Application of Multifractal Geometry on Financial Markets

Jan Korbel

Fakulta jaderná a fyzikálně inženýrská ČVUT, Praha

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- Our aim is to find generalizations of Brownian motion, that better describe real processes
- Common properties of different processes - (multi)fractal geometry

Concrete example - financial markets

Concrete example

- evolution of Lehman brothers' shares in 2008 - rapid fall before the begin of financial crisis
- in model with random walk extremely improbable
- many times larger standard deviation than normally
- need of other processes - modeling of extreme situations

Brownian motion - Definition

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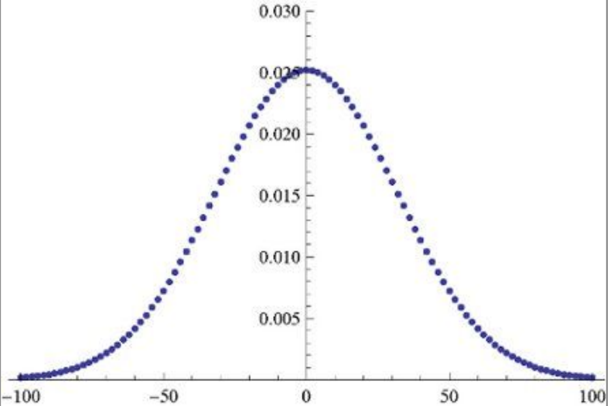
- limit for $n \to \infty$: according to Central limit theorem we get Gaussian distribution

random walk for n=1000



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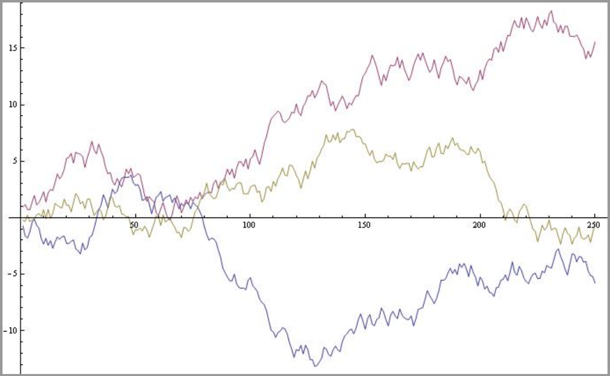
- Wiener process is almost nowhere differentiable!

representative trajectories of Wiener process



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where: $0 < \alpha \leq 2, \quad -1 \leq \beta \leq 1, \quad \gamma \geq 0, \quad c \in \mathbb{R},$

$$\omega(k,\alpha) = \begin{cases} \tan(\pi\alpha/2) & \text{if} \quad \alpha \neq 1 \\ \frac{2}{\pi}\ln|k| & \text{if} \quad \alpha = 1. \end{cases}$$

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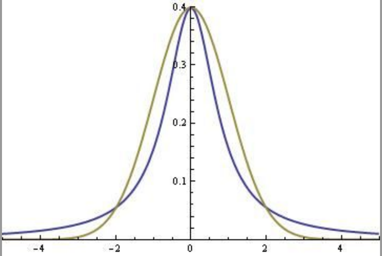
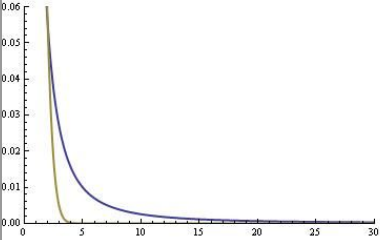
- for $\alpha < 2$ is variance infinite - class of α-stable Lévy distribution

Lévy distributuion





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- formal definition of fractal: fractal dimension is greater than topological

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- We define $f(\alpha)$ - multifractal spectrum:

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- For such α_0 for that $f'(\alpha_0) = 0$ holds, that $f(\alpha_0) = \dim_B(F)$

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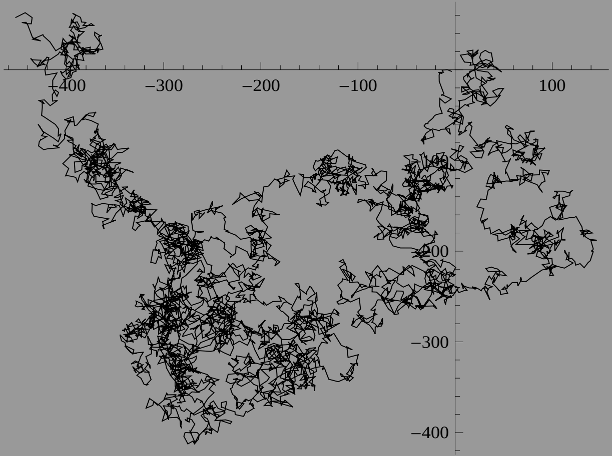
  3. $L_\alpha(t)$ is strictly α-stable process

Wiener process - example



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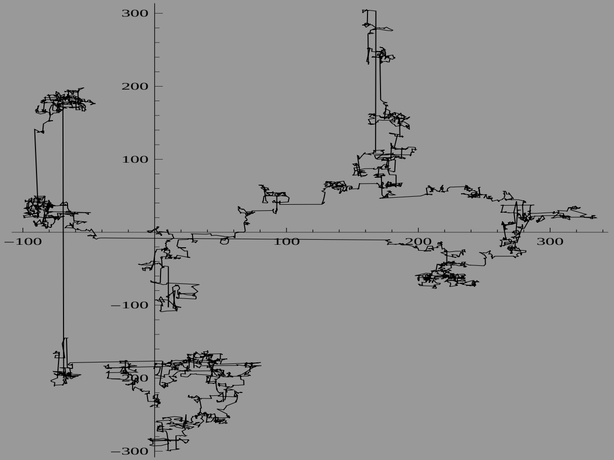

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- graph of random function $t \mapsto L_\alpha(t)$ has fractal dimension $\max\{1, 2 - \frac{1}{\alpha}\}$

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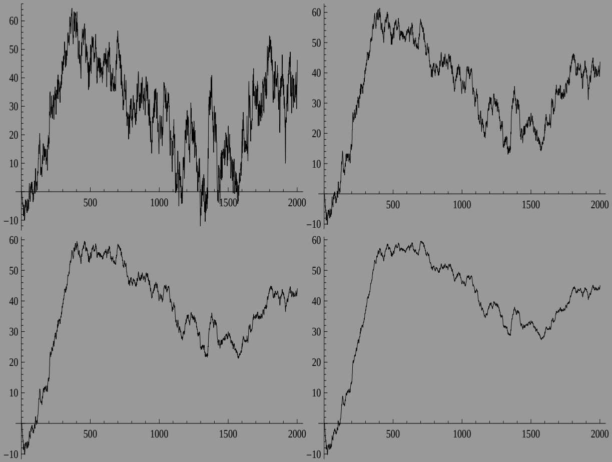
- graph of random function $t \mapsto W_H(t)$ has dimension $2 - H$ -analogue to Lévy process

ukázka fBM



fBM for $H = 0.3; 0.5; 0.6$ a 0.7

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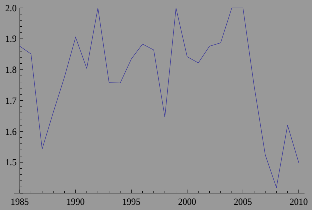
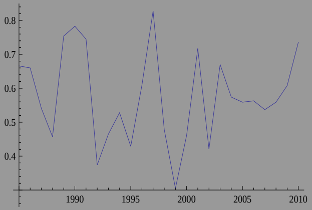
Estimation of α and H by index S&P 500 in years 1985-2010:



α parameter



Hurst exponent

Necessity of processes with time-dependent parameters

Generating of processes with time-dependent Hurst exponent

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  - Many trades are made just just the stock is open or before the stock is closed
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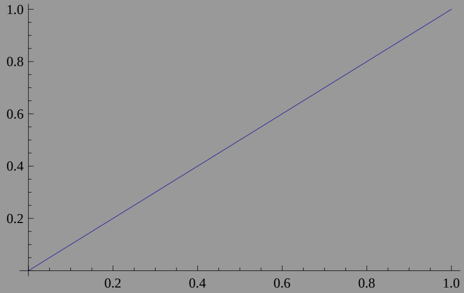
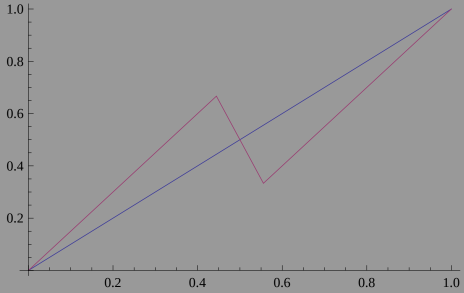
- generation of multifractal stochastic time using brownian patterns

Wiener fractal pattern



1.Initiator

Wiener fractal pattern

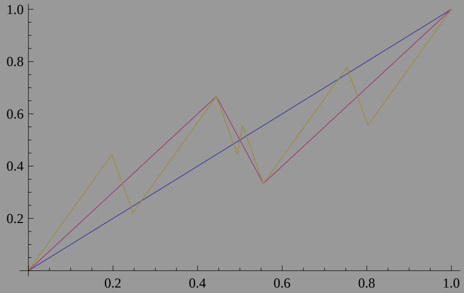


2.Generator

$$\Delta t = \Delta x^2$$

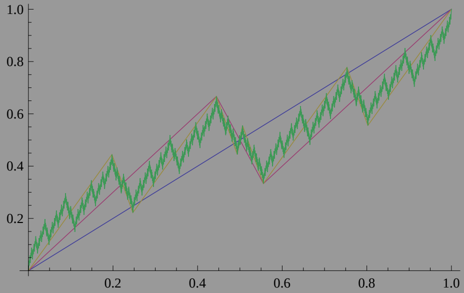
$$\Delta x = \{\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\}, \Delta t = \{\tfrac{4}{9}, \tfrac{1}{9}, \tfrac{4}{9}\}$$

Wiener fractal pattern



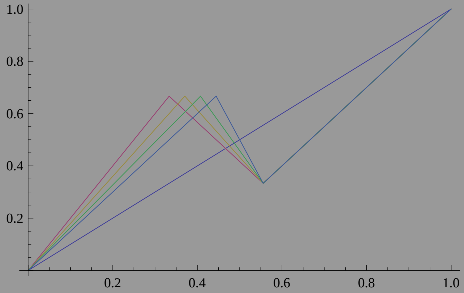
3.Recursive iteration

Wiener fractal pattern



4.Fractal structure

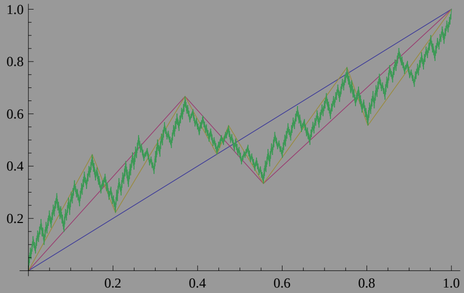
Multifractal pattern



other generators, random choice between generators in every iteration

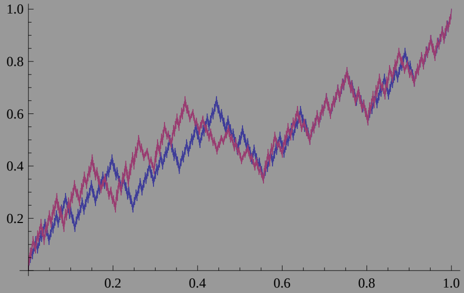
$$\Delta t = \Delta x^{H(t)}$$

Multifractal pattern



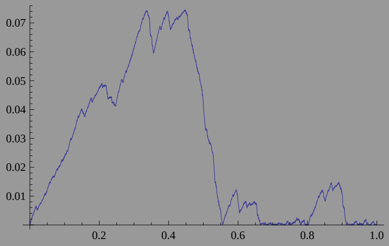
Multifractal pattern

Time as multifractal

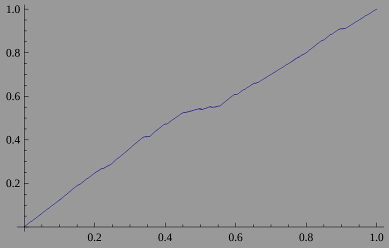


- Wiener pattern generates trading time
- Multifractal pattern generates clock time
- the shift of appropriate points in time generates dependence of both times

Time as multifractal

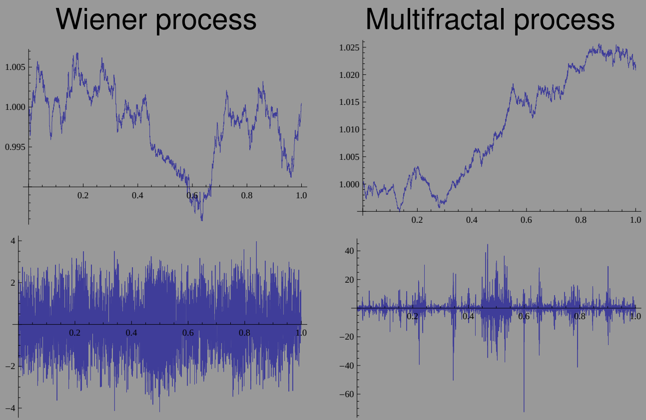


times difference



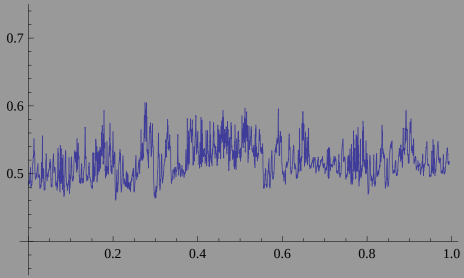
dependence of times

Processes generated by multifractal patterns



We can generate processes from view of trading time and transform them to clock time

Processes generated by fractal patterns



Hurst exponent

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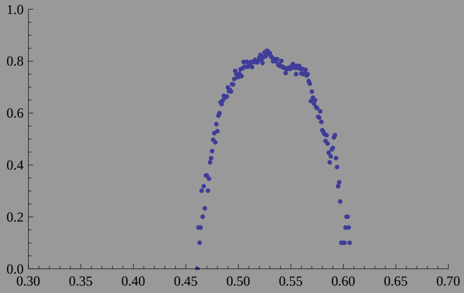

Multifractal spectrum

Conclusion

- Brownian motion an illustrative process, but it is not the best for modeling of complex processes
- better description - Lévy process, fractional Brownian motion...
- common properties of different processes - fractal geometry
- Multifractal processes - easy modeling of difficult processes

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