A crash introduction to orthogonal polynomials

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Introduction

The roots of the theory of orthogonal polynomials go back as far as to the end of the 18th century. The field of orthogonal polynomials was developed to considerable depths in the late 19th century from a study of continued fractions by P. L. Chebyshev, T. J. Stieltjes and others.

Some of the mathematicians who have worked on orthogonal polynomials include Hans Ludwig Hamburger, Rolf Herman Nevanlinna, Gábor Szegő, Naum Akhiezer, Arthur Erdélyi, Wolfgang Hahn, Theodore Seio Chihara, Mourad Ismail, Waleed Al-Salam, and Richard Askey.

The theory of orthogonal polynomials is connected with many other branches of mathematics. Selecting a few examples one can mention continued fractions, operator theory (Jacobi operators), moment problems, approximation theory and quadrature, stochastic processes (birth and death processes) and special functions.

Some biographical data as well as various portraits of mathematicians are taken from *Wikipedia*, the free encyclopedia, starting from the web page

- *http://en.wikipedia.org/wiki/Orthogonal_polynomials*

Most of the theoretical material has been adopted from the fundamental monographs due to Akhiezer and Chihara (detailed references are given below in the text).

Classical orthogonal polynomials

A scheme of classical orthogonal polynomials

- the Hermite polynomials
- the Laguerre polynomials, the generalized (associated) Laguerre polynomials
- the Jacobi polynomials, their special cases:
  - the Gegenbauer polynomials, particularly:
    - the Chebyshev polynomials
    - the Legendre polynomials

Some common features
In each case, the respective sequence of orthogonal polynomials, $\{\tilde{P}_n(x);\, n \geq 0\}$, represents an orthogonal basis in a Hilbert space of the type $\mathscr{H} = L^2(I, \varrho(x)\mathrm{d}x)$ where $I \subset \mathbb{R}$ is an open interval, $\varrho(x) > 0$ is a continuous function on I.

Any sequence of classical orthogonal polynomials $\{\tilde{P}_n(x)\}$, after having been normalized to a sequence of monic polynomials $\{P_n(x)\}$, obeys a recurrence relation of the type

$$P_{n+1}(x) = (x - c_n)P_n(x) - d_n P_{n-1}(x), \quad n \geq 0,$$

with $P_0(x) = 1$ and where we conventionally put $P_{-1}(x) = 0$. Moreover, the coefficients c_n, $n \geq 0$, are all real and the coefficients d_n, $n \geq 1$, are all positive (d_0 is arbitrary).

The zeros of $P_n(x)$ are real and simple and belong all to I, the zeros of $P_n(x)$ and $P_{n+1}(x)$ interlace, the union of the zeros of $P_n(x)$ for all $n \geq 0$ is a dense subset in I.

$$I = \begin{cases} \mathbb{R} & \text{for the Hermite polynomials} \\ (0, +\infty) & \text{for the generalized Laguerre polynomials} \\ (-1, 1) & \text{for the Jacobi (and Gegenbauer, Chebyshev, Legendre) polynomials} \end{cases}$$

Hermite polynomials

Charles Hermite: December 24, 1822 – January 14, 1901

References

• C. Hermite: *Sur un nouveau développement en série de fonctions*, Comptes Rendus des Séances de l'Académie des Sciences. Paris **58** (1864) 93-100

• P.L. Chebyshev: *Sur le développement des fonctions à une seule variable*, Bulletin physico-mathématique de l'Académie Impériale des sciences de St.-Pétersbourgh **I** (1859) 193-200
• P. Laplace: *Mémoire sur les intégrales définies et leur application aux probabilités*, Mémoires de la Classe des sciences, mathématiques et physiques de l'Institut de France **58** (1810) 279-347

Definition $(n = 0, 1, 2, \ldots)$

$$H_n(x) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!\,(n-2k)!}\,(2x)^{n-2k}$$

The Rodrigues formula

$$H_n(x) = (-1)^n \mathrm{e}^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\,\mathrm{e}^{-x^2} = \left(2x - \frac{\mathrm{d}}{\mathrm{d}x}\right)^n \cdot 1$$

Orthogonality

$$\int_{-\infty}^{\infty} H_m(x) H_n(x)\,\mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}\,2^n n!\,\delta_{m,n}$$

The Hermite polynomials form an orthogonal basis of $\mathscr{H} = L^2(\mathbb{R}, \mathrm{e}^{-x^2}\mathrm{d}x)$.

Recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad n \geq 0,$$

$H_0(x) = 1$ and, by convention, $H_{-1}(x) = 0$.

Differential equation

The Hermite polynomial $H_n(x)$ is a solution of Hermite's differential equation

$$y'' - 2xy' + 2ny = 0.$$

Laguerre polynomials and generalized (associated) Laguerre polynomials

Edmond Laguerre: April 9, 1834 – August 14, 1886

References

- E. Laguerre: *Sur l'intégrale* $\int_x^\infty \frac{e^{-x}}{x}\,dx$, Bulletin de la Société Mathématique de France **7** (1879) 72-81
- N. Y. Sonine: *Recherches sur les fonctions cylindriques et le développement des fonctions continues en séries*, Math. Ann. **16** (1880) 1-80

Definition $(n = 0, 1, 2, \ldots)$

$L_n(x) \equiv L_n^{(0)}(x)$,

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!}\, x^k, \quad L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!}$$

The Rodrigues formula

$$L_n(x) = \frac{\mathrm{e}^x}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(\mathrm{e}^{-x} x^n\right) = \frac{1}{n!} \left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)^n x^n$$

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}\mathrm{e}^x}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(\mathrm{e}^{-x} x^{n+\alpha}\right) = \frac{x^{-\alpha}}{n!} \left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)^n x^{n+\alpha}$$

Orthogonality $(\alpha > -1)$

$$\int_0^\infty L_m(x) L_n(x)\, \mathrm{e}^{-x}\, \mathrm{d}x = \delta_{m,n}$$

$$\int_0^\infty L_m^{(\alpha)}(x) L_n^{(\alpha)}(x)\, x^\alpha \mathrm{e}^{-x}\, \mathrm{d}x = \frac{\Gamma(n+\alpha+1)}{n!}\, \delta_{m,n}$$

The Laguerre polynomials form an orthonormal basis of $\mathscr{H} = L^2((0,\infty), \mathrm{e}^{-x}\mathrm{d}x)$, the generalized Laguerre polynomials form an orthogonal basis of $\mathscr{H} = L^2((0,\infty), x^\alpha \mathrm{e}^{-x}\mathrm{d}x)$.

Recurrence relation

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x), \quad n \geq 0,$$

more generally,

$$(n+1)L_{n+1}^{(\alpha)}(x) = (2n+\alpha+1-x)\,L_{n-1}^{(\alpha)}(x) - (n+\alpha)\,L_{n-1}^{(\alpha)}(x), \quad n \geq 0,$$

$L_0^{(\alpha)}(x) = 1$ and, by convention, $L_{-1}^{(\alpha)}(x) = 0$.

Differential equation

$L_n(x)$ is a solution of Laguerre's equation

$$x\,y'' + (1-x)\,y' + n\,y = 0,$$

more generally, $L_n^{(\alpha)}(x)$ is a solution of the second order differential equation

$$x\,y'' + (\alpha+1-x)\,y' + n\,y = 0.$$

Jacobi (hypergeometric) polynomials

Carl Gustav Jacob Jacobi: December 10, 1804 – 18 February 18, 1851

References

• C.G.J. Jacobi: *Untersuchungen über die Differentialgleichung der hypergeometrischen Reihe*, J. Reine Angew. Math. **56** (1859) 149-165

Definition $(n = 0, 1, 2, \ldots)$

$$P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+\beta+n+1)} \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)} \left(\frac{z-1}{2}\right)^m$$

The Rodrigues formula

$$P_n^{(\alpha,\beta)}(z) = \frac{(-1)^n}{2^n n!} (1-z)^{-\alpha}(1+z)^{-\beta} \frac{d^n}{dz^n} \left[(1-z)^{\alpha}(1+z)^{\beta}(1-z^2)^n\right]$$

Orthogonality $(\alpha, \beta > -1)$

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)\,(1-x)^{\alpha}(1+x)^{\beta}\,dx = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\,\Gamma(n+\alpha+\beta+1)\,n!}\,\delta_{m,n}$$

The Jacobi polynomials form an orthogonal basis of
$\mathscr{H} = L^2((-1,1),(1-x)^{\alpha}(1+x)^{\beta}\mathrm{d}x)$.

Recurrence relation

$$\begin{aligned}
&2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)\,P_{n+1}^{(\alpha,\beta)}(z)\\
&= (2n+\alpha+\beta+1)\Big((2n+\alpha+\beta+2)(2n+\alpha+\beta)\,z+\alpha^2-\beta^2\Big)P_n^{(\alpha,\beta)}(z)\\
&\quad -2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)\,P_{n-1}^{(\alpha,\beta)}(z)\ , \quad n \geq 0,
\end{aligned}$$

$P_0^{(\alpha,\beta)}(z) = 1$ and, by convention, $P_{-1}^{(\alpha,\beta)}(z) = 0$.

Differential equation

The Jacobi polynomial $P_n^{(\alpha,\beta)}$ is a solution of the second order differential equation

$$(1-x^2)y'' + (\beta - \alpha - (\alpha+\beta+2)\,x)y' + n(n+\alpha+\beta+1)y = 0.$$

Gegenbauer (ultraspherical) polynomials

Leopold Bernhard Gegenbauer: February 2, 1849 – June 3, 1903

References

- L. Gegenbauer: *Über einige bestimmte Integrale*, Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften. Mathematische-Naturwissenschaftliche Classe. Wien **70** (1875) 433-443
- L. Gegenbauer: *Über einige bestimmte Integrale*, Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften. Mathematische-Naturwissenschaftliche Classe. Wien **72** (1876) 343-354
- L. Gegenbauer: *Über die Functionen* $C_n^{\nu}(x)$, Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften. Mathematische-Naturwissenschaftliche Classe. Wien **75** (1877) 891-905

Definition $(n = 0, 1, 2, \ldots)$

$$C_n^{(\alpha)}(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha) k! (n-2k)!} (2z)^{n-2k}$$

The Gegenbauer polynomials are a particular case of the Jacobi polynomials

$$C_n^{(\alpha)}(z) = \frac{\Gamma(\alpha+1/2)\Gamma(2\alpha+n)}{\Gamma(2\alpha)\Gamma(n+\alpha+1/2)} P_n^{(\alpha-1/2,\alpha-1/2)}(z)$$

The Rodrigues formula

$$C_n^{(\alpha)}(z) = \frac{(-2)^n}{n!} \frac{\Gamma(n+\alpha)\Gamma(n+2\alpha)}{\Gamma(\alpha)\Gamma(2n+2\alpha)} (1-x^2)^{-\alpha+1/2} \frac{d^n}{dx^n} \left[(1-x^2)^{n+\alpha-1/2}\right]$$

Orthogonality $(\alpha, \beta > -1)$

$$\int_{-1}^{1} C_m^{(\alpha)}(x) C_n^{(\alpha)}(x) \, (1-x^2)^{\alpha-1/2} \, \mathrm{d}x = \frac{\pi 2^{1-2\alpha} \Gamma(n+2\alpha)}{(n+\alpha) \, n! \, \Gamma(\alpha)^2} \, \delta_{m,n}$$

The Gegenbauer polynomials form an orthogonal basis of
$\mathscr{H} = L^2((-1,1),(1-x^2)^{\alpha-1/2}\mathrm{d}x)$.
Recurrence relation

$$(n+1)C_{n+1}^{(\alpha)}(x) = 2x(n+\alpha)C_n^{(\alpha)}(x) - (n+2\alpha-1)C_{n-1}^{(\alpha)}(x), \quad n \geq 0,$$

$C_0^{(\alpha)}(x) = 1$ and, by convention, $C_{-1}^{(\alpha)}(x) = 0$.
Differential equation
Gegenbauer polynomials are solutions of the Gegenbauer differential equation

$$(1-x^2)y'' - (2\alpha+1)xy' + n(n+2\alpha)y = 0.$$

Chebyshev polynomials of the first and second kind

Alternative transliterations: Tchebycheff, Tchebyshev, Tschebyschow

Pafnuty Lvovich Chebyshev: May 16, 1821 – December 8, 1894

References

• P. L. Chebyshev: *Théorie des mécanismes connus sous le nom de parallélogrammes*, Mémoires des Savants étrangers présentés à l'Académie de Saint-Pétersbourg **7** (1854) 539–586

Definition $(n = 0, 1, 2, \ldots)$
$T_0(x) = 1$, $U_0(x) = 1$, and for $n > 0$,

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(n-k-1)!}{k!(n-2k)!} (2x)^{n-2k}, \quad U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k}$$

Moreover, for all $n \geq 0$,

$$T_n(\cos(\vartheta)) = \cos(n\vartheta), \quad U_n(\cos(\vartheta)) = \frac{\sin((n+1)\vartheta)}{\sin \vartheta}$$

The Chebyshev polynomials are a particular case of the Gegenbauer polynomials

$$T_n(x) = \frac{n}{2\alpha} C_n^{(\alpha)}(x)\Big|_{\alpha=0} \quad (\text{for } n \geq 1), \quad U_n(x) = C_n^{(1)}(x)$$

Orthogonality

$$\int_{-1}^{1} T_m(x) T_n(x) \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} (1 + \delta_{m,0}) \delta_{m,n}, \quad \int_{-1}^{1} U_m(x) U_n(x) \sqrt{1-x^2}\, dx = \frac{\pi}{2} \delta_{m,n}$$

The Chebyshev polynomials $\{T_n(x)\}$ form an orthogonal basis of
$\mathscr{H} = L^2((-1,1), (1-x^2)^{-1/2} \mathrm{d}x)$,
The Chebyshev polynomials $\{U_n(x)\}$ form an orthogonal basis of
$\mathscr{H} = L^2((-1,1), (1-x^2)^{1/2} \mathrm{d}x)$.
Recurrence relation

$$T_{n+1}(x) = (2 - \delta_{n,0})\, x T_n(x) - T_{n-1}(x), \quad U_{n+1}(x) = 2x U_n(x) - U_{n-1}(x),$$

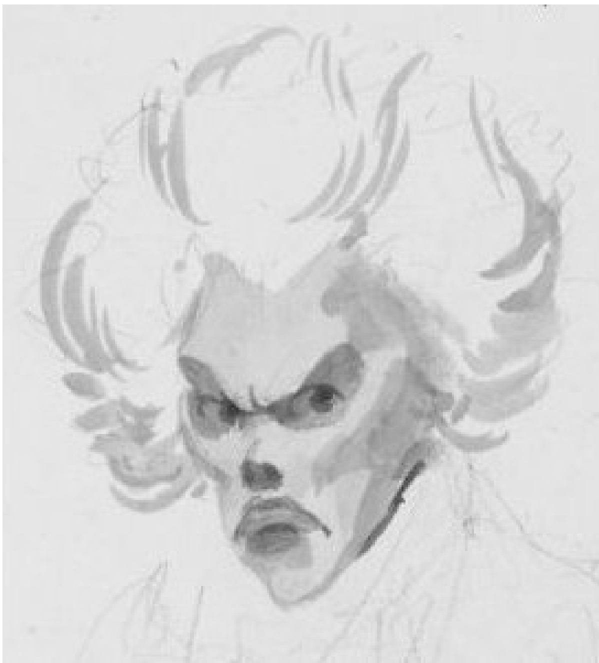
$T_0(x) = 1$, $U_0(x) = 1$ and, by convention, $T_{-1}(x) = 0$, $U_{-1}(x) = 0$.
Differential equation
The Chebyshev polynomial $T_n(x)$ is a solution of the Chebyshev differential equation

$$(1-x^2)\, y'' - x\, y' + n^2\, y = 0,$$

the Chebyshev polynomial $U_n(x)$ is a solution of the differential equation

$$(1-x^2)\, y'' - 3x\, y' + n(n+2)\, y = 0.$$

Legendre polynomials



Adrien-Marie Legendre: September 19, 1752 – January 10, 1833

References

- M. Le Gendre: *Recherches sur l'attraction des sphéroïdes homogènes*, Mémoires de Mathématiques et de Physique, présentés à l'Académie Royale des Sciences, par divers savans, et lus dans ses Assemblées **10** (1785) 411-435

Definition $(n = 0, 1, 2, \ldots)$

$$P_n(x) = 2^n \sum_{k=0}^{n} \binom{n}{k} \binom{(n+k-1)/2}{n} x^k$$

The Legendre polynomials are a particular case of the Gegenbauer polynomials

$$P_n(x) = C_n^{(1/2)}(x)$$

The Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^2 - 1)^n$$

Orthogonality

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{2}{2n+1}\, \delta_{m,n}$$

The Legendre polynomials form an orthogonal basis of $\mathscr{H} = L^2((-1,1), \mathrm{d}x)$.

Recurrence relation

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad n \geq 0,$$

$P_0(x) = 1$ and, by convention, $P_{-1}(x) = 0$.
Differential equation
Legendre polynomials are solutions to Legendre's differential equation,

$$\big((1-x^2)\,y'\big)' + n(n+1)y = 0.$$

Selected facts from the general theory

Basic monographs

- G. Szegő: *Orthogonal Polynomials*, AMS Colloquium Publications, vol. XXIII, 2nd ed., (AMS, Rhode Island, 1958) [first edition 1939]
- J. A. Shohat, J. D. Tamarkin: *The Problem of Moments*, Math. Surveys, no. I, 2nd ed., (AMS, New York, 1950) [first edition 1943]
- N. I. Akhiezer: *The Classical Moment Problem and Some Related Questions in Analysis*, (Oliver & Boyd, Edinburgh, 1965)
- T. S. Chihara: *An Introduction to Orthogonal Polynomials*, (Gordon and Breach, Science Publishers, New York, 1978)

The moment functional, an orthogonal polynomial sequence

Definition. A linear functional $\mathcal{L}$ on $\mathbb{C}[x]$ (the linear space of complex polynomials in the variable x) is called a *moment functional*, the number

$$\mu_n = \mathfrak{L}[x^n], \quad n = 0, 1, 2, \ldots,$$

is called a *moment of order* n.

Clearly, any sequence of moments $\{\mu_n\}$ determines unambiguously a moment functional $\mathfrak{L}$.

Definition. A moment functional $\mathfrak{L}$ is called *positive-definite*, if $\mathcal{L}[\pi(x)] > 0$ for every polynomial $\pi(x)$ that is not identically zero and is non-negative for all real x.

Theorem. *A moment functional $\mathfrak{L}$ is positive-definite if and only if its moments μ_n are all real and the determinants*

$$\Delta_n := \det(\mu_{j+k})_{j,k=0}^{n} = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}$$

are all positive, $n \geq 0$.

Remark. A real sequence $\{\mu_n;\, n \geq 0\}$ such that $\Delta_n > 0$, $\forall n \geq 0$, is said to be *positive.*

Definition. Given a positive-definite moment functional $\mathfrak{L}$, a sequence $\{\hat{P}_n(x);\, n \geq 0\}$ is called an *orthonormal polynomial sequence* with respect to the moment functional $\mathfrak{L}$ provided for all $m, n \in \mathbb{Z}_+$ ($\mathbb{Z}_+$ standing for non-negative integers),

(i) $\hat{P}_n(x)$ is a polynomial of degree n,

(ii) $\mathfrak{L}[\hat{P}_m(x)\hat{P}_n(x)] = \delta_{m,n}$.

Remark. Quite frequently, it is convenient to work with a sequence of orthogonal monic polynomials, which we shall denote $\{P_n(x)\}$, rather than with the orthonormal polynomial sequence $\{\hat{P}_n(x)\}$.

Theorem. *For every positive-definite moment functional $\mathfrak{L}$ there exists a unique monic orthogonal polynomial sequence $\{P_n(x)\}$.*

Remark. It can be shown that

$$\mathfrak{L}[P_n(x)^2] = \frac{\Delta_n}{\Delta_{n-1}}\,, \quad \forall n \geq 0$$

($\Delta_{-1} := 1$), and hence the polynomials

$$\hat{P}_n(x) = \sqrt{\frac{\Delta_{n-1}}{\Delta_n}}\, P_n(x)$$

are normalized. An explicit expression is known for the monic polynomials,

$$P_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix}.$$

The fundamental recurrence relation and Favard's theorem

Let $\mathfrak{L}$ be a positive-definite moment functional and let $\{\hat{P}_n(x)\}$ be the corresponding orthonormal polynomial sequence.

Obviously, $\{\hat{P}_0(x), \hat{P}_1(x), \dots, \hat{P}_n(x)\}$ is an orthonormal basis in the subspace of $\mathbb{C}[x]$ formed by polynomials of degree at most n. From the orthogonality it also follows that

$$\forall n \in \mathbb{N}, \forall \pi(x) \in \mathbb{C}[x],\, \deg \pi(x) < n \implies \mathfrak{L}[\hat{P}_n(x)\pi(x)] = 0.$$

Hence, for any $n = 0, 1, 2, \dots$,

$$x\hat{P}_n(x) = \sum_{k=0}^{n+1} a_{n,k}\hat{P}_k(x), \quad a_{n,k} = \mathfrak{L}[x\hat{P}_n(x)\hat{P}_k(x)] \;\; (a_{n,n+1} \neq 0).$$

But for $k < n-1$, $a_{n,k} = \mathfrak{L}[x\hat{P}_n(x)\hat{P}_k(x)] = \mathfrak{L}[\hat{P}_n(x)\big(x\hat{P}_k(x)\big)] = 0$. Put

$$\alpha_n = \mathfrak{L}[x\hat{P}_n(x)\hat{P}_{n+1}(x)],\ \beta_n = \mathfrak{L}[x\hat{P}_n(x)^2].$$

Necessarily, the coefficients α_n and β_n are all real and $\alpha_n = a_{n,n+1} \neq 0$.

We have found that the sequence $\{\hat{P}_n(x)\}$ fulfills the second-order difference relation

$$x\hat{P}_n(x) = \alpha_{n-1}\hat{P}_{n-1}(x) + \beta_n\hat{P}_n(x) + \alpha_n\hat{P}_{n+1}(x), \quad n \geq 0,$$

$\hat{P}_0(x) = 1$, and we put $\hat{P}_{-1}(x) = 0$ (so α_{-1} plays no role).

This observation can be rephrased in terms of the monic orthogonal polynomials. Let

$$c_n = \beta_n,\ d_n = \alpha_{n-1}^2$$

(d_0 may be arbitrary).

Theorem. *Let $\mathfrak{L}$ be a positive-definite moment functional and let $\{P_n(x)\}$ be the corresponding monic orthogonal polynomial sequence. Then there exist real constants c_n, $n \geq 0$, and positive constants d_n, $n \geq 1$, such that the sequence $\{P_n(x)\}$ obeys the three-term recurrence relation*

$$P_{n+1}(x) = (x - c_n)P_n(x) - d_nP_{n-1}(x), \quad n \geq 0,$$

with $P_0(x) = 1$ and where we conventionally put $P_{-1}(x) = 0$.

Remark. It is straightforward to see that

$$P_n(x) = \left(\prod_{k=0}^{n-1}\alpha_k\right)\hat{P}_n(x), \quad n \geq 0.$$

The opposite of the above theorem is also true.

Remark. If desirable, any positive-definite moment functional can be renormalized so that $\mathfrak{L}[1] = 1$.

Theorem (Favard's Theorem)**.** *Let c_n, $n \geq 0$, and d_n, $n \geq 1$, be arbitrary sequences of real and positive numbers, respectively, and let a sequence $\{P_n(x);\, n \in \mathbb{Z}_+\}$ be defined by the formula*

$$P_{n+1}(x) = (x - c_n)P_n(x) - d_nP_{n-1}(x), \quad \forall n \geq 0,\ P_{-1}(x) = 0,\ P_0(x) = 1.$$

Then there exists a unique positive-definite moment functional $\mathfrak{L}$ such that

$$\mathfrak{L}[1] = 1, \quad \mathfrak{L}[P_m(x)P_n(x)] = 0 \ \ \textit{for } m \neq n,\ m, n = 0, 1, 2, \ldots.$$

The zeros of an orthogonal polynomial sequence

Definition. Let $\mathcal{L}$ be a positive-definite moment functional and $E \subset \mathbb{R}$. The set E is called a *supporting set* for $\mathcal{L}$ if $\mathcal{L}[\pi(x)] > 0$ for every real polynomial $\pi(x)$ which is non-negative on E and does not vanish identically on E.

Theorem. *Let $\mathcal{L}$ be a positive-definite moment functional, $\{P_n(x);\, n \geq 0\}$ be the corresponding monic orthogonal polynomial sequence. For any n, the zeros of $P_n(x)$ are all real and simple, and the zeros of $P_n(x)$ and $P_{n+1}(x)$ interlace, i.e. between any two subsequent zeros of $P_{n+1}(x)$ there is exactly one zero of $P_n(x)$.*

On the contrary, if $2 \leq m < n$ then between any two zeros of $P_m(x)$ there is at least one zero of $P_n(x)$.

Moreover, if an interval I is a supporting set of $\mathcal{L}$ then the zeros of $P_n(x)$ are all located in the interior of I.

The Hamburger moment problem

Let $\{\mu_n;\, n = 0, 1, 2, \ldots\}$ be a sequence of moments defining a positive-definite moment functional $\mathfrak{L}$. Without loss of generality one can assume that $\mu_0 = 1$ meaning that $\mathfrak{L}$ is normalized, i.e. $\mathfrak{L}[1] = 1$.

One may ask whether $\mathfrak{L}$ can be defined with the aid of a probability measure $\mathrm{d}\sigma(x)$ on $\mathbb{R}$ where $\sigma(x)$ is a (cumulative) probability distribution, meaning that

$$\mathfrak{L}[\pi(x)] = \int_{-\infty}^{+\infty} \pi(x)\, \mathrm{d}\sigma(x), \quad \forall \pi(x) \in \mathbb{C}[x].$$

Obviously, this requirement can be reduced to

$$\int_{-\infty}^{+\infty} x^n\, \mathrm{d}\sigma(x) = \mu_n, \quad n = 0, 1, 2, \ldots$$

This problem is called the *Hamburger moment problem.* Provided one requires, in addition, $\mathrm{d}\sigma(x)$ to be supported on the half-line $[0, +\infty)$ or on the closed unit interval $[0, 1]$ one speaks about the *Stieltjes moment problem* or the *Hausdorff moment problem*, respectively. In what follows, we shall address the Hamburger moment problem only. This is to say that speaking about a moment problem we always mean the Hamburger moment problem.

The answer to the moment problem is always affirmative. On the other hand, the probability measure can, but need not be, unique. The moment problem is said to be *determinate* if there exists a unique probability measure solving the problem, and *indeterminate* in the opposite case.

Let, as before, $\alpha_n = \mathfrak{L}[x\hat{P}_n(x)\hat{P}_{n+1}(x)]$, $\beta_n = \mathfrak{L}[x\hat{P}_n(x)^2]$. Define a sequence of polynomials $\{Q_n(x)\}$ by the recurrence relation

$$xQ_n(x) = \alpha_{n-1}Q_{n-1}(x) + \beta_n Q_n(x) + \alpha_n Q_{n+1}(x), \;\; n \geq 1, \;\; Q_0(x) = 0, \;\; Q_1(x) = 1/\alpha_0.$$

Remark. $Q_n(x)$ is called a *polynomial of the second kind* ($Q_n(x)$ is of degree $n-1$) while $\hat{P}_n(x)$ is called a *polynomial of the first kind*. It is not difficult to verify that

$$Q_n(x) = \mathcal{L}_u \left[\frac{\hat{P}_n(x) - \hat{P}_n(u)}{x-u} \right]$$

(the moment functional acts in the variable u).

Remark. The Hamburger moment problem is known to be determinate if the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are bounded.

Theorem. *If for some $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\sum_{n=0}^{\infty} |\hat{P}_n(z)|^2 = \infty,$$

then the Hamburger moment problem is determinate. Conversely, this equality holds true for all $z \in \mathbb{C} \setminus \mathbb{R}$ if the Hamburger moment problem is determinate.

Theorem. *If for some $z \in \mathbb{C}$,*

$$\sum_{n=0}^{\infty} \left(|\hat{P}_n(z)|^2 + |Q_n(z)|^2 \right) < \infty,$$

then the Hamburger moment problem is indeterminate. Conversely, this inequality is fulfilled for all $z \in \mathbb{C}$ if the Hamburger moment problem is indeterminate.

The Nevanlinna parametrization

Let us focus on the indeterminate case. Then a natural question arises how to describe all solutions to the moment problem.

In case of the indeterminate moment problem, the following four series converge for every $z \in \mathbb{C}$, and, as one can show, the convergence is even locally uniform on $\mathbb{C}$. Hence these series define entire functions, the so called *Nevanlinna functions* A, B, C and D:

$$A(z) = z \sum_{n=0}^{\infty} Q_n(0) Q_n(z), \qquad B(z) = -1 + z \sum_{n=0}^{\infty} Q_n(0) \hat{P}_n(z),$$

$$C(z) = 1 + z \sum_{n=0}^{\infty} \hat{P}_n(0) Q_n(z), \qquad D(z) = z \sum_{n=0}^{\infty} \hat{P}_n(0) \hat{P}_n(z).$$

It is known that

$$A(z)D(z) - B(z)C(z) = 1.$$

Definition. *Pick functions* $\phi(z)$ are holomorphic functions on the open complex half-plane $\operatorname{Im} z > 0$, with values in the closed half-plane $\operatorname{Im} z \geq 0$. The set of Pick functions will be denoted by $\mathcal{P}$, and it is usually augmented by the constant function $\phi(z) = \infty$. Any function $\phi(z) \in \mathcal{P}$ is tacitly assumed to be extended to a holomorphic function on $\mathbb{C} \setminus \mathbb{R}$ by the formula

$$\phi(z) = \overline{\phi(\overline{z})} \quad \text{for } \operatorname{Im} z < 0.$$

Theorem (Nevanlinna)**.** *Let $A(z)$, $B(z)$, $C(z)$ and $D(z)$ be the Nevanlinna functions corresponding to an indeterminate moment problem. The following formula for the Stieltjes transform of a (probability) measure $d\sigma$,*

$$\int_{\mathbb{R}} \frac{d\sigma(x)}{z-x} = \frac{A(z)\phi(z) - C(z)}{B(z)\phi(z) - D(z)}\,, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

establishes a one-to-one correspondence between functions $\phi(z) \in \mathcal{P} \cup \{\infty\}$ and solutions $\sigma = \sigma_\phi$ of the moment problem in question.

Theorem (M. Riesz)**.** *Let σ_ϕ be a solution to an indeterminate moment problem corresponding to a function $\phi(z) \in \mathcal{P} \cup \{\infty\}$. Then the orthonormal set $\{\hat{P}_n(x);\, n = 0, 1, 2, \ldots\}$ is total and hence an orthonormal basis in the Hilbert space $L^2(\mathbb{R}, d\sigma_\phi)$ if and only if $\phi(z) = t$ is a constant function, with $t \in \mathbb{R} \cup \{\infty\}$.*

Remark. The solutions σ_t, $t \in \mathbb{R} \cup \{\infty\}$, from the theorem due to M. Riesz are referred to as *N-extremal.*

Proposition. *The Nevanlinna extremal solutions σ_t of a moment problem, with $t \in \mathbb{R} \cup \{\infty\}$, are all purely discrete and supported on the zero set*

$$\mathfrak{Z}_t = \{x \in \mathbb{R};\, B(x)t - D(x) = 0\}.$$

Hence

$$d\sigma_t = \sum_{x \in \mathfrak{Z}_t} \rho(x)\, \delta_x$$

where δ_x is the Dirac measure supported on $\{x\}$, and one has

$$\rho(x) := \sigma_t(\{x\}) = \left(\sum_{n=0}^{\infty} \hat{P}_n(x)^2 \right)^{-1} = \frac{1}{B'(x)D(x) - B(x)D'(x)}\,.$$

The associated Jacobi matrix

The recurrence relation

$$x\hat{P}_n(x) = \alpha_{n-1}\hat{P}_{n-1}(x) + \beta_n\hat{P}_n(x) + \alpha_n\hat{P}_{n+1}(x), \quad n \geq 0,$$

for an orthonormal polynomial sequence $\{\hat{P}_n(x)\}$ can be reinterpreted in the following way. Let M be an operator on $\mathbb{C}[x]$ acting via multiplication by x, i.e.

$$M\pi(x) = x\,\pi(x), \quad \forall \pi(x) \in \mathbb{C}[x].$$

The matrix of M with respect to the basis $\{\hat{P}_n(x)\}$ is a Jacobi (tridiagonal) matrix

$$\mathcal{J} = \begin{pmatrix} \beta_0 & \alpha_0 & & & \\ \alpha_0 & \beta_1 & \alpha_1 & & \\ & \alpha_1 & \beta_2 & \alpha_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

The matrix $\mathcal{J}$ clearly represents a well defined linear operator on the vector space of all complex sequences that we denote, for simplicity, by the same letter. According to the above recurrence relation, for every $z \in \mathbb{C}$, the sequence

$$(\hat{P}_0(z), \hat{P}_1(z), \hat{P}_2(z), \ldots)$$

represents a formal eigenvector of $\mathcal{J}$ corresponding to the eigenvalue z, i.e. a solution of the formal eigenvalue equation $\mathcal{J}f = zf$. Note that the formal eigenvector is unambiguous up to a scalar multiplier.

Let $\mathscr{D}$ be the subspace formed by those complex sequences which have at most finitely many nonzero elements. $\mathscr{D}$ is nothing but the linear hull of the canonical (standard) basis in $\ell^2(\mathbb{Z}_+)$. Clearly, $\mathscr{D}$ is $\mathcal{J}$-invariant. Denote by $\dot{J}$ the restriction $\mathcal{J}\big|_{\mathscr{D}}$. $\dot{J}$ is a symmetric operator on $\ell^2(\mathbb{Z}_+)$, and let $J_{\min}$ designate its closure. Furthermore, $J_{\max}$ is an operator on $\ell^2(\mathbb{Z}_+)$ defined as another restriction of $\mathcal{J}$, this time to the domain

$$\operatorname{Dom} J_{\max} = \{f \in \ell^2(\mathbb{Z}_+);\ \mathcal{J}f \in \ell^2(\mathbb{Z}_+)\}.$$

Clearly, $\dot{J} \subset J_{\max}$. Straightforward arguments based just on systematic application of definitions show that

$$(\dot{J})^* = (J_{\min})^* = J_{\max},\ (J_{\max})^* = J_{\min}.$$

Hence $J_{\max}$ is closed and $J_{\min} \subset J_{\max}$.

Since $\mathcal{J}$ is real and all formal eigenspaces of $\mathcal{J}$ are one-dimensional, the deficiency indices of $J_{\min}$ are equal and can only take the values either $(0,0)$ or $(1,1)$. The latter case happens if and only if for some and hence any $z \in \mathbb{C} \setminus \mathbb{R}$ one has

$$\sum_{n=0}^{\infty} |\hat{P}_n(z)|^2 < \infty.$$

Remark. A real symmetric Jacobi matrix $\mathcal{J}$ can also be regarded as representing a second-order difference operator on the discretized half-line. This point of view suggests that one can adopt various approaches and terminology originally invented for Sturm-Liouville differential operators. Following classical Weyl's analysis of admissible boundary conditions one says that $\mathcal{J}$ is *limit point* if the sequence $\{\hat{P}_n(z)\}$ is not square summable for some and hence any $z \in \mathbb{C} \setminus \mathbb{R}$, and $\mathcal{J}$ is *limit circle* in the opposite case. In other words, saying that $\mathcal{J}$ is limit point means the same as saying $\dot{J}$ is essentially self-adjoint. A good reference for these aspects is Subsections 2.4-2.6 in

• G. Teschl: *Jacobi Operators and Completely Integrable Nonlinear Lattices*, (AMS, Rhode Island, 2000)

Theorem. *The operator J_{min} is self-adjoint, i.e. $\dot{J}$ is essentially self-adjoint (equivalently, $J_{min} = J_{max}$) if and only if the Hamburger moment problem is determinate. In the indeterminate case, the self-adjoint extensions of J_{min} are in one-to-one correspondence with the N-extremal solutions of the Hamburger moment problem. If J_t is a self-adjoint extension of J_{min} for some $t \in \mathbb{T}^1$ (the unit circle in $\mathbb{C}$) then the corresponding probability measure (distribution) $\sigma = \sigma_t$ solving the moment problem is given by the formula*

$$\sigma_t(x) = \langle e_0, E_t\big((-\infty, x\,]\big) e_0 \rangle$$

where E_t is the spectral projection-valued measure for J_t and e_0 is the first vector of the canonical basis in $\ell^2(\mathbb{Z}_+)$. In particular, the measure σ_t is supported on the spectrum of J_t.

Remark. Let $\{e_n\}$ be the canonical basis in $\ell^2(\mathbb{Z}_+)$. One readily verifies that $\hat{P}_n(\mathcal{J})e_0 = e_n$, $\forall n$. Whence

$$\delta_{m,n} = \langle e_m, e_n \rangle = \langle \hat{P}_m(J_t)e_0, \hat{P}_n(J_t)e_0 \rangle = \int_{-\infty}^{\infty} \hat{P}_m(x)\hat{P}_n(x)\, \mathrm{d}\sigma_t(x).$$

Of course, the moments μ_n do not depend on t, and one has

$$\mu_n = \langle e_0, J_t^n e_0 \rangle = \big(\mathcal{J}^n\big)_{0,0}\,,\ n = 0, 1, 2, \ldots$$

Remark. In the indeterminate case, one infers from the construction of the Green function that the resolvent of any self-adjoint extension of J_{min} is a Hilbert-Schmidt operator.

Theorem. *Suppose the Hamburger moment problem is indeterminate. The spectrum of any self-adjoint extension J_t of J_{min} is simple and discrete. Two different self-adjoint extensions J_t have distinct spectra. Every real number is an eigenvalue of exactly one self-adjoint extension J_t.*

Continued fractions

Let $\{a_n\}$ and $\{b_n\}$ be complex sequences. A generalized infinite continued fraction

$$f = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \ddots}}}}$$

also frequently written in the form

$$f = \frac{a_1 \mid}{\mid b_1} + \frac{a_2 \mid}{\mid b_2} + \frac{a_3 \mid}{\mid b_3} + \cdots\,,$$

is understood here as a sequence of convergents

$$f_n = \frac{A_n}{B_n}, \quad n = 1, 2, 3, \ldots,$$

where the numerators and denominators, A_n and B_n, are given by the fundamental *Wallis recurrence formulas*

$$A_{n+1} = b_{n+1}A_n + a_{n+1}A_{n-1}, \quad B_{n+1} = b_{n+1}B_n + a_{n+1}B_{n-1},$$

with

$$A_{-1} = 1, \ A_0 = 0, \ B_{-1} = 0, \ B_0 = 1.$$

One says that a continued fraction is convergent if this is true for the corresponding sequence of convergents.

Definition. Let $\mathcal{L}$ be a positive-definite moment functional and

$$P_{n+1}(x) = (x - c_n)P_n(x) - d_nP_{n-1}(x), \quad n \geq 0,$$

be the fundamental recurrence relation defining the corresponding monic orthogonal polynomial sequence $\{P_n(x)\}$, with $P_{-1}(x) = 0$ and $P_0(x) = 1$. The monic polynomial sequence $\{P_n^{(1)}(x)\}$ defined by the recurrence formula

$$P_{n+1}^{(1)}(x) = (x - c_{n+1})P_n^{(1)}(x) - d_{n+1}P_{n-1}^{(1)}(x), \quad n \geq 0,$$

with $P_{-1}^{(1)}(x) = 0$ and $P_0^{(1)}(x) = 1$, is called the *associated (monic) polynomial sequence.*

Proposition. *Let $\{c_n;\, n = 0, 1, 2, \ldots\}$ and $\{d_n;\, n = 1, 2, 3, \ldots\}$ be a real and positive sequence, respectively. Let $\{P_n\}$ and $\{P_n^{(1)}\}$ designate the corresponding monic orthogonal polynomial sequence and the associated monic polynomial sequence, respectively. Then the convergents of the continued fraction*

$$f = \frac{1\,|}{|\,x - c_0} - \frac{d_1\,|}{|\,x - c_1} - \frac{d_2\,|}{|\,x - c_2} - \cdots$$

are

$$f_n = \frac{P_{n-1}^{(1)}(x)}{P_n(x)}, \quad n = 1, 2, 3, \ldots.$$

Remark. Recall that $c_n = \beta_n$ and $d_n = \alpha_{n-1}^2$ where α_n and β_n occur as entries in the associated Jacobi matrix. It is straightforward to verify that

$$\frac{P_{n-1}^{(1)}(x)}{P_n(x)} = \frac{Q_n(x)}{\hat{P}_n(x)}, \quad n = 0, 1, 2, \ldots.$$

Remark. It is worth of noting that the asymptotic expansion for large x of the convergents can be expressed in terms of the moments ($\mu_0 = 1$),

$$f_n = \frac{1}{x} + \frac{\mu_1}{x^2} + \ldots + \frac{\mu_{2n-1}}{x^{2n}} + O\left(\frac{1}{x^{2n+1}}\right), \quad \text{as } x \to \infty, \ n \in \mathbb{N}.$$

Gauss quadrature

Theorem (Gauss quadrature). *Let $\mathcal{L}$ be a positive-definite moment functional and $\{P_n(x)\}$ be the corresponding monic orthogonal polynomial sequence. Denote by $x_{n1} < x_{n2} < \ldots < x_{nn}$ the zeros of $P_n(x)$ ordered increasingly, $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ there exists a unique n-tuple of numbers A_{nk}, $1 \le k \le n$, such that for every polynomial $\pi(x)$ of degree at most $2n-1$,*

$$\mathcal{L}[\pi(x)] = \sum_{k=1}^{n} A_{nk}\, \pi(x_{nk}).$$

The numbers A_{nk} are all positive.

Remark. Let $\{P_n^{(1)}\}$ designate the associated monic polynomial sequence. Then for $n, k \in \mathbb{N}$, $k \le n$,

$$A_{nk} = \frac{P_{n-1}^{(1)}(x_{nk})}{P_n'(x_{nk})} = \left(\sum_{j=0}^{n-1} \hat{P}_j(x_{nk})^2\right)^{-1}.$$

One also has

$$A_{nk} = \mathcal{L}[l_{nk}(x)^2]$$

where

$$l_{nk}(x) = \frac{P_n(x)}{(x - x_{nk})\, P_n'(x_{nk})}\,.$$

Lommel polynomials – orthogonal polynomials with a discrete supporting set

The Lommel polynomials

The Lommel polynomials represent an example of an orthogonal polynomial sequence whose members are not known as solutions of a distinguished differential equation. On the other hand, the Lommel polynomials naturally arise within the theory of Bessel functions. The corresponding measure of orthogonality is supported on a discrete countable set rather than on an interval.

One of the fundamental properties of Bessel functions is the recurrence relation in the order

$$J_{\nu+1}(x) = \frac{2\nu}{x}\, J_\nu(x) - J_{\nu-1}(x).$$

As first observed by Lommel in 1871, this relation can be iterated which yields, for $n \in \mathbb{Z}_+$, $\nu \in \mathbb{C}$, $-\nu \notin \mathbb{Z}_+$ and $x \in \mathbb{C} \setminus \{0\}$,

$$J_{\nu+n}(x) = R_{n,\nu}(x) J_\nu(x) - R_{n-1,\nu+1}(x) J_{\nu-1}(x)$$

where

$$R_{n,\nu}(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} \frac{\Gamma(\nu+n-k)}{\Gamma(\nu+k)} \left(\frac{2}{x}\right)^{n-2k}$$

is the so called *Lommel polynomial*. But note that $R_{n,\nu}(x)$ is a polynomial in the variable x^{-1} rather than in x.

• E. von Lommel: *Zur Theorie der Bessel'schen Functionen*, Mathematische Annalen **4** (1871) 103-116.

As is well known, the Lommel polynomials are directly related to Bessel functions,

$$\begin{aligned} R_{n,\nu}(x) &= \frac{\pi x}{2}\left(Y_{-1+\nu}(x)J_{n+\nu}(x) - J_{-1+\nu}(x)Y_{n+\nu}(x)\right) \\ &= \frac{\pi x}{2\sin(\pi\nu)}\left(J_{1-\nu}(x)J_{n+\nu}(x) + (-1)^n J_{-1+\nu}(x)J_{-n-\nu}(x)\right). \end{aligned}$$

Furthermore, the Lommel polynomials obey the recurrence

$$R_{n+1,\nu}(x) = \frac{2\,(n+\nu)}{x}\,R_{n,\nu}(x) - R_{n-1,\nu}(x), \quad n \in \mathbb{Z}_+,$$

with the initial conditions $R_{-1,\nu}(x) = 0$, $R_{0,\nu}(x) = 1$.

The support of the measure of orthogonality for $\{R_{n,\nu+1}(x);\, n \geq 0\}$ turns out to coincide with the zero set of $J_\nu(z)$. Remember that $x^{-\nu}J_\nu(x)$ is an even function. Let $j_{k,\nu}$ stand for the k-th positive zero of $J_\nu(x)$ and put $j_{-k,\nu} = -j_{k,\nu}$ for $k \in \mathbb{N}$. The orthogonality relation reads

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{j_{k,\nu}^2}\,R_{n,\nu+1}(j_{k,\nu})R_{m,\nu+1}(j_{k,\nu}) = \frac{1}{2(n+\nu+1)}\,\delta_{m,n},$$

and is valid for all $\nu > -1$ and $m, n \in \mathbb{Z}_+$.

Let us also recall *Hurwitz' limit formula*

$$\lim_{n\to\infty} \frac{(x/2)^{\nu+n}}{\Gamma(\nu+n+1)}\,R_{n,\nu+1}(x) = J_\nu(x).$$

Lommel Polynomials in the variable ν

Lommel polynomials can also be addressed as polynomials in the parameter ν. Such polynomials are orthogonal, too, with the measure of orthogonality supported on the zero set of a Bessel function of the first kind regarded as a function of the order.

Let us consider a sequence of polynomials in the variable ν and depending on a parameter $u \neq 0$, $\{T_n(u;\nu)\}_{n=0}^\infty$, determined by the recurrence

$$u\,T_{n-1}(u;\nu) - n\,T_n(u;\nu) + u\,T_{n+1}(u;\nu) = \nu T_n(u;\nu), \quad n \in \mathbb{Z}_+,$$

with the initial conditions $T_{-1}(u;\nu) = 0$, $T_0(u;\nu) = 1$. It can be verified that

$$T_n(u;\nu) = R_{n,\nu}(2u),\ \forall n \in \mathbb{Z}_+.$$

The Bessel function $J_\nu(x)$ regarded as a function of ν has infinitely many simple real zeros which are all isolated provided $x > 0$. Below we denote the zeros of $J_{\nu-1}(2u)$ by $\theta_n = \theta_n(u)$, $n \in \mathbb{N}$, and restrict ourselves to the case $u > 0$ since $\theta_n(-u) = \theta_n(u)$.

The Jacobi matrix $J(u;\nu)$ corresponding to this case has the diagonal entries $\beta_n = -n$ and the weights $\alpha_n = u$, $n \in \mathbb{Z}_+$, and represents an unbounded self-adjoint operator with a discrete spectrum. The orthogonality measure for $\{T_n(u;\nu)\}$ is supported on the spectrum of $J(u;\nu)$, the orthogonality relation has the form

$$\sum_{k=1}^{\infty} \frac{J_{\theta_k}(2u)}{u\left(\partial_z\big|_{z=\theta_k} J_{z-1}(2u)\right)} R_{n,\theta_k}(2u) R_{m,\theta_k}(2u) = \delta_{m,n},\ \ m,n \in \mathbb{Z}_+.$$

Let us remark that initially this was Dickinson who formulated, in 1958, the problem of constructing the measure of orthogonality for the Lommel polynomials in the variable ν. Ten years later, Maki described such a construction.

• D. Dickinson: *On certain polynomials associated with orthogonal polynomials*, Boll. Un. Mat. Ital. **13** (1958) 116-124

• D. Maki: *On constructing distribution functions with application to Lommel polynomials and Bessel functions*, Trans. Amer. Math. Soc. **130** (1968), 281-297