Option Pricing Beyond Black-Scholes Based on Double-Fractional Diffusion

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Introduction

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- In finance are traded many derivatives - assets depending on some underlying assets
- Their price should be derived from the possible scenarios of underlying assets
- First option pricing model (Black and Scholes) was based on ordinary Brownian motion - 1973
- 1997 - Nobel prize in economics (Scholes, Merton)
- In times of financial crises, the model cannot catch the complex behavior of financial markets - large drops, sudden shocks
- These “black swans” can be better described by Lévy distributions and (double)-fractional diffusion

Option pricing

- *option* is a special asset which gives to the owner the right (option) to buy (call) or sell (put) an underlying asset for specified strike price K.
- buyer - long position, seller - short position
- seller takes the risk of losses - this is compensated by the option price
- Price of a call option at *maturity time* ($t = T$):

$$C(S, K) = \max\{S - K, 0\}$$

(if $S < K$ we can directly buy the underlying asset for price S)

- for $t < T$ we have

$$C(S_t, K, t) = e^{-r(T-t)} E[C(S, K)|\mathcal{F}_t] =$$
$$\int_{\mathbb{R}} \mathrm{d}y \;\; \max\left\{S_t e^{(t-T)(r+\mu)+y} - K, 0\right\} g(y, T - t)$$

- $g(y, \tau)$ is the probability distribution given by an appropriate stochastic model

Mathematical description of Double-fractional diffusion model

Stable distributions

- *Stable Hamiltonian* (logarithm of a characteristic function)

$$H_{\alpha,\beta}(p) = \ln \int_{\mathbb{R}} e^{ipx} L_{\alpha,\beta}(x)\,\mathrm{d}x =$$
$$i\bar{x}p - \bar{\sigma}^{\alpha}|p|^{\alpha}\left(1 - i\beta\mathrm{sign}(p)\omega(p,\alpha)\right)$$

  where $\alpha \in (0,2]$ - stability par., $\beta = [-1,1]$ - asymmetry par., $\sigma > 0$ - scale par., $\bar{x} \in \mathbb{R}$ - location par.

- for $\alpha < 2$ is decays polynomially as $1/|x|^{\alpha+1}$, except for extreme cases $\beta = \pm 1$, where an exponential decay is observed for left, resp. right tail.
- stable distributions with $\beta = -1$ are preferred for description of log-Lévy process $Y = e^{L_{\alpha,-1}}$, because all moments exist and are finite (= Laplace transform exists)

Double-fractional diffusion

- We consider a double-fractional diffusion equation

$$\left({}^{K}\partial_t^{\gamma} + \sigma^{\alpha} \sec\left(\frac{\pi\alpha}{2}\right) \mathfrak{D}_x^{\alpha}\right) g(x,t) = 0$$

- $\mathfrak{D}_x^{\alpha}$ - Riesz-Feller derivative, ${}^{K}\partial_t^{\gamma}$ - Caputo/RF derivative, $\alpha \in [1,2]$, $\gamma \in (0,\alpha]$.
- $\gamma = 1$ - (spatially) fractional diffusion - solution: stable distribution $L_{\alpha}(x,t)$
- for $\gamma \leq 1$ we need one initial condition $g(x,0) = \delta(x)$
  for $\gamma \in (1,2]$ we have another condition $\frac{\partial g}{\partial t}(x,t)|_{t=0} \equiv 0$.

Solution of double-fractional diffusion equation

- DFDE in Fourier-Laplace image ($x \stackrel{\mathcal{F}}{\to} p, t \stackrel{\mathcal{L}}{\to} s$)

$$(s^\gamma - H_{\alpha,-1}(p))\hat{\hat{g}}(p,s) = s^{\gamma-\kappa}$$

where $\kappa = \gamma$ for RF derivative, $\kappa = 1$ for Caputo derivative

- Mellin-Barnes representation of $g(x,t)$:

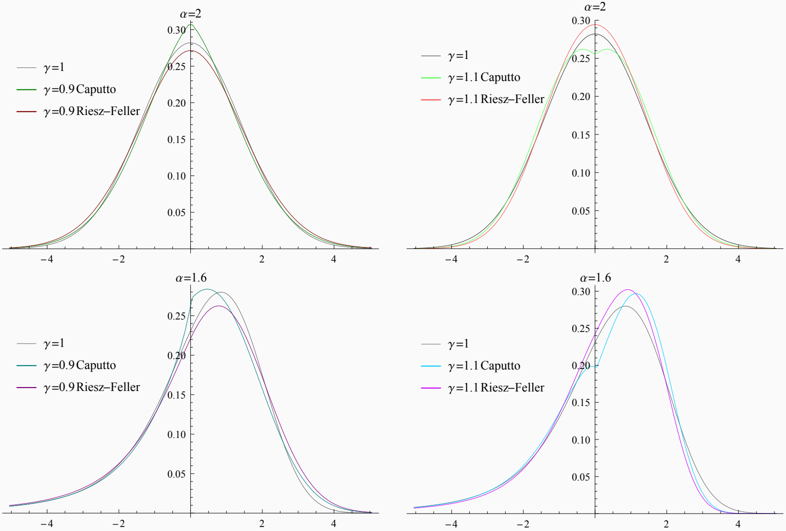
$$g^{DF}(x,t) = \frac{\Gamma(\kappa)}{2\alpha\pi i x} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{s}{\alpha}\right)\Gamma\left(1-\frac{s}{\alpha}\right)\Gamma(1-s)}{\Gamma\left(\kappa - \frac{\gamma}{\alpha}s\right)\Gamma\left(\frac{(\alpha-\theta)s}{2\alpha}\right)\Gamma\left(1-\frac{(\alpha-\theta)s}{2\alpha}\right)} \left[\frac{x}{(-\mu t^\gamma)^{1/\alpha}}\right]^s \mathrm{d}s.$$

- for $\gamma < 1$ it is possible to derive a composition rule, so the solution can be expressed as

$$g(x,t) = \int_0^\infty \mathrm{d}l\, g_\gamma(t,l)\, L_\alpha(l,x)$$

where $L_\alpha(l,x)$ is a stable distribution obtained from spatial fractional diffusion equation and $g_\gamma(t,l)$ is a smearing kernel

Graphs of double-fractional Green functions



Applications of double-fractional diffusion to option pricing

Double-fractional option pricing

- We fit the model with the option prices of S&P 500 in November 2008 ($\sim 10^5$ records)
- We minimize aggregated error $AE = \sum_{t,K} |\mathcal{O}_{model} - \mathcal{O}_{market}|$
- We compare Black-Scholes, Lévy-stable and Double-fractional model
- The parameter γ fluctuates from fast-diffusion mode ($\gamma > 1$) to slow-diffusion model ($\gamma < 1$)

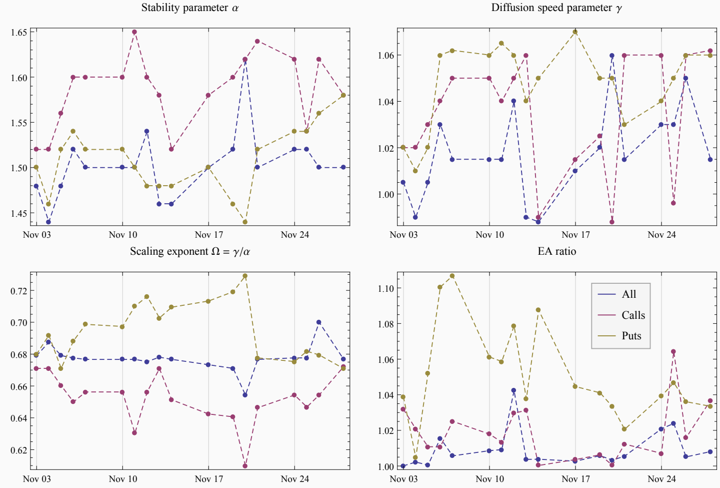
Model calibration for S&P 500 options traded in November 2008

All options			
parameter	Black-Scholes	Lévy stable	Double-fractional
σ	0.1696(0.027)	0.140(0.021)	0.143(0.030)
α	-	1.493(0.028)	1.503(0.037)
γ	-	-	1.017(0.019)
agg. error	8240(638)	6994(545)	6931(553)

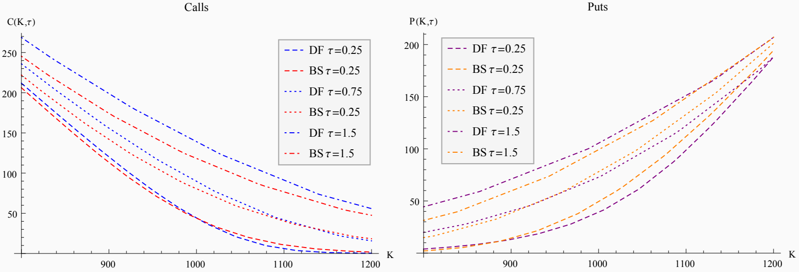
Call options			
parameter	Black-Scholes	Lévy stable	Double-fractional
σ	0.140(0.021)	0.118(0.026)	0.137(0.020)
α	-	1.563(0.041)	1.585(0.038)
γ	-	-	1.034(0.024)
agg. error	3882(807)	3610(812)	3550(828)

Put options			
parameter	Black-Scholes	Lévy stable	Double-fractional
σ	0.193(0.039)	0.163(0.034)	0.163(0.037)
α	-	1.493(0.031)	1.508(0.036)
γ	-	-	1.047(0.017)
agg. error	3741(711)	3114(591)	2968(594)

Estimated parameters day by day and aggregated error



Estimated call and put option prices for various maturity times



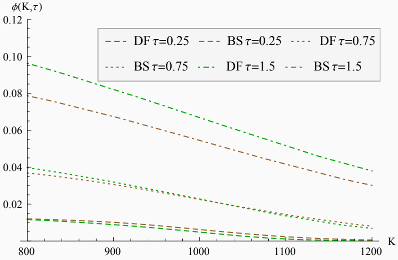
The options are generally not more expensive, the model only redistributes the risk

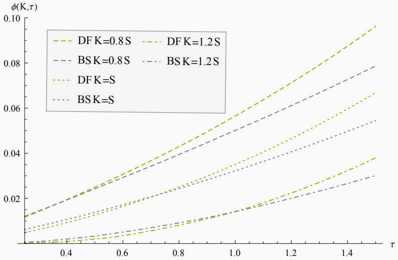
Optimal hedging strategies

- the risk coming from selling an option can be eliminated by appropriate hedging strategy
- we create a portfolio $\Pi(S,t) = C(S,t) - \phi(S,t)S(t)$ containing a short of the option and a fraction $\phi(S,t)$ of the underlying asset $S(t)$ used to hedge the option.
- optimal strategy $\phi^*(S,t)$ can be expressed as

$$\phi^*(S,t) = \frac{1}{\sigma^2}\int_{\mathbb{R}} \mathrm{d}S(S_{t_0} - S_t)\max\{S_t - K, 0\}g(S,T|S_t,t)$$

Optimal hedging strategies





Conclusions

- we have introduced a new model for option pricing based on double-fractional diffusion
- the model outperforms Black-Scholes model and also slightly Lévy-stable model, especially for separate put/call options
- we observe the transition between composite slow diffusion model to complex fast diffusion model
- further analysis and comparison with "regime-switching" models can possibly reveal new important results

- H. Kleinert, J. Korbel. Option Pricing Beyond Black-Scholes Based on Double-Fractional Diffusion. Physica A **449** (2016), 200-214.

Thank you for your attention.