# On q-non-extensive statistics with non-Tsallisian entropy

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# Introduction

- in statistical physics is the important concept of entropy
- for most systems, the Shannon-(Gibbs-Boltzmann) entropy is the most popular entropy
- in complex systems, other entropy forms describe the system better (Rényi, Tsallis)
- we combine the axiomatics of non–additive Tsallis entropy and self–similar Rényi entropy and obtain a new class of hybrid entropies

# Axiomatic definition of entropy

- information measure (entropy) represents the amount of uncertainty in the system
- exact form of entropy depends on statistical properties of the system
- the form also influences the thermodynamical properties of the system ($\#$ *of particles* $\to \infty$)
- the most common is the Shannon entropy (SE)
- SE can be defined through four axioms

# Axiomatic definition of entropy

1. *continuity*: $\mathcal{H}(\mathcal{P})$ is continuous in every argument,
2. *maximality*: $\mathcal{H}(\mathcal{P})$ takes maximal value for uniform distribution,
3. *expansibility*: $\mathcal{H}(p_1, \ldots, p_n, 0) = \mathcal{H}(p_1, \ldots, p_n)$.
4. *additivity*: $\mathcal{H}(A \cup B) = \mathcal{H}(A) + \mathcal{H}(B|A)$, where $\mathcal{H}(B|A) = \sum_k p_k \mathcal{H}(B|A = a_k)$.

From these four axioms can be determined the form of Shannon entropy

$$\mathcal{H}(\mathcal{P}) = -\sum_k p_k \ln p_k \,.$$

# Rényi entropy

## Entropy of multifractal systems

- Rényi entropy has tight relation to theory of multifractals and scaling exponents of the system
- Similarly to Shannon entropy, it is possible to find an *operational* definition of RE
- The defining axioms remain the same but the fourth is slightly changed:
  4. *Rényi additivity*: $\mathcal{I}_q(A \cup B) = \mathcal{I}_q(A) + \mathcal{I}_q(B|A)$, where $\mathcal{I}_q(B|A) = g^{-1}\left[\sum_k \rho_k(q) g[\mathcal{I}_q(B|A = a_k)]\right]$, $\rho_k(q) = p_k^q / \sum_j p_j^q$ and $g$ is positive and invertible function on $[0, \infty)$.

This leads to the definition of RE in the form:

$$\mathcal{I}_q(\mathcal{P}) = \frac{1}{1-q} \ln \left( \sum_k p_k^q \right)$$

# Tsallis entropy

## Entropy of non-extensive systems

- Tsallis entropy generalizes the concept of entropy to non-extensive systems
- these systems can be extensive in other entropy, for example in Tsallis entropy
- The additivity axiom is now changed as follows:
  4. *Tsallis additivity*: $\mathcal{S}_q(A \cup B) = \mathcal{S}_q(A) + \mathcal{S}_q(B|A) + (1-q)\mathcal{S}_q(A)\mathcal{S}_q(B|A)$, where $\mathcal{S}_q(B|A) = \sum_k \rho_k(q)\mathcal{S}_q(B|A = a_k)$ and $\rho_k(q) = p_k^q / \sum_j p_j^q$

We can deduce that Tsallis entropy has form

$$\mathcal{S}_q(\mathcal{P}) = \frac{1}{1-q}\left(\sum_k p_k^q - 1\right)$$

# Hybrid entropy

## Overlap between self–similarity and non–extensivity

- we combine axiomatic of Rényi entropy and Tsallis entropy
- the resulting entropy describes self–similar and non–extensive systems
- the additivity axiom is now changed as

  4. *J.-A. additivity*: $\mathcal{D}_q(A \cup B) = \mathcal{D}_q(A) + \mathcal{D}_q(B|A) + (1-q)D_q(A)\mathcal{D}_q(B|A)$, where $\mathcal{D}_q(B|A) = f^{-1}\left[\sum_k \rho_k(q) f[\mathcal{D}_q(B|A = a_k)]\right]$, $\rho_k(q) = p_k^q / \sum_j p_j^q$ is escort distribution and $f$ is a positive and invertible function on $[0, \infty)$.

These axioms define the hybrid entropy in form:

$$\mathcal{D}_q(\mathcal{P}) = \frac{1}{1-q}\left(e^{-(1-q)\sum_k \rho_k(q)\ln p_k} - 1\right) = \frac{1}{1-q}\left(e^{-(1-q)\langle \ln \mathcal{P}\rangle_q} - 1\right)$$

# MaxEnt distribution

- maximal entropy principle determines probability distribution that contains minimal amount of information under some given constraints
- we demand normalization condition $\sum_k p_k = 1$ and given expectation value of energy $\langle E \rangle_r = \sum_k \rho_k(r) E_k$.
- two most common choices are linear averaging ($r = 1$, $\langle E \rangle = \sum_k p_k E_k$) and $q$-averaging ($r = q$)
- for both cases we find the MaxEnt distributions and discuss their properties
- the maximization under constraints is done via the Lagrange multiplier method:

$$\mathcal{L}_{q,r}(\mathcal{P}) = \mathcal{D}_q(\mathcal{P}) - \Omega \left( \sum_k \rho_k(r) E_k - \langle E \rangle_r \right) - \Phi \left( \sum_k p_k - 1 \right)$$

# MaxEnt for $r = q$

- condition $\frac{\partial \mathcal{L}_{q,q}(\mathcal{P})}{\partial p_i} = 0$ leads to equation

$$\kappa p_i^{1-q} = q \ln p_i + \mathcal{E}_i$$

where $\kappa = \sum_j p_j^q$ and $\mathcal{E}_i = 1 + \frac{q \ln(-\Phi)}{1-q} + \frac{q\Omega}{-\Phi}(E_i - \langle E \rangle_q)$

- previous equation can be solved in terms of Lambert W-function defined as $W(x)e^{W(x)} = x$
- we obtain $p_i = \left[\frac{q}{\kappa(q-1)} W\left(\frac{\kappa(q-1)}{q} e^{(q-1)\mathcal{E}_i/q}\right)\right]^{1/(1-q)}$
- Lambert W-function is defined only on interval $[-1/e, \infty)$, so there are energy regions with zero probability - energy gaps

# MaxEnt for $r = 1$

- Similarly to previous case, from condition $\frac{\partial \mathcal{L}_{q,1}(\mathcal{P})}{\partial p_i} = 0$ we obtain

$$\kappa p_i^{1-q} = \frac{\Phi}{\Phi + \Omega(E_i - \langle E \rangle)} \left[ q \ln p_i - \frac{q \ln(-\Phi)}{q-1} + 1 \right]$$

- The solution of previous equation is

$$p_i = \left[ \frac{q\Phi}{(q-1)\kappa(\Phi + \Omega \Delta E_i)} W\left( -\frac{\kappa(q-1)}{\Phi q} \exp\left(\frac{q-1}{q}\right) \left(1 + \frac{\Omega}{\Phi}\Delta E_i\right)\right)\right]^{1/(1-q)}$$

- Also in this case we observe energy gaps and forbidden energies

# Maximality axiom and and (Schur-)concavity

- So far, we have not discussed the validity of maximality axiom
- We focus on the term $\langle \ln \mathcal{P} \rangle_q = \sum_k \rho_k(q) \ln p_k$
- For $q < 1/2$, $\langle \ln \mathcal{P} \rangle_q$ does not become maximal value for equal probabilities
- Thus, the hybrid entropy is properly defined only for $q \geq 1/2$
- One can also show that hybrid entropy is for $q \geq 1$ concave and for $q \geq 1/2$ Schur-concave (weaker version of concavity)

# Conclusions

- we have combined axiomatic of Rényi entropy and Tsallis entropy and obtained a new class of hybrid entropies
- it obeys maximality axiom for $q \geq 1/2$
- MaxEnt distribution is expressible in terms of Lambert W-function $\Rightarrow$ energy gaps
- Possible applications are in multifractal systems with non-complete energy spectrum
- Reference: Jizba, Korbel. On q-non-extensive statistics with non-Tsallisian entropy. (arXiv:1501.07386)

Thank you for your attention.