$$X = -i^n \left[\sqrt{k^2 - V_0} J_{n+1}(\sqrt{k^2 - V_0} R) J_n(kR) - k \overbrace{J_n(\sqrt{k^2 - V_0} R) J_{n+1}(kR)}^{\to 0} \right]$$

$$\sim -i^n \sqrt{k^2 - V_0} \left(\frac{\sqrt{k^2 - V_0} R}{2} \right)^{n+1} \frac{1}{(n+1)!} \left(\frac{kR}{2} \right)^n \frac{1}{n!} =$$

$$= \frac{-i^n k^n}{2^{2n+1} n! (n+1)!} \left(\sqrt{k^2 - V_0} \right)^{n+2} R^{2n+1} , \tag{5.67}$$

$$Y = \sqrt{k^2 - V_0} \left[\sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+1+j)!} \left(\frac{\sqrt{k^2 - V_0} R}{2} \right)^{2j+n+1} \right] \times$$

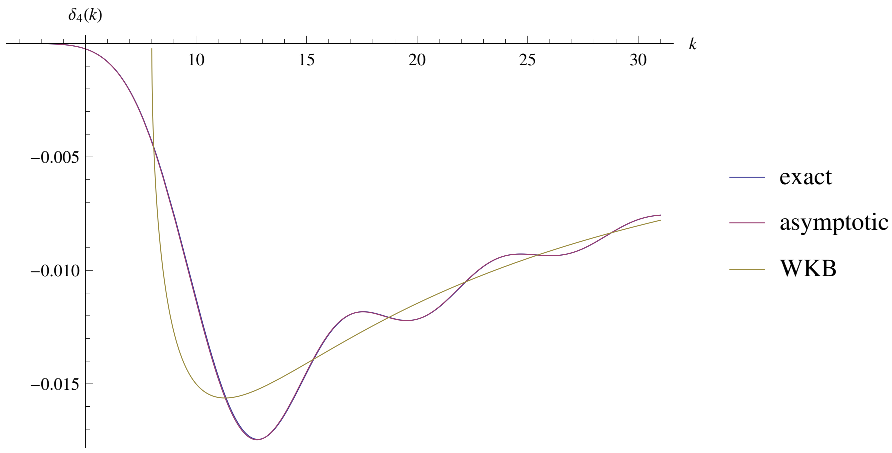

Figure 5.5: Comparison of $\delta_4(k)$.

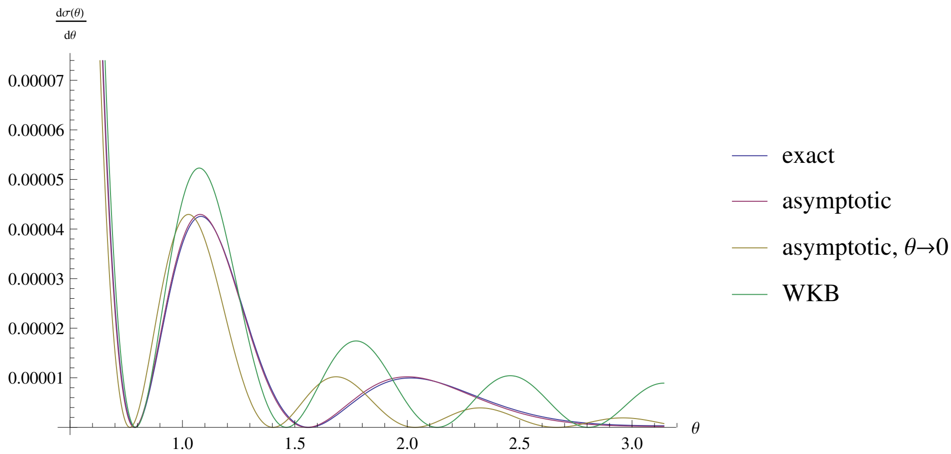

Figure 5.6: Comparison of the differential cross sections for θ indented from 0.

$$\times \left[\left(1 + \frac{2i}{\pi} \ln \frac{kR}{2}\right) \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!} \left(\frac{kR}{2}\right)^{2j+n} - \frac{i}{\pi} \sum_{j=0}^{n-1} \frac{(n-j-1)!}{j!} \left(\frac{kR}{2}\right)^{2j-n} + \right.$$

$$\left. -\frac{i}{\pi} \sum_{j=0}^{\infty} (\psi(j+1) + \psi(n+j+1)) \frac{(-1)^j}{j!(n+j)!} \left(\frac{kR}{2}\right)^{2j+n} \right] - k \left[\sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!} \left(\frac{\sqrt{k^2 - V_0}R}{2}\right)^{2j+n} \right] \times$$

$$\times \left[\left(1 + \frac{2i}{\pi} \ln \frac{kR}{2}\right) \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+1+j)!} \left(\frac{kR}{2}\right)^{2j+n+1} - \frac{i}{\pi} \sum_{j=0}^{n} \frac{(n-j)!}{j!} \left(\frac{kR}{2}\right)^{2j-(n+1)} + \right.$$

$$\left. -\frac{i}{\pi} \sum_{j=0}^{\infty} (\psi(j+1) + \psi(n+j+2)) \frac{(-1)^j}{j!(n+1+j)!} \left(\frac{kR}{2}\right)^{2j+n+1} \right] =$$

$$= \sqrt{k^2 - V_0}^{\,n+2} R^{2n+1} \frac{k^n}{2^{2n+1}} \left[\sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+1+j)!} \left(\frac{\sqrt{k^2 - V_0}R}{2}\right)^{2j} \right] \times$$

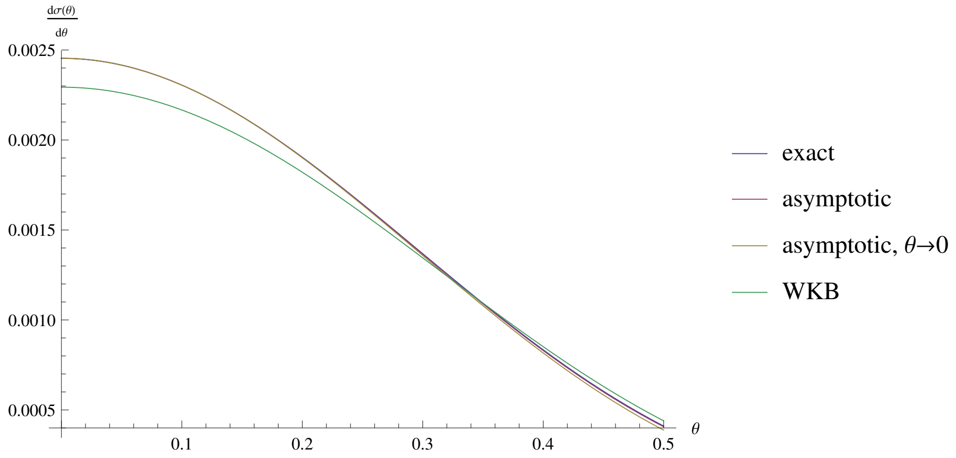

Figure 5.7: Comparison of the differential cross sections in the vicinity of 0.

$$
\begin{aligned}
&\times \left[\left(1 + \frac{2i}{\pi} \ln \frac{kR}{2}\right) \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!} \left(\frac{kR}{2}\right)^{2j} - \frac{i}{\pi} \sum_{j=0}^{n-1} \frac{(n-j-1)!}{j!} \left(\frac{kR}{2}\right)^{2j-2n} + \right. \\
&\left. -\frac{i}{\pi} \sum_{j=0}^{\infty} \left(\psi(j+1) + \psi(n+j+1)\right) \frac{(-1)^j}{j!(n+j)!} \left(\frac{kR}{2}\right)^{2j} \right] + \\
&-k \left(\sqrt{k^2 - V_0}\right)^n R^{2n+1} \frac{k^{n+1}}{2^{2n+1}} \left[\sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!} \left(\frac{\sqrt{k^2 - V_0} R}{2}\right)^{2j} \right] \times \\
&\times \left[\left(1 + \frac{2i}{\pi} \ln \frac{kR}{2}\right) \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+1+j)!} \left(\frac{kR}{2}\right)^{2j} - \frac{i}{\pi} \sum_{j=0}^{n} \frac{(n-j)!}{j!} \left(\frac{kR}{2}\right)^{2j-2(n+1)} + \right. \\
&\left. -\frac{i}{\pi} \sum_{j=0}^{\infty} \left(\psi(j+1) + \psi(n+j+2)\right) \frac{(-1)^j}{j!(n+1+j)!} \left(\frac{kR}{2}\right)^{2j} \right] \sim \\
&\overset{n \neq 0}{\sim} \sqrt{k^2 - V_0}^{\,n+2} R^{2n+1} \frac{k^n}{2^{2n+1}} \left[\sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+1+j)!} \left(\frac{\sqrt{k^2 - V_0} R}{2}\right)^{2j} \right] \times \\
&\times \left[\left(1 + \frac{2i}{\pi} \ln \frac{kR}{2}\right) - \frac{i}{\pi}(n-1)! \left(\frac{kR}{2}\right)^{-2n} - \frac{i}{\pi} \frac{\psi(1) + \psi(n+1)}{n!} \right] + \\
&-k \left(\sqrt{k^2 - V_0}\right)^n R^{2n+1} \frac{k^{n+1}}{2^{2n+1}} \left[\sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!} \left(\frac{\sqrt{k^2 - V_0} R}{2}\right)^{2j} \right] \times \\
&\times \left[\left(1 + \frac{2i}{\pi} \ln \frac{kR}{2}\right) - \frac{i}{\pi} n! \left(\frac{kR}{2}\right)^{-2(n+1)} \right] .
\end{aligned}
\tag{5.68}
$$

From here it should be already evident that

$$
C_n \to 0 \,, \qquad n \neq 0 \,, \tag{5.69}
$$

$$
D_n \to 0 \,, \qquad \forall n \,. \tag{5.70}
$$

Now let take a closer look at C_0.

$$C_0 \sim -\left[\left(1+\frac{2i}{\pi}\ln\frac{kR}{2}-\frac{2i}{\pi}\psi(1)\right)\sum_{j=0}^{\infty}\frac{(-1)^j}{j!(1+j)!}\left(\frac{\sqrt{k^2-V_0}R}{2}\right)^{2j}+\right.$$

$$\left.-k^2\left(\sqrt{k^2-V_0}\right)^{-2}\left(1+\frac{2i}{\pi}\ln\frac{kR}{2}-\frac{i}{\pi}\left(\frac{kR}{2}\right)^{-2}\right)\sum_{j=0}^{\infty}\frac{(-1)^j}{(j!)^2}\left(\frac{\sqrt{k^2-V_0}R}{2}\right)^{2j}\right]^{-1}\sim$$

$$\sim -\left[\left(1+\frac{2i}{\pi}\ln\frac{kR}{2}-\frac{2i}{\pi}\psi(1)\right)\left(1-\frac{\left(k^2-V_0\right)R^2}{8}\right)+\right.$$

$$\left.-k^2\left(\sqrt{k^2-V_0}\right)^{-2}\left(1+\frac{2i}{\pi}\ln\frac{kR}{2}-\frac{i}{\pi}\left(\frac{kR}{2}\right)^{-2}\right)\left(1-\frac{\left(k^2-V_0\right)R^2}{4}\right)\right]^{-1}\sim$$

$$\sim -\left[1+\frac{2i}{\pi}\ln\frac{kR}{2}-\frac{2i}{\pi}\psi(1)+\frac{i}{2\pi}\frac{\ln\frac{kR}{2}}{\ln\frac{R}{a}}+\right.$$

$$\left.-\frac{2ik^2}{\pi}\left(k^2-\frac{2}{R^2\ln\frac{R}{a}}\right)^{-1}\ln\frac{kR}{2}+\frac{4i}{\pi}\left(R^2k^2-\frac{2}{\ln\frac{R}{a}}\right)^{-1}-\frac{i}{\pi}\right]^{-1}\sim$$

$$\sim -\left[1+\frac{2i}{\pi}\ln\frac{kR}{2}-\frac{2i}{\pi}\psi(1)+\frac{i}{2\pi}\frac{\ln\frac{kR}{2}}{\ln\frac{R}{a}}-\frac{2i}{\pi}\ln\frac{R}{a}-\frac{i}{\pi}\right]^{-1} \tag{5.71}$$

$$C_0 \xrightarrow{R\to 0} \frac{\frac{i\pi}{2}}{\ln\frac{k}{2}-\frac{i\pi}{2}-\psi(1)-\frac{1}{4}-\ln a} \tag{5.72}$$

According to (5.10) and (4.21), the differential cross section is constant, independent of θ:

$$\frac{d\sigma(\theta)}{d\theta}=\frac{2}{\pi k}|C_0|^2\ , \tag{5.73}$$

and the total cross section is

$$\sigma=\frac{4}{k}|C_0|^2\ . \tag{5.74}$$

Conclusion

We have dealt with the stationary quantum scattering problem in two dimension for a spherically symmetric potential. For the finite cylindric potential we have derived the exact as well as the approximate solutions of the corresponding Schrödinger equation and cross sections, and provided numerical comparison of them. From the comparison we can see that the approximation of the exact expressions derived in the large energy asymptotic is better than that derived by the WKB approximation. Interesting is the difference between the classical total cross section and the total cross section derived by approximation for the large energy and the WKB approximation. The both approximations have given the same result, which does not agree with the general idea of classical mechanics becoming an approximation of quantum mechanics in the classical limit $k \to +\infty$ and it would be interesting to figure out the reason of such a behavior.

We have also solved the task of non-zero scattering on δ-potential, as it arises naturally from dealing with the finite cylindrical potential, due to fact it can be realized as a limit case of the finite cylindrical potential.

However, the intention of this paper was to deal with Aharonov-Bohm potential. Thus, this intention remains unfulfilled and should be a motivation for a future work.

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