Modeling Financial Time Series:

Multifractal Cascades and Rényi Entropy

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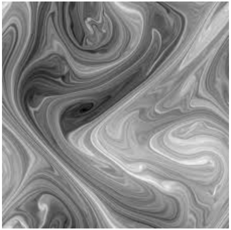

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MOTIVATION

Figure : Multifractals in nature

MOTIVATION

Figure : Multifractals in finance

MOTIVATION

- **scaling properties** are one of the most important quantifiers of complexity in many systems, e.g. financial time series
- presence of scaling exponents points to the inner structure of the system, described through fractal, or **multifractal analysis**
- multiscaling systems have a tight relation with Generalized dimensions and consequently with **Rényi entropy**
- **Multiplicative cascades** provide a successful approach of creating such systems
- aim is to create time series and compare it with real financial series trough multifractal analysis

Multifractal spectrum

- discrete time series $\{x_j\}_{j=1}^{N}$ in $\mathbb{R}^D$, j denotes discrete time moments with specific time lag s
- empirical probability of each region $K_i \subset \mathbb{R}^D$ is estimated as $p_i = \frac{\#\{x_j \in K_i\}}{N}$
- probabilities scale with the typical length as $p_i(s) \sim s^{\alpha}$
- regions with different scalings are identified and PDF of scaling exponents is assumed in form

$$p(\alpha)\mathrm{d}\alpha = \rho(\alpha)s^{-f(\alpha)}\mathrm{d}\alpha$$

- $f(\alpha)$ - **Multifractal spectrum** = fractal dimension of the subset with scaling exponent α

MULTIFRACTAL SPECTRUM OF S&P 500

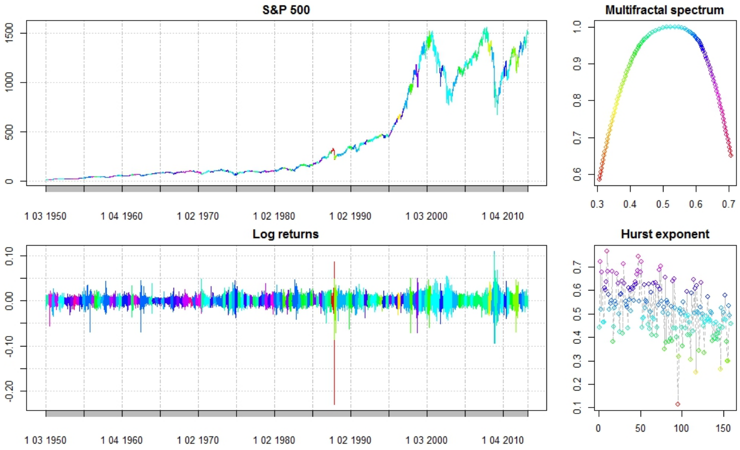


Figure : Multifractal spectrum of S&P 500

SCALING FUNCTION AND RÉNYI ENTROPY

- alternative way of describing multifractality is via **"Partition function"**: $Z_q(s) = \sum_i p_i^q(s) \sim s^{\tau(q)}$
- relation to $f(\alpha)$ is provided through Legendre transform $f(\alpha(q)) = q\alpha(q) - \tau(q)$
- **Scaling function** $\tau(q)$ has a tight relation to **Generalized dimension** $D_q = \frac{\tau(q)}{q-1}$ and **Rényi entropy**

$$S_q(s) = \frac{1}{q-1} \ln \sum_i p_i^q(s) = \frac{\ln Z_q(s)}{q-1}$$

- multifractality can be measured via estimation of Rényi entropy

Multifractal diffusion entropy analysis (MF-DEA)

- methods of multifractal spectrum estimation: **Detrended fluctuation analysis**, **Wavelets**, **Generalized Hurst exponent**, etc.
- in our case **Diffusion entropy analysis** method is used; it is based on self-similarity property of PDF - in monofractal case

$$p(x,t)\mathrm{d}x = \frac{1}{t^{\delta}} F\left(\frac{x}{t^{\delta}}\right) \mathrm{d}x,$$

  where δ is for monofractal Fractional Brownian motion equal to H.
- from relation for Shannon entropy (or more precisely Shannon divergence w.r.t. uniform distribution) we get

$$S(t) = -\int \mathrm{d}x\, p(x,t) \ln[p(x,t)] = A + \delta \ln t$$

MULTIFRACTAL DIFFUSION ENTROPY ANALYSIS

- in multifractal case, whole class of Rényi entropies is calculated, which gives a class of scaling exponents

$$S_q(t) = B_q + h(q) \ln t$$

- **Scaling function** $h(q)$ has a relation to multifractal spectrum and enables to classify multifractality
- PDF is estimated through **Fluctuation collection algorithm**
- all fluctuations over lag s are collected: $\tilde{x}_s(t) = \sum_{i=1}^{s} x_{i+t}$, and PDF $\mathcal{P}_s(t)$ is estimated

Multiplicative processes

- concept of **multiplicative processes** is based on the idea that typical quantity of the system is self-similarly compound of itself, measured at smaller scale
- cascade of scales is defined as $r_n = r_0 \prod_{i=1}^{n} l_j$;
  l_j - scale multipliers
- if typical quantity E is given by its density ϵ, aggregated quantity in the region Ω is measured as

$$E(\Omega) = \int_{\Omega} \epsilon(x) \mathrm{d}x$$

- multiplicative cascade defines the aggregated quantity at the scale r_n as $E(r_n) = E_0 \prod_{j=1}^{n} M_j$

Multifractal cascades

- there are many models of multiplicative cascades, both deterministic and stochastic
- our aim is to use such models that would sufficiently simulate the multiscaling nature of investigated systems
- this is the case of **Multifractal cascades**. A few examples:
  - **Binomial cascade (**BC**)** - deterministic model with $l_j = 1/2$ and $M_{\bullet,1} = m_1$, $M_{\bullet,2} = m_2$,
    we demand $m_1 + m_2 = 1$ - conservative cascade
  - **Microcanonical cascade (**μCC**)** - similar to Binomial cascade, but m_1 and m_2 are randomly shuffled (i.e. with $p = 0.5$ is $M_{\bullet,1} = m_1$)
    - $M_{\bullet,i}$ are not independent.
  - **Canonical cascade (**CC**)** - we demand the conservation on average $\sum_i \langle M_{\bullet,i} \rangle = 1$, hence in case of i.i.d. variables $\langle M_{j,i} \rangle = \frac{1}{l_j}$

Multifractal cascade

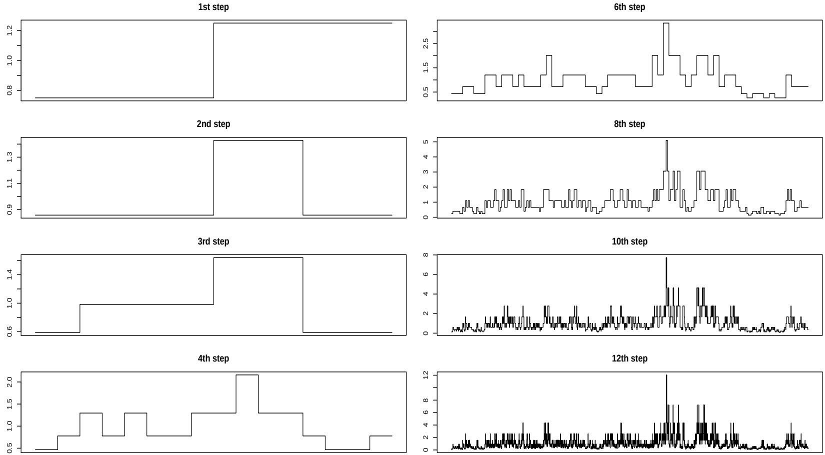
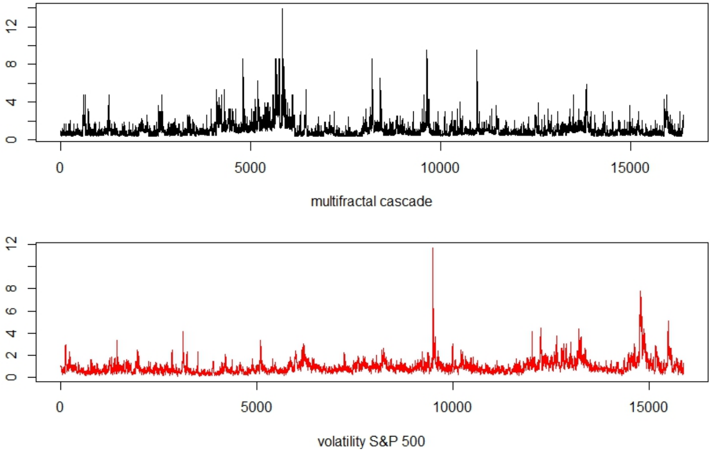


Figure : Generation of binomial cascade

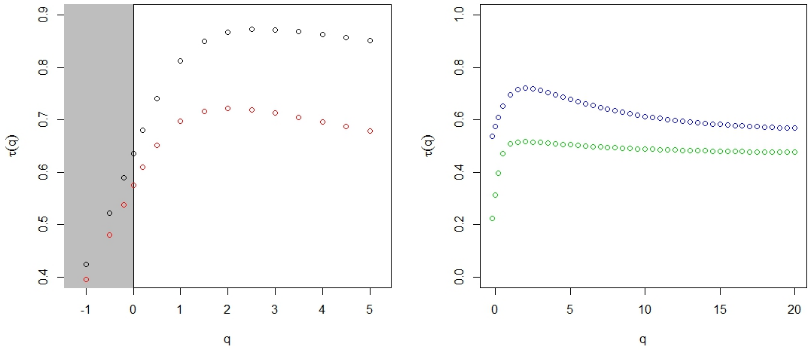
Numerical simulations of volatility as a multifractal cascade

- method of multifractal cascades is used in order to simulate **financial time series**
- **volatility** (σ_t) is a good candidate for a cascade modeling
- volatility is modeled as a canonical cascade with dyadic structure with binomial distribution, normalized such that $\langle \sigma_t \rangle = 1$
- daily returns can be modeled from the estimated volatility as $r_t = \sigma_0 \sigma_t \eta_t$, where η_t is a white noise (or more generally colored noise)
- when comparing simulated series to real data, statistical quantities as mean or variance bring us only a fraction of information about the series
- both series are therefore compared by MF-DEA

Simulated vs real volatility series



Multifractal spectrum of volatility and comparison to return spectrum



Conclusions

- many complex systems physics, economy, etc. can be sufficiently well described by **multifractals**
- self-similarity of these systems can be also modeled by **multiplicative cascades**
- interconnection of these two concepts brings a powerful tool in modeling such systems
- in case of financial markets, the **volatility** can be modeled as multifractal cascade
- there are many open questions in this field that need to be answered (spectrum for volatility, relation to other methods, parameter estimation, . . .)

Back to mountains!

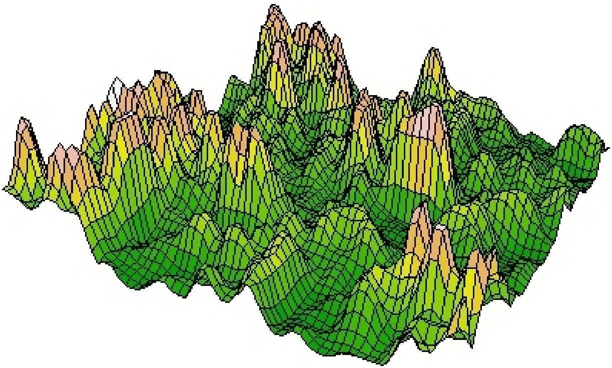


Figure : 2D multiplicative smoothed cascade as a terrain model

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Thank you for your attention.