Spectral analysis of the Hilbert matrix

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1 A few notes on the history

History of the Hilbert matrix is briefly explained in [8, Chp. IX]. According to this source, Hilbert proved his double series theorem in lectures on integral equations. The theorem asserts that there exists a positive constant M such that for any real square summable sequence $\{a_n\}$ one has

$$0 \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n-1} \leq M\sum_{m=1}^{\infty}a_m^2.$$

Hilbert's proof was later published by Weyl in his Göttingen dissertation in 1908. The optimal value of the constant, $M = \pi$, has been found by Schur in [15]. This way he effectively determined the norm of the Hilbert matrix operator $H = (H_{j,k})$ in $\ell^2(\mathbb{N})$, defined by

$$H_{j,k} = \frac{1}{j+k-1}\,, \quad j,k \in \mathbb{N}. \tag{1}$$

Hence $0 \leq H$ and $\|H\| = \pi$.

In 1950, Magnus noted that the spectrum of H contains the whole interval $[0,\pi]$ and, moreover, there are no embedded eigenvalues [11]. So the spectrum is purely continuous. Later on, in 1958, Rosenblum showed that H is unitarily equivalent to a comparatively simple integral operator on $L^2((0,\infty),\mathrm{d}x)$ [13], and, in addition, he found a Sturm-Liouville operator on the positive half-line commuting with that integral operator [14]. The spectral problem for the differential operator turned out to be solvable. Consequently, this made it possible to explicitly diagonalize H. One concluded that the Hilbert matrix operator is unitarily equivalent to the multiplication operator by the function $\pi/\cosh(\pi\tau)$ acting on $L^2((0,\infty),\mathrm{d}\tau)$ [14]. Particularly, $\operatorname{spec} H$ is purely absolutely continuous.

Leading principal submatrices H_n of H are usually also called Hilbert matrices. Despite of the mentioned complete solution of the spectral problem, the Hilbert matrix continued to attract a good deal of attention from the numerical point of view because the finite-dimensional Hilbert matrices H_n are canonical examples of ill-conditioned matrices, making them difficult to use in numerical computation [16]. To overcome certain computational problems Grünbaum constructed, in [7], a tridiagonal (Jacobi) matrix T_n commuting with H_n.

This fact inspired the author of these notes to consider the infinite-dimensional case as well and use a Jacobi matrix T commuting with H in an alternative approach leading to a diagonalization of the Hilbert matrix. This is possible, indeed, and this approach is outlined in full detail in Sections 3, 4, 5 and 6. Section 2 contains a sketchy summary of Rosenblum's result. Afterwards the author realized that practically the same approach, possibly differing in some details, had been carried out earlier by Otte, with results presented at the conference CDESFA 2005, Munich, Germany. The slides from the conference are currently available from Otte's homepage [12].

2 A solution due to Rosenblum

Recall that the Laguerre polynomials $\{L_n(x);\, n\in\mathbb{Z}_+\}$ form an orthonormal basis in $\mathscr{H}=L^2((0,\infty),\mathrm{e}^{-x}\mathrm{d}x)$. In [13], Rosenblum observed that the matrix with respect to this basis of the integral operator K,

$$Kf(x)=\int_0^\infty\frac{f(y)}{x+y}\,\mathrm{e}^{-y}\,\mathrm{d}y,\ \forall f\in\mathscr{H},$$

coincides with the Hilbert matrix. This is to say that the Hilbert matrix operator H, acting on $\ell^2(\mathbb{N})$, is unitarily equivalent to the integral operator K.

To see this one can employ the formula for the Laplace transform of a Laguerre polynomial,

$$\int_0^\infty \mathrm{e}^{-pt}L_n(t)\,\mathrm{d}t=(p-1)^n p^{-n-1},\quad \operatorname{Re}p>0,$$

see Eq. 7.414 ad 6 in [6]. Whence

$$\begin{aligned}\langle L_m,KL_n\rangle &= \int_0^\infty\int_0^\infty\frac{L_m(x)L_n(y)}{x+y}\,\mathrm{e}^{-x-y}\,\mathrm{d}x\mathrm{d}y\\ &= \int_0^\infty\left(\int_0^\infty \mathrm{e}^{-(t+1)x}L_m(x)\,\mathrm{d}x\right)\left(\int_0^\infty \mathrm{e}^{-(t+1)y}L_n(y)\,\mathrm{d}y\right)\mathrm{d}t\\ &= \int_0^\infty t^{m+n}(t+1)^{-m-n-2}\,\mathrm{d}t \;=\; \int_0^1 u^{m+n}\,\mathrm{d}u\\ &= \frac{1}{m+n+1},\quad m,n\in\mathbb{Z}_+.\end{aligned}$$

As a matter of fact, the problem addressed in [13], as well as in the subsequent paper [14], was somewhat more complex, involving a one-parameter family of generalized Hilbert matrices.

For the analysis to follow it is convenient to apply the unitary transform

$$L^2((0,\infty),\mathrm{e}^{-x}\mathrm{d}x)\to L^2((0,\infty),\mathrm{d}x):f(x)\mapsto\tilde{f}(x)=\mathrm{e}^{-x/2}f(x).$$

The operator K is transformed correspondingly. Its image, $\tilde{K}$, is an integral operator with the kernel

$$\tilde{\mathcal{K}}(x,y)=\frac{\mathrm{e}^{-(x+y)/2}}{x+y}.$$

Following [14], consider the formal elliptic second-order differential operator

$$\mathcal{L} = -\frac{\mathrm{d}}{\mathrm{d}x} x^2 \frac{\mathrm{d}}{\mathrm{d}x} + \frac{x^2-1}{4}.$$

It is straightforward to verify the identity

$$\mathcal{L}_x \tilde{\mathcal{K}}(x,y) - \mathcal{L}_y \tilde{\mathcal{K}}(x,y) = 0,$$

with indices indicating in which variable the differential operator is acting. This means that $\mathcal{L}$ and $\tilde{K}$ formally commute. To make this observation precise let us introduce, as usual, another operator, $\dot{L}$, acting as $\mathcal{L}$ on the domain $\operatorname{Dom}\dot{L}$ which coincides with the space of test function $\mathscr{D}\big((0,\infty)\big)$. Then

$$\langle \tilde{K}f, \dot{L}g\rangle - \langle \dot{L}f, \tilde{K}g\rangle = 0, \quad \forall f,g \in \operatorname{Dom}\dot{L}. \tag{2}$$

$\dot{L}$ is obviously symmetric and even positive. The last statement is implied by the inequality

$$\frac{1}{4}\int_0^\infty f(x)^2\,\mathrm{d}x \leq \int_0^\infty x^2 f'(x)^2\,\mathrm{d}x$$

which is valid for all real-valued C^1 functions on $[0,\infty)$ vanishing on a neighborhood of infinity. In fact,

$$x^2(f')^2 - \frac{1}{4}f^2 = \left(xf' + \frac{1}{2}f\right)^2 - \frac{1}{2}(xf^2)'.$$

Let $L_{\min}$ be the closure of $\dot{L}$, and $L_{\max}$ acts as $\mathcal{L}$ on

$$\operatorname{Dom}L_{\max} = \{f \in L^2\big((0,\infty)\big);\ \mathcal{L}f \in L^2\big((0,\infty)\big)\}$$

where $\mathcal{L}f$ should be understood in the distributional sense. It turns out that $\dot{L}$ is essentially self-adjoint, i.e. $L := L_{\min} = L_{\max}$ is self-adjoint. To see it, it suffices to show, according to the general theory of self-adjoint extensions, that the differential equation $\mathcal{L}f = -f/4$, i.e.

$$-(x^2 f')' + x^2 f/4 = 0, \tag{3}$$

has no square integrable solution on the positive half-line. This is the case, indeed, since for two independent solutions to (3) one can take $f_\pm(x) = \exp(\pm x)/x$. Moreover, from (2) it can be deduced that $\tilde{K}$ and L commute.

As one can readily check, the function

$$\psi_\tau(x) = \frac{\big(2\tau\sinh(\pi\tau)\big)^{1/2}}{\pi\sqrt{x}}\,K_{i\tau}\Big(\frac{x}{2}\Big),\ \ \tau > 0,$$

is a generalized eigenfunction of L corresponding to the eigenvalue τ^2. Here

$$K_\nu(z) = \int_0^\infty \mathrm{e}^{-z\cosh(t)}\cosh(\nu t)\,\mathrm{d}t$$

is the Macdonald (modified Bessel) function. The family $\{\psi_\tau;\, \tau > 0\}$ even represents a complete normalized set of generalized eigenfunction of L since the integral transform

$$L^2\big((0,\infty),\mathrm{d}x\big) \to L^2\big((0,\infty),\mathrm{d}\tau\big) \colon f(x) \mapsto \hat{f}(\tau) = \int_0^\infty \psi_\tau(x) f(x)\,\mathrm{d}x$$

is a unitary operator. This is in fact nothing but the Kontorovich-Lebedev transform [9]. The inverse transform has the form

$$f(x) = \int_0^\infty \psi_\tau(x)\hat{f}(\tau)\,\mathrm{d}\tau.$$

The differential operator L is diagonalized by this transform and becomes the multiplication operator by the function τ^2.

The image of $\tilde{K}$ by this transform commutes with this multiplication operator and so it should be itself, too, a multiplication operator (see, for instance, Proposition 1.9 of Supplement 1 in [4]). Actually, ψ_τ turns out to be a generalized eigenfunction of $\tilde{K}$ corresponding to the eigenvalue $\pi/\cosh(\pi\tau)$ as it follows from the formula [6, Eq. 6.627]

$$\int_0^\infty \frac{e^{-s-t}}{(s+t)\sqrt{t}}\, K_{i\tau}(t)\,\mathrm{d}t = \frac{\pi}{\cosh(\pi\tau)\sqrt{s}}\, K_{i\tau}(s), \quad s,\tau > 0,$$

meaning that

$$\int_0^\infty \tilde{\mathcal{K}}(x,y)\,\psi_\tau(y)\,\mathrm{d}y = \frac{\pi}{\cosh(\pi\tau)}\,\psi_\tau(x).$$

Thus an explicit diagonalization of the Hilbert matrix operator has been found.

In the remainder of these notes, an alternative approach to the diagonalization of H is described. In this case we stick to the discrete level while working with matrices only rather than with integral or differential operators.

3 A commuting Jacobi matrix and formal eigenvectors

Matrix entries in these notes are indexed by $j,k = 1,2,3,\ldots$. Recall definition (1) of the Hilbert matrix. Let T be a tridiagonal matrix with the entries

$$T_{n,n} = 2(n-1)n + \frac{3}{4},\ T_{n,n+1} = T_{n+1,n} = -n^2,\ T_{m,n} = 0 \text{ otherwise},\ m,n \in \mathbb{N}.$$

Then $TH = HT$. This property has been observed in [7].

T is positive but the proof is not so straightforward and will be discussed later, in Section 5. But it is rather easy to show that $\tilde{T} = T + 1/4I$ is positive and so T is semibounded. In fact,

$$\tilde{T}_{n,n} = 2\left(n-\frac{1}{2}\right)^2 + \frac{1}{2},\ \tilde{T}_{n,n+1} = \tilde{T}_{n+1,n} = -n^2,\ \tilde{T}_{m,n} = 0 \ \text{ otherwise},$$

and, for any real sequence $\{x_n\}$ with only finitely many nonzero members,

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tilde{T}_{m,n}x_m x_n &= \frac{1}{2}{x_1}^2+\sum_{n=1}^{\infty}\left(\left(n-\frac{1}{2}\right)x_n-\left(n+\frac{1}{2}\right)x_{n+1}\right)^2\\ &\quad +\frac{1}{4}\sum_{n=1}^{\infty}(x_n-x_{n+1})^2 \,\geq\, 0.\end{aligned}$$

For any $x\in\mathbb{C}$ there exists exactly one formal eigenvector $v(x)$ of T corresponding to the eigenvalue x and such that its first component equals 1. Any other formal eigenvector with the same eigenvalue is its multiple. Write

$$v(x)=\big(\hat{P}_0(x),\hat{P}_1(x),\hat{P}_2(x),\ldots\big). \tag{4}$$

Then $\hat{P}_n(x)$ is a polynomial of degree n, $n\in\mathbb{Z}_+$. The sequence $\{\hat{P}_n(x)\}$ is unambiguously determined by the recurrence $\hat{P}_0(x)=1$ and

$$\begin{aligned}&\left(\frac{3}{4}-x\right)\hat{P}_0(x)-\hat{P}_1(x)=0, \qquad (5)\\ &-n^2\,\hat{P}_{n-1}(x)+\left(2n(n+1)+\frac{3}{4}-x\right)\hat{P}_n(x)-(n+1)^2\hat{P}_{n+1}(x)=0\ \text{ for } n\geq 1.\end{aligned}$$

Put

$$\hat{P}_n(x)=\frac{1}{(n!)^2}\,P_n(x),\ n\in\mathbb{Z}_+.$$

Then $\{(-1)^nP_n(x);\,n\in\mathbb{Z}_+\}$ is a sequence of monic orthogonal polynomials obeying the recurrence rule ($P_{-1}(x)=0$ by definition)

$$P_0(x)=1,\ P_{n+1}(x)=\left(2n(n+1)+\frac{3}{4}-x\right)P_n(x)-n^4P_{n-1}(x),\ n=0,1,2,\ldots. \tag{6}$$

4 The Wilson polynomials, the continuous Hahn dual polynomials and an explicit form

The Wilson polynomials represent a very large and general four-parameter family of orthogonal polynomials sitting at a top position of the Askey hierarchical scheme for hypergeometric orthogonal polynomials, see [10]. Here is the definition

$$\frac{W_n\left(x^2;a,b,c,d\right)}{(a+b)_n(a+c)_n(a+d)_n}={}_4F_3(-n,n+a+b+c+d-1,a+ix,a-ix;a+b,a+c,a+d;1),$$

$n\in\mathbb{Z}_+$. We shall need just a particular case. Put $a=b=1/4$, $c=d=3/4$, and write for short

$$W_n(x^2)=W_n\!\left(x^2;\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}\right).$$

The respective orthogonality relation takes the form (see Eq. (9.1.2) in [10]),

$$\int_0^\infty \frac{x\sinh(2\pi x)}{\cosh^2(2\pi x)}\,W_m(x^2)W_n(x^2)\,\mathrm{d}x = \frac{1}{\pi}\,2^{-4n-3}((2n)!)^2\,(n!)^2\,\delta_{m,n}. \tag{7}$$

Let

$$P_n(z) = \frac{4^n n!}{(2n)!}\,W_n\Big(\frac{z}{4}\Big).$$

Then the sequence $\{P_n(x)\}$ obeys (6), see Eq. (9.1.5) in [10]. Similarly, the sequence $\{\hat{P}_n(x)\}$, with

$$\hat{P}_n(z) = \frac{1}{(n!)^2}\,P_n(z) = \frac{4^n}{n!\,(2n)!}\,W_n\Big(\frac{z}{4}\Big),$$

obeys (5). The orthogonality relation (7) rewritten in terms of $\hat{P}_n(z)$ reads

$$\int_0^\infty \hat{P}_m(x^2)\hat{P}_n(x^2)\rho(x)\,\mathrm{d}x = \delta_{m,n}$$

where

$$\rho(x) = \frac{2\pi x\sinh(\pi x)}{\cosh^2(\pi x)}. \tag{8}$$

It turns out that the polynomials $P_n(z)$ coincide with the continuous dual Hahn polynomials representing a three-parameter subclass of the class of Wilson polynomials. Here is the definition

$$\frac{S_n(x^2;a,b,c)}{(a+b)_n(a+c)_n} = {}_3F_2(-n,a+ix,a-ix;a+b,a+c;1). \tag{9}$$

We shall need just a particular case. Put $a=b=c=1/2$, and write for short

$$S_n(x^2) = S_n\bigg(x^2;\frac{1}{2},\frac{1}{2},\frac{1}{2}\bigg).$$

From recurrence (9.3.5) in [10] it follows that $P_n(z)=S_n(z)$ for all n, indeed. In view of (9), we have the explicit formula

$$\begin{aligned}\hat{P}_n(z) &= \frac{1}{(n!)^2}\,S_n(z) \;=\; {}_3F_2\bigg(-n,\frac{1}{2}+i\sqrt{z},\frac{1}{2}-i\sqrt{z};1,1;1\bigg)\\ &= \sum_{k=0}^{n}\frac{(-1)^k}{(k!)^2}\binom{n}{k}\prod_{j=0}^{k-1}\bigg(\Big(j+\frac{1}{2}\Big)^2+z\bigg).\end{aligned} \tag{10}$$

Remark. The results of [2] imply that the sequence of coefficients of the power series expansion at $z=0$ of the function

$$f_a(z) = (1-z)^{a-1}\,{}_2F_1(a,a;1;z) = \sum_{n=0}^\infty A_n(a)z^n.$$

is a formal eigenvector of H and hence of T as well. Then $A_0(a)=1$, $A_1(a)=1-a+a^2$, ... Recall that $\hat{P}_1(x)=\frac{3}{4}-x$, see (5). Hence the sequence $\big(A_0(a),A_1(a),A_2(a),\dots\big)$ is a formal eigenvector of T corresponding to the eigenvalue $-(1/2-a)^2$, and so it holds true

$$\hat{P}_n\left(-\left(\frac{1}{2}-a\right)^2\right)=A_n(a),\ \forall n\in\mathbb{Z}_+.$$

Choosing $a=\frac{1}{2}+i\sqrt{x}$ one arrives at the following formula, valid for all $n\in\mathbb{Z}_+$,

$$\begin{aligned}\hat{P}_n(x) &= \sum_{k=0}^{n}\frac{\big(\frac{1}{2}-i\sqrt{x}\big)_{n-k}}{(n-k)!}\left(\frac{\big(\frac{1}{2}+i\sqrt{x}\big)_k}{k!}\right)^2\\ &= {}_3F_2\left(\frac{1}{2}-i\sqrt{x},\frac{1}{2}+i\sqrt{x},-n;1,1;1\right),\end{aligned}\tag{11}$$

in agreement with (10).

Let us recall that the Bateman orthogonal polynomials are defined as [3]

$$F_n(z)={}_3F_2\left(-n,n+1,\frac{1+z}{2};1,1;1\right),\ n\in\mathbb{Z}_+.$$

The Bateman polynomials are directly related to the Touchard polynomials $Q_n(x)$ by a simple transformation of variable, see [5, 17]. On the other hand, the Bateman polynomials are a special case of the symmetric continuous Hahn polynomials, see [10]. By comparison with (11) one has, for any $m\in\mathbb{Z}_+$,

$$\hat{P}_n\left(-\left(\frac{1}{2}+m\right)^2\right)=F_m(-2n-1),\ \forall n\in\mathbb{Z}_+.$$

5 The matrix operator T is positive and essentially self-adjoint

Proposition. *T is positive on the linear hull of the canonical basis in $\ell^2(\mathbb{N})$.*

Proof. In order to verify that the matrix T is positive on that domain it suffices to show all its leading principal minors to be positive. For any $N\in\mathbb{N}$ denote by $T^{(N)}$ the $N\times N$ truncation of T, i.e. the Nth leading principal submatrix of T, and put $\Delta_N=\det T^{(N)}$. By definition, $\Delta_0=1$. One may make use of the fact that the sequence $v(0)$, with $v(x)$ defined in (4), formally belongs to the kernel of T. Using Crammer's rule one obtains at once the recurrence rule

$$\hat{P}_{N-1}(0)\Delta_N=N^2\hat{P}_N(0)\Delta_{N-1},\quad\forall N\in\mathbb{N}.$$

Thus to show that all Δ_N are positive it suffices to check that the values $\hat{P}_N(0)$ are all positive, and this is the same as saying that all values $W_N(0)$ are positive. But this is obvious from formula (9.1.13) in [10] for a generating function of the Wilson polynomials. In fact, in our particular case that formula claims that

$$ {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;t\right)^2 = \sum_{n=0}^{\infty} \frac{W_n(0)t^n}{(n!)^3}\,. $$

Of course, one notes that the coefficients of the hypergeometric power series on the LHS are all positive. □

Corollary. *T is essentially self-adjoint.*

Proof. One can show that T is essentially self-adjoint by verifying that for some and hence any $x>0$ the formal eigenvector $v(-x)$, as defined in (4), is not a square summable sequence. But from (10) it follows that $\hat{P}_n(-1/4)=1$ for all $n\in\mathbb{Z}_+$. □

6 The resulting unitary transform of the Hilbert matrix

Let us denote $\mathscr{H}=L^2((0,\infty),\rho(x)\mathrm{d}x)$ where $\rho(x)$ is given in (8). Let $\{e_n;\,n\in\mathbb{N}\}$ stand for the standard basis in $\ell^2(\mathbb{N})$. Then $\{\hat{P}_n(x^2);\,n\in\mathbb{Z}_+\}$ is an orthonormal basis in $\mathscr{H}$, and there exists a unique unitary mapping $U:\ell^2(\mathbb{N})\to\mathscr{H}$ such that $(Ue_n)(x)=\hat{P}_{n-1}(x^2)$, $\forall n\in\mathbb{N}$. We wish to describe the operator UHU^{-1} explicitly.

We know that the matrix operator T on $\ell^2(\mathbb{N})$, with $\operatorname{Dom}T=\operatorname{span}\{e_n;\,n\in\mathbb{N}\}$, is essentially self-adjoint and formally commutes with H. This is to say that

$$ \langle Tu,Hv\rangle=\langle u,HTv\rangle,\quad \forall u,v\in\operatorname{Dom}T. $$

Hence the closure $\overline{T}=T_{\max}$ is self-adjoint, and one immediately deduces that

$$ H\operatorname{Dom}(\overline{T})\subset\operatorname{Dom}(\overline{T})\ \text{ and }\ \overline{T}Hv=H\overline{T}v,\quad \forall v\in\operatorname{Dom}(\overline{T}). $$

Moreover, $\overline{T}$ is positive and, consequently, the operator $(\overline{T}+1)^{-1}$ is bounded and commutes with H.

We know that, for any $x>0$, the sequence $v(x^2)$ defined in (4) is a formal eigenvector of T with the eigenvalue x^2. This fact implies at once that, $\forall n\in\mathbb{Z}_+$,

$$ (U\overline{T}U^{-1}\hat{P}_n)(x^2)=T_{n,n+1}\hat{P}_{n-1}(x^2)+T_{n+1,n+1}\hat{P}_n(x^2)+T_{n+2,n+1}\hat{P}_{n+1}(x^2)=x^2\hat{P}_n(x^2). $$

Hence $U\overline{T}U^{-1}$ acts on $\mathscr{H}$ as a multiplication operator by the function x^2. Moreover, the bounded operator UHU^{-1} commutes with $U(\overline{T}+1)^{-1}U^{-1}$ which itself is a multiplication operator by $(x^2+1)^{-1}$. This is a general fact that in that case UHU^{-1} is necessarily a multiplication operator by a measurable and almost everywhere bounded function $h(x)$ (see, for instance, Proposition 1.9 of Supplement 1 in [4] or, alternatively, Lemma 6.4 in [18]). The function $h(x)$ can be computed explicitly from the equation

$$ h(x)=h(x)\hat{P}_0(x^2)=UHU^{-1}\hat{P}_0(x^2)=(UHe_1)(x^2)=\sum_{n=0}^{\infty}\frac{1}{n+1}\,\hat{P}_n(x^2). $$

The above sum converges in $\mathscr{H}$ and therefore almost everywhere on $(0,+\infty)$. Rewriting the sum in terms of the continuous dual Hahn polynomials instead of the polynomials $\hat{P}_n(x^2)$, one has

$$h(x)=\sum_{n=0}^{\infty}\frac{S_n(x^2)}{(n+1)!\,n!}\,.$$

This sum can be evaluated with the aid of formula (9.3.12) from [10] for a generating function, claiming that, in our particular case,

$$\sum_{n=0}^{\infty}\frac{S_n\left(x^2\right)}{(n!)^2}\,t^n=(1-t)^{-1/2+ix}\,{}_2F_1\!\left(\frac{1}{2}+ix,\frac{1}{2}+ix;1;t\right).$$

One may carry out integration in t from 0 to 1 to obtain

$$\begin{aligned} h(x) &= \sum_{n=0}^{\infty}\left(\frac{\left(\frac{1}{2}+ix\right)_n}{n!}\right)^2 B\!\left(\frac{1}{2}+ix,n+1\right)\\ &= \sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}+ix\right)_n}{\left(n+\frac{1}{2}+ix\right)n!}=\left(\frac{1}{2}+ix\right)^{-1}{}_2F_1\!\left(\frac{1}{2}+ix,\frac{1}{2}+ix;\frac{3}{2}+ix;1\right). \end{aligned}$$

By Eq. (15.1.20) in [1] and the reflection formula for the gamma function,

$${}_2F_1(z,z;1+z;1)=\Gamma(1+z)\Gamma(1-z)=\frac{\pi z}{\sin(\pi z)}.$$

So we conclude that $h(x)=\pi/\cosh(\pi x)$, i.e.

$$UHU^{-1}\psi(x)=\frac{\pi}{\cosh(\pi x)}\,\psi(x),\quad \forall\psi\in\mathscr{H}.$$

This is to say that we have diagonalized the Hilbert matrix H regarded as a Hermitian operator on $\ell^2(\mathbb{N})$.

Theorem. *The Hilbert matrix operator H has a pure absolutely continuous spectrum, and one has* $\operatorname{spec}H=[0,\pi]$.

Remark. It also follows that, $\forall m,n\in\mathbb{Z}_+$,

$$\frac{1}{m+n+1}=\langle e_{m+1},He_{n+1}\rangle=\int_0^{\infty}h(x)\hat{P}_m(x^2)\hat{P}_n(x^2)\rho(x)\,\mathrm{d}x.$$

Using explicit expressions one has, in terms of the continuous dual Hahn polynomials,

$$\int_0^{\infty}\frac{t\sinh(t)}{\cosh^3(t)}\,S_m\!\left(\frac{t^2}{\pi^2}\right)S_n\!\left(\frac{t^2}{\pi^2}\right)\mathrm{d}t=\frac{(m!\,n!)^2}{2(m+n+1)}\,.$$

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