Multifractal Processes and Their Applications

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Outline

- History & Motivation
- Brownian motion
- Beyond classical diffusion: memory and scaling
- Fractal geometry
- Multifractal processes

Brief history overview

- Diffusion is a transport phenomenon that has been studied since 18th century
- 1827 - discovered Brownian motion on a pollen grain in the water
- 1900 - Louis Bachelier: Théorie de la spéculation - First application of Brownian motion in financial markets
- 60's - Benoit Mandelbrot: Fractals and self-similarity - description of irregular objects
- 90's - econophysics - application of physical models into financial markets

Econophysics

- Why econophysics? - necessity of modeling and analyzing complex processes as financial time series
- Presence of various phenomena - memory, crash, economic cycles, financial crisis...
- Aim: generalization of models based on random walk (discrete version of Brownian motion)
- Multiscaling: general phenomenon that enables to model many different processes

Brownian motion: different approaches

- Brownian motion is a well known transport phenomenon that has many applications in different fields
- It can be described with many formalisms such as Random walk, Langevin equation, theory of stochastic processes, etc.
- It is advantageous to introduce a few of possible definitions and show the relations between them
- First diffusion description - Adolf Fick

Fick's law

$$\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2}$$

Solution: $\phi(x,t) = \frac{1}{\sqrt{2Dt}} \exp\left[-\frac{(x-x_0)^2}{2Dt}\right]$

Basic approach: Random walk

- We begin with a walker that can do a step to the right with probability p and to the left with probability $1-p$
- After n steps we get a binomial distribution

$$p(m,n) = \frac{n!}{\left(\frac{n+m}{2}\right)!\left(\frac{n-m}{2}\right)!} p^{\frac{n+m}{2}} (1-p)^{\frac{n-m}{2}}$$

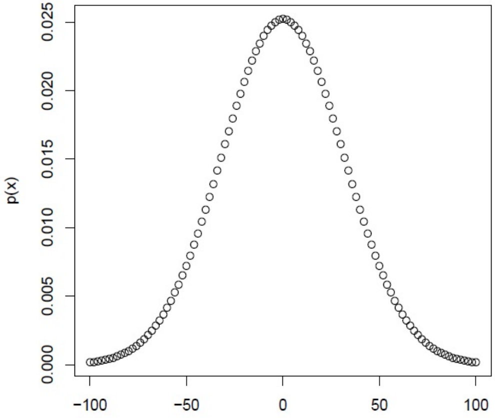
- For long times $n \to \infty$ around the expectation value $\mathrm{E}(m) = n(2p-1)$ we get that

$$p(m,n) \approx \frac{1}{\sqrt{2\pi np(1-p)}} \exp\left[-\frac{(m-\mathrm{E}(m))^2}{8npq}\right]$$

- For long times in the center part of the distribution we get a gaussian distribution, which describes classical diffusion

Central part of distribution of random walk

for $n = 1000$, $p = \frac{1}{2}$



Physical approach: Langevin equation

- We generalize a classical Newton's law for systems in the contact with heat bath (presence of random fluctuations)
- Newton equation

$$m\ddot{x}(t) - F = 0 \tag{1}$$

- We add a random force and because of conservation of physical laws we have to add a friction forces too

Langevin equation

$$m\ddot{x}(t) + \frac{\partial U}{\partial x} + \gamma \dot{x}(t) = \eta(t) \tag{2}$$

- $-\frac{\partial U}{\partial x}$ - external forces
- $-\gamma \dot{x}(t)$ - friction forces
- $\eta(t)$ - fluctuation forces with $\langle \eta(t) \rangle = 0,$

Diffusion equation

- Alternative representation od Langevin equation is through probability distribution of the system $p(x,t)$

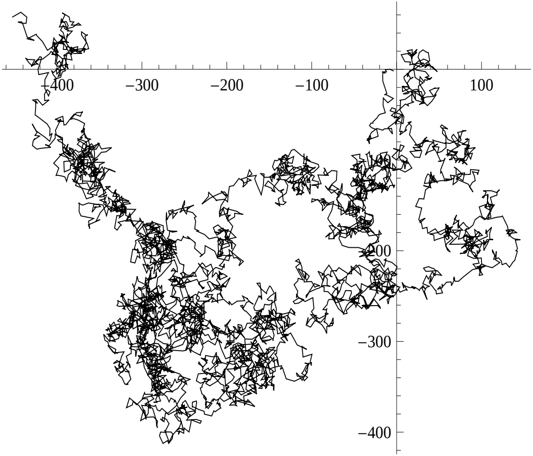
Diffusion equation for free particle

$$\frac{\partial p(x,t)}{\partial t} = \frac{D}{\gamma^2}\frac{\partial^2 p(x,t)}{\partial x^2} \tag{3}$$

- The equation is formally the same as Fick's equation for concentration
- For one localized particle at time 0 we get a Gaussian function

$$p(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{(x-x_0)^2}{4Dt}\right) \tag{4}$$

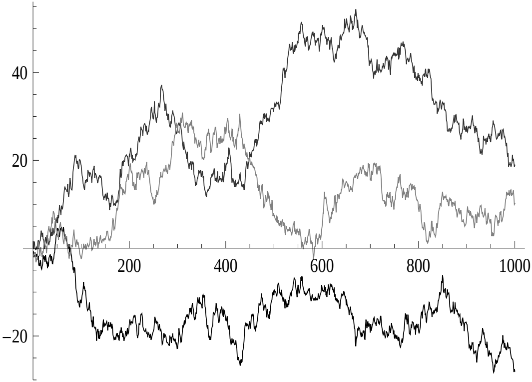
Diffusion in 2D



Mathematical approach: Wiener process

- Another possibility is to use a formalism of stochastic processes
- A stochastic process $W(t)$ (for $t \in [0, \infty]$) is called Wiener process, if
  - $W(0) \stackrel{a.s.}{=} 0$
  - For every t, s are increments $W(t) - W(s)$ dependent only on |t-s| with distribution: $W(t) - W(s) \sim \mathcal{N}(0, |t-s|)$.
  - for different values are increments not correlated.
- The Wiener process also obeys diffusion equation
- All formalisms lead to the main property of diffusion: $|\Delta W(t)| = t^{\frac{1}{2}}$

Sample paths of Wiener process in 1D



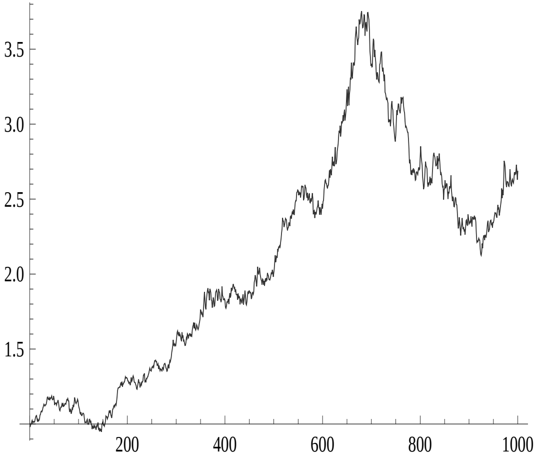
Remark: Diffusion on financial markets

- On financial markets is observed a modified version of diffusion
- We demand that price $S(t)$ is always positive
- From empirical observations $r(t) = \log(S(t)) - \log(S(t-1))$ has a normal distribution - increment of Wiener process
- the price is then defined as

Geometric Brownian motion

$$S(t) = S_0 \exp\left(\sum_t r(t)\right)$$

Sample path of geometric Brownian motion



Beyond classical diffusion

- The theory of Brownian motion is an elegant simple theory, but cannot describe systems with more complex behavior
- Generalizations of Brownian motion: introduction of memory and large fluctuation
- Typical scales for Brownian motion: for space - variance, for time - correlation
- Both have their typical values (scales) - in generalizations these typical scales vanish

Fractional Brownian Motion

- We generalize Brownian motion by introduction of non-trivial correlations
- For Brownian motion is the covariance element

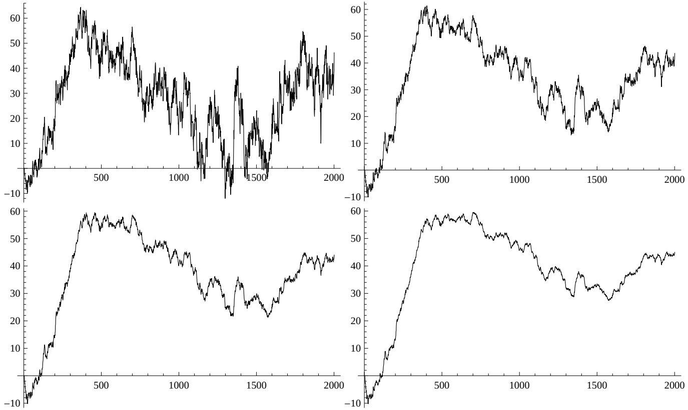
$$\mathrm{E}[W(t)W(s)] = \min\{s,t\} = \frac{1}{2}(s+t-|s-t|) \qquad (5)$$

- We introduce a generalization $W_H(t)$ with the same properties, but covariance

$$E[W_H(t)W_H(s)] = \frac{1}{2}(s^{2H}+t^{2H}-|s-t|^{2H}) \qquad (6)$$

- Standard deviation scales as $|\Delta W_H(t)| \propto t^H$
- For $H = \frac{1}{2}$ we have Brownian motion, for $H < \frac{1}{2}$ sub-diffusion, for $H > \frac{1}{2}$ super-diffusion

Sample functions of fBM for H=0.3, 0.5, 0.6, 0.7.



Lévy distributions

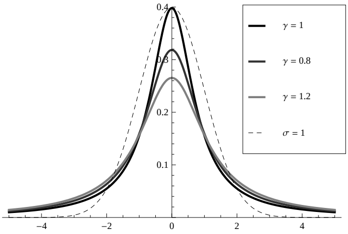
- Gaussian distribution has special property - it is a stable distribution
- Such distributions are limits in long time for stochastic processes driven by independent increments with given distribution
- Lévy distributions - class of stable distributions with polynomial decay

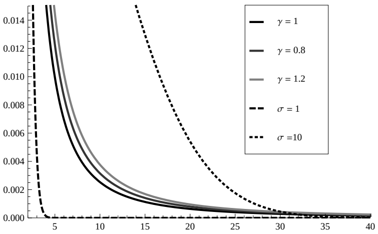
$$L_\alpha(x) \simeq \frac{l_\alpha}{|x|^{1+\alpha}} \quad \text{for} \quad |x| \to \infty \tag{7}$$

for $\alpha \in (0, 2)$

- The variance for these distributions is infinite
- The distribution has sharper peak and fatter tails (= heavy tails)

Difference between Gaussian distribution and Cauchy distribution ($\alpha = 1$)



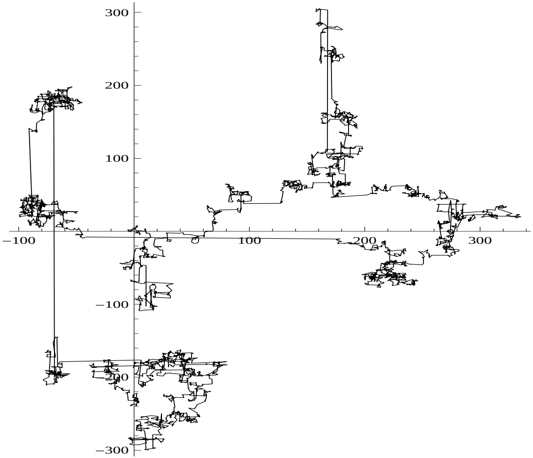


Lévy flights

- Lévy flight $L_\alpha(t)$ is a stochastic process that the same properties as Brownian motion, but its increments have Lévy distribution
- Because of infiniteness of variance, scaling properties are expressed via sum of random variables
  - For Brownian motion: $a^{1/2}W(t) + b^{1/2}W(t) \stackrel{d}{=} (a+b)^{1/2}W(t)$
  - For Lévy flight: $a^{1/\alpha}L_\alpha(t) + b^{1/\alpha}L_\alpha(t) \stackrel{d}{=} (a+b)^{1/\alpha}L_\alpha(t)$
- α-th fractional moment $\mathrm{E}(|X|^\alpha) = \int x^\alpha p(x)\mathrm{d}x$ of increment is equal to

$$\mathrm{E}(|L_\alpha(t_1) - L_\alpha(t_2)|^\alpha) \sim |t_1 - t_2|. \tag{8}$$

Lévy flight in 2D



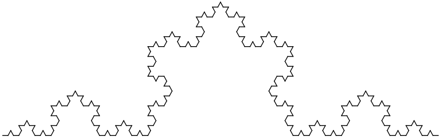
Fractal dimension

- Many objects in nature exhibit inner structure that is present at any scale
- These object fill the space more than regular curves, surfaces, etc - fractals
- The robustness of fractals is measured by a generalization of the dimension
  - we measure by how many squares with side l can be the object covered
  - a curve is covered by $N(l) = Al^{-1}$ squares
  - a surface is covered by $N(l) = Bl^{-2}$ squares etc.
  - for a dimension we have $-D \ln l = \ln N(l) - \ln B$

Fractal dimension

$$\dim F = -\lim_{l \to 0} \frac{\ln N(l)}{\ln l}$$

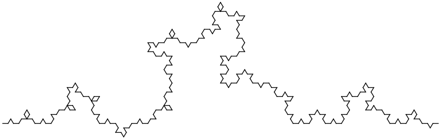
Example: Koch curve



- The Koch curve is generated iteratively
- We begin with a line, we remove a middle part of the line and replace it by two lines with the angle of 60°
- We repeat it to infinity
- Fractal dimension: we cover the object by squares of length $l = 3^{-k}$, the number of squares is $N(l) = 4^k$

$$\dim F = -\lim_{k\to\infty} \frac{\ln 4^k}{\ln 3^{-k}} = -\lim_{k\to\infty} \frac{k \ln 4}{-k \ln 3} = \frac{\ln 4}{\ln 3} > 1$$

Fractal dimension of random fractals



- We can calculate the dimension of random objects from its statistical properties (random Koch curve has the same dimension as Koch curve)
- Fractal dimension of stochastic processes:
    - Random processes in $2D$: dimension of Wiener process - 2, dimension of Lévy process - $\max\{1, \alpha\}$
    - Random functions $t \mapsto X(t)$: Wiener function - $\frac{3}{2}$, Lévy function - $\max\{1, 2 - \frac{1}{\alpha}\}$, fBM - $2 - H$
- Hurst exponent - gives scaling between space and time increments $|\Delta x| \propto \Delta t^H$, relation to fractal dimension

Multifractal spectrum

- Assumption of one scaling index (even in a statistical meaning) seems to be quite restrictive for many processes occurring in nature
- We relax the condition of one scaling exponent, processes can have scaling exponents locally different ($|\Delta x| \propto \Delta t^{H(t)}$)
- We would like to capture relative strengths of fractal dimensions, we divide the fractal into subsets F_α that scale with an exponent α

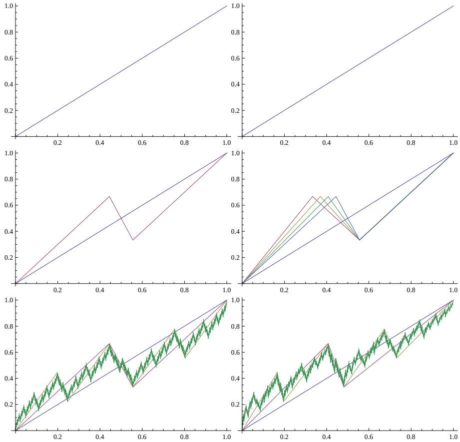
Multifractal spectrum

$$f(\alpha) = -\lim_{\delta \to 0} \frac{\ln N[F_\alpha](\delta)}{\ln \delta}$$

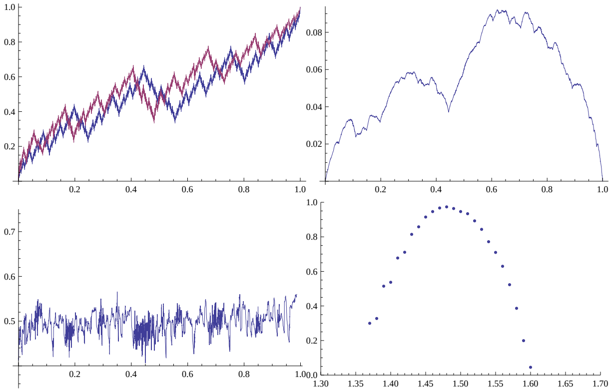
Multifractal deformations

- The basic model of price evolution is based on geometric Brownian motion
$$\ln S(t) = \mu t + \sigma W(t)$$
- We generalize this model by introduction of time deformation, where we consider an existence of two times: trading time on the market and real physical clock time. The transformation between them is given by a time deformation $\theta(t)$, so
$$\ln S(t) = \mu t + \sigma W[\theta(t)]$$
- The time deformation is constructed as a multifractal process, which enables us to estimate the multifractal spectrum and from it scaling properties of the process

Generation of a multifractal patterns



Multifractal deformation and its spectrum



Applications to financial markets

- MMAR - Multifractal model of asset returns: $\ln S(t) = \sigma W[\theta(t)]$
- MSM - Markov switching multifractal: $\ln S(t) = W(t, \sigma(t))$
  - $\sigma^2(t) = \sigma^2 \left(\prod_{j=1}^{k} M_k(t)\right)$
  - $M_k(t)$ - state variables driven by a Markov process, they determine the final volatility
  - special choice of the process: For every time t_n the variable $M_k(t_n)$ is either updated from given distribution M with intensity γ_k or remains the same value as in t_{n-1}
  - γ_k is chosen approximately geometrically, M is binomial, M_k - representation of economic cycles, the process depends on 4 variables
  - in limit of continuous time and countably many state variables we become a time deformation

Benefits of MSM

- With a few parameters, we can analyze and predict complex time series on financial markets
- Compared to other models commonly used on financial markets (ARMA, GARCH,...) has MSM better results
- State variables have nice interpretation
- MSM is related to multifractal deformations and multiscaling processes

Conclusions

- Brownian motion is a phenomenon that has many applications
- It provides an elegant description of various systems
- In case of complex processes, with memory or large fluctuations, the description fails
- More appropriate models: fBM, Lévy flight
- Multifractal processes: good description of real models on financial markets

Thank you for attention.