Methods and Techniques for Multifractal Spectrum Estimation in Financial Time Series

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- We examine different techniques of multifractality estimation, especially Multifractal detrended analysis and Multifractal entropy analysis

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- Multiple scalings can be analyzed trough various techniques as Multifractal spectrum
- We examine different techniques of multifractality estimation, especially Multifractal detrended analysis and Multifractal entropy analysis
- We discuss both theoretical and practical properties of the techniques and compare them on both model and empirical time series

Multifractal Analysis

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- We wish to identify structure and strenght of α's in the system
- For this end we analyze the probability density in a form

$$p(\alpha, s)\mathrm{d}\alpha = \rho(\alpha) s^{-f(\alpha)} \mathrm{d}\alpha$$

- $f(\alpha)$ - Mutlifractal spectrum - measure of strength of each exponent

Multifractal Analysis

Scaling Function

Multifractal Analysis

Scaling Function

- A dual description is provided by the Partition function

$$Z(q, s) = \sum_i p_i^q \sim s^{\tau(q)}$$

- Function $\tau(q)$ is called Scaling function
- Relation to the $f(\alpha)$ is given by Legendre transform

$$\tau(q) = [q\alpha(q) - f(\alpha(q))]$$

where $\alpha(q)$ is such that it maximizes $\tau(q)$

Multifractal Analysis

Generalized Dimension

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Generalized Dimension

- Partition function is related to Rényi entropy $S(q) = \frac{\ln Z(q,l)}{q-1}$, for $q \to 1$ it reduces to the Shannon entropy
- Scaling exponent of the Rényi entropy is called Generalized dimension and is related to $\tau(q)$ as

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- Another measure of multifractality is a Generalized Hurst exponent defined as

$$\langle |x(t+s) - x(t)|^q \rangle \sim s^{qH(q)}$$

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- We divide a series into intervals of length s, calculate the local linear (quadratic,...) trends y_ν and calculate the Fluctuation function

$$F(\nu, s) = \sum_{i=1}^{s} [Y(s(\nu - 1) + i) - y_\nu(i)]$$

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- We calculate the Total fluctuation function as a generalized average of all fluctuation functions and this function is assumed to have scaling function $h(q)$, so

$$F(q,s) = \left\{ \frac{1}{N_s} \sum_{\nu=1}^{N_s} [F(\nu,s)^2]^{q/2} \right\}^{1/q} \sim s^{h(q)}$$

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- Exponent $\tau(q)$ can be calculated from $h(q)$ as $\tau(q) = qh(q) - 1$
- Method can be problematic, because it is based on assumption, that the expression

$$|p_s(\nu)| = |\sum_{k=(\nu-1)s-1}^{\nu s} (x_k - \langle x \rangle)|$$

is proper measure, which is not always true

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  - i.e., generalized Gaussian scaling
- The exponent δ is possible to express from the entropy

$$S(t) = -\int \mathrm{d}x\; p(x,t)\ln[p(x,t)] = A + \delta \ln t$$

- For the Fractional Brownian motion (monofractal) is δ equal to the Hurst exponent $\delta = H$

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$$S_q(t) = B_q + H(q) \ln t$$

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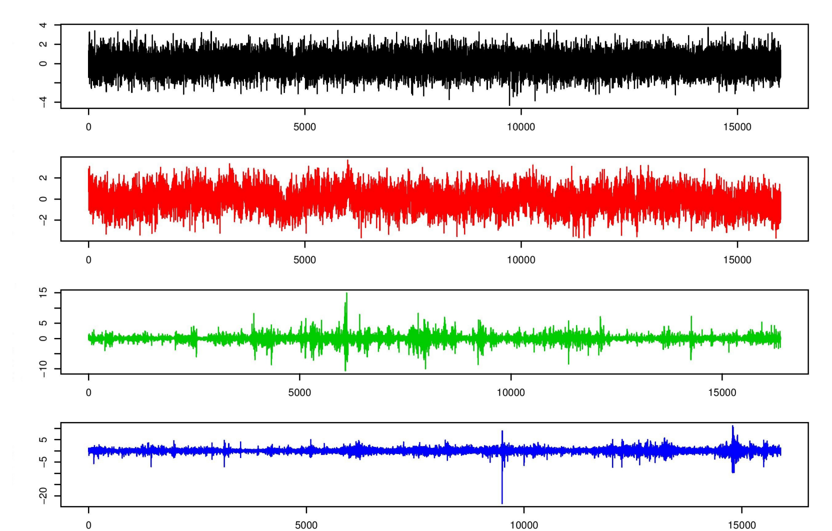
- For the monofractal series is the MFDFA scaling function equal to $\tau_M(q) = qH - 1$, whereas the MFDEA scaling function $\tau_R(q) = H(q)(q-1) = H \cdot (q-1)$, so

$$\tau_M(q) = \frac{q}{q-1}\tau_R(q) - 1$$

Numerical Comparison of Multifractal Techniques

Investigated Time Series - which one is real?



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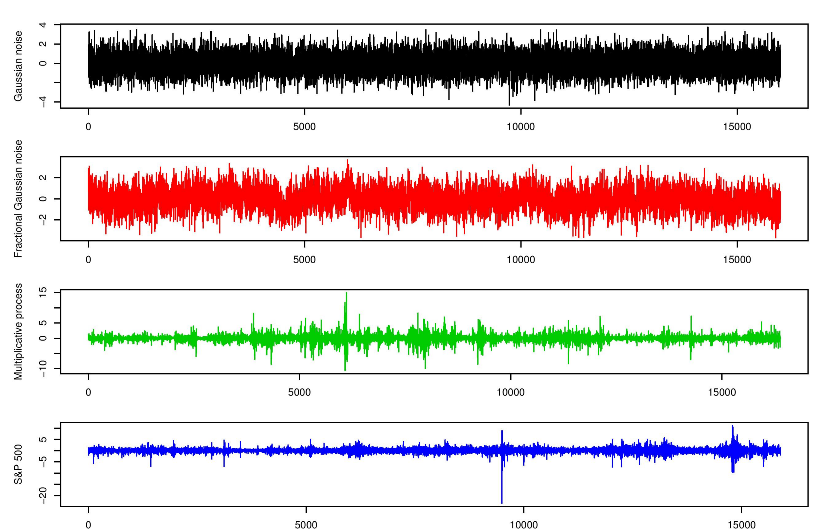
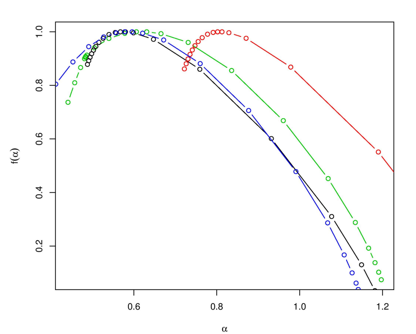
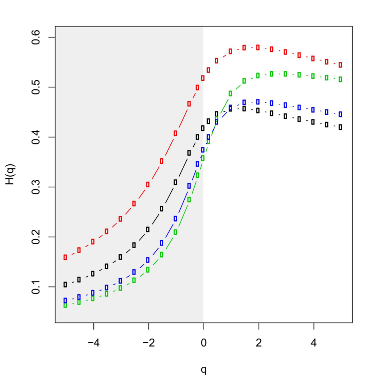


Numerical comparison of multifractal techniques

Multifractal spectrum, Renyi entropy





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- Properties of MFDFA and MFDEA were discussed
- Benefit of MFDEA is its interpretation as a Generalized dimension and also applicability to heavy-tailed processes
- Recently, there was published a discussion paper [1] that points to the problematic mathematical background and necessity to deeper discussion

[1] A. Yu. Morozov. Comment on 'multifractal diffusion entropy analysis on stock volatility in financial markets' [Physica A 391 (2012) 5739-5745].

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- Properties of MFDFA and MFDEA were discussed
- Benefit of MFDEA is its interpretation as a Generalized dimension and also applicability to heavy-tailed processes
- Recently, there was published a discussion paper [1] that points to the problematic mathematical background and necessity to deeper discussion
- Benefit of MDFDA is the elegance of the algorithm, computational effectiveness and its interpretation as a scaling exponent of the probability density

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Thank you for your attention.