Space-time fractional diffusion and its applications in finance

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Outline

Introduction

Anomalous diffusion in finance

- Financial markets are complex systems with many non-trivial phenomena
- Prices S_t are described via log-returns $R_t = \ln S_t - \ln S_0$
- Returns can be modeled via diffusion processes $R_t \sim \mathcal{N}(0, \sigma^2 t)$
- Prices are described as *geometric Brownian motion* $S_t = e^{R_t} = e^{\sum_{i=1}^{t} r_i}$
- Brownian motion cannot describe more complex phenomena (jumps, memory), so we introduce several classes of generalized anomalous diffusion which describe these phenomena more accurately

Derivative pricing

- In finance are traded many derivatives - assets, whose price depends on underlying asset - futures, forwards, CDF, options...
- Their price should be derived from the possible scenarios of underlying assets
- First option pricing model (Black and Scholes) was based on ordinary Brownian motion - 1973
- 1997 - Nobel prize in economics (Scholes, Merton)
- In financial crises or in complex markets, the model cannot catch realistic market dynamics - large drops, sudden shocks, memory effects
- We investigate these markets in the framework of several classes of anomalous diffusion

Mathematical description of Space-time diffusion model

Stable distributions

- $L_{\alpha,\beta;\bar{x},\sigma}$ - class of distributions form-invariant w.r.t. convolution
- limiting distributions for sums of i. i. d. variables with no constraint on variance ($\sigma^2 \leqq +\infty$)
- Stable Hamiltonian (logarithm of a characteristic function)

$$H_{\alpha,\beta;\bar{x},\sigma}(k) = \ln \int_{\mathbb{R}} e^{ikx} L_{\alpha,\beta;\bar{x},\sigma}(x)\,\mathrm{d}x$$
$$= i\bar{x}k - \sigma^\alpha |k|^\alpha \left(1 - i\beta \mathrm{sign}(k)\omega(k,\alpha)\right)$$

  with $\omega(k,\alpha) = \tan(\alpha\pi/2)$ for $\alpha \neq 1$ and $\omega(k,1) = 2/\pi \ln|k|$.
- Parameters: $\alpha \in (0,2]$ - shape, $\beta = [-1,1]$ - asymmetry

  $\sigma > 0$ - scale, $\bar{x} \in \mathbb{R}$ - location
- standard distribution $L_{\alpha,\beta}(x) = \frac{1}{\sigma} L_{\alpha,\beta;\bar{x},\sigma}\left(\frac{x-\bar{x}}{\sigma}\right)$
- for $\alpha < 2$ is decays polynomially as $1/|x|^{\alpha+1}$,
- extreme cases $\beta = \pm 1$:
  - $\alpha \in (1,2)$ - exponential decay for left, resp. right tail.
  - $\alpha \in (0,1]$ - bounded support from left, resp. from right.

Stable distributions

- stable distributions with $\beta = -1$ are preferred in financial applications for description of log-Lévy process $Y = e^{L_{\alpha,-1}}$, because all moments exist and are finite = Laplace transform exists

$$\int L_{\alpha,-1}(x) e^{x} \, \mathrm{d}x = e^{-\sigma^{\alpha} \sec(\frac{\pi\alpha}{2})}$$

- alternative representation of stable Hamiltonian

$$\mathcal{H}_{\alpha,\theta,\bar{x},c}(k) = i\bar{x}k - c|k|^{\alpha} e^{i \operatorname{sign}(k)\theta \frac{\pi}{2}}$$

- c, θ - functions of α, β and σ
- $\theta \leq \min\{\alpha, 2-\alpha\}$ - *Feller-Takayasu diamond*

Fractional calculus

- The aim is to generalize diffusion equation $\frac{\partial}{\partial t} g(x,t) = \frac{\partial^2}{\partial x^2} g(x,t)$ to obtain more complex diffusion processes
- fractional derivatives - generalization for non-natural orders: not unique
- Caputo derivative
  $${}_{x_0}^{*}\mathcal{D}_x^{\nu} f(x) := \frac{1}{\Gamma(\lceil \nu \rceil - \nu)} \int_{x_0}^{x} \frac{f^{\lceil \nu \rceil}(y)}{(x-y)^{\nu+1-\lceil \nu \rceil}} \mathrm{d}y$$
  - preserves derivative of polynomials: ${}^{*}\mathcal{D}_x^{\nu} x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)} x^{\mu-\nu}$.
  - because of lower integral bound x_0, it is convenient for time derivative
  - in the following we consider $x_0 = 0$

Fractional calculus

- Riesz-Feller derivative
  $\mathfrak{D}_x^\nu f(x) = \int_{\mathbb{R}} \frac{e^{ikx}}{2\pi} (-ik)^\nu \mathcal{F}[f](k)\,\mathrm{d}k$
  - preserves derivative of exponentials: $\mathfrak{D}_x^\nu \exp(\lambda x) = \lambda^\nu \exp(\lambda x)$
  - in Fourier image, the R.-F. derivative corresponds to stable Hamiltonian with $\beta = -1$
  - Riesz-Feller derivative can be defined via Caputo derivative for $x_0 \to -\infty$
  - R.-F. pseudo-derivative operator ${}^\theta\mathfrak{D}^\nu$:
    $\mathcal{F}[{}^\theta\mathfrak{D}^\nu f(x)](k) = \mathcal{H}_{\nu,\theta}(k) f(k)$
  - Due to the connection with stable distribution, it is convenient for space derivative

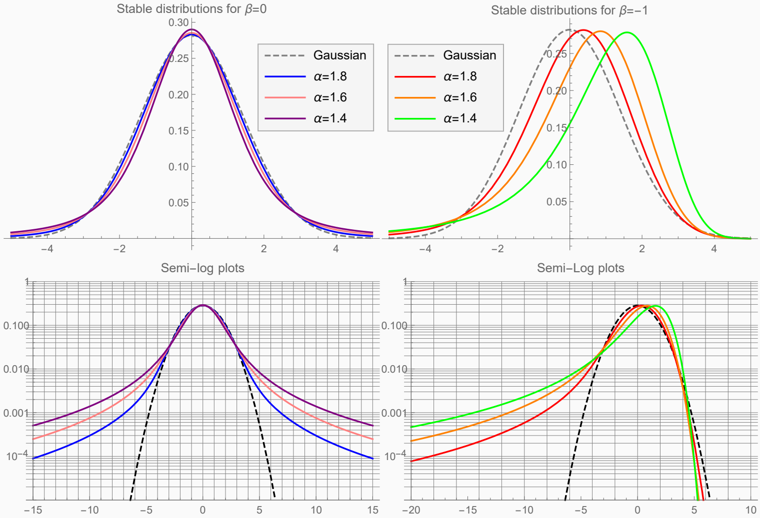
Space-fractional diffusion and Lévy flight

- Space fractional diffusion equation in 1D

$$\left(\partial_t - {}^{\theta}\mathfrak{D}_x^{\alpha}\right) g_\alpha^\theta(x,t) = 0$$

- Solution: Lévy-flight $g_\alpha^\theta(x,t) = 1/t^{1/\alpha} L_{\alpha,\theta}(x/t^{1/\alpha})$
- continuous sample paths for $\alpha > 1$
- for $\alpha < 2$ are some moments infinite/indeterminable ($E[|x|^\delta] < +\infty$ for $\delta < \alpha$)
- scaling exponent $1/\alpha$ - self-similarity

Graphs of stable distribution



Space-time double-fractional diffusion[1]

- Generalization of space-fractional diffusion for fractional time derivative - non-Markovian
- Space-time fractional diffusion equation in 1D

$$\left({}^{*}\mathcal{D}_t^{\gamma} - \mathfrak{D}_x^{\alpha}\right) g_{\alpha,\gamma}^{\theta}(x,t) = 0$$

- Parameter space: $\alpha \in [1,2]$ - continuity of sample paths
  $\gamma \in (0,\alpha]$ - probabilistic interpretation $(g(x,t) \geq 0)$
- for $\gamma \leq 1$ we have one initial condition $g(x,0) = \delta(x)$
  for $\gamma \in (1,2]$ we have another condition $\frac{\partial g}{\partial t}(x,t)|_{t=0} \equiv 0$.

[1] F. Mainardi, Yu. Luchko, G. Pagnini, Frac. Calc. Appl. Anal. 4(2), 153 (2001)

Solution of double-fractional diffusion equation

- Solution is obtained through Fourier-Laplace image ($x \overset{\mathcal{F}}{\to} p, t \overset{\mathcal{L}}{\to} s$)

$$(s^\gamma - H_{\alpha,-1}(p))\hat{\hat{g}}(p,s) = s^{\gamma-1}$$

- inverse Laplace transform:

$$\hat{g}(p,t) = E_\gamma(\mathcal{H}_{\alpha,\theta}(k)t^\gamma)$$

- Mittag-Leffler function: $E_\gamma(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\gamma n+1)}$
- Mellin representation: $E_\gamma(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-\gamma s)}(-z)^{-s}\,\mathrm{d}s$
- Mellin-Barnes integral rep. of $g_{\alpha,\gamma}^\theta(x,t)$:

$$g_{\alpha,\gamma}^\theta(x,t) = \frac{1}{2\alpha\pi i x}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{s}{\alpha}\right)\Gamma\left(1-\frac{s}{\alpha}\right)\Gamma(1-s)}{\Gamma\left(1-\frac{\gamma}{\alpha}s\right)\Gamma\left(\frac{(\alpha-\theta)s}{2\alpha}\right)\Gamma\left(1-\frac{(\alpha-\theta)s}{2\alpha}\right)}\left[\frac{x}{(t^\gamma)^{1/\alpha}}\right]^s \mathrm{d}s.$$

Smearing kernel representation[2]

- for $\gamma < 1$ it is possible to derive a composition rule, so the solution can be expressed as

$$g(x,t) = \int_0^\infty \mathrm{d}l \, g_\gamma(t,l) \, g_\alpha(l,x)$$

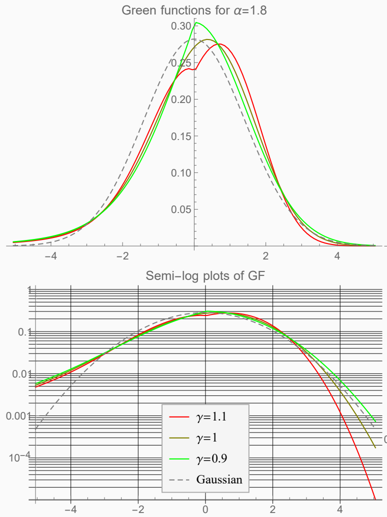
where the kernels are solutions of fractional equations

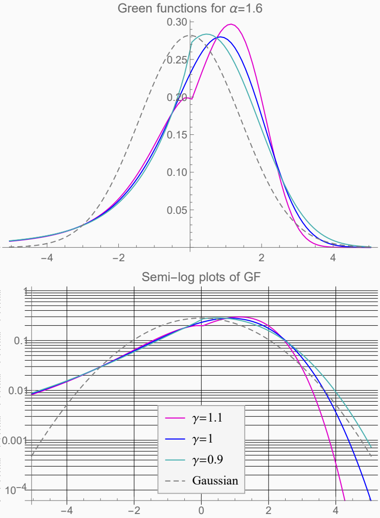
$$\frac{\partial g_\gamma(t,l)}{\partial l} = {}_0^*\mathcal{D}_t^\gamma \, g_\gamma(t,l)$$
$$\frac{\partial g_\alpha^\theta(l,x)}{\partial l} = {}^\theta\mathcal{D}_x^\alpha \, g_\alpha^\theta(l,x)$$

- $g_\alpha^\theta(l,x) = L_{\alpha,\theta}(l,x)$ - stable distribution
- $g_\gamma(t,l) = \left(\frac{t}{l\gamma}\right) \frac{1}{l^{1/\gamma}} L_{\gamma,1}\left(\frac{t}{l^{1/\gamma}}\right)$ - smearing kernel
- Path integral representation: it is possible to rewrite the smearing-kernel representation into a double path integral

[2] H. Kleinert, V. Zatloukal, Phys. Rev. E 88, 052106 (2013)

Graphs of double-fractional Green functions





Space-time fractional diffusion of varying order

with Yu. Luchko

- One of important aspects of financial markets is switching between different regimes - conjuncture vs crisis
- Long-term scaling properties remain stable and characteristic for each stock
- This requires time-dependent description by fractional diffusion of varying order: we have intervals $T_i = (t_i, t_{i+1})$
- dynamics described by a space-time fractional diffusion in each interval

$$\left({}_{t_i}^{*}\mathcal{D}_t^{\gamma_i} - \mathfrak{D}_x^{\Omega\gamma_i}\right) g_i(x,t) = 0$$

with initial condition $g_i(x,t_i) = g_{i-1}(x,t_i)$, $g_0(x,0) = f(x)$.
For $\gamma_i > 1$ we add another condition $\frac{\partial g_i(x,t)}{\partial t}|_{t=t_i} = 0$.

- the dynamics is given by convolution of g_i

$$g(x,t) = f(x) * g_0(x, t_1 - t_0) * \cdots * g_i(x, t - t_i) \qquad \text{for } t \in T_i$$

Space-time fractional diffusion of varying order

- stable parameter is determined by other parameters: $\alpha_i = \Omega\gamma_i$
- $\Omega = \frac{\gamma_i}{\alpha_i}$ remains constant as describes the scaling $g(x,t) = \frac{1}{t^\Omega} g\left(\frac{x}{t^\Omega}\right)$
- Estimation of Ω - scaling methods
    - Diffusion entropy analysis[3]:
      $S(t) = -\int g(x,t)\ln[g(x,t)]\mathrm{d}x = S(1) + \Omega \ln t$
    - Entropy production rate: $R(t) = \frac{\mathrm{d}S(t)}{\mathrm{d}t} = \frac{\Omega}{t}$
- Connection to regime-switching volatility models
  absolute moment for $\theta = \alpha - 2$ ($\beta = -1$) are

$$E[|x|^s] = \frac{1 + \csc\left(\frac{\pi s}{\Omega\gamma}\right)\sin\left(\pi s\left(1 - \frac{1}{\Omega\gamma}\right)\right)}{\gamma} \frac{\Gamma(s)}{\Gamma\left(\frac{\pi}{\Omega}\right)} \propto \frac{1}{\gamma}$$

[3]see e.g., P. Jizba, *J. K.*, Physica A 413, 348 (2014)

Diffusion in a temporally abnormal period

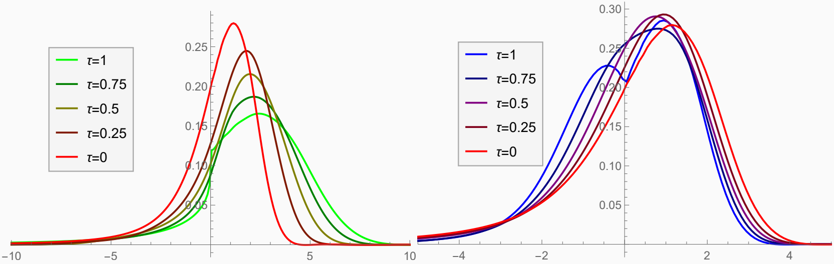
- Space-time fractional diffusion of varying order can be used for description of temporally abnormal period - e.g. crisis
- We distinguish two intervals
    - short-term behavior affected by immediate dynamics
    - long-term behavior characterized by scaling properties

  Described by $g(x,t)$ as overlap between space-time fractional diffusion ($t \leq \tau$) and Lévy flight ($t \to \infty$). Ω is the system-characterized scaling exponent

$$g_{\gamma,\theta,\tau,\sigma}(x,t) = \begin{cases} g^{\theta}_{\Omega\gamma,\gamma}(x/\sigma, t), & t \leq \tau, \\ \left[g^{\theta}_{\Omega\gamma,\gamma}(\tau) * L_{\Omega,\theta}(t-\tau)\right](x/\sigma), & t > \tau, \end{cases}$$

- In financial applications $\theta = \alpha - 2$
- Good approximation of models with more intervals - PDF converges to stable distribution

Green function of fractional diffusion of varying order



Green functions for $t = 1$, $\alpha = 1.6$, $\beta = -1$ and different τ for $\gamma = 0.9$ (left) and $\gamma = 1.1$ (right)

Applications of anomalous fractional diffusion to option pricing

Option pricing

- *option* is a special asset which gives to the owner the right (option) to buy (call) or sell (put) an underlying asset for specified strike price K.
- European options: the option can be exercised only at a certain maturity time T
- buyer - long position, seller - short position
- seller takes the risk of losses - this is compensated by the option price
- Price of a call option at *maturity time* ($t = T$):

$$C(S, K) = \max\{S - K, 0\}$$

(if $S < K$ we can directly buy the underlying asset for price S)

Option pricing

- Call option for $t < T$

$$C(S_t, K, t) = e^{-r(T-t)} E[C(S_T, T|S_t, t)]_{\mathbb{Q}} =$$
$$\int_{\mathbb{R}} \mathrm{d}y \;\; \max\left\{S_t e^{(t-T)(r+\mu)+y} - K, 0\right\} g(y, T-t)$$

- Put option $P(S_t, t) = C(S_t, t) - S_t + Ke^{-r(T-t)}$
- $g(y, \tau)$ is the probability distribution given by an appropriate stochastic model
- $\mathbb{Q}$ is the *equivalent risk-neutral measure* which is reflected by presence of μ in the option pricing formula
- μ can be calculated as

$$\mu = \ln \int e^x g(x, 1) \mathrm{d}x$$

the integral has to converge - only for $\theta = \alpha - 2$ - exponential decay

Comparison of fractional option pricing models

with H. Kleinert[6], Yu. Luchko[7]

- We fit the model with the option prices of S&P 500 in November 2008 ($\sim 10^5$ records)
- We minimize aggregated error over all available maturity times T and all strike prices K

$$AE = \sum_{t \in T, K \in \mathcal{K}} |\mathcal{O}_{model} - \mathcal{O}_{market}|$$

- We compare Black-Scholes[4], Lévy-stable[5], Double-fractional[6] and 2-period Varying order[7] model
- We do the analysis for all options and separately for call and put options

[4] F. Black, M. Scholes, J. Polit. Econ. 81(3), 1973

[5] P. Carr, L. Wu, J. Fin. 58(2), 2003

[6] H. Kleinert, *J. K.*, Physica A 449, 2016

[7] *J. K.*, Yu. Luchko, Frac. Calc. Apl. Anal. 19(6), 2016

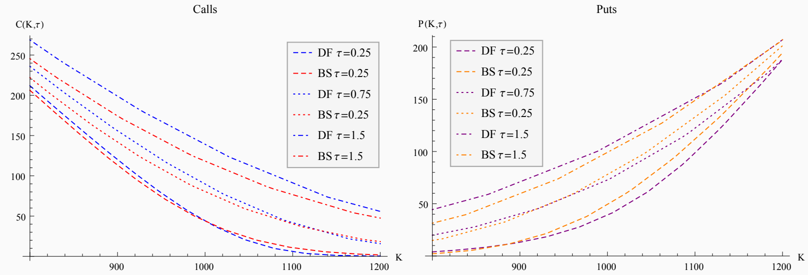
Model calibration for S&P 500 options traded in November 2008

All options				
par.	Black-Scholes	Lévy stable	Double-fractional	Varying order
σ	0.1696(0.027)	0.140(0.021)	0.143(0.030)	0.132(0.019)
α	-	1.493(0.028)	1.503(0.037)	$1.50 \cdot \gamma$
γ	-	-	1.017(0.019)	0.905(0.040)
τ	-	-	-	0.072(0.025)
AE	8240(638)	6994(545)	6931(553)	4794(584)

Call options				
par.	Black-Scholes	Lévy stable	Double-fractional	Varying order
σ	0.140(0.021)	0.118(0.026)	0.137(0.020)	0.079(0.017)
α	-	1.563(0.041)	1.585(0.038)	$1.50 \cdot \gamma$
γ	-	-	1.034(0.024)	0.809(0.016)
τ	-	-	-	0.118(0.067)
AE	3882(807)	3610(812)	3550(828)	1437(293)

Put options				
par.	Black-Scholes	Lévy stable	Double-fractional	Varying order
σ	0.193(0.039)	0.163(0.034)	0.163(0.037)	0.174(0.072)
α	-	1.493(0.031)	1.508(0.036)	$1.50 \cdot \gamma$
γ	-	-	1.047(0.017)	0.961(0.092)
τ	-	-	-	0.578(0.728)
AE	3741(711)	3114(591)	2968(594)	2161(466)

Estimated call and put option prices for various maturity times



Risk redistribution among strike prices and maturity times

Optimal hedging strategies

- the risk coming from selling an option can be eliminated by appropriate hedging strategy
- we create a portfolio $\Pi(S,t) = C(S,t) - \phi(S,t)S(t)$ containing a short of the option and a fraction $\phi(S,t)$ of the underlying asset $S(t)$ used to hedge the option.
- optimal strategy $\phi^*(S,t)$ can be expressed as

$$\phi^*(S,t) = \frac{1}{\sigma^2}\int_{\mathbb{R}} \mathrm{d}S(S_{t_0} - S_t)\,\mathrm{max}\{S_t - K, 0\}g(S, T|S_t, t)$$

Ongoing research and perspectives

Series formula for the fractional option pricing models

with J.-P. Aguilar and C. Coste

- Calculation of option prices driven by fractional diffusion requires knowledge of advanced mathematical concepts - stable distributions, Mellin calculus, etc.
- Alternatively it is possible to express the price though residue series
- we express payoff function as

$$[Se^{(r+\mu)\tau+y} - K]^+ = \frac{K}{2i\pi}\int_{c_s-i\infty}^{c_s+i\infty} -\frac{e^{-(r+\mu)\tau s-ys}}{s(s+1)}\left(\frac{S}{K}\right)^{-s} ds$$

- Together with Mellin-Barnes representation of $\exp(-\mu s\tau)$ it is possible to rewrite the option price as

$$C_{(\alpha,\gamma,\theta)}(S,K,\tau) = \frac{Ke^{-r\tau}}{\alpha}\frac{1}{(2i\pi)^2}\int_{\underline{c}+i\mathbb{R}^2} \omega_{\alpha,\gamma,\theta}(\underline{t})$$

where $\omega_{\alpha,\gamma,\theta}(\underline{t})$ is a complex 2-form.

Series formula for the fractional option pricing models

- It is possible to derive a residue formula

$$\frac{1}{(2i\pi)^2}\int_{\underline{c}+i\mathbb{R}^2}\omega(\underline{t}) = \sum_{z_k\in\Pi}\mathrm{Res}_{z_k}\omega$$

where Π is an appropriate cone in $\mathbb{C}^2$.
- Example: residue summation for totally asymmetric space-time fractional diffusion

$$C_\alpha(S,K,\tau) = \frac{1}{\alpha}\sum_{\substack{n\geq -1\\ m\geq 0}}\frac{2^{\frac{1+n}{\alpha}-m}(S-(-1)^m K e^{-r\tau})}{(1+n-m)!m!\Gamma(1-\frac{\gamma}{\alpha}(1+n))}[\log]^{1+n-m}\Sigma^{-1-n+\alpha m}\tau^{\frac{1-\gamma}{\alpha}(1+n)}$$

where $\tau = T-t$, $[\log] = \log\frac{S}{K}+r\tau$ and $\Sigma = \sigma(-\tau^\gamma\sec\frac{\pi\alpha}{2})^{1/\alpha}$
- For other models is the calculation analogous, but technically more complicated

Series formula for the fractional option pricing models

Convergence of the series for $S = K = 4000$, $\alpha = 1.9$, $\gamma = 1$, $\sigma = 0.25$, $\tau = 1$ year [8]

	-1	0	1	2	3	4	5
0	20.9477	0.6412	-0.0017	-0.0002	0.0000	0.0000	0.0000
1		466.1127	-2.5408	-0.3926	0.0030	0.0002	0.0000
2			-0.0231	-0.0071	0.0000	0.0000	0.0000
3				-1.7287	0.0390	0.0044	0.0000
4					0.0001	0.0000	0.0000
5						0.0058	0.0003
6							0.0000
Price	20.948	497.702	485.136	483.007	483.050	483.060	483.060

[8] taken from J.-P. Aguilar, C. Coste, Non-Gaussian analytic option pricing: a closed formula for the Lévy-stable model. arXiv:1609.00987.

Option pricing with arbitrary asymmetry θ

- So far, we have anticipated that all fractional models have extreme asymmetry - $\beta = -1 \Rightarrow \theta = \alpha - 2$
- Nevertheless, there are examples of assets with both positive and negative jumps - commodities, etc.
- Option pricing of such assets cannot be done within the classic scheme of risk-neutral measure
- one needs to generalize the option pricing scheme

Time-dependent fractional diffusion

- We have considered a special class of time-dependent fractional diffusion
- general time-dependent fractional diffusion

$$\left({}_{t_0}^{*}\mathcal{D}_t^{\gamma(t)} + \mu[{}^{\theta}\mathcal{D}_x^{\Omega\,\gamma(t)}]\right) g(x,t) = 0\,,\ \ \mu < 0 \tag{1}$$

time-dependent Caputo derivative

$$({}_{t_0}^{*}\mathcal{D}^{\gamma(t)} f)(t) = \frac{1}{\Gamma(\lceil\gamma(t)\rceil - \gamma(t))} \int_{t_0}^{t} \frac{f^{\lceil\gamma(t)\rceil}(s)}{(t-s)^{\gamma(t)+1-\lceil\gamma(t)\rceil}} \mathrm{d}s. \tag{2}$$

- Solution of this equation is very complicated and the techniques are not well developed

Pricing of exotic options

- European options is not the only type of options which is traded on financial markets
- Actually, there are options which are more popular
- American put option: the right to sell the underlying option any time from now to maturity
- There is an optimal excercise price $S_f(t)$
- Dynamics: the same - (generalized) Black-Scholes, but different boundary conditions[9]

$$V(S_f(t), t) = K - S_f$$

$$\frac{\partial V}{\partial S}(S_f(t), t) = -1 \qquad \text{continuity in prices}$$

- *Physics works with different potentials, options with different boundary conditions*

[9] S.-P. Zhu, Int. J. Theor. Appl. Fin. 9(7), 1141 (2006)

Conclusions

- Financial markets are a complex system with non-trivial phenomena - sudden jumps, seasonal changes, memory effects
- We have discussed several models based on fractional diffusion which can be used for description of these phenomena
- These models are particularly useful in option pricing
- Some of the properties (regime switching, memory,. . .) are even more general and can be used beyond the framework of fractional diffusion

Thank you for your attention.